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Fractional Evolution Equations with Infinite Time Delay in Abstract Phase Space

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Abstract: In the presented research, the uniqueness and existence of a mild solution for a fractional system of semilinear evolution equations with infinite delay and an infinitesimal generator operator are demonstrated. The generalized Liouville–Caputo derivative of non-integer-order $1 < \alpha \leq 2$ and the parameter $0 < \rho < 1$ are used to establish our model. The ρ-Laplace transform and strongly continuous cosine and sine families of uniformly bounded linear operators are adapted to obtain the mild solution. The Leray–Schauder alternative theorem and Banach contraction principle are used to demonstrate the mild solution's existence and uniqueness in abstract phase space. The results are applied to the fractional wave equation.

Keywords: generalized Liouville–Caputo fractional derivative; ρ-Laplace transformation; infinite time delay; mild solution; Leray–Schauder alternative

MSC: 34A08; 34A12; 34G99; 34K99; 34A60

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1. Introduction

In the past few years, it has been noted that nonlinear fractional differential equations have great value in many scientific branches such as physics, biology, engineering, control theory, fractal dynamics, signal processing, etc. For further information, see [1–5]. This is due to the fast expansion of fractional calculus theory and its vast scope of applications [6–9].

Several phenomena are mathematically modeled as evolution equations, and thus, their solutions represent continuous or discrete functions and provide the distinctive qualitative and quantitative properties of these phenomena [10,11]. Furthermore, the discussion of evolution equations, especially with time delay, is usually a difficult task, which has led to the reluctance of many authors to approach the topic and the lack of contributions to it, such that the research on it is relatively sparse despite its utmost importance. Of course, the study of time delay is rather widespread. Many changes and operations depend not only on the current position, but also on the previous position. Thus, it is obligatory to investigate the time delay influence in the mathematical modeling of fractional evaluation equations.

Many scientists are curious about the presence of solutions and their uniqueness and have studied and investigated them (see [12–14] and the references therein). Benchohra et al. [15] substantiated the existence and uniqueness of a solution for fractional differential equations with infinite delay, and the outcomes were obtained by using fractional calculus, the Banach fixed-point theorem, and the nonlinear alternative of the Leray–Schauder type. Zhou and Jiao [16] established sufficient conditions for the existence of a mild solution to fractional neutral evolution equations. The approach used was the analytic semigroup and nonlinear Krasnoselkii fixed-point theorem. Norouzi et al. [17] proved the existence and uniqueness solution for a ψ-Hilfer neutral fractional semilinear equation with infinite delay, and the outcomes were obtained by the semigroup operator, Banach fixed-point theorem, and nonlinear alternative of the Leray–Schauder type. For more contributions to the literature, see [18–22] and the references therein.

The fractional wave equations are derived by substituting the ordinary second derivative with a fractional one of order lying between one and two. They have drawn attention principally in the dynamical theory of linear viscoelasticity, in the description of the propagation of stress waves in viscoelastic media, and in the description of the power-law attenuation with the frequency when sound waves travel through inhomogeneous media. Furthermore, they are used as a model for the oscillation of a cable made of special smart materials [23]. For the first time, Schneider and Wyss [24] introduced the fractional wave equation as

$$ {}_c\mathcal{D}_0^{\alpha} u(\tau, x) = \frac{\partial^2}{\partial x^2} u(\tau, x) + f(\tau, x), \qquad \tau > 0, x \in \mathbb{R} $$

where ${}_c\mathcal{D}_0^{\alpha}$ is the Caputo fractional derivative of order $1 < \alpha \le 2$. They presented the corresponding Green function in closed form for arbitrary space dimensions in terms of Fox functions and exhibited their properties.

Keyantuo et al. [25] introduced the representation of the inhomogeneous Cauchy problem:

$$ {}_c\mathcal{D}_0^{\alpha} u(\tau, x) = A u(\tau, x) + f(\tau, x), \qquad \tau > 0, x \in \Omega $$

where $1 < \alpha \le 2$ and A is a closed linear operator. They proved an explicit representation of a mild solution in terms of the integrated cosine family. Their results have been applied to several examples, which include:

- Elliptic operators with Dirichlet-, Neumann-, or Robin-type boundary conditions on L^p-spaces and on the space of continuous functions with a specific form of the operator A.
- Elliptic operators in one-dimension and on general L^p spaces with

$$ A = a(x) \frac{\partial^2}{\partial x^2} u(\tau, x) + b(x) \frac{\partial}{\partial x} u(\tau, x) + c(x) u(\tau, x). $$

- The fractional-order Schrodinger-like equation with the operator $A = e^{i\theta}\Delta$ where Δ is a realization of the Laplace operator and $\frac{\pi}{2} < \theta < (1 - \frac{\alpha}{4})\pi$.

Most of the contributions are concerned with the Caputo fractional derivatives and applying the classical Laplace transform [26–29].

Inspired by the mentioned works and as a continuation of the contributions to the literature, we considered the following fractional semilinear evolution equation with infinite time delay:

$$ \begin{cases} {}_c^{\rho}\mathcal{D}_0^{\alpha} u(\tau) = A u(\tau) + f(\tau, u(\tau), u_\tau), & \tau \in= [0, a], \\ u(\tau) = \phi(\tau) & \tau \in (-\infty, 0], \\ \lim\limits_{\tau \to 0^+} \tau^{-\rho} u'(\tau) = \tau_o & \tau_0 \in \mathbb{X}. \end{cases} \tag{1} $$

where ${}_c^{\rho}\mathcal{D}_0^{\alpha}$ is the generalized Liouville–Caputo fractional derivative of order $1 < \alpha \le 2$ and type $0 < \rho < 1$. The function $f : [0, a] \times \mathbb{X} \times \mathcal{P}_{\mathfrak{h}} \to \mathbb{X}$ is a continuous function where $\mathbb{X}$ is a Banach space equipped with the norm $\| \cdot \|_{\mathbb{X}}$, $\phi(\tau) \in \mathcal{P}_{\mathfrak{h}}$, where $\mathcal{P}_{\mathfrak{h}}$ is the admissible phase space, which will be decided upon later. Furthermore, u_τ represents the state function's history up to the present time τ, i.e., $u_\tau(\theta) = u(\tau + \theta)$ for all $\theta \in (-\infty, 0]$.

Let A be an infinitesimal generator of a strongly continuous cosine family $\{\mathfrak{C}(\tau)\}_{\tau \ge 0}$ of uniformly bounded linear operators defined on infinite Banach space $\mathbb{X}$ ($D(A) \subseteq \mathbb{X}$). The Banach space of continuous and bounded functions from $(-\infty, a]$ into $\mathbb{X}$ provided with the topology of uniform convergence is denoted by $\mathcal{C} = \mathcal{C}_a((-\infty, a], \mathbb{X})$ with the norm

$$ \|u\|_{\mathcal{C}} = \sup_{\tau \in (-\infty, a]} |u(\tau)| $$

and let $(\mathcal{B}(\mathbb{X}), \|.\|_{\mathcal{B}(\mathbb{X})})$ be the Banach space of all linear and bounded operators from $\mathbb{X}$ to $\mathbb{X}$. As $\{\mathfrak{C}(\tau)\}_{\tau\geq 0}$ is the cosine family on $\mathbb{X}$, then there exists $M \geq 1$ where

$$\|\mathfrak{C}(\tau)\| \leq M. \tag{2}$$

2. Preliminaries

This section is divided into five sections: The first section provides the principles and important results in fractional calculus that will be needed for the study. The second subsection is devoted to a recap of the ρ-Laplace transform and its most important properties used to find the solution for the evaluation equations. The third subsection gives a quick rundown of the cosine family operator, which is employed in this article. The fourth subsection begins with a definition of abstract phase space. The final subsection introduces some other requirements.

2.1. The Generalized Liouville–Caputo Fractional Derivative

Katugampola [30] introduced a new approach for the fractional integral, depending on a parameter $\rho \in \mathbb{R} - \{0\}$, being a generalization of both Riemann–Liouville and Hadamard fractional integrals. Based on this new fractional integral, he introduced a new approach for fractional derivative of the Riemann type and Caputo type. Furthermore, he provided a numerical example and noticed that the characteristics of the fractional derivative are highly affected by the value of ρ. Thus, this approach provides a new direction for control applications.

Let $\mathrm{AC}^n(I)$ be the space of all absolutely continuous functions determined by the interval $I = [a, b]$, $a, b \in \mathbb{R}$. Suppose the parameter $0 < \rho < 1$ and $\Gamma(\cdot)$ denotes the Gamma function.

