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Article

Effects of the Wiener Process on the Solutions of the Stochastic Fractional Zakharov System

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Abstract: We consider in this article the stochastic fractional Zakharov system derived by the multiplicative Wiener process in the Stratonovich sense. We utilize two distinct methods, the Riccati–Bernoulli sub-ODE method and Jacobi elliptic function method, to obtain new rational, trigonometric, hyperbolic, and elliptic stochastic solutions. The acquired solutions are helpful in explaining certain fascinating physical phenomena due to the importance of the Zakharov system in the theory of turbulence for plasma waves. In order to show the influence of the multiplicative Wiener process on the exact solutions of the Zakharov system, we employ the MATLAB tools to plot our figures to introduce a number of 2D and 3D graphs. We establish that the multiplicative Wiener process stabilizes the solutions of the Zakharov system around zero.

Keywords: fractional Zakharov system; stochastic Zakharov system; Riccati–Bernoulli sub-ODE method; Jacobi elliptic function method

MSC: 60H15; 60H10; 35A20; 83C15; 35Q51

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1. Introduction

In 1972, Zakharov [1] developed the Zakharov system. It is a group of coupled nonlinear wave equations that explains the interaction of high-frequency Langmuir (dispersive) and low-frequency ion-acoustic (roughly nondispersive) waves. In one dimension, the Zakharov system can be authored as

$$\begin{aligned} v_{tt} - v_{xx} + (|u|^2)_{xx} &= 0, \\ iu_t + u_{xx} + 2uv &= 0, \end{aligned} \tag{1}$$

where $v : \Omega \times \mathbb{R}^+ \to \mathbb{R}$ denotes the plasma density as determined by its equilibrium value, and $u : \Omega \times \mathbb{R}^+ \to \mathbb{C}$ denotes the high-frequency electric field's envelope. The Zakharov system is similar to nonlinear Schrödinger equations and significant in plasma turbulence theory. As a result, the Zakharov system has piqued the interest of many physicists and mathematicians, and has been extensively studied both theoretically and numerically [2–6]. To solve system problems (1), researchers have used a variety of methods. For example, Song et al. [7] introduced unbounded wave solutions, kink wave solutions, and periodic wave solutions by utilizing bifurcation theory method. Wang and Li [8] used the extended F-expansion method to obtain periodic wave solutions. Javidi et al. [9] applied the variational iteration technique to obtain solitary wave solutions. Taghizadeh et al. [10] obtained some exact solutions using infinite series method. Hong et al. [11] obtained a few new doubly periodic solutions utilizing the Jacobian elliptic function expansion method.

In recent years, the fractional derivatives are utilized to describe numerous physical phenomena in engineering applications, signal processing, electromagnetic theory, finance, physics, mathematical biology, and various scientific studies, see for instance [12–17]. For instance, the fractional derivative is utilized in control theory, controller tuning, optics, seismic wave analysis, dynamical system, signal processing, and viscoelasticity.

On the other hand, the benefits of taking random effects into consideration in predicting, simulating, analyzing and modeling complex phenomena has been extensively distinguished in biology, engineering, physics, geophysical, chemistry, climate dynamics, and other fields [18–21]. Stochastic partial differential equations (SPDEs) are suitable mathematical equations for complicated systems subject to noise or random influences. Normally, random influences can be thought of as a simple estimate of turbulence in fluids. Therefore, we have to generalize the Zakharov system by taking into account more elements due to some important effects such as ion nonlinearities and transit-time damping.

To achieve a higher level of qualitative agreement, we consider here the following stochastic fractional-space Zakharov system (SFSZS) with multiplicative noise in the Stratonovich sense:

$$iu_t + \mathbb{T}^{\alpha}_{xx}u + 2uv + i\sigma u \circ \mathcal{W}_t = 0, \tag{2}$$

$$v_{tt} - \mathbb{T}^{\alpha}_{xx}v + \mathbb{T}^{\alpha}_{xx}(|u|^2) = 0, \tag{3}$$

where $\mathbb{T}^{\alpha}$ is the conformable fractional derivative (CFD) [22], $\mathcal{W}(t)$ is standard Wiener process (SWP).

In [23,24], the stochastic dissipative Zakharov system are obtained by utilizing the global-random attractors provided with normal topology, while in [25], the uniqueness and existence of solutions of the Zakharov system with stochastic term are obtained by applying the method of Galerkin approximation.

