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Interaction Behaviours between Soliton and Cnoidal Periodic Waves for Nonlocal Complex Modified Korteweg–de Vries Equation

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Abstract: The reverse space-time nonlocal complex modified Kortewewg–de Vries (mKdV) equation is investigated by using the consistent tanh expansion (CTE) method. According to the CTE method, a nonauto-Bäcklund transformation theorem of nonlocal complex mKdV is obtained. The interactions between one kink soliton and other different nonlinear excitations are constructed via the nonauto-Bäcklund transformation theorem. By selecting cnoidal periodic waves, the interaction between one kink soliton and the cnoidal periodic waves is derived. The specific Jacobi function-type solution and graphs of its analysis are provided in this paper.

Keywords: nonlocal modified Korteweg–de Vries equation; consistent tanh expansion method; parity-time symmetry

MSC: 35J60; 35N05; 35L05

1. Introduction

Physical systems exhibiting parity-time ($\mathcal{PT}$)-symmetries have received increasing attention since a family of non-Hermitian $\mathcal{PT}$-symmetric Hamiltonians with a real constant was first shown by Bender and Boettcher to admit entirely real spectra [1,2]. The study of $\mathcal{PT}$ symmetry in mathematics and physics can offer great research value and strong prospects for dynamical systems. $\mathcal{PT}$-symmetric nonlinear systems have become a major focus of nonlinear science, such as soliton theory, fluid mechanics, hydrodynamics and optical theory. Some effective methods have been developed to derive exact solutions of nonlinear integrable systems, such as the inverse scattering transform method [3,4], the dressing method [5], the Hirota direct method [6,7], the Darboux transformations [8–10] and the Bäcklund transformations [11–13], etc.

The modified Kortewewg–de Vries (mKdV) equation, which describes the evolutions of weakly dispersive wavelets in shallow water, is widely studied. The integrable nonlocal nonlinear Schrödinger equation proposed by Ablowitz and Musslimani [14] attracted many researchers because of its special property. Ablowitz and Musslimani proposed some new nonlocal nonlinear integrable equations, including the reverse space-time nonlocal complex mKdV equation [15]. In these new types of nonlocal equations; in addition to the terms at space-time point (x,t), there are terms at mirror image point $(-x,-t)$. The self-induced potential of the nonlocal complex mKdV equation is $V(x,t) = u(x,t)u^*(-x,-t)$ [15]. The PT-symmetry for the nonlocal complex mKdV equation amounts to the invariance of the self-induced potential in the case of classical optics, i.e., $V(x,t) = V^*(-x,-t)$, under the combined effect of parity and time reversal symmetry. A family of traveling solitary wave solutions including soliton, kink, periodic and singular solutions of the nonlocal mKdV equation is discussed [16].

The interaction between solitons and a periodic cnoidal wave of the Korteweg–de Vries equation and the cubic Schrödinger equation is discussed by using the inverse scattering technique [17,18]. Rogue waves on a periodic background and the nonlinear superposition of the two periodic solutions of mKdV equation are obtained by using the Darboux transformation [19,20]. The soliton excitation of the circular vortex motion can be constructed based on localized-induction approximation equations [21,22]. Recently, the consistent tanh expansion (CTE) method has been proposed to identify CTE-solvable systems [23,24]. The interaction between one soliton and other different nonlinear excitations such as cnoidal periodic waves can be obtained by using the CTE method. The method has been valid for classical integrable nonlinear systems, including the nonlinear Schrödinger system [25], the Broer–Kaup system [26], the higher-order KdV equation [27], etc. [28–30]. The application of the CTE method to nonlocal integrable systems with $\mathcal{PT}$-symmetric is deficient. Applying the CTE method to nonlocal $\mathcal{PT}$-symmetric integrable systems is innovative and convenient. In this paper, the CTE method is used to investigate the $\mathcal{PT}$-symmetric nonlocal complex mKdV equation and can construct the interaction solution of the soliton and cnoidal periodic waves.

This paper is organized as follows. In Section 2, a nonauto-Bäcklund transformation theorem is obtained by using the CTE method. The interactions between one kink soliton and other different nonlinear excitations are constructed by the nonauto-Bäcklund transformation theorem. Section 3 discusses the interaction between one kink soliton, and the Jacobi-elliptic function types are explicitly discussed both with analytical and graphical methods. Sections 4 and 5 include simple discussions and provide conclusions.