Definition 1 ([30]). *Consider $f : [0, \infty) \to \mathbb{R}$ to be an absolutely continuous function, $n \in \mathbb{N}$ and $n - 1 < \alpha \leq n$. The Liouville–Caputo generalized derivative of the function f is expressed in the form*

$$ {}^{\rho}_{c}\mathcal{D}^{\alpha}_{a} f(t) = \frac{1}{\Gamma(1-\alpha)} \int_a^t \left(\frac{t^\rho - s^\rho}{\rho}\right)^{n-\alpha-1} f^{(n)}(s) \frac{ds}{s^{1-\rho}}, \quad t > a.$$

Definition 2 ([30]). *The generalized left fractional integrals of the function f are*

$$ {}^{\rho}\mathcal{I}^{\alpha}_{a} f(t) = \frac{1}{\Gamma(\alpha)} \int_a^t \left(\frac{t^\rho - s^\rho}{\rho}\right)^{\alpha-1} f(s) \frac{ds}{s^{1-\rho}}, \ t > a, \ \alpha > 0$$

provided the integral exists.

Lemma 1 ([31,32]). *Consider that $n \in \mathbb{N}$, $n - 1 < \alpha \leq n$, $0 < \rho \leq 1$, and $f \in \mathbb{X}^p_c(a, b)$. Following that, there is*

- ${}^{\rho}_{c}\mathcal{D}^{\alpha}_{a}\mathfrak{s} = 0$ *where $\mathfrak{s}$ is a constant,*
- ${}^{\rho}_{c}\mathcal{D}^{\alpha}_{a}(t^\rho - a^\rho)^\beta = \dfrac{\rho^\nu \Gamma(\beta+1)}{\Gamma(\beta-\alpha+1)}(t^\rho - a^\rho)^{\beta-\alpha}, \quad \alpha > 0, \beta > -1, \beta \neq 0, 1, \cdots, n-1,$
- ${}^{\rho}_{c}\mathcal{D}^{\alpha}_{a}\, {}^{\rho}I^{\nu}_{a} f(t) = {}^{\rho}I^{\nu-\alpha}_{a} f(t), \qquad \nu \geq \alpha,$
- ${}^{\rho}I^{\alpha}_{a}\, {}^{\rho}_{c}\mathcal{D}^{\alpha}_{a} f(t) = f(t) - \displaystyle\sum_{r=0}^{n-1} c_r (t^\rho - a^\rho)^r$ *where $c_r, r = 0, 1, \cdots, n-1$ are constants.*

2.2. ρ-Laplace Transform

Definition 3 ([33]). *The Laplace transform of a function $h : [0, \infty) \to \mathbb{R}$ is defined as*

$$\mathcal{L}\{h(\tau)\}(s) = H(s) = \int_0^\infty h(\tau) e^{-s\tau} d\tau \qquad \text{for } s > 0.$$

Furthermore, if $H(s) = \mathcal{L}\{h(\tau)\}(s)$ *and* $K(s) = \mathcal{L}\{k(\tau)\}(s)$*, then*

$$\mathcal{L}\left\{\int_0^\tau f(\tau - r)g(r)dr\right\}(s) = H(s)K(s).$$

Definition 4 ([33])**.** *Suppose* $h : [0, \infty) \to \mathbb{R}$ *is a real function. Then, the* ρ*-Laplace transform is denoted and characterized by*

$$\mathcal{L}_\rho\{h(\tau)\}(s) = \mathcal{L}\{h((\rho\tau)^{\frac{1}{\rho}})\}(s),$$

where $\mathcal{L}\{h\}$ *denotes the standard Laplace transform of* h*.*

Theorem 1 ([33])**.** *In the Liouville–Caputo sense, the* ρ*-Laplace transform of the generalized L-C fractional derivative is represented as:*

$$\mathcal{L}_\rho\{{}^cD^{\alpha,\rho}h(\tau)\}(s) = s^\alpha\mathcal{L}_\rho\{h(\tau)\}(s) - s^{\alpha-1}h(0),\ \rho > 0.$$

The ρ*-Laplace transform of the function* h *is presented in the format*

$$\mathcal{L}_\rho\{h(\tau)\}(s) = \int_0^\infty e^{-s\frac{\tau^\rho}{\rho}}h(\tau)\frac{d\tau}{\tau^{1-\rho}},\ \rho > 0$$

and

$$\mathcal{L}_\rho\{h(\tau)\}(s) = \mathcal{L}\{h((\rho\tau)^{\frac{1}{\rho}})\}(s),\ \rho > 0$$

where $\mathcal{L}\{h\}$ *denotes the standard Laplace transform of* h*.*

Definition 5 ([34])**.** *Assume* h *and* g *are two piecewise continuous functions on* $[0, \mathcal{T}]$ *and exponential order* $e^{c\frac{\rho\tau}{\rho}}$*. Then, the* ρ*-convolution of* h *and* g *is given by*

$$(h *_\rho g)(\tau) = \int_0^\tau h((\tau^\rho - s^\rho)^{\frac{1}{\rho}})g(s)\frac{ds}{s^{1-\rho}},\ \rho > 0.$$

Theorem 2 ([34])**.** *Assume* h *and* g *are two piecewise continuous functions on* $[0, \mathcal{T}]$ *and exponential order* $e^{c\frac{\rho\tau}{\rho}}$*. Then,*

$$h *_\rho g = g *_\rho h,\ \rho > 0$$

also,

$$\mathcal{L}_\rho\{h *_\rho g\}(s) = \mathcal{L}_\rho\{h\}\mathcal{L}_\rho\{g\}(s).$$

Lemma 2 ([33])**.** *We have:*

(1) $\mathcal{L}_\rho\{1\}(s) = \frac{1}{s},\ \ s > 0;$

(2) $\mathcal{L}_\rho\{\tau^p\}(s) = \rho^{\frac{p}{\rho}}\frac{\Gamma(1+\frac{p}{\rho})}{s^{1+\frac{p}{\rho}}},\ \ p \in \mathbb{R}^+, s > 0.$

2.3. Cosine Family Operator

Definition 6. *Consider* $\{\mathfrak{C}(t)\}_{t\in\mathbb{R}}$ *to be a one-parameter family of bounded linear operators mapping the Banach space* $\mathbb{X} \to \mathbb{X}$*, which is referred to as a strongly continuous cosine family if and only if:*

(i) $\mathfrak{C}(0) = I;$

(ii) $\mathfrak{C}(s - t) + \mathfrak{C}(s + t) = 2\mathfrak{C}(s)\mathfrak{C}(t)$ *for all* $s, t \in \mathbb{R};$

(iii) $\mathfrak{C}(t)x$ *is a continuous on* $\mathbb{R}$ *for any* $x \in \mathbb{X}$*.*

The sine family $\{S(t)\}_{t\in R}$ *is correlated with the strongly continuous cosine family* $\{\mathfrak{C}(t)\}_{t\in\mathbb{R}}$*, and it is characterized by*

$$S(t)x = \int_0^t \mathfrak{C}(s)xds, \quad x \in \mathbb{X}, t \in \mathbb{R}.$$

Lemma 3 ([35])**.** *Consider* A *to be an infinitesimal generator of a strongly continuous cosine family* $\{\mathfrak{C}(t)\}_{t\in\mathbb{R}}$ *on Banach space* $\mathbb{X}$ *such that* $\|\mathfrak{C}(t)\|_{L_b} \leq Me^{\xi|t|}, t \in \mathbb{R}$*. Then, for* $Re\lambda > \xi, \lambda^2 \in \rho(A)$*, we have*

$$\lambda R(\lambda^2; A)x = \int_0^\infty e^{-\lambda t}\mathfrak{C}(t)xdt, \qquad R(\lambda^2; A)x = \int_0^\infty e^{-\lambda t}S(t)xdt, \quad x \in \mathbb{X}$$

where the operator $R(\lambda; A) = (\lambda I - A)^{-1}$ *is the resolvent of the operator* A *and* $\lambda \in \varrho(A)$ *the resolvent set of* A*.*

A is defined by

$$Ax = \frac{d^2}{dt^2}\mathfrak{C}(0)x, \ \ \forall \ \ x \in \mathcal{D}(A)$$

where $\mathcal{D}(A) = \{x \in \mathbb{X} : \mathfrak{C}(t)x \in \mathcal{C}^2(\mathbb{R}, \mathbb{X})\}$. Clearly, the infinitesimal generator A is a densely defined operator in $\mathbb{X}$ and closed.