The novelty of this paper is to construct the exact fractional stochastic solutions of the SFSZS (2)–(3). This study is the first one to obtain the exact solutions of the SFSZS (2)–(3). We use two distinct methods including the Jacobi elliptic function and the Riccati–Bernoulli sub-ODE to achieve a wide range of solutions, including hyperbolic, trigonometric, rational, and elliptic functions. Besides that, we employ Matlab tools to plot 3D and 2D graphs for some of the analytical solutions developed in this study to check the effect of the Wiener process on the solutions of SFSZS (2)–(3).

The following is how the paper is arranged. In Section 2, we define the CFD and Wiener process and we state some features about them. To obtain the wave equation of SFSZS (2)–(3), we use a suitable wave transformation in Section 3. In Section 4, we apply two different methods to construct the exact solutions of SFSZS (2)–(3). In Section 5, we study the effect of the SWP on the obtained solutions. Finally, we present the paper's conclusion.

2. Preliminaries

In this section, we introduce some definitions and features for CFD, which are reported in [22] and SWP.

Definition 1. *Assume $f : (0, \infty) \to \mathbb{R}$; hence, the CFD of f of order α is defined as*

$$\mathbb{T}^{\alpha}_{x} f(x) = \lim_{h \to 0} \frac{f(x + hx^{1-\alpha}) - f(x)}{h}.$$

Theorem 1. *Let $f, g : (0, \infty) \to \mathbb{R}$ be differentiable, and also α differentiable functions; then, the next rule holds:*

$$\mathbb{T}^{\alpha}_{x}(f \circ g)(x) = x^{1-\alpha} g'(x) f'(g(x)).$$

Let us state some properties of the CFD:

1. $\mathbb{T}_x^\alpha[af(x)+bg(x)] = a\mathbb{T}_x^\alpha f(x) + b\mathbb{T}_x^\alpha g(x), \ \ a,b \in \mathbb{R}$,
2. $\mathbb{T}_x^\alpha[C] = 0$, C is a constant,
3. $\mathbb{T}_x^\alpha[x^\hbar] = \hbar x^{\hbar-\alpha}$, $\hbar \in \mathbb{R}$,
4. $\mathbb{T}_x^\alpha g(x) = x^{1-\alpha}\frac{dg}{dx}$,

In the next definition, we define standard Wiener process $\mathcal{W}(t)$:

Definition 2. *stochastic process* $\{\mathcal{W}(t)\}_{t\geq 0}$ *is called a Wiener process if it satisfies*

1. $\mathcal{W}(0) = 0$,
2. $\mathcal{W}(t)$, $t \geq 0$ *is continuous function of* t,
3. *For* $t_1 < t_2$, $\mathcal{W}(t_1) - \mathcal{W}(t_2)$ *is independent*,
4. $\mathcal{W}(t_2) - \mathcal{W}(t_1)$ *has a Gaussian distribution with mean* 0 *and variance* $t_2 - t_1$.

We know the stochastic integral $\int_0^t \Theta d\mathcal{W}$may be interpreted in a variety of ways [26]. The Stratonovich and Itô interpretations of a stochastic integral are widely used. The stochastic integral is Itô (denoted by $\int_0^t \Theta d\mathcal{W}$) when it is evaluated at the left-end, while a Stratonovich stochastic integral (denoted by $\int_0^t \Theta \circ d\mathcal{W}$) is one that is calculated at the midpoint. The next is the relationship between the Stratonovich and Itô integral:

$$\int_0^t \Theta(\tau, Z_\tau) d\mathcal{W}(\tau) = \int_0^t \Theta(\tau, Z_\tau) \circ d\mathcal{W}(\tau) - \frac{1}{2}\int_0^t \Theta(\tau, Z_\tau)\frac{\partial \Theta(\tau, Z_\tau)}{\partial z} d\tau, \quad (4)$$

where Θ is supposed to be sufficiently regular and $\{Z_t, t \geq 0\}$ is a stochastic process.

