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The Stability of Functional Equations with a New Direct Method

Dongwen Zhang [1], Qi Liu [1], John Michael Rassias [2] and Yongjin Li [1,*]

[1] Department of Mathematics, Sun Yat-Sen University, Guangzhou 510275, China; zhangdw25@mail2.sysu.edu.cn (D.Z.); liuq325@mail2.sysu.edu.cn (Q.L.)
[2] Department of Mathematics and Informatics, National and Kapodistrian University of Athens, 15342 Attikis, Greece; jrassias@primedu.uoa.gr
* Correspondence: stslyj@mail.sysu.edu.cn; Tel.:+86-13802930356

Abstract: We investigate the Hyers–Ulam stability of an equation involving a single variable of the form $\|f(x) - \alpha f(k^n(x)) - \beta f(k^{n+1}(x))\| \leqslant u(x)$ where f is an unknown operator from a nonempty set X into a Banach space Y, and it preserves the addition operation, besides other certain conditions. The theory is employed and stability theorems are proven for various functional equations involving several variables. By comparing this method with the available techniques, it was noticed that this method does not require any restriction on the parity, on the domain, and on the range of the function. Our findings suggest that it is very much easy and more appropriate to apply the proposed method while investigating the stability of functional equations, in particular for several variables.

Keywords: Hyers–Ulam stability; functional equations; approximation; the direct method; the convergence series

MSC: 39B52; 39B62; 39B82

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1. Introduction

In the past few decades, the Hyers–Ulam stability analysis of functional equations has attracted many researchers, and a number of research articles (of good quality) can be found in the topic; for instance, see [1–9] and the references cited therein. Being an emerging field for researchers, various methods (that is the direct method, the fixed point method, and so on) have been developed and applied to a variety of functional equations [10–13].

Generally, proving the stability results of functional equations by using the direct method requires one of the following two conditions: $\|f(x) - \frac{1}{a}f(ax)\| \leqslant u(x)$ or $\|f(x) - af(\frac{1}{a}x)\| \leqslant u(\frac{x}{a})$. Depending on certain assumptions, we need different distinctions to choose one approximate approach to solve some particular problems. The author in [14] studied the stability of a functional equation involving a single variable while imposing various conditions on the underlying constants. The results were then used to prove the stability of a pair of functional equations in several variables by using a direct technique. This work not only generalized and shortened the known available methods, but also improved the approximating constants to a great extent. Before studying the idea of [14], Forti in [15] presented a direct method for proving the Hyers–Ulam stability of functional equations. The latter literature is considered as a special case of the former article. In fact, these two research articles provided a solid background for the subject matter and opened different doors of research related to the field. These research papers can be used for obtaining stability results related to orthogonal additivity [10], the functional equation of Drygas [16], and the functional equation with the quadratic property [17] without repeating the same procedure.

It is important to mention that the proposed problem arises in various applications, and the same functional equation may be extended to a class of functions involving several variables. However, the available approaches cannot be applied directly to these functional equations, and thus, modifications of these methods are necessary. Keeping in view the importance of the problem, in Section 2, the authors try to solve another class of functional equations involving a single variable. We extend this approach, and various functional inequalities are constructed, which definitely helped us perform the stability analysis of a class of functional equations involving several variables.

The main theme of the present work is to analyze the stability of the solution of the functional equation form:

$$\|f(x) - \alpha f(k^n(x)) - \beta f(k^{n+1}(x))\| \leqslant u(x)$$

where $n = 1$. The solution of the problem was checked for preserving the addition operation, and we show that the solution satisfies a few identities from which the optimum conditions can be obtained. Based on our main results, the authors solved a number of functional equations, and to the best of our knowledge, these problems are novel and not previously attempted by researchers. The authors have great concern about solving more general forms of the inequalities in the near future. For the interest of the readers, the authors provide a problem in the last paragraph of Section 3.

2. Main Results

Theorem 1. *Assume that X is a nonempty set and $(Y, \|\cdot\|)$ is a Banach space. Suppose that the mapping $f : X \to Y$ satisfies the following inequality:*

$$\|f(x) - \alpha f(k(x)) - \beta f(k^2(x))\| \leqslant u(x), \quad x \in X, \tag{1}$$

where α and β are complex numbers, and the mapping $k : X \to X$, $u : X \to [0, \infty)$ is arbitrarily given, while the series $\sum_{n=0}^{\infty} |\alpha_n| u(k^n(x))$ with:

$$\alpha_0 := 1, \quad \alpha_n := \sum_{i=0}^{n} a^i b^{n-i}, \quad n \in \mathbb{N},$$
$$\beta_0 := 0, \quad \beta_n := \beta\alpha_{n-1}, \qquad n \in \mathbb{N}$$

where $ab = -\beta$, $b = \alpha - a$ (and where k^n denotes the $n-$th iterate of the mapping k), is convergent for all $x \in X$. Then, there exists a uniquely determined function $g : X \to Y$ such that:

$$g(x) = \alpha_n g(k^n(x)) + \beta_n g(k^{n+1}(x)), \tag{2}$$

and:

$$\|f(x) - g(x)\| \leqslant \sum_{i=0}^{\infty} |\alpha_i| u\left(k^i(x)\right) \quad x \in X, n \in \mathbb{N}. \tag{3}$$

Proof. The proof is divided into two steps. The first step of the proof is to study the existence of the limiting function by using an approximation argument. The second step of the proof is used to prove the property (2) of the limiting function $g(x)$ to prove the uniqueness of the limiting function. First of all, we made an attempt to prove the following inequality with $n > m$:

$$\left\| f(k^m(x)) - \alpha_n f(k^n(x)) - \beta_n f\left(k^{n+1}(x)\right) \right\| \leqslant \gamma_n(x), \tag{4}$$

where:

$$\gamma_n(x) := \sum_{i=m}^{n-1} |\alpha_i| u\left(k^i(x)\right), \quad x \in X, \ m, \ n \in \mathbb{N}.$$