2.4. Abstract Phase Space $\mathcal{P}_{\mathfrak{h}}$

We illustrate the abstract phase space $\mathcal{P}_{\mathfrak{h}}$ in a convenient way [15,36]. Let $\mathfrak{h} = \mathcal{C}((-\infty, 0], [0, \infty))$ with $\int_{-\infty}^0 \mathfrak{h}(t)dt < \infty$. Then, for any $c > 0$, we have that

$$\mathcal{P} = \{\mathfrak{A} : [-c, 0] \to \mathbb{X} \ , \ \ \mathfrak{A} \text{ is bounded and measurable}\}$$

and establish the space $\mathcal{P}$ with

$$\|\mathfrak{A}\|_{\mathcal{P}} = \sup_{s\in[-c,0]} |\mathfrak{A}(s)|, \quad \text{for all} \quad \mathfrak{A} \in \mathcal{P}.$$

Let us define the space:

$$\mathcal{P}_{\mathfrak{h}} = \left\{\mathfrak{A} : (-\infty, 0] \to \mathbb{X} \text{ such that for any } \ c > 0, \mathfrak{A}|_{[-c,0]} \in \mathcal{P} \text{ and } \int_{-\infty}^0 \mathfrak{h}(t) \sup_{t\leq s\leq 0} \mathfrak{A}(s)dt < \infty\right\}.$$

If $\mathcal{P}_{\mathfrak{h}}$ is equipped with the norm

$$\|\mathfrak{A}\|_{\mathcal{P}_{\mathfrak{h}}} = \int_{-\infty}^0 \mathfrak{h}(t) \sup_{t\leq s\leq 0} \|\mathfrak{A}(s)\|dt, \ \ \forall\, \mathfrak{A} \in \mathcal{P}_{\mathfrak{h}},$$

then $\left(\mathcal{P}_{\mathfrak{h}}, \|\cdot\|_{\mathcal{P}_{\mathfrak{h}}}\right)$ is a Banach space.

Forthwith, we take into consideration the space

$$\overline{\mathcal{P}}_{\mathfrak{h}} = \left\{v : (-\infty, a] \to \mathbb{X} \ \text{ such that } v|_{[0,a]} \text{is continuous, } \ v|_{(-\infty,0]} = \phi \in \mathcal{P}_{\mathfrak{h}}\right\},$$

which has the norm

$$\|x\|_{\overline{\mathcal{P}}_{\mathfrak{h}}} = \sup_{s\in[0,a]} \|v(s)\| + \|\phi\|_{\mathcal{P}_{\mathfrak{h}}}.$$

Definition 7 ([37])**.** *If* $v : (-\infty, a] \to \mathbb{X}, \ a > 0$*, such that* $\phi \in \mathcal{P}_{\mathfrak{h}}$*, the following circumstances are true* $\forall \tau \in [0, a]$*:*

1. $v_\tau \in \mathcal{P}_{\mathfrak{h}}$*;*

2. *There are two functions $\mu_1(\tau), \mu_2(\tau) > 0$ such that $\mu_1(\tau) : [0,\infty) \to [0,\infty)$ is a continuous function and $\mu_2(\tau) : [0,\infty) \to [0,\infty)$ is a locally bounded function, which are independent of $v(\cdot)$, whereas*

$$\|v_\tau\|_{\mathcal{P}_\mathfrak{h}} \leq \mu_1(\tau) \sup_{0<s<\tau} \|v(s)\| + \mu_2(\tau)\|\phi\|_{\mathcal{P}_\mathfrak{h}};$$

3. $\|v(\tau)\| \leq H\|v_\tau\|_{\mathcal{P}_\mathfrak{h}}$, *where $H > 0$ is constant.*

2.5. Some Other Basic Requirements

In the sequel, to present our results, we need the following:

Definition 8. *Suppose that $\tau > 0$; Mainardi's Wright-type function is defined as*

$$M_\varrho(\tau) = \sum_{n=0}^{\infty} \frac{(-\tau)^n}{n!\Gamma(1-\varrho(n+1))}, \quad \varrho \in (0,1), \quad \tau \in \mathbb{C}$$

and achieves

$$M_\varrho(\tau) \geq 0, \qquad \int_0^\infty \theta^\xi M_\varrho(\theta)d\theta = \frac{\Gamma(1+\xi)}{\Gamma(1+\varrho\xi)}, \qquad \xi > -1.$$

Lemma 4 (Gronwall-type inequality [38])**.** *Let $\nu : [a,b] \to [0,\infty)$ be a real function and $\omega : [a,b] \to [0,\infty)$ be a non-negative, locally integrable function on $[a,b]$, and there are a nondecreasing continuous function $g : [a,b] \to [0,\infty)$ and $\alpha > 0$ such that*

$$\nu(t) \leq \omega(t) + \frac{g(t)}{\rho^\alpha}\int_0^t (t^\rho - s^\rho)^{\alpha-1} ds^\rho.$$

Then,

$$\nu(t) \leq \omega(t) + \int_0^t \sum_{r=1}^{\infty} \frac{(g(t)\Gamma(\alpha))^r}{\rho^{r\alpha}\Gamma(r\alpha)}(t^\rho - s^\rho)^{r\alpha-1}\omega(s)ds^\rho.$$

In addition, if ω is nondecreasing, then

$$\nu(t) \leq \omega(t)\mathbb{E}_\alpha\left(g(t)\Gamma(\alpha)\left(\frac{t^\rho - a^\rho}{\rho}\right)^\alpha\right)$$

where

$$\mathbb{E}_\alpha(x) = \sum_{r=0}^{\infty} \frac{x^r}{\Gamma(\alpha r+1)}, \qquad \alpha > 0, \ x \in \mathbb{R}$$

denotes the Mittag–Leffler function.

A fascinating topological fixed-point theorem that has been greatly applied while dealing with the nonlinear equations is the following variant of the nonlinear alternative due to Leray and Schauder [39].

Theorem 3 (Nonlinear alternative Leray–Schauder Theorem)**.** *Let H be a Banach space, $D \subset H$ be a closed and convex set, and U be an open subset of D such that $0 \in U$. Suppose that the operator $\mathcal{T} : \overline{U} \to D$ is a continuous and compact map (that is, $\mathcal{T}(\overline{U})$ is a relatively compact subset of D). Then, either:*

1. *The operator $\mathcal{T}$ has a fixed point $x^* \in \overline{U}$;*
2. *There are $x \in \partial U$ (the boundary of U in D) and $\delta \in (0,1)$ such that $x = \delta\mathcal{T}(x)$.*

3. Setting of the Mild Solution

It is appropriate to recast the fractional semilinear problem (1) in the corresponding integral equation according to Definitions 1 and 2.

$$u(\tau) = \frac{\rho^{1-\alpha}}{\Gamma(\alpha)} \int_0^{\tau} \left(\tau^{\rho} - s^{\rho}\right)^{\alpha-1} \left\{ Au(s) + f(s) \right\} \frac{ds}{s^{1-\rho}} + \phi(0) + \frac{\tau_o}{\rho}\tau^{\rho} \tag{3}$$

as long as the integral in (3) exists.

First, we show the following lemma before giving a definition of the mild solution of (1).

Lemma 5. *Let (1) hold. Then, we have*

$$u(\tau) = \begin{cases} \mathfrak{C}_q(\tau)\phi(0) + \int_0^{\tau} \mathfrak{C}_q(s)\tau_o \frac{ds}{s^{1-\rho}} + \int_0^{\tau} \left(\frac{\tau^{\rho}-s^{\rho}}{\rho}\right)^{q-1} S_q(\tau,s) f(s) \frac{ds}{s^{1-\rho}} & \tau \in [0,a], \\ \phi(\tau) & \tau \in (-\infty, 0]. \end{cases}$$

where $1/2 < q = \frac{\alpha}{2} < 1$,

$$\mathfrak{C}_q(\tau) = \int_0^{\infty} M_q(\theta)\mathfrak{C}\left(\left(\frac{\tau^{\rho}}{\rho}\right)^q \theta\right) d\theta,$$
$$S_q(\tau,s) = q\int_0^{\infty} \theta M_q(\theta) S\left(\left(\frac{\tau^{\rho}-s^{\rho}}{\rho}\right)^q \theta\right) d\theta, \qquad 0 < s < \tau$$

where M_q *is a probability density function defined by Lemma 4.*

Proof. Let $p > 0$ be a variable of the ρ-Laplace transform. Suppose that

$$u(p) = \int_0^{\infty} e^{-p\frac{\tau^{\rho}}{\rho}} u(\tau) \frac{d\tau}{\tau^{1-\rho}}, \qquad f(p) = \int_0^{\infty} e^{-p\frac{\tau^{\rho}}{\rho}} f(\tau, u(\tau)) d\frac{d\tau}{\tau^{1-\rho}}.$$

Let $p^{\alpha} \in \varrho(A)$. Now, applying ρ-Laplace transforms to (3), we have

$$\begin{aligned} u(p) &= p^{-\alpha}\left[Au(p) + f(p)\right] + p^{-1}\phi(0) + p^{-2}\tau_o \\ &= (p^{\alpha} - A)^{-1}\left[f(p) + p^{\alpha-1}\phi(0) + p^{\alpha-2}\tau_o\right] \\ &= p^{q-1}\int_0^{\infty} e^{-p^q t}\mathfrak{C}(t)\phi(0)dt + p^{q-2}\int_0^{\infty} e^{-p^q t}\mathfrak{C}(t)\tau_o dt + \int_0^{\infty} e^{-p^q t} S(t) f(p) dt. \end{aligned}$$