3. Wave Equation for SFSZS

To acquire the wave equation for the SFSZS (2)–(3), the next wave transformation is applied:

$$u(x,t) = \varphi(\mu)e^{(i\theta - \sigma\mathcal{W}(t) - \sigma^2 t)}, \ \ \mu = k(\frac{1}{\alpha}x^\alpha - \lambda t) \text{ and } \theta = \frac{\lambda}{2\alpha}x^\alpha + \rho t, \quad (5)$$

where φ is a deterministic function and k, λ, ρ are nonzero constants. Plugging Equation (5) into Equation (2) and using

$$\begin{aligned} \frac{du}{dt} &= (-\lambda k\varphi' + i\rho\varphi - \sigma\varphi\mathcal{W}_t - \frac{1}{2}\sigma^2\varphi)e^{(i\theta - \sigma\mathcal{W}(t) - \sigma^2 t)}, \\ &= (-\lambda k\varphi' + i\rho\varphi - \sigma\varphi \circ \mathcal{W}_t)e^{(i\theta - \sigma\mathcal{W}(t) - \sigma^2 t)}, \\ \mathbb{T}_{xx}^\alpha &= (k^2\varphi'' + i\lambda k\varphi' - \frac{1}{4}\lambda^2\varphi)e^{(i\theta - \sigma\mathcal{W}(t) - \sigma^2 t)}, \end{aligned} \quad (6)$$

where we used (4). We obtain, for the real part,

$$k^2\varphi'' - (\frac{1}{4}\lambda^2 + \rho)\varphi + 2\varphi v = 0. \quad (7)$$

Now, we suppose

$$v(x,t) = \psi(\mu),$$

where ψ is real deterministic function, to obtain

$$v_t = -\lambda k\psi', \ \ v_{tt} = \lambda^2 k^2\psi'', \ \mathbb{T}_{xx}^\alpha v = k^2\psi''. \quad (8)$$

Substituting Equation (8) into Equation (3), we attain

$$(\lambda^2 - 1)\psi'' + (\varphi^2)'' e^{(-2\sigma\mathcal{W}(t) - 2\sigma^2 t)} = 0. \quad (9)$$

Taking expectation $\mathbb{E}(\cdot)$ on both sides, we have

$$(\lambda^2 - 1)\psi'' + (\varphi^2)'' e^{-2\sigma^2 t}\mathbb{E}(e^{-2\sigma\mathcal{W}(t)}) = 0. \tag{10}$$

Since $\mathcal{W}(t)$ is standard Gaussian process; hence, $\mathbb{E}(e^{\hbar\mathcal{W}(t)}) = e^{\frac{\hbar^2}{2}t}$ for any real constant $\hbar$. Now, Equation (10) has the form

$$(\lambda^2 - 1)\psi'' + (\varphi^2)'' = 0, \tag{11}$$

Integrating Equation (11) twice and putting the constants of integration equal zero yields

$$(\lambda^2 - 1)\psi + \varphi^2 = 0. \tag{12}$$

Hence, Equation (12) becomes

$$\psi = \frac{-\varphi^2}{(\lambda^2 - 1)}. \tag{13}$$

Putting Equation (13) into Equation (7), we obtain the next wave equation

$$\varphi'' - \gamma_1\varphi^3 - \gamma_2\varphi = 0, \tag{14}$$

where

$$\gamma_1 = \frac{2}{k^2(\lambda^2 - 1)} \quad \text{and} \quad \gamma_2 = \frac{1}{4k^2}(\lambda^2 + 4\rho). \tag{15}$$

4. The Analytical Solutions of the SFSZS

To find the solutions of Equation (14), we utilize two different methods: Riccati–Bernoulli sub-ODE [27] and the Jacobi elliptic function method [28]. Therefore, we acquire the analytical solutions of the SFSZS (2)–(3).

4.1. Riccati–Bernoulli Sub-ODE Method

Assume the following Riccati–Bernoulli equation:

$$\varphi' = \hbar_1\varphi^2 + \hbar_2\varphi + \hbar_3, \tag{16}$$

where $\hbar_1, \hbar_2, \hbar_3$ are undefined constants and $\varphi = \varphi(\mu)$.

Differentiating Equation (16) with respect to μ, we obtain

$$\varphi'' = 2\hbar_1\varphi\varphi' + \hbar_2\varphi',$$

and using Equation (16) yields

$$\varphi'' = 2\hbar_1^2\varphi^3 + 3\hbar_1\hbar_2\varphi^2 + (2\hbar_1\hbar_3 + \hbar_2^2)\varphi + \hbar_2\hbar_3. \tag{17}$$

Substituting (17) into (14), we have

$$(2\hbar_1^2 - \gamma_1)\varphi^3 + 3\hbar_1\hbar_2\varphi^2 + (2\hbar_1\hbar_3 + \hbar_2^2 - \gamma_2)\varphi + \hbar_2\hbar_3 = 0.$$

Equating each coefficient of $\varphi^i (i = 0, 1, 2, 3)$ to zero, we achieve the next algebraic equations

$$\hbar_2\hbar_3 = 0,$$

$$(2\hbar_1\hbar_3 + \hbar_2^2 - \gamma_2) = 0,$$

$$3\hbar_1\hbar_2 = 0,$$

$$2\hbar_1^2 - \gamma_1 = 0.$$

When the above equations are solved, the result is

$$\hbar_1 = \pm\sqrt{\frac{1}{2}\gamma_1},\ \hbar_2 = 0,\ \hbar_3 = \frac{\gamma_2}{2\hbar_1} = \pm\frac{\gamma_2}{\sqrt{2\gamma_1}}. \tag{18}$$

There are numerous solutions to the Riccati–Bernoulli Equation (16) depending on $\hbar_1$ and $\hbar_3$.