Before studying of the inequality (4), due to it being related to certain parameters, then it is necessary to claim that, for any $m, n \in \mathbb{N}$:

$$\alpha_{n+m} = \alpha_m \alpha_n + \beta_m \alpha_{n-1}, \quad m,\ n \in \mathbb{N} \tag{5}$$

$$\alpha_{n+1} = \alpha\alpha_n + \beta_n, \quad \beta_{n+1} = \beta\alpha_n = \alpha\beta_n + \beta\beta_{n-1}. \tag{6}$$

Equation (6) is easily obtained by observation. Therefore, we straightforwardly prove Equation (5) by induction. If $m = 2$, then we have:

$$\alpha_{n+2} = \alpha^2\alpha_n + \alpha\beta\alpha_{n-1} + \beta\alpha_n = \alpha_n\alpha_2 + \beta_2\alpha_{n-1}.$$

If $m = 3$, then we achieve:

$$\begin{aligned}\alpha_{n+3} =& \alpha^3\alpha_n + \alpha^2\beta_n + \alpha\beta_{n+1} + \beta_{n+2} \\ =& \alpha_n(\alpha_3 + 2\beta\alpha) + (\alpha^2 + \beta)\beta_n \\ =& \alpha_n\alpha_3 + \alpha_{n-1}\beta_3.\end{aligned}$$

By induction, we can set that:

$$\alpha_{n+i} = \alpha_i\alpha_n + \alpha_{n-1}\beta_i$$

holds for some $i \in \mathbb{N}$. We prove that:

$$\begin{aligned}\alpha_{n+i+1} &= \alpha(\alpha_i\alpha_n + \alpha_{n-1}\beta_i) + \beta(\alpha_{i-1}\alpha_n + \alpha_{n-1}\beta_{i-1}) \\ &= \alpha_n(\alpha\alpha_i + \beta\alpha_{i-1}) + (\beta\beta_{i-1} + \alpha\beta_i)\alpha_{n-1} \\ &= \alpha_n\alpha_{i+1} + \beta\alpha_{n-1}(\beta_{i-1} + \alpha\alpha_{i-1}) \\ &= \alpha_n\alpha_{i+1} + \alpha_{n-1}\beta_{i+1}.\end{aligned}$$

The claim is proven. First, it is easy to see that (1) can give (4) for $n = 1$ with exchanging $k^m(x)$ for x, and by induction, later, we set that (4) holds for some $n \in N$. We prove that in the case for $n + 1$ by making use of (1) and (6):

$$\begin{aligned}&\|f(k^m(x)) - \alpha_{n+1}f\left(k^{n+1}(x)\right) - \beta_{n+1}f\left(k^{n+2}(x)\right)\| \\ &\quad\leqslant \left\|f(k^m(x)) - \alpha_n f(k^n(x)) - \beta_n f\left(k^{n+1}(x)\right)\right\| \\ &\qquad + \left\|\alpha_n f(k^n(x)) + \beta_n f\left(k^{n+1}(x)\right) - (\alpha\alpha_n + \beta_n)f\left(k^{n+1}(x)\right) - \beta_{n+1}f\left(k^{n+2}(x)\right)\right\| \\ &\quad\leqslant \sum_{i=m}^{n-1}|\alpha_i|u\left(k^i(x)\right) + |\alpha_n|\left\|f(k^n(x)) - \alpha f\left(k^{n+1}(x)\right) - \beta f\left(k^{n+2}(x)\right)\right\| \\ &\quad\leqslant \sum_{i=m}^{n-1}|\alpha_i|u\left(k^i(x)\right) + |\alpha_n|u(k^n(x)) \\ &\quad= \sum_{i=m}^{n}|\alpha_i|u\left(h^i(x)\right).\end{aligned}$$

Since the series $\sum_{i=0}^{\infty}|\alpha_i|u\left(h^i(x)\right)$ is convergent for every $x \in X$, combined with (4) and by virtue of the completeness of the space Y, the mapping can be well defined as:

$$g(x) := \lim_{n\to\infty}\left[\alpha_n f(k^n(x)) + \beta_n f\left(k^{n+1}(x)\right)\right], \quad x \in X. \quad (*)$$

Combined with (6), we prove the following properties of the mapping g:

$$\begin{aligned} g(x) &= \lim_{n\to\infty}\Big[\alpha_{n+1}f\Big(k^{n+1}(x)\Big)+\beta_{n+1}f\Big(k^{n+2}(x)\Big)\Big] \\ &= \lim_{n\to\infty}[(\alpha\alpha_n+\beta\alpha_{n-1})f\Big(k^{n+1}(x)\Big)+(\alpha\beta_n+\beta\beta_{n-1})f\Big(k^{n+2}(x)\Big)] \\ &= \lim_{n\to\infty}\Big[\alpha(\alpha_n f\Big(k^{n+1}(x)\Big)+\beta_n f\Big(k^{n+2}(x)\Big)) \\ &\quad +\beta(\alpha_{n-1}f\Big(k^{n+1}(x)\Big)+\beta_{n-1}f\Big(k^{n+2}(x)\Big))\Big] \\ &= \alpha g(k(x))+\beta g(k^2(x)). \end{aligned}$$

Furthermore, we prove the more general property of g:

$$g(x)=\alpha_n g(k^n(x))+\beta_n g\Big(k^{n+1}(x)\Big),\quad \textit{for all } x\in X \textit{ and } n\in\mathbb{N}. \tag{7}$$