Let $\theta \in (0,\infty)$, $q \in (\frac{1}{2}, 1)$, and $\Psi_q(\theta) = \frac{q}{\theta^{q+1}} M_q(\theta^{-q})$. Then,

$$\int_0^{\infty} e^{-p\theta}\Psi_q(\theta) d\theta = e^{-p^q}, for \;\; q \in (\frac{1}{2}, 1).$$

which can be used to calculate the first term replacing t with s^q as

$$\begin{aligned} p^{q-1}\int_0^\infty e^{-p^q t}\mathfrak{C}(t)\phi(0)dt =& q\int_0^\infty (ps)^{q-1}e^{-(ps)^q}\mathfrak{C}(s^q)\phi(0)ds \\ =& \frac{-1}{p}\int_0^\infty \frac{d}{ds}\left(e^{-(ps)^q}\right)\mathfrak{C}(s^q)\phi(0)ds \\ =& \int_0^\infty\int_0^\infty \theta\Psi_q(\theta)e^{-ps\theta}\mathfrak{C}(s^q)\phi(0)d\theta\, ds \\ =& \int_0^\infty e^{-p\frac{\tau^\rho}{\rho}}\left\{\int_0^\infty \Psi_q(\theta)\mathfrak{C}\left(\left(\frac{\tau^\rho}{\rho\theta}\right)^q\right)\phi(0)d\theta\right\}\frac{d\tau}{\tau^{1-\rho}} \\ =& \int_0^\infty e^{-p\frac{\tau^\rho}{\rho}}\left\{\int_0^\infty M_q(\theta)\mathfrak{C}\left(\left(\frac{\tau^\rho}{\rho}\right)^q\theta\right)\phi(0)d\theta\right\}\frac{d\tau}{\tau^{1-\rho}} \\ =& \int_0^\infty e^{-p\frac{\tau^\rho}{\rho}}\left\{\mathfrak{C}_q(\tau)\phi(0)\right\}\frac{d\tau}{\tau^{1-\rho}}. \end{aligned}$$

In addition, since $\mathcal{L}_\rho[1](p) = p^{-1}$, we obtain

$$p^{-1}p^{q-1}\int_0^\infty e^{-p^q t}\mathfrak{C}(t)\tau_o dt = \int_0^\infty e^{-p\frac{\tau^\rho}{\rho}}\left\{\int_0^\tau \mathfrak{C}_q(s)\tau_o\frac{ds}{s^{1-\rho}}\right\}\frac{d\tau}{\tau^{1-\rho}}.$$

Furthermore, we have that

$$\begin{aligned} \int_0^\infty e^{-p^q t}S(t)f(p)dt =& q\int_0^\infty e^{-(ps)^q}S(s^q)s^{q-1}f(p)ds \\ =& q\int_0^\infty\int_0^\infty e^{-ps\theta}\Psi_q(\theta)S(s^q)s^{q-1}f(p)d\theta ds \\ =& q\int_0^\infty\int_0^\infty e^{-p\frac{x^\rho}{\rho}}\Psi_q(\theta)S\left(\left(\frac{x^\rho}{\theta\rho}\right)^q\right)\left(\frac{x^\rho}{\theta\rho}\right)^{q-1}f(p)\frac{d\theta}{\theta}\frac{dx}{x^{1-\rho}} \\ =& q\int_0^\infty\int_0^\infty\int_0^\infty e^{-p\frac{\tau^\rho}{\rho}}e^{-p\frac{x^\rho}{\rho}}\Psi_q(\theta)S\left(\left(\frac{x^\rho}{\theta\rho}\right)^q\right)\left(\frac{x^\rho}{\theta\rho}\right)^{q-1}f(\tau)\frac{d\theta}{\theta}\frac{dx}{x^{1-\rho}}\frac{d\tau}{\tau^{1-\rho}} \\ =& q\int_0^\infty\int_0^\infty\int_0^\infty e^{-p\frac{x^\rho+\tau^\rho}{\rho}}M_q(\theta)S\left(\left(\frac{x^\rho}{\rho}\right)^q\theta\right)\left(\frac{x^\rho}{\rho}\right)^{q-1}f(\tau)\theta d\theta\frac{dx}{x^{1-\rho}}\frac{d\tau}{\tau^{1-\rho}} \\ =& q\int_0^\infty e^{-p\frac{t^\rho}{\rho}}\left\{\int_0^\infty\int_0^t \theta M_q(\theta)S\left(\left(\frac{t^\rho-\tau^\rho}{\rho}\right)^q\theta\right)\left(\frac{t^\rho-\tau^\rho}{\rho}\right)^{q-1}f(\tau)d\theta\frac{d\tau}{\tau^{1-\rho}}\right\}\frac{dt}{t^{1-\rho}} \\ =& \int_0^\infty e^{-p\frac{t^\rho}{\rho}}\left\{\int_0^t\left(\frac{t^\rho-\tau^\rho}{\rho}\right)^{q-1}S_q(t,\tau)f(\tau)\frac{d\tau}{\tau^{1-\rho}}\right\}\frac{dt}{t^{1-\rho}}. \end{aligned}$$

In conclusion, we can write

$$\begin{aligned} \int_0^\infty e^{-p\frac{\tau^\rho}{\rho}}u(\tau)\frac{d\tau}{\tau^{1-\rho}} =& \int_0^\infty e^{-p\frac{\tau^\rho}{\rho}}\left\{\mathfrak{C}_q(s)\phi(0) + \int_0^\tau \mathfrak{C}_q(s)\tau_o\frac{ds}{s^{1-\rho}}\right. \\ &\left. + \int_0^\tau\left(\frac{\tau^\rho - s^\rho}{\rho}\right)^{q-1}S_q(\tau,s)f(s)\frac{ds}{s^{1-\rho}}\right\}\frac{d\tau}{\tau^{1-\rho}}. \end{aligned}$$

By applying the inverse ρ-Laplace transform, we obtain the desired result. □

Definition 9. *A function $u(\tau) \in (\mathcal{C}(-\infty,a];\mathbb{X})$ is said to be the mild solution of* (1) *if it satisfies*

$$u(\tau) = \begin{cases} \mathfrak{C}_q(\tau)\phi(0) + \int_0^\tau \mathfrak{C}_q(s)\tau_o\frac{ds}{s^{1-\rho}} + \int_0^\tau\left(\frac{\tau^\rho - s^\rho}{\rho}\right)^{q-1}S_q(\tau,s)f(s,u_s)\frac{ds}{s^{1-\rho}} & \tau\in[0,a], \\ \phi(\tau), & \tau\in(-\infty,0]. \end{cases}$$

Remark 1. *From the linearity of $\mathfrak{C}(\tau)$ and $S(\tau)$ for all $\tau \geq 0$, it is clearly deduced that $\mathfrak{C}_q(\tau)$ and $S_q(\tau,s)$ are also linear operators where $0 < s < \tau$.*

Lemma 6. *The following estimates for $\mathfrak{C}_q(\tau)$ and $S_q(\tau,s)$ are verified for any fixed $\tau \geq 0$ and $0 < s < \tau$:*

$$\left|\mathfrak{C}_q(\tau)x\right| \leq M|x| \qquad \text{and} \qquad \left|S_q(\tau,s)x\right| \leq M_1|x|$$

where M is defined in (2) and $M_1 = \frac{Ma^{q\rho}}{\rho^q\Gamma(2q)}$.

Proof. For any $x \in \mathbb{X}$ and fixed $\tau \geq 0$, according to Lemma 4 and Inequality (2), direct calculation gives:

$$|\mathfrak{C}_q(\tau)x| = \left|\int_0^\infty M_q(\theta)\mathfrak{C}\left(\left(\frac{\tau^\rho}{\rho}\right)^q\theta\right)xd\theta\right| \leq M|x|. \tag{4}$$

In addition, for any fixed $0 \leq \tau < a$ and $0 < s < \tau$, we have

$$\begin{aligned}\left|S_q(\tau,s)x\right| =& \left|q\int_0^\infty \theta M_q(\theta)S\left(\left(\frac{\tau^\rho - s^\rho}{\rho}\right)^q\theta\right)xd\theta\right| \\ &\leq q\int_0^\infty \theta M_q(\theta)\int_0^{\left(\frac{\tau^\rho - s^\rho}{\rho}\right)^q\theta}|\mathfrak{C}(t)x|dtd\theta \\ &\leq qM|x|\left(\frac{\tau^\rho - s^\rho}{\rho}\right)^q\int_0^\infty \theta^2 M_q(\theta)d\theta \\ &= \frac{M|x|}{\Gamma(2q)}\left(\frac{\tau^\rho - s^\rho}{\rho}\right)^q.\end{aligned}$$

Since $0 < s < \tau \leq a$, then

$$\left(\frac{\tau^\rho - s^\rho}{\rho}\right)^q \leq \left(\frac{\tau^\rho}{\rho}\right)^q \leq \frac{a^{q\rho}}{\rho}$$

which implies that

$$\left|S_q(\tau,s)x\right| \leq \frac{Ma^{q\rho}|x|}{\rho^q\Gamma(2q)} = M_1|x|. \tag{5}$$

