



Tracking Control for Triple-Integrator and Quintuple-Integrator Systems with Single Input Using Zhang Neural Network with Time Delay Caused by Backward Finite-Divided Difference Formulas for Multiple-Order Derivatives

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Abstract: Tracking control for multiple-integrator systems is regarded as a fundamental problem associated with nonlinear dynamic systems in the physical and mathematical sciences, with many applications in engineering fields. In this paper, we adopt the Zhang neural network method to solve this nonlinear dynamic problem. In addition, in order to adapt to the requirements of real-world hardware implementations with higher-order precision for this problem, the multiple-order derivatives in the Zhang neural network method are estimated using backward finite-divided difference formulas with quadratic-order precision, thus producing time delays. As such, we name the proposed method the Zhang neural network method with time delay. Moreover, we present five theorems to describe the convergence property of the Zhang neural network method with time delay derived from the backward finite-divided difference formulas with quadratic-order error pattern of the Zhang neural network method with quadratic-order precision, which specifically demonstrate the effect of the time delay. Finally, tracking controllers with quadratic-order precision for multiple-integrator systems are constructed using the Zhang neural network method with time delay, and two numerical experiments are presented to substantiate the theoretical results for the Zhang neural network methods with and without time delay.

Keywords: time delay; Zhang neural network; backward finite-divided difference formula; tracking control

MSC: 93C10; 93C15; 93C95

1. Introduction

In real-world engineering applications, many control systems are characterized by nonlinearity, which are mathematically described as nonlinear dynamic systems [1–3]. The problem of tracking control for a given nonlinear system with single input, which focuses on studying their dynamic behaviors, can be formulated as follows [4–8]:

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, u, t), y = h(\mathbf{x}),$$
(1)

where $\mathbf{x} \in \mathbb{R}^n$ is the state vector of the system; $y \in \mathbb{R}$ and $u \in \mathbb{R}$ are the output and control input of the system, respectively; and y_d is the desired path. The target of the above tracking control problem for a given nonlinear system with single input is to make the tracking error $e_{tr} = y - y_d$ equal to zero for a suitable initial state. In particular, multiple-integrator (MI) systems are usually considered as fundamental nonlinear dynamic systems in the physical



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and mathematical sciences, having many applications in the engineering fields [9,10], such as secure communication [11–13], electrical engineering [14], and economics [15]. Moreover, the tracking control of MI systems has contributed significantly to their real-world implementation and has attracted a lot of attention in related fields [16–19].

Some methods/algorithms which construct a controller to track the desired path $y_d(t)$ accurately have been designed to address the tracking control problem for MI systems [20–31]. These methods/algorithms mainly include the input–output linearization (IOL) method [21], variants of the IOL method (e.g., approximate IOL, a switching scheme of approximate IOL, and exact IOL) [22–24], and neural network-based dynamic methods [25–31]. The IOL method and its variants are usually adopted to tackle the problem of tracking control for nonlinear systems; however, they always face problems related to singularities and instability at some key points, causing the nonlinear system to fall outside of a well-defined relative degree. Thus, more than two controllers with single linear output should generally be constructed, which increases the complexity of the corresponding algorithms [23,24]. Therefore, neural network-based dynamic methods have been adopted to tackle with the problem of tracking control for nonlinear systems; [32–39].