In a similar way, by induction, we set that Equation (7) is true for all natural numbers k with $k \le n$ with some $n \in N$. Let us calculate it in the case for $k = n+1$:

$$\begin{aligned} &\alpha_{n+1}g\Big(k^{n+1}(x)\Big)+\beta_{n+1}g\Big(k^{n+2}(x)\Big)) \\ &=(\alpha\alpha_n+\beta\alpha_{n-1})g\Big(k^{n+1}(x)\Big)+(\alpha\beta_n+\beta\beta_{n-1})g\Big(k^{n+2}(x)\Big) \\ &=\alpha(\alpha_n g\Big(k^{n+1}(x)\Big)+\beta_n g\Big(k^{n+2}(x)\Big))+\beta(\alpha_{n-1}g\Big(k^{n+1}(x)\Big)+\beta_{n-1}g\Big(k^{n+2}(x)\Big)) \\ &=\alpha g(k(x))+\beta g\Big(k^2(x)\Big) \\ &=g(x). \end{aligned}$$

In the second step of the proof, we prove the uniqueness of the mapping g by using the above Equation (7). Let us suppose that $\widetilde{g}: X \to Y$ is another approximating mapping satisfying (2) and (3). Therefore, let us first prove the inequality together with Equation (6):

$$\begin{aligned} &\|f(k^m(x))-\alpha_m(\alpha_n f(k^{n+m}(x))+\beta_n f\Big(k^{m+n+1}(x)\Big)) \\ &-\beta_m(\alpha_{n-1}f(k^{n+m}(x))+\beta_{n-1}f\Big(k^{m+n+1}(x)\Big))\| \\ =&\|f(k^m(x))-(\alpha_m\alpha_n+\alpha_{n-1}\beta_m)f(k^{n+m}(x))-(\alpha_m\beta_n+\beta_m\beta_{n-1})f\Big(k^{n+m+1}(x)\Big)\| \\ =&\|f(k^m(x))-\alpha_{n+m}f(k^{n+m}(x))-\beta_{n+m}f\Big(k^{n+m+1}(x)\Big)\| \\ \leqslant&\sum_{i=m}^{m+n-1}|\alpha_i|u\Big(k^i(x)\Big), \end{aligned}$$

and letting $n \to \infty$, we have for any $m \in \mathbb{N}$:

$$\|f(k^m(x))-\alpha_m g(k^m(x))-\beta_m g\Big(k^{m+1}(x)\Big)\|\leqslant \sum_{i=m}^{\infty}|\alpha_i|u\Big(k^i(x)\Big),$$

then we can rewrite:

$$\begin{aligned} \|g(x)-\widetilde{g}(x)\| \leqslant&\|f(k^m(x))-\alpha_m g(k^m(x))-\beta_m g\Big(k^{m+1}(x)\Big)\| \\ &+\|f(k^m(x))-\alpha_m\widetilde{g}(k^m(x))-\beta_m\widetilde{g}\Big(k^{m+1}(x)\Big)\| \\ \leqslant&2\sum_{i=m}^{\infty}|\alpha_i|u(k^i(x)) \end{aligned}$$

for any $x \in X$ and $n \in \mathbb{N}$, which yields $g = \widetilde{g}$ in X as $m \to \infty$ by the convergence series $\sum_{n=0}^{\infty} |\alpha_n| u(k^n(x))$.

Finally, we can assume $\alpha_{n+1} - a\alpha_n = b(\alpha_n - a\alpha_{n-1})$ with $a + b = \alpha,\ -ab = \beta$ to prove the form of the expression of the sequence (α_n). Thus, we achieve:

$$\alpha_{n+1} - a\alpha_n = b^n(\alpha_1 - a\alpha_0) = b^{n+1},$$

and:

$$\alpha_n = a^{n-1}\alpha + a^{n-2}b + \cdots + ab^{n-1} + b^n = \sum_{i=0}^{n} a^i b^{n-i}.$$

This completes the proof. □

Remark 1. *In contrast to the paper [14,15], the parameters α, β in Equation (1) can be relaxed to the complex field C, that is $\alpha, \beta \in C$, and the mediate function $k(x)$ does not need to be even or odd (the mediate function in [14] is $h(x)$, which is even, and $\beta = 0$ in [15]). In fact, the limiting function $g(x)$ is even if the mediate function is even. Conversely, the limiting function is not odd even if the mediate function is odd. In a similar way as the proof, we give the following corollary to prove that the sequences (α_n) and (β_n) in the literature have similar properties to the corresponding sequences in [14]. At the same time, we present two other functional equations for application. See the following:*

$$\|f(x) - \alpha f(k^m(x)) - \beta f(k^{2m}(x))\| \leqslant u(x), \quad x \in X, \tag{8}$$

$$\|f(x) - \alpha f(\sqrt{k}(x)) - \beta f(\sqrt{k}(x))\| \leqslant u(x), \quad x \in X. \tag{9}$$

We skip their proof, leaving it to the reader. On the other hand, it is worth noting that the system of $a + b = \alpha, ab = -\beta$ is seemingly always led to two different series $\sum_{j=0}^{\infty} |\alpha_j| u(k^j(x))$ (see the application in Section 3). In fact, two different series may not always appear in the general case since the sequence (α_n) is symmetric with respect to the parameters a and b. Now, we construct the states α_2, α_3. and α_4 in the following, which are to improve the evaluative effects of the sequence (α_n):

$$\alpha_2 = \alpha^2 + \beta = (a+b)^2 - ab = a^2 + ab + b^2,$$

$$\alpha_3 = \alpha^3 + 2\alpha\beta = (a+b)((a+b)^2 - 2ab) = a^3 + a^2b + ab^2 + b^3,$$

and:

$$\alpha_4 = \alpha^4 + 3\alpha^2\beta + \beta^2 = (a+b)^2((a+b)^2 - 3ab) + a^2b^2 = a^4 + a^3b + a^2b^2 + ab^3 + b^4.$$