This ends the proof. □

Lemma 7. *The operators $\mathfrak{C}_q(\tau)$ and $S_q(s,\tau)$ are strongly continuous for every $0 < s < \tau$ and $\tau > 0$.*

Proof. It is known that the cosine family $\{\mathfrak{C}(\tau)\}_{\tau\geq 0}$ of uniformly bounded linear operators based on infinite Banach space $\mathbb{X}$ is strongly continuous, which implies that for any $x \in \mathbb{X}$ and $\tau_1, \tau_2 \in \mathbb{R}$, it follows that $|\mathfrak{C}(\tau_2)x - \mathfrak{C}(\tau_1)x| \to 0$ as $\tau_2 \to \tau_1$. Hence, let $\tau_1 < \tau_2$. Then, we obtain:

$$\left|\mathfrak{C}_q(\tau_2)x - \mathfrak{C}_q(\tau_1)x\right| \leq \int_0^\infty M_q(\theta)\left|\left[\mathfrak{C}\left(\left(\frac{\tau_2^\rho}{\rho}\right)^q\theta\right) - \mathfrak{C}\left(\left(\frac{\tau_1^\rho}{\rho}\right)^q\theta\right)\right]x\right|d\theta \to\ 0 \text{ as } \ \tau_2 \to \tau_1.$$

On other hand, from the strongly continuous sine family operator, we have

$$\left|S_q(\tau_2,s)x - S_q(\tau_1,s)x\right| \leq \int_0^\infty q\theta M_q(\theta)\left|\left[S\left(\left(\frac{\tau_2^\rho - s^\rho}{\rho}\right)^q\theta\right) - S\left(\left(\frac{\tau_1^\rho - s^\rho}{\rho}\right)^q\theta\right)\right]x\right|d\theta \to 0$$

as $\tau_2 \to \tau_1$. This ends the proof. □

Lemma 8. *Assume that $\mathfrak{C}(\tau)$ and $S(\tau,s)$ are compact for every $0 < s < \tau$. Then, the operators $\mathfrak{C}_q(\tau)$ and $S_q(s,\tau)$ are compact for every $0 < s < \tau$.*

Proof. Let

$$V_r = \{x \in C(-\infty,a],\mathbb{X}); \|x\| \leq r,\ r > 0\}.$$

Clearly, V_r is a closed, bounded, and convex subset in $\mathcal{C}((-\infty, a], \mathbb{X})$. That is, to verify any $\tau \in (-\infty, a]$, the set

$$\Omega(\tau) = \{\mathfrak{C}_q(\tau)x, x \in V_r\}$$

is relatively compact on $\mathbb{X}$. In view of the previous two lemmas, the set Ω is bounded on V_r and is equicontinuous, by which we conclude that the set Ω is relatively compact on $\mathbb{X}$. Therefore, in conjunction with the Arzela–Ascoli theorem, the operator $\mathfrak{C}_q(\tau)$ is compact. Furthermore, in a similar way, we can deduce that the operator $S_q(\tau, s)$ is a compact operator for all $0 < s < \tau \leq a$. $\square$

4. The Main Results

Define the operator $\mathscr{N} : \overline{\mathcal{P}}_{\mathfrak{h}} \to \overline{\mathcal{P}}_{\mathfrak{h}}$ as follows:

$$\mathscr{N}(u)(\tau) = \begin{cases} \mathfrak{C}_q(\tau)\phi(0) + \int_0^\tau \mathfrak{C}_q(s)\tau_o \frac{ds}{s^{1-\rho}} + \int_0^\tau S_q(\tau,s) f(s,u,u_s) \left(\frac{\tau^\rho - s^\rho}{\rho}\right)^{q-1} \frac{ds}{s^{1-\rho}}, & \tau \in [0,a], \\ \phi(\tau), & \tau \in (-\infty, 0]. \end{cases}$$

Let $x(\cdot) : (-\infty, a] \to \mathbb{X}$ be the function denoted by

$$x(\tau) = \begin{cases} 0, & \tau \in (0,a], \\ \phi(\tau), & \tau \in (-\infty, 0]. \end{cases}$$

After that, $x(0) = \phi(0)$. We indicate the function defined by $\mathfrak{z}$ for each $z \in \mathcal{C}([0,a], \mathbb{X})$ with $z(0) = 0$, by

$$\mathfrak{z}(\tau) = \begin{cases} z(\tau), & \tau \in [0,a], \\ 0, & \tau \in (-\infty, 0]. \end{cases}$$

If $u(\cdot)$ satisfies that $u(\tau) = \mathscr{N}(u)(\tau)$ for all $\tau \in (-\infty, a]$, we can decompose $u(\tau) = \mathfrak{z}(\tau) + x(\tau)$, $\tau \in (-\infty, a]$; it is denoted as $u_\tau = \mathfrak{z}_\tau + x_\tau$ for every $\tau \in (-\infty, a]$, and the function $z(\cdot)$ satisfies

$$z(\tau) = \mathfrak{C}_q(\tau)\phi(0) + \int_0^\tau \mathfrak{C}_q(s)\tau_o \frac{ds}{s^{1-\rho}} + \int_0^\tau S_q(\tau,s) f(s, \mathfrak{z} + x, \mathfrak{z}_s + x_s) \left(\frac{\tau^\rho - s^\rho}{\rho}\right)^{q-1} \frac{ds}{s^{1-\rho}}.$$

Set the space $\Theta = \{z \in \mathcal{C}([0,a], \mathbb{X}), z(0) = 0\}$ equipped with the norm

$$\|z\|_\Theta = \underset{\tau \in [0,a]}{sup} \|z(\tau)\|.$$

Therefore, $(\Theta, \|\cdot\|_\Theta)$ is a Banach space. Assume that the operator G is defined as follows: Let the operator $G : \Theta \to \Theta$ be formulated as follows:

$$G(z)(\tau) = \mathfrak{C}_q(\tau)\phi(0) + \int_0^\tau \mathfrak{C}_q(s)\tau_o \frac{ds}{s^{1-\rho}} + \int_0^\tau S_q(\tau,s) f(s, \mathfrak{z} + x, \mathfrak{z}_s + x_s) \left(\frac{\tau^\rho - s^\rho}{\rho}\right)^{q-1} \frac{ds}{s^{1-\rho}}.$$

That the operator $\mathscr{N}$ seems to have a fixed point is equivalent to G having a fixed point; thus, we proceed to prove that G has a fixed point.

Lemma 9. *Let $\mu_1^* = \sup_{\tau \in [0,a]} \mu_1(\tau)$ and $\mu_2^* = \sup_{\tau \in [0,a]} \mu_2(\tau)$ where $\mu_1(\cdot)$ and $\mu_2(\cdot)$ are defined in Definition (7). Assume that the assumption $(\mathcal{H}_1)$ is satisfied with $M_2 = \max_{\tau \in [0,a]} |f(\tau, 0, 0)|$. Then,*

$$\begin{aligned} \|f(\tau, \mathfrak{z} + x, \mathfrak{z}_\tau + x_\tau)\| &\leq (C_1 H + C_2)\left(\mu_1(\tau)\|z\|_\Theta + \mu_2(\tau)\|\phi\|_{\mathcal{P}_{\mathfrak{h}}}\right) + M_2 \\ &\leq (C_1 H + C_2)\left(\mu_1^*\|z\|_\Theta + \mu_2^*\|\phi\|_{\mathcal{P}_{\mathfrak{h}}}\right) + M_2. \end{aligned}$$

Proof. In view of Definition 7 and assumption $(\mathcal{H}_1)$, we obtain

$$\begin{aligned}
\|f(\tau,\mathfrak{z}+x,\mathfrak{z}_\tau+x_\tau)\| &= \|f(\tau,\mathfrak{z}+x,\mathfrak{z}_\tau+x_\tau)-f(\tau,0,0)+f(\tau,0,0)\| \\
&\le \|f(\tau,\mathfrak{z}+x,\mathfrak{z}_\tau+x_\tau)-f(\tau,0,0)\|+\|f(\tau,0,0)\| \\
&\le C_1\|\mathfrak{z}+x\|_\Theta + C_2\|\mathfrak{z}_\tau+x_\tau\|_{\mathcal{P}_\mathfrak{h}} + M_2 \\
&\le (C_1H+C_2)\|\mathfrak{z}_\tau+x_\tau\|_{\mathcal{P}_\mathfrak{h}} + M_2 \\
&\le (C_1H+C_2)\left(\mu_1(\tau)\sup_{0<\tau<a}\|\mathfrak{z}(\tau)\| + \mu_2(\tau)\|\phi\|_{\mathcal{P}_\mathfrak{h}}\right) + M_2 \\
&= (C_1H+C_2)\left(\mu_1(\tau)\|z\|_\Theta + \mu_2(\tau)\|\phi\|_{\mathcal{P}_\mathfrak{h}}\right) + M_2 \\
&\le (C_1H+C_2)\left(\mu_1^*\|z\|_\Theta + \mu_2^*\|\phi\|_{\mathcal{P}_\mathfrak{h}}\right) + M_2.
\end{aligned}$$