First case: If $\frac{\hbar_3}{\hbar_1} = 0$, then Riccati–Bernoulli Equation (16) has the solution

$$\varphi(\mu) = \frac{-1}{\hbar_1\mu + C}.$$

Hence, the SFSZS (2)–(3) has the analytical solutions

$$u(x,t) = \varphi(\mu)e^{(i\theta-\sigma\mathcal{W}(t)-\sigma^2 t)} = \frac{-1}{\hbar_1(\frac{k}{\alpha}x^\alpha - k\lambda t) + C}e^{(i\theta-\sigma\mathcal{W}(t)-\sigma^2 t)}, \tag{19}$$

$$v(x,t) = \frac{-\varphi^2}{(\lambda^2-1)} = \frac{-1}{(\lambda^2-1)\Big(\hbar_1(\frac{k}{\alpha}x^\alpha - k\lambda t) + C\Big)^2}. \tag{20}$$

Second case: If $\frac{\hbar_3}{\hbar_1} > 0$, then the Riccati–Bernoulli equation (16) has the solution

$$\varphi(\mu) = \sqrt{\frac{\hbar_3}{\hbar_1}}\tan\Big(\sqrt{\frac{\hbar_3}{\hbar_1}}(\hbar_1\mu + C)\Big),$$

or

$$\varphi(\mu) = -\sqrt{\frac{\hbar_3}{\hbar_1}}\cot\Big(\sqrt{\frac{\hbar_3}{\hbar_1}}(\hbar_1\mu + C)\Big).$$

Therefore, SFSZSs (2)–(3) have the following solutions:

$$u(x,t) = e^{(i\theta-\sigma\mathcal{W}(t)-\sigma^2 t)}\sqrt{\frac{\hbar_3}{\hbar_1}}\tan\Big(\sqrt{\frac{\hbar_3}{\hbar_1}}(\hbar_1(\frac{k}{\alpha}x^\alpha - k\lambda t) + C)\Big), \tag{21}$$

$$v(x,t) = \frac{-\hbar_3}{(\lambda^2-1)\hbar_1}\tan^2\Big(\sqrt{\frac{\hbar_3}{\hbar_1}}(\hbar_1(\frac{k}{\alpha}x^\alpha - k\lambda t) + C)\Big), \tag{22}$$

or

$$u(x,t) = -e^{(i\theta-\sigma\mathcal{W}(t)-\sigma^2 t)}\sqrt{\frac{\hbar_3}{\hbar_1}}\cot\Big(\sqrt{\frac{\hbar_3}{\hbar_1}}(\hbar_1(\frac{k}{\alpha}x^\alpha - k\lambda t) + C)\Big), \tag{23}$$

$$v(x,t) = \frac{-\hbar_3}{(\lambda^2-1)\hbar_1}\cot^2\Big(\sqrt{\frac{\hbar_3}{\hbar_1}}(\hbar_1(\frac{k}{\alpha}x^\alpha - k\lambda t) + C)\Big), \tag{24}$$

respectively.

Third case: If $\frac{\hbar_3}{\hbar_1} < 0$ and $|\varphi| < \sqrt{-\frac{\hbar_3}{\hbar_1}}$, then Riccati–Bernoulli Equation (16) has the solution

$$\varphi(\mu) = -\sqrt{\frac{-\hbar_3}{\hbar_1}}\tanh\Big(\sqrt{\frac{-\hbar_3}{\hbar_1}}(\hbar_1\mu + C)\Big).$$