Remark 2. *As a matter of the fact, our first result can also be formulated in the β-homogeneous F-space, the sequentially complete linear topological space, and so on. For simplicity, we do not repeat them one by one. If $\alpha = 1, \beta = 0$ and $k(x) = x$, the result may become trivial and meaningless. Therefore, we may suppose that α is an element in C different from $0, 1$ in Theorem 1. In contrast with the paper [14], we may address this similar method to solve our model, which is stated in the following.*

Corollary 1. *Assume that X is a nonempty set and $(Y, \|\cdot\|)$ is a Banach space. Suppose that the mapping $f : X \to Y$ satisfies the following inequality:*

$$\|f(x) - \alpha f(k(x)) - \beta f(k^2(x))\| \leqslant u(x), \quad x \in X, \tag{10}$$

where α and β are real constants, and the mapping $k : X \to X,\ \ u : X \to [0, \infty)$ is arbitrarily given, while the series $\sum_{n=0}^{\infty} |\alpha_n| u(k^n(x))$ with:

$$\alpha_0 := 1, \quad \alpha_n := \sum_{i=0}^{n} a^i b^{n-i}, \quad n \in \mathbb{N},$$

$$\beta_0 := 0, \quad \beta_n := \beta\alpha_{n-1}, \qquad n \in \mathbb{N}$$

where $ab = -\beta$, $b = \alpha - a$ (and where k^n denotes the $n-th$ iterate of the mapping k), is convergent for all $x \in X$. Then, there exists a uniquely determined function $g : X \to Y$ such that:

$$g(x) = \alpha_n g(k^n(x)) + \beta_n g(k^{n+1}(x)), \tag{11}$$

and:

$$\|f(x) - g(x)\| \leqslant \sum_{i=0}^{\infty} |\alpha_i| u\left(k^i(x)\right) \quad x \in X, n \in \mathbb{N}. \tag{12}$$

Proof. We skip the same part of the proof of Theorem 1. From the definition of sequences (α_n) and (β_n), we also would like to have:

$$\alpha_m \alpha_n \beta_m \alpha_{n-1} \geq 0.$$

Then:

$$g(x) - \widetilde{g}(x) = \alpha_n[g(k^n(x)) - \widetilde{g}(k^n(x))] + \beta_n\left[g\left(k^{n+1}(x)\right) - \widetilde{g}\left(k^{n+1}(x)\right)\right], \quad x \in X,$$

and on account of (5) and (6), we can rewrite:

$$\begin{aligned}
\|g(x) - \widetilde{g}(x)\| &\leqslant 2|\alpha_n| \sum_{i=0}^{\infty} |\alpha_i| u\left(k^{i+n}(x)\right) + 2|\beta_n| \sum_{i=0}^{\infty} |\alpha_i| u\left(k^{i+n+1}(x)\right) \\
&= 2\sum_{i=1}^{\infty} [(|\alpha_i \alpha_n| + |\alpha_{i-1}\beta_n|) u\left(k^{i+n}(x)\right) + 2|\alpha_n| u(k^n(x))] \\
&= 2\sum_{i=1}^{\infty} [(|\alpha_i \alpha_n + \alpha_{i-1}\beta_n|) u\left(k^{i+n}(x)\right) + 2|\alpha_n| u(k^n(x))] \\
&= 2\sum_{j=n}^{\infty} |\alpha_j| u\left(k^j(x)\right)
\end{aligned}$$

for any $x \in X$ and $n \in \mathbb{N}$, which yields $g = \widetilde{g}$ in X as $n \to \infty$. This completes the proof. □

An extended conclusion can be stated in the following, which discusses several cases in detail in Theorem 1.

Theorem 2. *Assume that X is a nonempty set and $(Y, \|\cdot\|)$ is a Banach space. Suppose that the mapping $f : X \to Y$ satisfies the conditions of Theorem 1. Then, there are several cases to discuss:*

(1) If the condition $\alpha = 0$ or $\beta = 0$ holds true, then there exist results that were stated in [15] and also a special case in Theorem 1, which will not be repeated;

(2) If the condition is either the situation $\alpha = \beta = 1$ and $k(x) = x$ or $k(x) = -x$ or the situation $\alpha = \beta = 0$ with an arbitrary function k established, then there is an exactly unique trivial solution of Equation (3): If the other cases are establish, then there exist results stated in Theorem 1, which will not be repeated.

3. Applications

An F-space is called β-homogeneous if $\|tx\| = t^\beta \|x\|$ for all $x \in X$ and all $t \in C$ (see the definition in [18]). In this section of Theorem 3, β_1, β_2 are considered as positive real numbers with $\beta_1 \leq 1$ and $\beta_2 \leq 1$. Furthermore, X is supposed as a β_1-homogeneous F-space, while Y is assumed as a β_2-homogeneous F-space. In fact, there was also a similar solution of the functional inequality in the Banach space in [14]. The following functional inequality was originally derived from the inner product space [19,20].

Theorem 3. *Let $f : X \to Y$ be a function such that, for some $K \geq 0$ and $r < \frac{\beta_2}{\beta_1}$:*

$$\begin{aligned} &\|f(x+y+z) + f(x) + f(y) + f(z) - f(x+y) - f(z+y) - f(x+z)\| \\ &\leqslant K(\|x\|^r + \|y\|^r + \|z\|^r) \end{aligned} \tag{13}$$

for all $x, y, z \in X$. Then, there exists a unique mapping $\psi_1 : X \to Y$ such that:

$$\|f(x) - \psi_1(x)\| \leqslant \frac{(2 + 2^{r\beta_1} + 3 \cdot 2^{\beta_2})2^{\beta_1 r}K}{(2^{\beta_2} - 2^{\beta_1 r})8^{\beta_2}}\|x\|^r$$

for all $x \in X$. Moreover, ψ_1 is the unique solution of the following equation satisfying the above equality:

$$\psi_1(x+y+z) + \psi_1(x) + \psi_1(z) + \psi_1(y) = \psi_1(x+y) + \psi_1(z+y) + \psi_1(x+z)$$

for all $x, y, z \in X$.