As an effective approach in the neural network-based dynamic methods category, the Zhang neural network (ZNN) has been proposed to deal with the problem of tracking control for nonlinear dynamic systems. In [32,33], Zhang et al. addressed the problems of tracking control for the Lu chaotic (LC) system with multiple additive inputs and a doubleintegrator (DI) system with linear and nonlinear outputs by combining the ZNN and the gradient dynamics (GD) method. In [34], Zhang et al. proposed a variant of the ZNN approach to tackle the singularity-conquering problem in tracking control for the singleinput LC system and the mixed inputs modified Lorenz chaotic system. In [40], Zhang et al. developed a method, named Zhang gradient control (ZGC), to solve an MI system with linear and nonlinear outputs. In [41], Jin et al. derived a group of controllers based on the ZGC method for a modified Lorenz chaotic system with an additive input or a mixture of additive and multiplicative inputs. In [42], Li et al. developed a group of effective controllers using the zeroing dynamics (ZD) method, which is related to the ZNN method, in order to tackle the synchronization problem of the chaotic system by considering model uncertainty, parameter perturbation, and external noise disturbance. In [43], Huang et al. proposed a method using the ZNN method and the three-step Zhang et al. discretization (i.e., ZeaD) formulas at four points to solve the synchronization problem of discrete LC system with a single input. In [44], Ling et al. presented a group of controllers based on the ZNN method to solve the synchronization problem of the Genesio Chaotic (GC) system with or without noise disturbance. In [45], Zhang et al. developed a generalized Zhang equivalency (ZE), also related to the ZNN method, in order to revisit the problem of tracking control for the single additive input LC system. In [46], Li et al. constructed a type of ZNN-based controller to track MI systems with noise disturbance.

The above studies have verified the effectiveness and accuracy of ZNN methods. For the controllers that were constructed using ZNN-based methods, the derivatives employed in the design process, such as $\dot{y}_d(t)$, $\ddot{y}_d(t)$, $\ddot{y}_d(t)$, and $\ddot{y}_d(t)$, are often computed directly using accurate derivative formulas; thus, the problem of tracking control for MI systems with high accuracy can be achieved in the case with no time delay. However, as is well-known, time delays always appear in real-world applications, especially with the implementation of different types of neural networks [47–61]. For instance, in [48], Stamov et al. studied the stability problem of Bidirectional Associative Memory (BAM) Cohen–Grossberg-type impulsive neural networks with time-varying delays by using manifold notions. In [49], Chanthorn et al. reported the robust dissipativity of Hopfield-type complex-valued neural network models incorporated with time-varying delays and linear fractional uncertainties using the multiple integral inequality method. In the same year, Chanthorn et al. presented the robust stability of complex-valued stochastic neural networks with time-varying delays and parameter uncertainties by exploiting the real–imaginary separate-type activation function [50]. In [55], Akhmet et al. studied the unpredictable oscillations problem in Hopfield-type neural networks with time delay and advanced arguments that can be applied to neuroscience. In [56], Rajchakit et al. studied the finite-time synchronization problem of Clifford-valued neural networks with finite-time distributed delays using the number field transformation method. In [61], Sun and Liu presented an adaptive synchronization control and synchronization-based parameter identification method for fractional chaotic neural networks with time-varying delays. Although the above mentioned research works studied the stability properties of the control problem for different neural network systems with time-varying delay, they did not attempt to understand the effect of time delay in the controller design procedure for different nonlinear dynamic systems. As a canonical type of neural network, the influence of time delay caused by approximating the derivatives in the ZNN method needs to be studied, in order to deal with the problem of tracking control for nonlinear systems. In our previous work [62], we have investigated ZNN methods without time delay and the first-order error pattern of ZNN methods with time delay derived from the backward finite-divided difference rules with linear-order precision. However, higher-order precision tracking controllers for nonlinear systems constructed based on ZNN methods with time delay should be developed, in order to satisfy the demands of real-world engineering applications.

In this paper, the multiple-order derivatives in the ZNN method are estimated using backward finite-divided difference formulas (BFDDFs) with quadratic-order precision, which produce time delays; as such, we call the proposed method the order-*N* ZNN method with time delay. To check the validity and precision of the order-*N* ZNN method with time delay, in terms of solving the problem of tracking control for nonlinear systems, both theoretical results and systematic proofs are presented. In addition, numerical experiments considering the problem of tracking control for a triple-integrator (TI) system with non-linear output function (NOF) and a quintuple-integrator (QI) system with linear output function (LOF) are conducted, in order to substantiate the theoretical results of the order-*N* ZNN methods with and without time delay.