This ends the proof. □

Lemma 10. *Let $\mu_1^* = \sup_{\tau\in[0,a]}\mu_1(\tau)$ and $\mu_2^* = \sup_{\tau\in[0,a]}\mu_2(\tau)$ where $\mu_1(\cdot)$ and $\mu_2(\cdot)$ are defined in Definition (7). Assume that the assumption $(\mathcal{H}_2)$ is satisfied with $\eta_1 = \sup_{\tau\in[0,a]}\eta_1(\tau)$ and $\eta_2 = \sup_{\tau\in[0,a]}\eta_2(\tau)$. Then,*

$$\|f(\tau,\mathfrak{z}+x,\mathfrak{z}_\tau+x_\tau)\| \le \ell(\tau) \le \ell$$

where

$$\ell(\tau) = (\eta_1(\tau)H+\eta_2(\tau))\left(\mu_1(\tau)\|z\|_\Theta + \mu_2(\tau)\|\phi\|_{\mathcal{P}_\mathfrak{h}}\right)$$

and

$$\ell = \sup_{0\le\tau\le a}\ell(\tau) = (\eta_1H+\eta_2)\left(\mu_1^*\|z\|_\Theta + \mu_2^*\|\phi\|_{\mathcal{P}_\mathfrak{h}}\right).$$

Proof. In the same way as in Lemma 9, we can easily reach the desired result. □

We now make the following assumptions:

$(\mathcal{H}_1)$ $f : [0,a]\times\mathbb{X}\times\mathcal{P}_\mathfrak{h} \to \mathbb{X}$ is a continuous function, and there exist two constants $C_1, C_2 \ge 0$ such that for any $(\tau,u,u_\tau),(\tau,v,v_\tau) \in [0,a]\times\mathbb{X}\times\mathcal{P}_\mathfrak{h}$, we have

$$\|f(\tau,u,u_\tau)-f(\tau,v,v_\tau)\| \le C_1\|u-v\|_\mathbb{X} + C_2\|u_\tau-v_\tau\|_{\mathcal{P}_\mathfrak{h}}.$$

$(\mathcal{H}_2)$ There is a positive constant $\mathcal{L}_\alpha < 1$ that ensures

$$\mathcal{L}_\alpha = \frac{M_1a^{\rho q}}{q\rho^q}(C_1H + C_2\mu_1^*)$$

where $\mu_1^* = \sup_{\tau\in[0,a]}\mu_1(\tau)$, which is mentioned in Definition 7.

$(\mathcal{H}_3)$ $f : [0,a]\times\mathbb{X}\times\mathcal{P}_\mathfrak{h} \to \mathbb{X}$ is a continuous function, and there exist two continuous functions $\eta_i(\tau) : [0,a] \to [0,\infty)$, where $i = 1,2$ ensures that for any $(\tau,u,u_\tau) \in [0,a]\times\mathbb{X}\times\mathcal{P}_\mathfrak{h}$, we have

$$\|f(\tau,u,u_\tau)\| \le \eta_1(\tau)\|u\|_\mathbb{X} + \eta_2(\tau)\|u_\tau\|_{\mathcal{P}_\mathfrak{h}}.$$

$(\mathcal{H}_4)$ There exists a positive number $\mathcal{N}$ ensuring $\mathcal{N}/\sigma > 1$ where

$$\sigma = M\left(\|\phi(0)\| + \|\tau_o\|\frac{a^\rho}{\rho}\right) + \frac{\Lambda M_1a^{q\rho}}{q\rho^q}\mathbb{E}_q\left(g\Gamma(q)\left(\frac{a^\rho}{\rho}\right)^q\right).$$

Here,

$$\Lambda = \sup_{0\le\tau\le a}\Lambda(\tau) = (\eta_1 H + \eta_2)\left(\mu_2^*\|\phi\|_{\mathcal{P}_{\mathfrak{h}}} + \mu_1^*\left(\|\phi(0)\| + \frac{\|\tau_0\|a^\rho}{\rho}\right)\right),$$
$$g = \sup_{0\le\tau\le a} g(\tau) = M_1\mu_1^*(\eta_1 H + \eta_2).$$

where $\mu_i^* = \sup_{\tau\in[0,a]}\mu_i(\tau), i = 1,2$, $\eta_i = \sup_{\tau\in[0,a]}\eta_i(\tau), i = 1,2$ and

$$\Lambda(\tau) = (\eta_1(\tau)H + \eta_2(\tau))\left(\mu_2(\tau)\|\phi\|_{\mathcal{P}_{\mathfrak{h}}} + \mu_1(\tau)\left(\|\phi(0)\| + \frac{\|\tau_0\|a^\rho}{\rho}\right)\right),$$
$$g(\tau) = M_1\mu_1(\tau)(\eta_1(\tau)H + \eta_2(\tau)).$$

Theorem 4. *On the basis of the assumptions* $(\mathcal{H}_1)$ *and* $(\mathcal{H}_2)$*, the fractional semilinear problem* (1) *has a unique mild solution on* $(-\infty, a]$.

Proof. To demonstrate that G maps bounded sets of Θ into bounded sets in Θ, for any $\mathcal{L} \ge 0$, we set

$$\Theta_{\mathcal{L}} = \{z \in \Theta : \|z\|_\Theta \le \mathcal{L}\}$$

where

$$\mathcal{L} \ge \frac{M\left(\|\phi(0)\| + \|\tau_o\|\frac{a^\rho}{\rho}\right) + \frac{a^q M_1}{q\rho^q}\left[(C_1 H + C_2)\left(\mu_2^*\|\phi\|_{\mathcal{P}_{\mathfrak{h}}}\right) + M_2\right]}{1 - \mathcal{L}_\alpha}.$$

Then, for any $z \in \Theta_{\mathcal{L}}$, we obtain

$$\begin{aligned}\|G(z)(\tau)\| &\le \|\mathfrak{C}_q(\tau)\phi(0)\| + \left\|\int_0^\tau \mathfrak{C}_q(s)\tau_o\frac{ds}{s^{1-\rho}}\right\| + \left\|\int_0^\tau S_q(\tau,s)f(s,\mathfrak{z}+x,\mathfrak{z}_s+x_s)\left(\frac{\tau^\rho - s^\rho}{\rho}\right)^{q-1}\frac{ds}{s^{1-\rho}}\right\|\\ &\le M\left(\|\phi(0)\|_{\mathcal{P}_{\mathfrak{h}}} + \|\tau_0\|\frac{a^\rho}{\rho}\right) + \frac{M_1}{\rho^{q-1}}\int_0^\tau(\tau^\rho - s^\rho)^{q-1}\|f(s,\mathfrak{z}+x,\mathfrak{z}_s+x_s)\|\frac{ds}{s^{1-\rho}}.\end{aligned}$$

Using the result in Lemma 9 gives

$$\begin{aligned}\|G(z)(\tau)\| \le\, & M\left(\|\phi(0)\|_{\mathcal{P}_{\mathfrak{h}}} + \|\tau_o\|\frac{a^\rho}{\rho}\right)\\ &+ \frac{M_1\left((C_1 H + C_2)\left(\mu_1^*\|z\|_\Theta + \mu_2^*\|\phi\|_{\mathcal{P}_{\mathfrak{h}}}\right) + M_2\right)}{\rho^q}\int_0^\tau(\tau^\rho - s^\rho)^{q-1}ds^\rho\\ &\le M\left(\|\phi(0)\|_{\mathcal{P}_{\mathfrak{h}}} + \|\tau_o\|\frac{a^\rho}{\rho}\right) + \frac{a^q}{q\rho^q}M_1\left[(C_1 H + C_2)\left(\mu_1^*\mathcal{L} + \mu_2^*\|\phi\|_{\mathcal{P}_{\mathfrak{h}}}\right) + M_2\right] \le \mathcal{L}.\end{aligned}$$