Proof. From (x, x, x) in (13), we have:

$$\|f(3x) + 3f(x) - 3f(2x)\| \leqslant 3K(\|x\|^r),$$

and then:

$$\|2f(3x) + 6f(x) - 6f(2x)\| \leqslant 3 \cdot 2^{\beta_2}K(\|x\|^r). \tag{14}$$

From $(x, x, 2x)$ in (13), we have:

$$\|f(4x) + 2f(x) - 2f(3x)\| \leqslant (2 + 2^{r\beta_1})K(\|x\|^r). \tag{15}$$

Combining (14) and (15), we obtain:

$$\|f(x) - \frac{3}{4}f(2x) + \frac{1}{8}f(4x)\| \leqslant (2 + 2^{r\beta_1} + 3 \cdot 2^{\beta_2})K\|x\|^r/8^{\beta_2}.$$

Let $u(x) = K_1\|x\|^r = (2 + 2^{r\beta_1} + 3 \cdot 2^{\beta_2})K\|x\|^r/8^{\beta_2}$. We apply the results obtained in Section 2 with $\alpha = \frac{3}{4}$, $\beta = \frac{-1}{8}$ and $k(x) = 2x$. We can easily compute that either $(a, b) = (\frac{1}{4}, \frac{1}{2})$ or $(a, b) = (\frac{1}{2}, \frac{1}{4})$. Thus:

$$\alpha_n = [\sum_{i=0}^{n} (\frac{1}{4})^i (\frac{1}{2})^{n-i}]^{\beta_2} = [\frac{1}{2^n}(1 - \frac{1}{2^n})]^{\beta_2}.$$

Therefore:

$$\sum_{n=0}^{\infty} |\alpha_n| u(2^n x) = \sum_{n=0}^{\infty} [\frac{1}{2^n}(1 - \frac{1}{2^n})]^{\beta_2} 2^{\beta_1 r n} K_1\|x\|^r \leq \frac{2^{\beta_2}}{2^{\beta_2} - 2^{\beta_1 r}} K_1\|x\|^r.$$

Therefore, the series $\sum_{i=0}^{\infty} |\alpha_n| u(k^n x)$ is convergent for all $x \in X$.

Related to the above results, there exists a uniquely determined mapping $\psi_1 = g : X \to Y$ satisfying (2) and (3).

In order to show that g satisfies the last equation of Theorem 3, it is easy to see that:

$$\begin{aligned}
&\Big\|\alpha_n\Big[f(2^n(x+y+z))+f(2^nx)+f(2^ny)+f(2^nz)\\
&-f(2^n(x+y))-f(2^n(z+y))-f(2^n(x+z))\Big]\\
&+\beta_n\Big[f(2^{n+1}(x+y+z))+f(2^{n+1}x)+f(2^{n+1}y)+f(2^{n+1}z)\\
&-f(2^{n+1}(x+y))-f(2^{n+1}(z+y))-f(2^{n+1}(x+z))\Big]\Big\|\\
\leqslant&(|\alpha_n|+|\beta_n|)K(\|x\|^r+\|y\|^r+\|z\|^r)
\end{aligned}$$

which tends to zero as $n\to\infty$ by the convergence series $\sum_{i=0}^{\infty}|\alpha_n|u(k^i(x))$, and this completes the proof. □

In Theorem 3, the problem can also be derived in the Banach space with $\alpha_n = \frac{1}{2^n}(1-\frac{1}{2^n})$ and $\sum_{i=0}^{\infty}|\alpha_n|u(k^nx) = \varepsilon$. For the Euler–Lagrange equation, we provide another approach to prove its stability in comparison to that from [11] in completing the stability of the title functional equation.

Theorem 4. *Assume that $(X,+)$ is a group and $(Y,\|\cdot\|)$ is a Banach space. Suppose that the mapping $f: X\to Y$ such that for all $x,y,z\in X$ and some $\varepsilon>0$:*

$$\|f(x+y+z)+f(x-y+z)+f(x+y-z)+f(x-y-z)-4f(x)-4f(y)-4f(z)\|\leqslant\varepsilon. \tag{16}$$

Then, there exists a uniquely determined function $g: X\to Y$ such that:

$$g(x)=\frac{1}{8}g(2x)-\frac{1}{32}g(4x)-\frac{5}{21}f(0),\quad x\in X$$

and:

$$\|f(x)-g(x)\|\leqslant\frac{4\varepsilon}{21}\quad x\in X.$$

In particular, if X is commutative, then g is also the solution of the following equality:

$$g(x+y+z)+g(x-y+z)+g(x+y-z)+g(x-y-z)=4g(x)+4g(y)+4g(y),$$

for all $x,y\in X$.

Proof. We show that $(x,x,-2x)$ in (16) and obtain:

$$\|f(4x)-3f(-2x)+f(2x)-8f(x)+f(0)\|\leqslant\varepsilon,\quad x\in X. \tag{17}$$

From $(0,x,x)$ in (16), we have:

$$\|f(2x)-2f(0)+f(-2x)-8f(x)\|\leqslant\varepsilon,\quad x\in X. \tag{18}$$

Consequently, (17) and (18) yield that:

$$\|f(4x)+4f(2x)-32f(x)-5f(0)\|\leqslant4\varepsilon,\quad x\in X.$$

Without loss of generality, we may assume that $f(0) = 0$. Let $u(x) = \frac{\varepsilon}{8}$. We apply the results obtained in Section 2 with $\alpha = \frac{1}{8}$, $\beta = \frac{1}{32}$, and $k(x) = 2x$. We can easily compute that either $(a, b) = (\frac{-1}{8}, \frac{1}{4})$ or $(a, b) = (\frac{1}{4}, \frac{-1}{8})$. Thus:

$$\alpha_n = \sum_{i=0}^{n} (\frac{-1}{8})^i (\frac{1}{4})^{n-i} = \frac{2}{4^n}(1 + \frac{(-1)^n}{2^{n+1}}).$$

Therefore,

$$\sum_{n=0}^{\infty} |\alpha_n| u(2^n x) = \frac{4\varepsilon}{21}.$$

Therefore, the series $\sum_{i=0}^{\infty} |\alpha_n| u(k^n x)$ is convergent for all $x \in X$.