The remainder of this paper is organized into four sections, and the main contents are illustrated in Figure 1. Section 2 presents five theorems for order-*N* ZNN methods with and without time delay. Section 3 investigates the problem of tracking control for a TI system with NOF. Numerical experiments substantiate the theoretical results of order-*N* ZNN methods with and without time delay. Section 4 investigates the problem of tracking control for the QI system with LOF. Again, numerical experiments substantiate the theoretical results of order-*N* ZNN methods with and without time delay. Finally, in Section 5, we present the conclusion of this paper. The contributions of our study can be summarized as follows:

- To study the impact of the time delay caused by the multiple-order derivative approximation on the order-*N* ZNN method. To investigate this, the order-*N* ZNN model is transformed into a non-homogeneous time delay differential equation by using BFDDFs with quadratic-order precision.
- Five theorems, together with rigorous mathematical proofs, are illustrated to describe the convergence properties of the order-*N* ZNN method without time delay and the quadratic-order error pattern of the order-*N* ZNN method with time delay derived from the backward finite-divided difference formulas with quadratic-order precision, which specifically demonstrate the effect of the time delay.
- The ZNN method with time delay is successfully adopted to solve the problem of tracking control for the TI system with NOF and the QI system with LOF, and the corresponding numerical experiment results substantiate the quadratic-order error pattern of order-*N* ZNN methods with time delay derived from the backward finite-divided difference formulas with quadratic-order precision.



Figure 1. Structure of main content of this paper.

2. Multiple-Order ZNN Methods with and without Time Delay

In this section, the backward finite-divided difference formulas are revisited, as a method for approximating derivatives of various orders for target functions. Based on the uniform order of truncation errors for backward finite-divided difference formulas, the multiple-order ZNN methods with and without time delay are developed, and we provide five theorems about the order-*N* ZNN methods with and without time delay, together with theoretical proofs.

2.1. Backward Finite-Divided Difference Formulas

As mentioned in the introduction, time delays always appear in the real-world applications. In particular, as the time-dependent states of a system may be measured using sensors, or produced by complex transformations including Fourier and Laplace transforms, among others, their derivatives are difficult to obtain directly. To address this, we adopt the BFDDFs, which only need historical states to approximate the derivatives, which leads to multiple-step time delay. BFDDFs computed by the Lagrange interpolation polynomial play an important role in theoretical analyses and numerical experiments. For convenience, we list several BFDDFs and their corresponding truncation errors in the following Table 1, where z(t) is a desired function or a desired time-continuous matrix, and τ is the sampling time delay. Specifically, we use the formulas in Table 1 to construct the controller for MI systems with a single input and investigate the effect of time delay in the ZNN method. We believe that other BFDDFs with uniform-order truncation error can also be adopted to construct controllers for nonlinear system with single input.

Table 1. Backward finite-divided difference formulas for order-*n* derivative functions.

Order-n Derivative Function	Backward Finite-Divided Difference Formula	Truncation Error
$\dot{z}(t)$	$(3z(t) - 4z(t - \tau) + z(t - 2\tau))/2\tau$	$O(\tau^2)$
$\ddot{z}(t)$	$(2z(t) - 5z(t - \tau) + 4z(t - 2\tau) - z(t - 3\tau))/\tau^2$	$O(\tau^2)$
$\ddot{z}(t)$	$(5z(t) - 18z(t - \tau) + 24z(t - 2\tau) - 14z(t - 3\tau) + 3z(t - 4\tau))/2\tau^3$	$O(\tau^2)$
$\ddot{z}(t)$	$(3z(t) - 14z(t - \tau) + 26z(t - 2\tau) - 24z(t - 3\tau) + 11z(t - 4\tau) - 2z(t - 5\tau)) / \tau^4$	$O(\tau^2)$

2.2. Order-N ZNN Methods with and without Time Delay

In order to solve the problem of tracking control for nonlinear systems and to optimize the solution to other real-world problems, the order-*N* ZNN methods with and without time delay are constructed step-by-step in this subsection.