Now, we show that the operator G is a contraction map. Indeed, consider $z, z^* \in \Theta$. Then, for any $\tau \in [0,a]$, we have

$$\begin{aligned}\|G(z)(\tau) - G(z^*)(\tau)\| &\le \frac{M_1}{\rho^q}\int_0^\tau(\tau^\rho - s^\rho)^{q-1}\left[C_1\|\mathfrak{z}(s) - \mathfrak{z}^*(s)\|_\Theta + C_2\|\mathfrak{z}_s - \mathfrak{z}^*{}_s\|_{\mathcal{P}_{\mathfrak{h}}}\right]ds^\rho\\ &\le \frac{M_1}{\rho^q}\int_0^\tau(\tau^\rho - s^\rho)^{q-1}(C_1 H + C_2)\|\mathfrak{z}_s - \mathfrak{z}^*{}_s\|_{\mathcal{P}_{\mathfrak{h}}}ds^\rho\\ &\le \frac{M_1}{\rho^q}(C_1 H + C_2\mu_1^*)\|z - z^*\|_\Theta\int_0^\tau(\tau^\rho - s^\rho)^{q-1}ds^\rho\\ &\le \frac{M_1 a^{\rho q}}{q\rho^q}(C_1 H + C_2\mu_1^*)\|z - z^*\|_\Theta\\ &= \mathcal{L}_\alpha\|\mathfrak{z} - \mathfrak{z}^*\|_\Theta.\end{aligned}$$

As a result of the preceding, we conclude that

$$\|G(z) - G(z^*)\|_\Theta \leq \mathcal{L}_\alpha \|\mathfrak{z} - \mathfrak{z}^*\|_\Theta$$

in view of the given condition $\mathcal{L}_\alpha < 1$. Hence, taking into consideration the assumption $(\mathcal{H}_1)$, we may derive that G has a unique fixed point using Banach's contraction mapping principle and that (9) is the mild solution to the problem (1) on $(-\infty, a]$. $\square$

Theorem 5. *Under the assumptions $(\mathcal{H}_3)$ and $(\mathcal{H}_4)$, the fractional semilinear problem (1) has at least one mild solution on $(-\infty, a]$.*

Proof. We seek to employ the Leray–Schauder alternative theorem, which is based on satisfying the four phases that follow the operator $G : \Theta \to \Theta$ defined as above:

- **Step (1):** The continuity of the operator G due to the continuity of the function f according to the assumption $(\mathcal{H}_2)$ and the continuity of the operators $\mathfrak{C}_q(\tau)$ and $S_q(\tau, s)$ according to Lemma 7 is clear.
- **Step (2):** To show that the operator $G : \Theta \to \Theta$ is bounded on a bounded subset of Θ, let the set $\Theta_\mathcal{L}$ be defined as in Theorem 4, and take $z \in \Theta_\mathcal{L}$; by the assumption $(\mathcal{H}_2)$ and Lemma 10, we have

$$\begin{aligned}
\|G(z)(\tau)\|_\Theta &\leq M\left(\|\phi(0)\| + \|\tau_o\|\frac{a^\rho}{\rho}\right) + \frac{M_1}{\rho^q}\int_0^\tau (\tau^\rho - s^\rho)^{q-1}\ell(s)ds^\rho \\
&\leq M\left(\|\phi(0)\| + \|\tau_o\|\frac{a^\rho}{\rho}\right) + \frac{M_1\ell}{\rho^q}\int_0^\tau (\tau^\rho - s^\rho)^{q-1}ds^\rho \\
&= M\left(\|\phi(0)\| + \|\tau_o\|\frac{a^\rho}{\rho}\right) + \frac{a^{\rho q}M_1\ell}{q\rho^q}
\end{aligned}$$

  which implies the boundedness of the operator G on the bounded subset $\Theta_\mathcal{L}$.
- **Step (3):** To demonstrate that the set $\{G(z) : z \in \Theta\}$ is equicontinuous, let $z \in \Theta$ and $\tau_1, \tau_2 \in [0, a]$ with $\tau_1 > \tau_2$. Then,

$$\begin{aligned}
\|G(z)(\tau_1) - G(z)(\tau_2)\|_\Theta &\leq \|\mathfrak{C}_q(\tau_1) - \mathfrak{C}_q(\tau_2)\|\|\phi(0)\| + \|\mathfrak{C}_q(s)\|\|\tau_0\|\int_{\tau_2}^{\tau_1} s^{\rho-1}ds \\
&+ \frac{1}{\rho^q}\int_0^{\tau_2}\left\|\left(\tau_1^\rho - s^\rho\right)^{q-1}S_q(\tau_1, s) - \left(\tau_2^\rho - s^\rho\right)^{q-1}S_q(\tau_2, s)\right\|\ell(s)ds^\rho \\
&+ \frac{M_1\ell}{\rho^q}\int_{\tau_2}^{\tau_1}\left(\tau_1^\rho - s^\rho\right)^{q-1}ds^\rho \\
&\leq \|\mathfrak{C}_q(\tau_1) - \mathfrak{C}_q(\tau_2)\|\|\phi(0)\| + M\|\tau_0\|\frac{\tau_1^\rho - \tau_2^\rho}{\rho} + \frac{M_1\ell}{\rho^q}\frac{(\tau_1^\rho - \tau_2^\rho)^q}{q} \\
&+ \frac{1}{\rho^q}\int_0^{\tau_2}\left\|\left(\tau_1^\rho - s^\rho\right)^{q-1}S_q(\tau_1, s) - \left(\tau_2^\rho - s^\rho\right)^{q-1}S_q(\tau_2, s)\right\|\ell(s)ds^\rho.
\end{aligned}$$

  Now, we have

$$\begin{aligned}
&\int_0^{\tau_2}\left\|\left(\tau_1^\rho - s^\rho\right)^{q-1}S_q(\tau_1, s) - \left(\tau_2^\rho - s^\rho\right)^{q-1}S_q(\tau_2, s)\right\|\ell(s)ds^\rho \\
&\leq \int_0^{\tau_2}\left(\tau_1^\rho - s^\rho\right)^{q-1}\|S_q(\tau_1, s) - S_q(\tau_2, s)\|\ell(s)ds^\rho \\
&+ \int_0^{\tau_2}\left(\left(\tau_2^\rho - s^\rho\right)^{q-1} - \left(\tau_1^\rho - s^\rho\right)^{q-1}\right)\|S_q(\tau_2, s)\|\ell(s)ds^\rho \\
&\leq \int_0^{\tau_2}\left(\tau_1^\rho - s^\rho\right)^{q-1}\|S_q(\tau_1, s) - S_q(\tau_2, s)\|\ell(s)ds^\rho + \frac{\ell M_1}{q}\left(\tau_1^\rho - \tau_2^\rho\right)^q.
\end{aligned}$$

Due to the compactness of the operators $\mathfrak{C}_q(s)$ and $S_q(\tau, s)$ (see Lemma 7), we deduce that $\|G(z)(\tau_1) - G(z)(\tau_2)\| \to 0$ as $\tau_2 \to \tau_1$. Consequently, the set $\{G(z) : z \in \Theta\}$ is equicontinuous.

- **Step(4):** (A priori bounds) We now argue that there is an open set $\mathcal{U} \subseteq \Theta$ with $z \neq \lambda G(z)$ for $\lambda \in (0,1)$ and $z \in \partial U$. To do this: Let $z \in \Theta$ and $z = \lambda G(z)$ for some $0 < \lambda < 1$. Then, for each $\tau \in [0, a]$ using the result obtained in **Step (2)**, we have

$$\|z\|_\Theta = \lambda \|G(z)\|_\Theta \leq \|G(z)\|_\Theta \leq M\left(\|\phi(0)\| + \|\tau_o\| \frac{a^\rho}{\rho}\right) + \frac{M_1}{\rho^q} \int_0^\tau (\tau^\rho - s^\rho)^{q-1} \ell(s) ds^\rho.$$

Hence, the inequality above and the definition of ℓ give

$$\begin{aligned}
\ell(\tau) &= (\eta_1(\tau)H + \eta_2(\tau))\Big(\mu_2(\tau)\|\phi\|_{\mathcal{P}_\mathfrak{h}} + \mu_1(\tau)\|z\|_\Theta\Big) \\
&\leq (\eta_1(\tau)H + \eta_2(\tau))\Big(\mu_2(\tau)\|\phi\|_{\mathcal{P}_\mathfrak{h}} + \mu_1(\tau)\Big[\Big(\|\phi(0)\| + \|\tau_o\| \frac{a^\rho}{\rho}\Big) + \frac{M_1}{\rho^q} \int_0^\tau (\tau^\rho - s^\rho)^{q-1} \ell(s) ds^\rho\Big]\Big) \\
&= \Lambda(\tau) + \frac{g(\tau)}{\rho^q} \int_0^\tau (\tau^\rho - s^\rho)^{q-1} \ell(s) ds^\rho.
\end{aligned}$$

Using the Gronwall-type inequality (Lemma 4), then,

$$\begin{aligned}
\ell(\tau) &\leq \Lambda(\tau) + \int_0^\tau \sum_{r=1}^{\infty} \frac{(g(s)\Gamma(q))^r}{\rho^{rq}\Gamma(rq)} (\tau^\rho - s^\rho)^{rq-1} \Lambda(s) ds^\rho \\
&\leq \Lambda \mathbb{E}_q\left(g(\tau)\Gamma(q)\left(\frac{\tau^\rho}{\rho}\right)^q\right) \leq \Lambda \mathbb{E}_q\left(g\Gamma(q)\left(\frac{a^\rho}{\rho}\right)^q\right).
\end{aligned}$$

Hence

$$\begin{aligned}
\|z\|_\Theta &\leq M\left(\|\phi(0)\| + \|\tau_o\| \frac{a^\rho}{\rho}\right) + \frac{\Lambda M_1}{\rho^q} \mathbb{E}_q\left(g\Gamma(q)\left(\frac{a^\rho}{\rho}\right)^q\right) \int_0^\tau (\tau^\rho - s^\rho)^{q-1} ds^\rho \\
&\leq M\left(\|\phi(0)\| + \|\tau_o\| \frac{a^\rho}{\rho}\right) + \frac{\Lambda M_1 a^{q\rho}}{q\rho^q} \mathbb{E}_q\left(g\Gamma(q)\left(\frac{a^\rho}{\rho}\right)^q\right) = \sigma.
\end{aligned}$$

Set

$$\mathcal{U} = \{z \in \Theta : \|z\|_\Theta < \mathcal{N} + 1\}.$$

The operator $G : \bar{\mathcal{U}} \to \Theta$ is obviously continuous and completely continuous. According to the assumption $(\mathcal{H}_4)$ and from the selection of $\mathcal{U}$, there is no $z \in \partial\mathcal{U}$ such that $z = \lambda G(z)$ for $\lambda \in (0,1)$.