It remains to interpret that if X is commutative, then for all $x, y, z \in X$ and $n \in N$, we have by (2):

$$\begin{aligned}
&\|\alpha_n[f(2^n(x+y+z)) + f(2^n(x-y+z)) + f(2^n(x+y-z)) \\
&+ f(2^n(x-y-z)) - 4f(2^n x) - 4f(2^n y) - 4f(2^n z)] \\
&+ \beta_n[f(2^{n+1}(x+y+z)) + f(2^{n+1}(x-y+z)) + f(2^{n+1}(x+y-z)) + \\
&f(2^{n+1}(x-y-z)) - 4f(2^{n+1}x) - 4f(2^{n+1}y) - 4f(2^{n+1}z)]\| \\
&\leqslant (|\alpha_n| + |\beta_n|)\varepsilon
\end{aligned}$$

which tends to zero as $n \to \infty$ by the convergence series $\sum_{i=0}^{\infty} |\alpha_i| u(k^i(x))$ and the relation between (α_n) and (β_n). According to the commutativeness of X, we prove that the last equation of Theorem 4 holds for all $x \in X$. This completes the proof. □

In fact, the stability problem for the equation can also be solved in the $\beta-$homogeneous space and the more direct method derived in [11]. The Drygas equation has been solved by many papers in [16], and also, an interesting property of the equation is that it is symmetric to y. Applying the property, we solve the following stability problem.

Theorem 5. *Assume that $(X, +)$ is a group and $(Y, \|\cdot\|)$ is a Banach space. Suppose that the mapping $f : X \to Y$ such that for all $x, y \in X$ and some $\varepsilon > 0$:*

$$\|f(x+y) + f(x-y) - 2f(x) - f(y) - f(-y)\| \leqslant \varepsilon, \quad x, y \in X. \tag{19}$$

Then, there exists a unique mapping $g : X \to Y$ such that:

$$g(x) = \frac{3}{4}g(2x) - \frac{1}{8}g(4x), \quad x \in X$$

and:

$$\|f(x) - g(x)\| \leqslant 2\varepsilon \quad x \in X.$$

In particular, if X is commutative, then g also satisfies:

$$g(x+y) + g(x-y) = 2g(x) + g(y) + g(-y), \quad x, y \in X. \tag{20}$$

Proof. Without loss of generality, we may assume that $f(0) = 0$. Substitute in the sequel (x, x) in (19) in order to obtain:

$$\|f(2x) + f(0) - 3f(x) - f(-x)\| \leqslant \varepsilon, \quad x \in X. \tag{21}$$

From $(3x, x)$ in (19), we have:

$$\|f(4x) + f(2x) - 2f(3x) - f(x) - f(-x)\| \leqslant \varepsilon, \quad x \in X. \tag{22}$$

Consequently, (21) and (22) yield that:

$$\|f(4x) + 2f(x) - 2f(3x)\| \leqslant 2\varepsilon, \quad x \in X. \tag{23}$$

From $(2x, x)$ in (19), we have:

$$\|f(3x) - 2f(2x) - f(-x)\| \leqslant \varepsilon, \quad x \in X. \tag{24}$$

Combined with the inequality (21), we have:

$$\|f(3x) - 3f(2x) + 3f(x)\| \leqslant 2\varepsilon, \quad x \in X, \tag{25}$$

and this inequality together with (23) give:

$$\|f(4x) - 6f(2x) + 8f(x)\| \leqslant 6\varepsilon, \quad x \in X. \tag{26}$$

The rest of the proof is the same as in the previous Theorem 3 if $\beta_1 = \beta_2 = 1$ and $u(x) = \varepsilon$. $\square$

Theorem 6. *Assume that $(X, +)$ is a unique two-divisible group and $(Y, \|\cdot\|)$ is a Banach space. Suppose that the mapping $f : X \to Y$ satisfies the condition:*

$$\|f(x + dy) + df(x - y) - f(x - dy) - df(x + y)\| \leqslant \varepsilon, \quad x, y \in X, \tag{27}$$

where d is an integer with $d > 1$ and ε is a nonnegative constant. Then, there exists a uniquely determined function $g : X \to Y$ satisfying (2) and such that:

$$\|f(x) - g(x)\| \leqslant \frac{3d^2 + 6d + 1}{2(1 - d^2)(1 + d)}\varepsilon, \quad x \in X. \tag{28}$$

In particular, if X is commutative, then g satisfies:

$$g(x + dy) + dg(x - y) = g(x - dy) + dg(x + y), \quad x, y \in X.$$

Proof. Observe first that without loss of generality, we may assume that $f(0) = 0$. Currently, substitute $(-x, -x)$ and $(-dx, x)$ in the place of (x, y) in (27), in order to have:

$$\|f((d-1)x) + df(-2x) - f(-(d+1)x)\| \leqslant \varepsilon, \quad x \in X,$$
$$\|af(-(d+1)x) - f(-2dx) - df(-(d-1)x)\| \leqslant \varepsilon, \quad x \in X.$$

From the above two inequalities, this leads to:

$$\left\|df((d-1)x) + d^2 f(-2x) - f(-2dx) - df(-(d-1)x)\right\| \leqslant (1 + d)\varepsilon, \quad x \in X,$$

and this inequality together with the one with x changed for $-x$ can obtain:

$$\left\|d^2 f(2x) + d^2 f(-2x) - f(2dx) - f(-2dx)\right\| \leqslant 2(1 + d)\varepsilon, \quad x \in X.$$