To construct the ZNN method, the error function e_1 captured from the actual trajectory and the desired path is formulated as

$$e_1 = z - z_d, \tag{2}$$

where *z* is the actual trajectory of the nonlinear system and z_d is the desired path. Then, the ZNN design formula [25,29–31] is adopted as

$$\dot{e}_1 = -\lambda_1 e_1,\tag{3}$$

where $\lambda_1 > 0 \in \mathbb{R}$ is the ZNN design parameter, which is adopted to control the convergence rate of ZNN solution [25,29]. Equation (3) is reformulated as the corresponding standard form $\dot{e}_1 + \lambda_1 e_1 = 0$, which is the so-called order-1 ZNN model without time delay. However, when the order-1 ZNN model without time delay is approximated using the BFDDFs given in Table 1, it has a truncation error of $O(\tau^2)$. Therefore, the order-1 ZNN model without time delay is formulated into the following order-1 ZNN model with time delay:

$$\dot{e}_1 + \lambda_1 e_1 = O(\tau^2). \tag{4}$$

In addressing the problem of tracking control for MI systems, one needs to use the ZNN method several times to design suitable controllers; that is, one must use the ZNN design formula to construct an order-N ZNN model without time delay and with time delay, as follows. Let e_2 denote the second error function, which is represented using a linear combination of e_1 and its derivative \dot{e}_1 , in the without time delay case, as

$$e_2 = \dot{e}_1 + \lambda_1 e_1. \tag{5}$$

The ZNN design formula is adopted again for the second error function e_2 ,

$$\dot{e}_2 = -\lambda_2 e_2,\tag{6}$$

where $\lambda_2 > 0 \in \mathbb{R}$ is the ZNN design parameter. Substituting Equation (5) into Equation (6), one obtains the following order-2 ZNN model without time delay, which is described by a constant coefficient homogeneous linear ordinary differential equation of order 2:

$$\ddot{e}_1 + (\lambda_1 + \lambda_2)\dot{e}_1 + \lambda_1\lambda_2e_1 = 0.$$
⁽⁷⁾

When the order-2 ZNN model without time delay is approximated by the BFDDFs mentioned in Table 1, one obtains the following order-2 ZNN model with time delay:

$$\ddot{e}_1 + (\lambda_1 + \lambda_2)\dot{e}_1 + \lambda_1\lambda_2e_1 = O(\tau^2).$$
(8)

Let e_3 denote the third error function, which is represented using a linear combination of e_2 and its derivative \dot{e}_2 , in the without time delay case, as

$$e_3 = \dot{e}_2 + \lambda_2 e_2. \tag{9}$$

Substituting Equation (5) into Equation (9), one obtains the following relation between e_3 and e_1 :

$$e_3 = \ddot{e}_1 + (\lambda_1 + \lambda_2)\dot{e}_1 + \lambda_1\lambda_2e_1. \tag{10}$$

The ZNN design formula is used again for the third error function e_3 ,

ė

$$_{3}=-\lambda_{3}e_{3}, \tag{11}$$

where $\lambda_3 > 0 \in \mathbb{R}$ is the ZNN design parameter. By substituting Equation (10) into Equation (11), one obtains the order-3 ZNN model without time delay, as follows:

$$\ddot{\vec{e}}_1 + (\lambda_1 + \lambda_2 + \lambda_3)\ddot{\vec{e}}_1 + (\lambda_1\lambda_2 + \lambda_2\lambda_3 + \lambda_1\lambda_3)\dot{\vec{e}}_1 + \lambda_1\lambda_2\lambda_3\vec{e}_1 = 0.$$
(12)

When the order-3 ZNN model without time delay is approximated by the BFDDFs mentioned in Table 1, one obtains the following order-3 ZNN model with time delay:

$$\ddot{e}_1 + (\lambda_1 + \lambda_2 + \lambda_3)\ddot{e}_1 + (\lambda_1\lambda_2 + \lambda_2\lambda_3 + \lambda_1\lambda_3)\dot{e}_1 + \lambda_1\lambda_2\lambda_3e_1 = O(\tau^2).$$
(13)

Let e_4 denote the forth error function. The relation between the error function e_4 and e_3 , in the without time delay case, is formulated as follows:

$$e_4 = \dot{e}_3 + \lambda_3 e_3. \tag{14}$$

Substituting Equation (10) into Equation (14), one can express the error function e_4 as the following linear combination of different order derivatives of e_1 :

$$e_4 = \ddot{e}_1 + (\lambda_1 + \lambda_2 + \lambda_3)\ddot{e}_1 + (\lambda_1\lambda_2 + \lambda_2\lambda_3 + \lambda_1\lambda_3)\dot{e}_1 + \lambda_1\lambda_2\lambda_3e_1.$$
(15)