We derive that G has a fixed point z in U as a result of the nonlinear alternative of the Leray–Schauder type. Thus, on $(-\infty, a]$, the problem (1) has at least one mild solution on $(-\infty, a]$. □

5. An Application

Consider the following fractional wave equation with infinite time delay:

$$\begin{cases}
{}^\rho_c\mathcal{D}^\alpha_0 u(\tau, x) = \dfrac{\partial^2}{\partial x^2} u(\tau, x) + f(\tau, x, u, u_\tau), & \tau \in [0,1],\ x \in [0, \pi] \\
u(\tau, x) = \cos 3\tau, & \tau \in (-\infty, 0],\ x \in [0, \pi], \\
\lim\limits_{\tau \to 0^+} \tau^{-\rho} u'(\tau) = 3e^{-x}, & x \in [0, \pi], \\
u(\tau, 0) = u(\tau, \pi) = 0, & \tau \in [0, 1].
\end{cases} \tag{6}$$

Let the space $\mathbb{X} = C([0,1] \times \mathbb{R}, \mathbb{R})$ be the space of all continuous functions equipped with the norm $\|u\|_{\mathbb{X}} = \sup_{\tau \in [0,1]} \sup_{x \in [0,\pi]} |u(\tau, x)|$, and the operator $A : D(A) \subseteq \mathbb{X} \to \mathbb{X}$ is defined as $A = \frac{\partial^2}{\partial x^2} u(\tau, x)$ with a domain

$$D(A) = \left\{ u \in \mathbb{X} | \frac{\partial}{\partial x} u, \frac{\partial^2}{\partial x^2} u \in \mathbb{X} \right\}.$$

Obviously, the operator A is densely defined in $\mathbb{X}$ and is the infinitesimal generator of a resolvent cosine family $\mathfrak{C}(\tau), \tau > 0$ on $\mathbb{X}$.

Here, we take $\alpha = \frac{3}{2}$, which implies $q = \frac{3}{4}$, and the parameter $\rho = \frac{3}{4}$, $\tau_0 = 3e^{-x}$ $A = \frac{\partial^2}{\partial x^2}$, $x \in [0, \pi]$, $H = \frac{1}{16}$, $\mu_1(\tau) = \tau^{\frac{1}{4}} \to \mu_1^* = 1$, $\mu_2(\tau) = \frac{1}{\tau^2+2} \to \mu_2^* = \frac{1}{2}$, and $\|\mathfrak{C}_q(\tau)\| \leq 1, \|S_q(\tau, s)\| \leq 1.4001$ for all $0 < s < \tau \leq 1$.

Let $h(s) = e^{3s}, s < 0$, then $\int_{-\infty}^{0} h(s) ds = \frac{1}{3}$, and we define

$$\|\phi\|_{\mathcal{P}_{\mathfrak{h}}} = \int_{-\infty}^{0} e^{3s} \sup_{\tau \leq s \leq 0} \|\phi(s)\| d\tau$$

Then, we can say

$$\|\phi\|_{\mathcal{P}_{\mathfrak{h}}} = \| \cos 3\tau \|_{\mathcal{P}_{\mathfrak{h}}} = \frac{1}{3}.$$

Case I: Banach fixed-point theorem. To outline Theorem 4, we take

$$f(\tau, u, u_\tau) = (1 + \tau \sin u) + \frac{\tau^{\frac{7}{2}} u_\tau}{12}$$

Clearly, $f : [0,1] \times \mathbb{X} \times \mathcal{P}_{\mathfrak{h}} \to \mathbb{X}$ is continuous and satisfies, for $u, v \in \mathbb{X}$ and $u_\tau, v_\tau \in \mathcal{P}_{\mathfrak{h}}$, that

$$\|f(\tau, u, u_\tau) - f(\tau, v, v_\tau)\| \leq \frac{\tau^{\frac{7}{2}}}{12} \|u_\tau - v_\tau\|_{\mathcal{P}_{\mathfrak{h}}} + \tau \|\sin u - \sin v\|_{\mathbb{X}} \leq \frac{1}{12} \|u_\tau - v_\tau\|_{\mathcal{P}_{\mathfrak{h}}} + \|u - v\|_{\mathbb{X}}$$

which implies that $C_1 = 1$ and $C_2 = \frac{1}{12}$. Observe that the condition $(\mathcal{H}_1)$ holds true. To check the presumption of Theorem 4, we have $\mathcal{L}_\alpha = 0.3378 < 1$. Thus, all assumptions of this theorem are satisfied. Therefore, the fractional wave Equation (6) has a unique mild solution on $(-\infty, 1]$.

Case II: Leray–Schauder nonlinear alternative theorem. To realize Theorem 5, we assume

$$f(\tau, u, u_\tau) = \frac{\tau^2 \sin(\pi\tau)}{1 + \sqrt{\tau}} \sin u + \frac{\tau^{\frac{8}{3}} \cos(\pi\tau)}{4 + \sqrt{\tau}} u_\tau$$

It is not uncomplicated to show that

$$\|f(\tau, u, u_\tau)\| \leq \frac{\tau^2 \sin(\pi\tau)}{1 + \sqrt{\tau}} \|u\|_{\mathbb{X}} + \frac{\tau^{\frac{8}{3}}}{4 + \sqrt{\tau}} \|u_\tau\|_{\mathcal{P}_{\mathfrak{h}}}$$

which leads to $\eta_1(\tau) = \frac{\tau^2}{1+\sqrt{\tau}}$ and $\eta_2(\tau) = \frac{\tau^{\frac{8}{3}} \cos(\pi\tau)}{4+\sqrt{\tau}}$, and so, $\eta_1 = 1$ and $\eta_2 = \frac{1}{4}$. Therefore, $\Lambda = 1.61458$, $g = 0.43753125$, and using the Mathematica software, we obtain $\mathbb{E}_q(0.665269) \cong 2.21482$. Now, we can calculate $\sigma = 13.2833$. Then, we have to take $\mathcal{N} > 13.2833$. As a result, all conditions of Theorem 5 are satisfied, and this leads to the existence of at least one mild solution of the fractional wave Equation (6) on $(-\infty, 1]$.

6. Conclusions

In the current study, we investigated the system of fractional evolution equations with infinite delay. Our findings were based on current functional analysis approaches. The generalized Liouville–Caputo derivative, which is affiliated with many well-known fractional derivatives, was the fractional derivative used in our model, and we used the ρ-Laplace transform to obtain the integral equivalent equation and propose the mild solution of the system by considering unbounded operator A as the generator of the strongly continuous cosine family.

We achieved two outcomes in the case of the problem (1), and we had two result: The first arguments involve the existence and uniqueness of the solution, whereas the second concerns the existence of solutions for the given problem. The first result is based on a Banach fixed-point theorem and gives a criterion for ensuring a unique solution to the problem at hand by necessitating the use of a nonlinear function $f(\tau, u, u_\tau)$ to meet the traditional Lipschitz condition. The second argument is based on a nonlinear Leray–Schauder alternative, which permits the nonlinearity $f(\tau, u, u_\tau)$ to behave like $\|f(\tau, u, u_\tau)\| \leq \eta_1(\tau)\|u\|_{\mathbb{X}} + \eta_2(\tau)\|u_\tau\|_{\mathcal{P}_\mathfrak{h}}$ where η_i is defined in $(\mathcal{H}_3)$. In the scenario of simple assumptions, the fixed-point theory's instruments were selected. For our purposes, they are straightforward to implement and expand the scope of the given outcomes. Finally, a numerical examples was given to demonstrate our point by studying a function that meets all of the predetermined criteria.

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