Thus, the SFSZS (2)–(3) have the following analytical solutions:

$$u(x,t) = -e^{(i\theta-\sigma\mathcal{W}(t)-\sigma^2 t)}\sqrt{\frac{-\hbar_3}{\hbar_1}}\tanh\Big(\sqrt{\frac{-\hbar_3}{\hbar_1}}(\hbar_1(\frac{k}{\alpha}x^\alpha - k\lambda t) + C)\Big), \tag{25}$$

$$v(x,t) = \frac{-\hbar_3}{(\lambda^2-1)\hbar_1}\tanh^2\Big(\sqrt{\frac{-\hbar_3}{\hbar_1}}(\hbar_1(\frac{k}{\alpha}x^\alpha - k\lambda t) + C)\Big). \tag{26}$$

Fourth case: If $\frac{\hbar_3}{\hbar_1} < 0$ and $\varphi^2 > \frac{-\hbar_3}{\hbar_1}$, then Riccati–Bernoulli Equation (16) has the solution

$$\varphi(\mu) = -\sqrt{\frac{-\hbar_3}{\hbar_1}}\coth\Big(\sqrt{\frac{-\hbar_3}{\hbar_1}}(\hbar_1\mu + C)\Big).$$

Consequently, the analytical solutions of the SFSZS (2)–(3) are

$$u(x,t) = -e^{(i\theta-\sigma\mathcal{W}(t)-\sigma^2 t)}\sqrt{\frac{-\hbar_3}{\hbar_1}}\coth\Big(\sqrt{\frac{\hbar_3}{\hbar_1}}(\hbar_1(\frac{k}{\alpha}x^\alpha - k\lambda t) + C)\Big), \tag{27}$$

$$v(x,t) = \frac{-\hbar_3}{(\lambda^2-1)\hbar_1}\coth^2\Big(\sqrt{\frac{\hbar_3}{\hbar_1}}(\hbar_1(\frac{k}{\alpha}x^\alpha - k\lambda t) + C)\Big), \tag{28}$$

where $\hbar_1$ and $\hbar_2$ are defined in Equation (18).

4.2. The Jacobi Elliptic Function Method

Assuming that the solutions to Equation (14) are of the form

$$\varphi(\mu) = a + bsn(\delta\mu), \tag{29}$$

where $sn(\delta\mu) = sn(\delta\mu, m)$, for $0 < m < 1$, is the Jacobi elliptic sine function and $a,\ b, \delta$ are unknown constants. Differentiate Equation (29) two times and we have

$$\varphi''(\mu) = -(m^2+1)b\delta^2 sn(\delta\mu) + 2m^2 b\delta^2 sn^3(\delta\mu). \tag{30}$$

Substituting Equations (29) and (30) into Equation (14), we attain

$$(2m^2b\delta^2 - \gamma_1 b^3)sn^3(\delta\mu) - 3\gamma_1 ab^2 sn^2(\delta\mu)$$
$$-[(m^2+1)b\delta^2 + 3\gamma_1 a^2 b + \gamma_2 b]sn(\delta\mu) - (\gamma_1 a^3 + a\gamma_2) = 0.$$

Setting each coefficient of $[sn(\delta\mu)]^n (n = 0,1,2,3)$ equal to zero, we attain

$$\gamma_1 a^3 + a\gamma_2 = 0,$$
$$(m^2+1)b\delta^2 + 3\gamma_1 a^2 b + \gamma_2 b = 0,$$
$$3\gamma_1 ab^2 sn^2 = 0,$$

and

$$2m^2 b\delta^2 - \gamma_1 b^3 = 0.$$

Solving the above equations, we have

$$a = 0,\ b = \pm\sqrt{\frac{-2m^2\gamma_2}{(m^2+1)\gamma_1}}\ \delta^2 = \frac{-\gamma_2}{(m^2+1)}.$$

Hence, the solution of Equation (14), by using (29), has the form

$$\varphi(\mu) = \pm\sqrt{\frac{-2m^2\gamma_2}{(m^2+1)\gamma_1}}sn(\frac{-\gamma_2}{(m^2+1)}\mu).$$

Therefore, the analytical solutions of the SFSZS (2)–(3) are

$$u(x,t) = \pm\sqrt{\frac{-2m^2\gamma_2}{(m^2+1)\gamma_1}}sn\Big(\sqrt{\frac{-\gamma_2}{(m^2+1)}}(\frac{k}{\alpha}x^\alpha - k\lambda t)\Big)e^{(i\theta-\sigma\mathcal{W}(t)-\sigma^2 t)}, \tag{31}$$

$$v(x,t) = \frac{k^2 m^2 \gamma_2}{(m^2+1)} sn^2\Big(\sqrt{\frac{-\gamma_2}{(m^2+1)}}(\frac{k}{\alpha}x^\alpha - k\lambda t)\Big), \tag{32}$$