2. CTE Method for the Nonlocal Complex mKdV System

The reverse space-time nonlocal complex mKdV equation reads as follows [15]:

$$u_t(x,t) - 6\alpha u(x,t)u^*(-x,-t)u_x(x,t) + u_{xxx}(x,t) = 0, \tag{1}$$

where $u = u(x,t)$ is a complex function of real variables x and t, α is an arbitrary constant and $*$ denotes complex conjugation. The self-induced potential $V(x,t) = u(x,t)u^*(-x,-t)$ of (1) satisfies the $\mathcal{PT}$-symmetry condition $V(x,t) = V^*(-x,-t)$. The nonauto-Bäcklund transformations and the soliton phenomenology of the standard mKdV equation are systematically studied [31].

For the nonlocal complex mKdV system, one can take the generalized truncated tanh expansion form by using leading order analysis:

$$u = u_0 + u_1 \tanh(f), \tag{2}$$

where u_0 and u_1 are arbitrary functions of (x,t). f satisfies constraint $f(x,t) = f^*(-x,-t)$.

By substituting (2) into the nonlocal complex mKdV system (1), a complicated polynomial with respect to $\tanh(f)$ is obtained. Collecting coefficients of the powers of $\tanh^4(f)$ and $\tanh^3(f)$, we derive the following.

$$\alpha u_1(x,t)u_1^*(-x,-t) - f_x^2 = 0, \tag{3}$$

$$\alpha u_1^2(x,t)u_0^*(-x,-t)f_x + \alpha u_0(x,t)u_1^*(-x,-t)u_1(x,t)f_x - \alpha u_1(x,t)u_1^*(-x,-t)u_{1,x}(x,t) + u_1(x,t)f_x f_{xx} + u_{1,x}(x,t)f_x^2 = 0. \tag{4}$$

Substituting u_1 obtained by solving (3) into (4) and further solving for u_0, a set of solutions for u_1 and u_0 is derived as follows.

$$u_1 = \frac{1}{\sqrt{-\alpha}} f_x, \qquad u_0 = -\frac{1}{2\sqrt{-\alpha}} \frac{f_{xx}}{f_x}. \tag{5}$$

Substituting (5) into the complicated polynomial obtained before and collecting the coefficients of $\tanh^2(f)$, $\tanh^1(f)$ and $\tanh^0(f)$ via symbolic computation with the help of Maple, we obtain the following three over-determined systems.

$$\frac{3}{2}f_{xx}^2 + 2f_x^4 - f_x f_{xxx} - f_x f_t = 0,\tag{6}$$

$$\frac{3}{2}\frac{f_{xx}^3}{f_x^2} - 3\frac{f_{xx}f_{xxx}}{f_x} - 6f_x^2 f_{xx} + f_{xt} + f_{xxxx} = 0,\tag{7}$$

$$-\frac{3}{2}f_{xx}^2 - 2f_x^4 - \frac{21}{4}\frac{f_{xx}^2 f_{xxx}}{f_x^3} + \frac{9}{4}\frac{f_{xx}^4}{f_x^4} + f_x f_t + 4f_x f_{xxx} + 3f_{xx}^2 - \frac{1}{2}\frac{f_{xxt}}{f_x} + \frac{1}{2}\frac{f_{xx}f_{xt}}{f_x^2} - \frac{1}{2}\frac{f_{xxxxx}}{f_x} + \frac{2f_{xxxx}f_{xx}}{f_x^2} + \frac{3}{2}\frac{f_{xxx}^2}{f_x^2} = 0.\tag{8}$$

Moreover, the above three Equations (6)–(8) are consistent each other, meaning that if *f* satisfies one of the equations, it will be a solution for other two equations. According to above analysis, we derive the following nonauto-Bäcklund transformation theorem.

Nonauto-Bäcklund transformation theorem. If one finds that solution *f* satisfies (6), then *u* is obtained with the following:

$$u(x,t) = -\frac{1}{2\sqrt{-\alpha}}\frac{f_{xx}}{f_x} + \frac{1}{\sqrt{-\alpha}}f_x \tanh(f),\tag{9}$$

which is a solution of the reverse space-time nonlocal complex mKdV system (1).

The Miura transform is known as the transformation connection the solutions between KdV equation and mKdV equation. This nonauto-Bäcklund transformation can be treated as a form of Miura transformation. According to the above theorem, the exact solutions of the nonlocal complex mKdV system (1) are obtained by solving (6). Here are some interesting examples.