By virtue of the unique two-divisibility of X, we have:

$$\left\|d^2 f(x) + d^2 f(-x) - f(dx) - f(-dx)\right\| \leqslant 2(1 + d)\varepsilon, \quad x \in X.$$

Substitute $(0, x)$ in (27) in order to know:

$$\|f(dx) + df(-x) - f(-dx) - df(x)\| \leqslant \varepsilon, \quad x \in X.$$

The above equation multiplied by d together with the above equation yield that:

$$\left\|2d^2f(x)-(d+1)f(dx)+(d-1)f(-dx)\right\|\leqslant(2+3d)\varepsilon,\quad x\in X. \tag{29}$$

We exchange dx for x in (29):

$$\left\|2d^2f(dx)-(d+1)f(d^2x)+(d-1)f(-d^2x)\right\|\leqslant(2+3d)\varepsilon,\quad x\in X. \tag{30}$$

From $(0,-dx)$ in (27), we have:

$$\|f(-d^2x)+df(dx)-f(d^2x)-df(-dx)\|\leqslant\varepsilon,\quad x,y\in X, \tag{31}$$

and by virtue of (29), (30), and (31), we can obtain:

$$\|\frac{2d^3}{d-1}f(x)-\frac{2d^2+2d}{d-1}f(dx)+\frac{2}{d-1}f(d^2x)\|\leqslant\frac{3d^2+6d+1}{d-1}\varepsilon,\quad x,y\in X. \tag{32}$$

Let $u(x)=\frac{3d^2+6d+1}{2d^3}\varepsilon$. We apply the results obtained in Section 2 with $\alpha=\frac{1+d}{d^2}$, $\beta=\frac{1}{d^3}$ and $k(x)=dx$. We can easily compute that either $(a,b)=(\frac{1}{d},\frac{1}{d^2})$ or $(a,b)=(\frac{1}{d^2},\frac{1}{d})$. Thus:

$$\alpha_n=\sum_{i=0}^{n}(\frac{1}{d})^i(\frac{1}{d^2})^{n-i}=\frac{1}{d^{2n}}\frac{1-d^{n+1}}{1-d}.$$

Therefore:

$$\sum_{n=0}^{\infty}|\alpha_n|u(2^nx)=\frac{3d^2+6d+1}{2(1-d^2)(1-d)}\varepsilon.$$

Therefore, the series $\sum_{i=0}^{\infty}|\alpha_n|u(k^nx)$ is convergent for all $x\in X$.

It remains to show that if X is commutative, then g satisfies the last equation of Theorem 6. By (27) applied for $(d^nx,d^{n+1}y)$ and $(d^{n+1}x,d^{n+2}y)$, for all $x,y\in X$ and $n\in\mathbb{N}$, we can obtain:

$$\begin{aligned}
&\|\alpha_n[f\left(d^nx+d^{n+1}y\right)+df(d^nx-d^ny)-f\left(d^nx-d^{n+1}y\right)\\
&\quad-df(d^nx+d^ny)]-\beta_n[f\left(d^{n+1}x+d^{n+2}y\right)+df\left(d^{n+1}x-d^{n+1}y\right)\\
&\quad-f\left(d^{n+1}x-d^{n+2}y\right)-df\left(d^{n+1}x+d^{n+1}y\right)]\|\\
&\leqslant(\alpha_n+\beta_n)\varepsilon.
\end{aligned}$$

Letting n tend to infinity, by the commutativity of X and the convergence sequences (α_n) and (β_n), we prove the last equation of Theorem 6 holds true. □

In Equation (27), the stability for the problem can be derived in [14,21]. In particular, $d=3$ in Theorem 6 can be presented in the following as a corollary.

Corollary 2. *Assume that $(X,+)$ is a unique two-divisible group and $(Y,\|\cdot\|)$ is a Banach space. Suppose that the mapping $f:X\to Y$ satisfies the condition:*

$$\|f(x+3y)+3f(x-y)-f(x-3y)-3f(x+y)\|\leqslant\varepsilon,\quad x,y\in X$$

where $\varepsilon>0$. Then, there exists a uniquely determined function $g:X\to Y$ such that:

$$\|f(x)-g(x)\|\leqslant\frac{23}{16}\varepsilon,\quad x\in X.$$

In particular, if X is commutative, then g satisfies:

$$g(x+3y)+3g(x-y)=g(x-3y)+3g(x+y), \quad x,y\in X.$$

Equation $9f\left(\frac{x+y+z}{3}\right)+f(x)+f(y)+f(z)=4\left[f\left(\frac{x+y}{2}\right)+f\left(\frac{y+z}{2}\right)+f\left(\frac{z+x}{2}\right)\right]$ *was treated, as well as the solutions of it were given in [14].*

Theorem 7. *Assume that* $(X,+)$ *is a group uniquely divisible by two and by three, and let* $(Y,\|\cdot\|)$ *be a Banach space. Given an* $\varepsilon\geqslant 0$*, assume that* $f:X\to Y$ *satisfies for all* $x,y,z\in X$ *the condition:*

$$\left\|9f\left(\frac{x+y+z}{3}\right)+f(x)+f(y)+f(z)-4\left[f\left(\frac{x+y}{2}\right)+f\left(\frac{y+z}{2}\right)+f\left(\frac{z+x}{2}\right)\right]\right\|\leqslant\varepsilon. \tag{33}$$