One uses the ZNN design formula for the forth error function e_4 ,

$$\dot{e}_4 = -\lambda_4 e_4,\tag{16}$$

where $\lambda_4 > 0 \in \mathbb{R}$ is the ZNN design parameter. Then, one obtains the following order-4 ZNN model without time delay, as follows:

$$\ddot{e}_1 + m_1 \ddot{e}_1 + m_2 \ddot{e}_1 + m_3 \dot{e}_1 + m_4 e_1 = 0, \tag{17}$$

where $m_1 = \lambda_1 + \lambda_2 + \lambda_3 + \lambda_4$, $m_2 = \lambda_1\lambda_2 + \lambda_1\lambda_3 + \lambda_1\lambda_4 + \lambda_2\lambda_3 + \lambda_2\lambda_4 + \lambda_3\lambda_4$, $m_3 = \lambda_1\lambda_2\lambda_3 + \lambda_1\lambda_2\lambda_4 + \lambda_1\lambda_3\lambda_4 + \lambda_2\lambda_3\lambda_4$, and $m_4 = \lambda_1\lambda_2\lambda_3\lambda_4$. When the order-4 ZNN model without time delay is approximated by the BFDDFs mentioned in Table 1, one obtains the following order-4 ZNN model with time delay:

$$\ddot{e}_1 + m_1 \ddot{e}_1 + m_2 \ddot{e}_1 + m_3 \dot{e}_1 + m_4 e_1 = O(\tau^2).$$
(18)

Generally, the N^{th} error function e_N is defined by the following linear combination of the $(N-1)^{\text{th}}$ -order function e_{N-1} and its first derivative:

$$e_N = \dot{e}_{N-1} + \lambda_{N-1} e_{N-1}.$$
 (19)

The ZNN design formula is adopted for the error function e_N ,

$$\dot{e}_N = -\lambda_N e_N,$$
 (20)

where $\lambda_N > 0 \in \mathbb{R}$ is the ZNN design parameter. By expressing the N^{th} error function e_N as the linear combination of the error function e_1 and its derivatives with the highest order N, one obtains the following order-N ZNN model without time delay:

$$e_1^{(N)} + n_1 e_1^{(N-1)} + \dots + n_i e_1^{(i)} + \dots + n_{N-1} \dot{e}_1 + n_N e_1 = 0,$$
 (21)

where $e_1^{(i)}$ is the *i*th-order derivative of the error function e_1 , and the coefficient of corresponding high-order derivative is given by $n_i = \sum_{1 \le l_1 < l_2 < \cdots < l_i \le n} \lambda_{l_1} \lambda_{l_2} \cdots \lambda_{l_i}$, with Card $\{l_1, l_2, \cdots, l_i\} = i$. Assuming that each order derivative of the desired path can be approximated using BFDDFs with truncation error $O(\tau^2)$, one obtains the order-*N* ZNN model with time delay:

$$e_1^{(N)} + n_1 e_1^{(N-1)} + \dots + n_i e_1^{(i)} + \dots + n_{N-1} \dot{e}_1 + n_N e_1 = O(\tau^2).$$
(22)

For better illustration and understanding, in Figure 2, we present a flowchart of the controller design using the ZNN methods with and without time delay for the tracking control of a nonlinear system with single input.



Figure 2. Flowchart of controller design using ZNN methods with and without time delay for tracking control of nonlinear system with single input.

2.3. Error Analysis of Order-N ZNN Methods with and without Time Delay

In this subsection, we introduce five theorems to describe the error properties of the order-*N* ZNN methods with and without time delay.

Theorem 1. Assume that there exists a group of backward finite-divided difference formulas with truncation error $O(\tau^2)$. As the time t becomes large enough (i.e., $t \gg 0$), the relation between the solution error $e_1(t)$ of the order-1 ZNN model with time delay (4) and the truncation error $O(\tau^2)$ is formulated as $e_1(t) \approx O(\tau^2)/\lambda_1 = O(\tau^2)$, where λ_1 is the ZNN design parameter. In other words, the steady-state solution error $e_1(t)$ has a positive proportional relationship with the truncation error $O(\tau^2)$.