A quite trivial solution of (6) has the following form:

$$f = i(k_0 x + w_0 t), \quad w_0 = -2k_0^3,\tag{10}$$

where k_0 is a free constant, and w_0 is determined by dispersion relations. Substituting the trivial solution (10) into (9), one kink soliton solution of the nonlocal complex mKdV system yields the following.

$$u = -\frac{1}{\sqrt{-\alpha}}k_0 \tan(k_0 x - 2k_0^3 t).\tag{11}$$

Some nontrivial solutions of the mKdV equation can be derived from a quite trivial solution of (10). To find interaction solutions between one kink soliton and other nonlinear excitations, we assume the interaction solution form as follows:

$$f = i(k_0 x + w_0 t) + F(X), \quad X = kx + wt,\tag{12}$$

where k_0, w_0, k and w are all free constants. Substituting expression (12) into (6), (6) becomes the following.

$$F_X^4 + \frac{4ik}{k_0}F_X^3 - \frac{12kk_0^2 + w}{2k^3}F_X^2 - \frac{i(8k_0^3 k + kw_0 + k_0 w)}{2k^4}F_X - \frac{1}{2}F_X F_{XXX} + \frac{3}{4}F_{XX}^2 - \frac{ik_0}{2k}F_{XXX} + \frac{2k_0^4 + k_0 w_0}{2k^4} = 0.\tag{13}$$

Then, the following equation is obtained by using transformation $F_X = F_1$.

$$F_1^4 + \frac{4ik_0}{k}F_1^3 - \frac{12kk_0^2 + w}{2k^3}F_1^2 - \frac{i(8kk_0^3 + kw_0 + k_0 w)}{2k^4}F_1 + \frac{3}{4}F_{1,X}^2 - \frac{1}{2}(F_1 + \frac{ik_0}{k})F_{1,XX} + \frac{k_0(2k_0^3 + w_0)}{2k^4} = 0.\tag{14}$$

The CTE method is valid in many classical integrable systems. For the interaction between soliton and Jacobi periodic waves in classical integrable systems, one can obtain the standard Jacobi-elliptic function equation [32]. One only obtains Equation (14) rather than the standard Jacobi-elliptic function equation. In order to obtain the Jacobi periodic wave solution of (14), we assume that Equation (14) has a Jacobian elliptic function solution as $F_1(X) = c_1 S_n(c_2 X, m)$ [33]. Hence, the solution expressed by (9) is just the explicit exact

interaction between one kink soliton and cnoidal periodic waves. To show more clearly this form of solution, we offer one special case for solving (14).

3. Interaction between Soliton and Cnoidal Periodic Waves

According to above analysis, the solution of (13) has the following form:

$$F(X) = \int c_1 S_n(c_2 X, m) dX = \frac{c_1 \ln[D_n(c_2 X, m) - m C_n(c_2 X, m)]}{c_2 m},\tag{15}$$

where S_n, C_n and D_n are the Jacobian-elliptic functions with modulus m. Verified by Maple's symbolic calculation, (15) satisfies constraint $f(x, t) = f^*(-x, -t)$ and is a real even function. By substituting the undetermined parameter solution (15) into (13) and using symbolic computation with the help of Maple, the parameters satisfy the following.

$$c_1 = -\frac{c_2 m}{2}, \quad c_2 = 2\frac{ik_0}{k}, \quad w_0 = -2k_0^3(3m^2 + 1), \quad w = 2k_0^2 k(m^2 - 5).\tag{16}$$

The interaction between one kink soliton and the cnoidal wave of the nonlocal complex mKdV system (1) has the the following form.

$$u = \frac{2}{\sqrt{-\alpha}(c_1 S_n + ik_0)} \left[\tanh\left(\frac{c_1 \ln(D_n - m C_n) + ic_2 m(k_0 x + w_0 t)}{c_2 m} \right) \left(\frac{c_1^2 k^2}{2} S_n^2 + ic_1 k k_0 S_n - \frac{k_0^2}{2} \right) - \frac{c_1 c_2 k^2}{4} C_n D_n \right].\tag{17}$$

The parameters c_1, c_2, w_0 and w have been given in (16).