Then, there exists a uniquely determined function $g:X\to Y$ *satisfying (2) and such that:*

$$\|f(x)-g(x)\|\leqslant\varepsilon, \quad x\in X.$$

In particular, if X is commutative, then g satisfies:

$$9g\left(\frac{x+y+z}{3}\right)+g(x)+g(y)+g(z)=4\left[g\left(\frac{x+y}{2}\right)+g\left(\frac{y+z}{2}\right)+g\left(\frac{z+x}{2}\right)\right]$$

for all $x,y,z\in X$*.*

Proof. Substitute $(6x,0,0)$, $(6x,6x,0)$, $(0,2x,4x)$ in the place of (x,y,z), respectively, in (33) in order to obtain:

$$\|9f(2x)+f(6x)-2f(0)-8f(3x)\|\leqslant\varepsilon, \quad x\in X,$$

$$\|9f(4x)-2f(6x)+f(0)-8f(3x)\|\leqslant\varepsilon, \quad x\in X,$$

$$\|6f(2x)+f(4x)+f(0)-4f(3x)-4f(x)\|\leqslant\varepsilon, \quad x\in X.$$

Hence:

$$\|f(x)-\frac{3}{4}f(2x)+\frac{1}{8}f(4x)+\frac{3}{8}f(0)\|\leqslant\frac{3}{8}\varepsilon, \quad x\in X.$$

We do not repeat the calculation procedures, which are similar to the proof in Theorem 3. It remains to show that if X is commutative, then g satisfies the last equation. By (33) applied for $(2^n x, 2^{n+1}y)$ and $(2^{n+1}x, 2^{n+2}y)$, for all $x,y\in X$ and $n\in\mathbb{N}$, we can obtain:

$$\begin{aligned}
&\Bigg\|\alpha_n\left\{9f\left(\frac{2^n(x+y+z)}{3}\right)+f(2^n x)+f(2^n y)+f(2^n z)\right\}\\
&\quad-4\alpha_n\left\{f\left(\frac{2^n(x+y)}{2}\right)+f\left(\frac{2^n(y+z)}{2}\right)+f\left(\frac{2^n(z+x)}{2}\right)\right\}\\
&\quad-\beta_n\left\{9f\left(\frac{2^{n+1}(x+y+z)}{3}\right)+f(2^{n+1}x)+f(2^{n+1}y)+f(2^{n+1}z)\right\}\\
&\quad-4\beta_n\left\{f\left(\frac{2^{n+1}(x+y)}{2}\right)+f\left(\frac{2^{n+1}(y+z)}{2}\right)+f\left(\frac{2^{n+1}(z+x)}{2}\right)\right\}\Bigg\|\\
&\leqslant(\alpha_n+\beta_n)\varepsilon\\
&=\left(\frac{1}{2^n}(1-\frac{1}{2^n})-\frac{1}{8}\cdot\frac{1}{2^{n-1}}(1-\frac{1}{2^{n-1}})\right)\varepsilon.
\end{aligned}$$

Letting n tend to infinity, by the commutativity of X and the convergence sequences (α_n) and (β_n), we prove the last equation holds true. □

In the future, the functional inequality:

$$\|f(x) - \alpha f(k^n(x)) - \beta f(k^{n+1}(x))\| \leqslant u(x)$$

for $x \in X$ and some $n \in N$ with $n > 1$ will be further explored. In fact, the more general forms:

$$\|f(x) - \alpha f(k^n(x)) - \beta f(k^m(x)) - \gamma f(k^l(x))\| \leqslant u(x)$$

may be discussed to solve the stability problem for functional equations in several variables. Finally, a useful example of the quartic functional equation is stated:

$$\|f(x+2y) + f(x-2y) + 6f(x) - 4[f(x+y) + f(x-y) + 6f(y)\| \leq \varepsilon$$

for all $x, y \in X$. Substitute $(2x, x)$, (x, x), $(x, -x)$ in the above equation to obtain:

$$\|f(4x) - 4f(3x) + 6f(x) + 6f(2x) - 28f(x)\| \leq \varepsilon, \tag{34}$$

$$\|\|f(3x) - 4f(2x) + f(-x) - 18f(x)\| \leq \varepsilon, \tag{35}$$

$$\|f(3x) - 23f(-x) - 4f(2x) - 6f(x)\| \leq \varepsilon \tag{36}$$

an obvious fact that eliminates $f(-x)$ combining (35) and (36) and also later eliminates $f(3x)$. Consequently, we can solve the functional inequality:

$$\|F(f(4x), f(2x), f(x))\| \leq m\varepsilon$$

for some integer $m \in N$ and $x \in X$ by using Theorem 1. In parallel with this method, we eliminate $f(-3x)$ together with (35) and (36), and replacing $2x$ by x, we obtain functional inequality:

$$\|F(f(x), f(\frac{x}{2}), f(\frac{x}{2}))\| \leq n\varepsilon$$

for some integer $n \in N$ and $x \in X$ and solving this by the method in [14]. We listed it as follows:

1. In Theorem 3, we achieve that the approximating constant is ε for $\beta_1 = \beta_2 = 1$. However, the approximating constant was 3ε in [1,17] and 2ε in [14];

2. In Theorem 4, the approximating constant is more than the approximating constant obtained in [11];

3. In Theorem 5, we achieve that the approximating constant is 2ε. However, the approximating constant was 3ε in [1,16,22], and also, there was ε in [14];

4. In Theorem 6, we achieve that the approximating constant is $\frac{3d^2+6d+1}{2(1-d^2)(1+d)}\varepsilon$, and there was $\frac{2+3d}{2d(d-1)}\varepsilon$ in [14]. Furthermore, the more concrete approximating constant were presented as $\frac{4}{3}\varepsilon$ in [23], $\frac{11}{16}\varepsilon$ in [14], as well as $\frac{23}{16}\varepsilon$ in Corollary 2 in our work;

5. The approximating constant is $\frac{1}{6}\varepsilon$ in [14] and ε in Theorem 7 in the literature and $\frac{4}{27}\varepsilon$ in [13,24].

To summarize, comparing these five application results for the approximating constant, it is hard to say which method achieves the best approximating constant in the theory of functional equations. Now, this may also be considered as an open problem for this research field.

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