for $\gamma_2 < 0$ and $\gamma_1 > 0$. When $m \to 1$, the solutions (31)–(32) transfer into

$$u(x,t) = \pm\sqrt{\frac{-\gamma_2}{\gamma_1}} \tanh\Big(\sqrt{\frac{-\gamma_2}{2}}(\frac{k}{\alpha}x^\alpha - k\lambda t)\Big) e^{(i\theta - \sigma \mathcal{W}(t) - \sigma^2 t)}, \tag{33}$$

$$v(x,t) = -\frac{k^2 \gamma_2}{2} \tanh^2\Big(\sqrt{\frac{-\gamma_2}{2}}(\frac{k}{\alpha}x^\alpha - k\lambda t)\Big). \tag{34}$$

Analogously, we can replace sn in (29) by cn and dn in order to obtain the solutions of Equation (14), respectively, as follows:

$$\varphi(\mu) = \pm\sqrt{\frac{-2m^2\gamma_2}{(2m^2-1)\gamma_1}} cn(\frac{-\gamma_2}{(2m^2-1)}\mu),$$

and

$$\varphi(\mu) = \pm\sqrt{\frac{2m^2\gamma_2}{(2-m^2)\gamma_1}} dn(\frac{-\gamma_2}{(2-m^2)}\mu).$$

Consequently, the solutions of the SFSZS (2)–(3) have the following forms:

$$u(x,t) = \pm\sqrt{\frac{-2m^2\gamma_2}{(2m^2-1)\gamma_1}} cn\Big(\sqrt{\frac{-\gamma_2}{(2m^2-1)}}(\frac{k}{\alpha}x^\alpha - k\lambda t)\Big) e^{(i\theta - \sigma \mathcal{W}(t) - \sigma^2 t)}, \tag{35}$$

$$v(x,t) = \frac{k^2 m^2 \gamma_2}{(2m^2-1)} cn^2\Big(\sqrt{\frac{-\gamma_2}{(2m^2-1)}}(\frac{k}{\alpha}x^\alpha - k\lambda t)\Big), \tag{36}$$

for $\frac{\gamma_2}{(2m^2-1)} < 0$, $\gamma_1 > 0$, and

$$u(x,t) = \pm\sqrt{\frac{-2m^2\gamma_2}{(2m^2-1)\gamma_1}} dn\Big(\sqrt{\frac{-\gamma_2}{(2m^2-1)}}(\frac{k}{\alpha}x^\alpha - k\lambda t)\Big) e^{(i\theta - \sigma \mathcal{W}(t) - \sigma^2 t)}, \tag{37}$$

$$v(x,t) = \frac{k^2 m^2 \gamma_2}{(2-m^2)} dn^2\Big(\sqrt{\frac{-\gamma_2}{(2-m^2)}}(\frac{k}{\alpha}x^\alpha - k\lambda t)\Big), \tag{38}$$

for $\gamma_2 < 0$, $\gamma_1 > 0$, respectively. When $m \to 1$, the solutions (35)–(36) and (37)–(38) transfer into

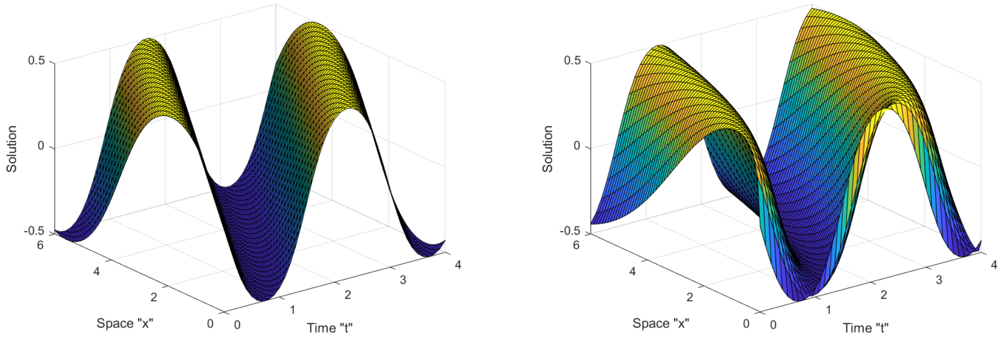
$$u(x,t) = \pm\sqrt{\frac{-2\gamma_2}{\gamma_1}} \operatorname{sech}\Big(\sqrt{-\gamma_2}(\frac{k}{\alpha}x^\alpha - k\lambda t)\Big) e^{(i\theta - \sigma \mathcal{W}(t) - \sigma^2 t)}, \tag{39}$$

$$v(x,t) = k^2 m^2 \gamma_2 \operatorname{sech}^2\Big(\sqrt{-\gamma_2}(\frac{k}{\alpha}x^\alpha - k\lambda t)\Big), \tag{40}$$

for $\gamma_2 < 0$, $\gamma_1 > 0$.