We select the parameters as $\alpha = -1, k_0 = 0.4i, m = 0.4$ in Figures 1–3. Figures 1 and 2 plot the interaction solution between one kink soliton and the cnoidal wave in the patterns of three-dimensional and wave along x-axis. Field u exhibits one kink soliton propagating on the cnoidal wave's background. Figure 3 plots the status-only soliton or cnoidal wave at $t = 0$. The superpose status is just the interaction between one kink soliton and the cnoidal waves, which are depicted in Figure 2. The changes before and after superposition are displayed visually. There are some nonlinear waves including interactions between solitary waves and the cnoidal periodic waves, which can be described in certain ocean phenomena.
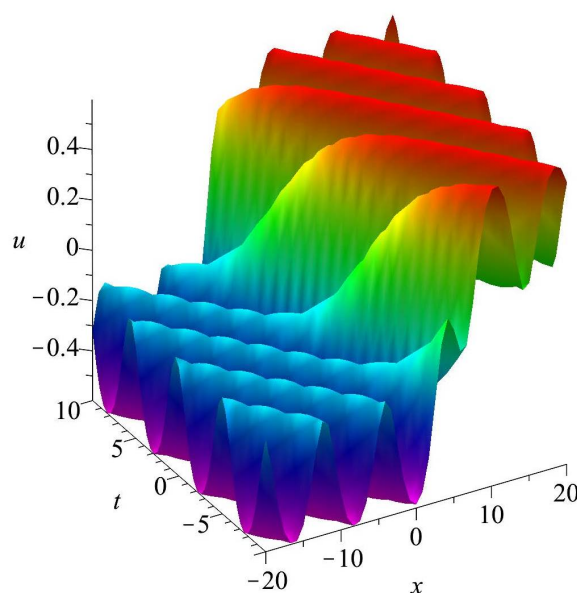
Figure 1. Plot of one kink soliton on the cnoidal wave background expressed by (17) of the nonlocal mKdV equation in three dimensions.
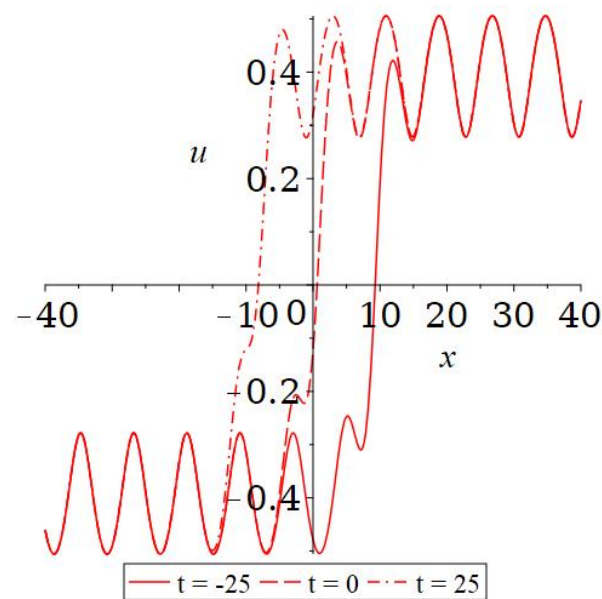
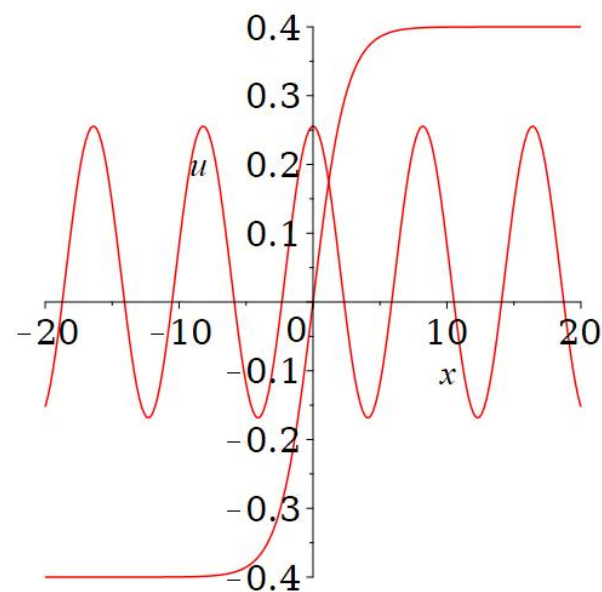**Figure 2.** One dimensional image followed by $t = -25, 0, 25$.

Figure 3. Plot of separate state for one kink soliton or the cnoidal wave expressed by (10) and (15) of the nonlocal mKdV equation at $t = 0$.

4. Discussion

Kuznetsov and Mikhailov discussed the interaction between solitons and a periodic cnoidal wave of the Korteweg–de Vries equation [17]. Gorshkov and Ostrovsky investigated the interaction between soliton and a periodic wave via the direct perturbation method [34]. The interaction between the Jacobi elliptic periodic wave and kink soliton for the complex mKdV equation is directly obtained by the CTE method in this paper. Compared with the previous two methods, the CTE method can obtain this type solution more directly and conveniently. Other reverse space-time nonlocal system is worthy of study by using the CTE method.

5. Conclusions

The reverse space-time nonlocal complex mKdV equation is investigated by using the CTE method. A nonauto-Bäcklund transformation theorem is constructed by using

the CTE method. The interactions between one kink soliton and the cnoidal waves are derived by means of the nonauto-Bäcklund transformation theorem. The dynamics of the interactions are studied both with analytical and graphical methods. These types of interaction solutions can describe certain oceanic phenomena. The method is valid and promising for the $\mathcal{PT}$-symmetry models. The interactions between solitons and the cnoidal waves can be obtained by symmetry reductions related by nonlocal symmetry [27]. Symmetry reductions related by the nonlocal symmetry of the nonlocal complex mKdV equation will be studied in the future.

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