Proof of Theorem 1. The order-1 ZNN model with time delay is a non-homogeneous linear differential Equation (4), as follows:

$$\dot{e}_1 + \lambda_1 e_1 = O(\tau^2).$$

According to the solution structure of non-homogeneous linear ordinary differential equations [63,64], the solution of (4) is expressed as the summation of a special solution of it and the general solution of its corresponding homogeneous equation. Thus, the solution to (4) is given as follows:

$$e_1(t) = C_1 \exp(-\lambda_1 t) + \exp(-\lambda_1 t) \int_0^t O(\tau^2) \exp(\lambda_1 \iota) d\iota = C_1 \exp(-\lambda_1 t) + O(\tau^2) / \lambda_1,$$

$$e_1(t) \approx O(\tau^2) / \lambda_1 = O(\tau^2).$$

The proof is completed. \Box

Theorem 2. Assume that there exists a group of backward finite-divided difference formulas with truncation error $O(\tau^2)$. As the time t becomes large enough (i.e., $t \gg 0$), the relation between the solution error $e_1(t)$ of the order-2 ZNN model with time delay (8) and the truncation error $O(\tau^2)$ is formulated as $e_1(t) \approx O(\tau^2)/(\lambda_1\lambda_2) = O(\tau^2)$, where λ_1 and λ_2 are the ZNN design parameters. In other words, the steady-state solution error $e_1(t)$ has a positive proportional relationship with the truncation error $O(\tau^2)$.

Proof of Theorem 2. The order-2 ZNN model with time delay is a non-homogeneous linear differential equation with constant coefficients (8), as follows:

$$\ddot{e}_1 + (\lambda_1 + \lambda_2)\dot{e}_1 + \lambda_1\lambda_2e_1 = O(\tau^2).$$

The corresponding homogeneous linear differential equation of (8), that is also the order-2 ZNN model without time delay (7), is formulated by setting the free term to zero, as follows:

$$\ddot{e}_1 + (\lambda_1 + \lambda_2)\dot{e}_1 + \lambda_1\lambda_2e_1 = 0,$$

and its characteristic equation is the following algebraic equation:

$$p^2 + (\lambda_1 + \lambda_2)p + \lambda_1\lambda_2 = 0.$$
⁽²³⁾

The corresponding characteristic roots of (23) are $p_1 = -\lambda_1$ and $p_2 = -\lambda_2$. According to the solution structure of non-homogeneous linear ordinary differential equations [63,64], the solution of (8) is expressed as the summation of its special solution and the general solution of its corresponding homogeneous equation. As the free term of (8) can be rewritten as $O(\tau^2) = O(\tau^2) \exp(0 \cdot t)$, and the characteristic roots of (7) satisfy $p_1 = -\lambda_1 < 0$, $p_2 = -\lambda_2 < 0$, the special solution of (8) is assumed to be of the form $C_2 \exp(0 \cdot t)$, which is equal to the constant C_2 . Substituting C_2 into the equation (8), one obtains the following relation:

$$\lambda_1 \lambda_2 C_2 = O(\tau^2);$$

that is, $C_2 = O(\tau^2) / \lambda_1 \lambda_2$. As there are two characteristic roots when solving Equation (23), the general solution of (8) is associated with the relationship between λ_1 and λ_2 .

Case I If $\lambda_1 = \lambda_2$, the general solution of the order-2 ZNN model without time delay is $\tilde{e}_1(t) = (c_1 + c_2 t) \exp(-\lambda_1 t)$. Then, the general solution of the order-2 ZNN model with time delay is given by the following function:

$$e_1(t) = (c_1 + c_2 t) \exp(-\lambda_1 t) + O(\tau^2) / \lambda_1 \lambda_2.$$
(24)

Case II If $\lambda_1 \neq \lambda_2$, the general solution of the order-2 ZNN model without time delay is $\tilde{e}_1(t) = c_1 \exp(-\lambda_1 t) + c_2 \exp(-\lambda_2 t)$. Then, the general solution of the order-2 ZNN model with time delay is given by the following function:

$$e_1(t) = c_1 \exp(-\lambda_1 t) + c_2 \exp(-\lambda_2 t) + O(\tau^2) / \lambda_1 \lambda_2.$$
(25)