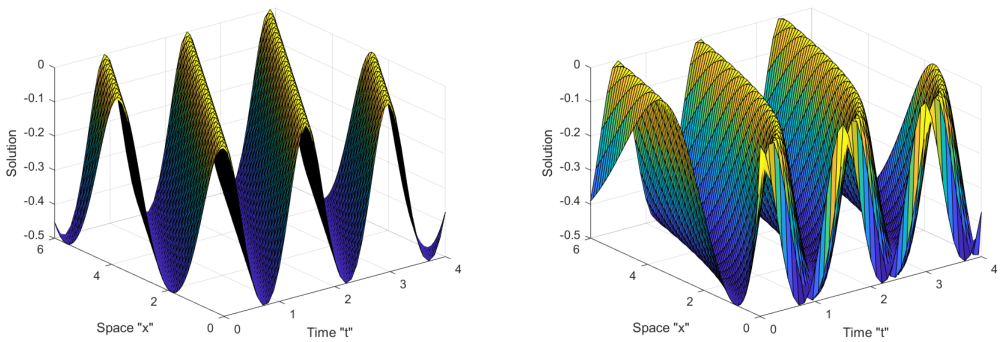
5. The Influence of Noise on SFSZS Solutions

The influence of the noise on the analytical solution of the SFSZS (2)–(3) is addressed here. Fix the parameters $k = 1, \rho = 1$, $m = 0.5$, and $\lambda = 3$. We introduce a number of simulations for various values of σ (noise intensity) and α (fractional derivative order). We employ the MATLAB tools to plot our figures. In Figures 1 and 2, if $\sigma = 0$, we see that the surface fluctuates for different values of α:



$\sigma = 0,\ \alpha = 1$ $\sigma = 0,\ \alpha = 0.5$

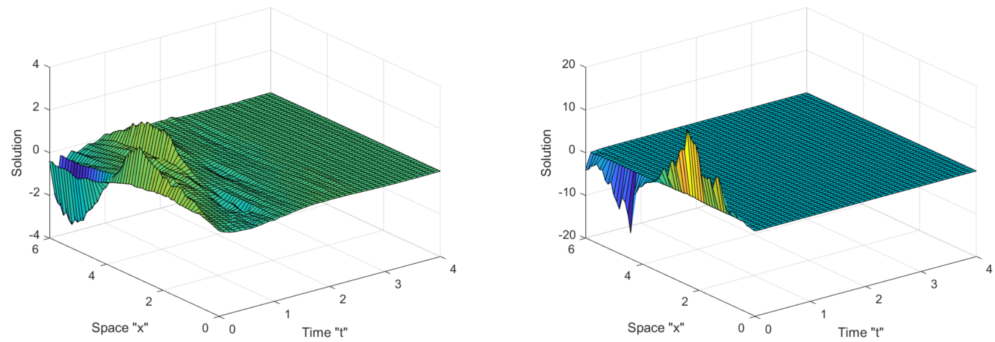
Figure 1. 3D graphs of the solution (31).



$\sigma = 0,\ \alpha = 1$ $\sigma = 0,\ \alpha = 0.5$

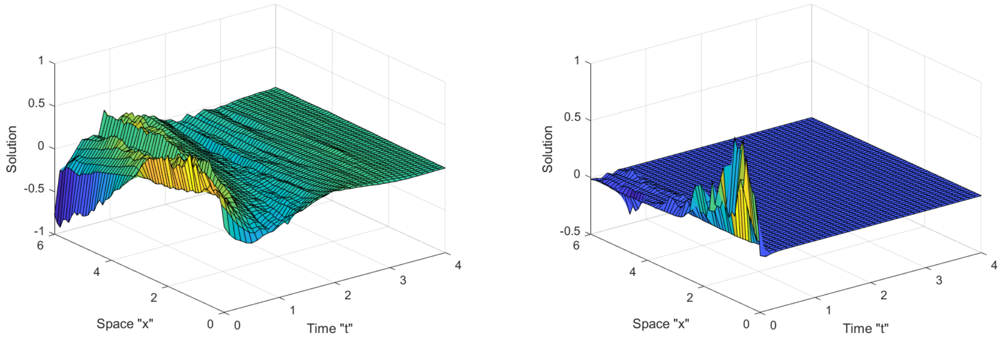
Figure 2. 3D graphs of the solution (32).

In the following Figures 3–5, we can see that after minor transit patterns, the surface becomes considerably flattered when noise is included and its strength is increased $\sigma = 1,\ 2$.



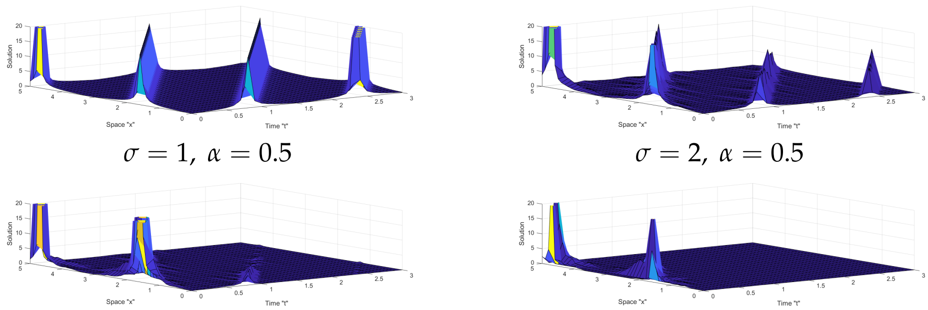
$\sigma = 1,\ \alpha = 1$ $\sigma = 2,\ \alpha = 1$

Figure 3. 3D graphs of the solution (31) with $\alpha = 1$.



$\sigma = 1,\ \alpha = 0.5$ $\sigma = 2,\ \alpha = 0.5$

Figure 4. 3D graphs of the equation (31) with $\alpha = 0.5$.



$\sigma = 1,\ \alpha = 0.5$ $\sigma = 2,\ \alpha = 0.5$

$\sigma = 1,\ \alpha = 0.5$ $\sigma = 2,\ \alpha = 0.5$

Figure 5. 3D graphs of the equation (21) with $\alpha = 1$.

In Figure 6, we introduce 2D plots of the u in (31) with $\sigma = 0, 0.5, 1, 2$ and $\alpha = 1$, which emphasize the results above.

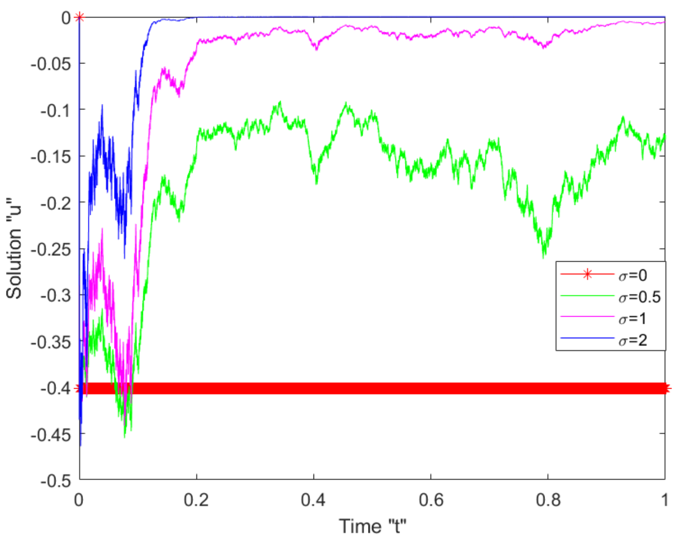


Figure 6. 2D graphs of the u in (31).

From Figures 1–6, we deduce the following:

1. The surface expands as the fractional order α increases;
2. Multiplicative Wiener process stabilizes the solutions of SFSBE around zero.

6. Conclusions

In this article, we provided a wide range of exact solutions of the stochastic fractional Zakharov system (2)–(3). We applied two different methods such as the Riccati–Bernoulli sub-ODE method and Jacobi elliptic function method to attain rational, trigonometric, hyperbolic, and elliptic stochastic fractional solutions. Such solutions are critical for comprehending certain essential, fundamental, complex phenomena. The solutions obtained will be extremely useful for further studies such as fiber applications, spatial plasma, quasi particle theory, coastal water motion, and industrial research. Finally, the effect of multiplicative Wiener process on the exact solution of Zakharov system (2)–(3) is demonstrated. In future research, we can address the fractional-time Zakharov system (2)–(3) with multidimensional multiplicative noise.

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