When $t \to +\infty$, both the first term $(c_1 + c_2 t) \exp(-\lambda_1 t)$ of function (24) and the first term $c_1 \exp(-\lambda_1 t) + c_2 \exp(-\lambda_2 t)$ of function (25) approach zero. Hence, as $t \to +\infty$, the first error function can be approximated as

$$e_1(t) \approx O(\tau^2) / \lambda_1 \lambda_2 = O(\tau^2).$$

The proof is completed. \Box

Theorem 3. Assume that there exists a group of backward finite-divided difference formulas with truncation error $O(\tau^2)$. As the time t becomes large enough (i.e., $t \gg 0$), the relation between the solution error $e_1(t)$ of the order-3 ZNN model with time delay (13) and the truncation error $O(\tau^2)$ is formulated as $e_1(t) \approx O(\tau^2)/(\lambda_1\lambda_2\lambda_3) = O(\tau^2)$, where λ_1 , λ_2 , and λ_3 are the ZNN design parameters. In other words, the steady-state solution error $e_1(t)$ has a positive proportional relationship with the truncation error $O(\tau^2)$.

Proof of Theorem 3. The order-3 ZNN model with time delay is a constant coefficient non-homogeneous linear differential Equation (13), as follows:

$$\ddot{e}_1 + (\lambda_1 + \lambda_2 + \lambda_3)\ddot{e}_1 + (\lambda_1\lambda_2 + \lambda_2\lambda_3 + \lambda_1\lambda_3)\dot{e}_1 + \lambda_1\lambda_2\lambda_3e_1 = O(\tau^2).$$

The corresponding homogeneous linear differential equation of (13), which is the order-3 ZNN model without time delay (12), is formulated by setting the free term to zero, as follows:

$$\ddot{e}_1 + (\lambda_1 + \lambda_2 + \lambda_3)\ddot{e}_1 + (\lambda_1\lambda_2 + \lambda_2\lambda_3 + \lambda_1\lambda_3)\dot{e}_1 + \lambda_1\lambda_2\lambda_3e_1 = 0,$$

and its corresponding characteristic function is the following polynomial equation with degree 3:

$$p^{3} + (\lambda_{1} + \lambda_{2} + \lambda_{3})p^{2} + (\lambda_{1}\lambda_{2} + \lambda_{2}\lambda_{3} + \lambda_{1}\lambda_{3})p + \lambda_{1}\lambda_{2}\lambda_{3} = 0.$$
(26)

Then, the corresponding characteristic roots of (26) are $p_1 = -\lambda_1$, $p_2 = -\lambda_2$, and $p_3 = -\lambda_3$. According to the solution structure of non-homogeneous linear ordinary differential equations [63,64], the solution of (13) is expressed as the summation of its special solution and the general solution of its corresponding homogeneous equation. As the free term of (13) can be rewritten as $O(\tau^2) = O(\tau^2) \exp(0 \cdot t)$, and the characteristic roots of (12) satisfy $p_1 = -\lambda_1 < 0$, $p_2 = -\lambda_2 < 0$, $p_3 = -\lambda_3 < 0$, the special solution of (13) is assumed to be of the form $C_3 \exp(0 \cdot t)$, which is equal to the constant C_3 . Substituting C_3 into Equation (13), one obtains the following relation:

$$\lambda_1 \lambda_2 \lambda_3 C_3 = O(\tau^2);$$

that is, $C_3 = O(\tau^2) / \lambda_1 \lambda_2 \lambda_3$. Since there are three characteristic roots by solving Equation (26), the general solution of (13) is associated with the relationship between λ_1 , λ_2 , and λ_3 .

Case I If $\lambda_1 = \lambda_2 = \lambda_3$, the general solution of the order-3 ZNN model without time delay is $\tilde{e}_1(t) = (c_1 + c_2t + c_3t^2) \exp(-\lambda_1t)$. Then, the general solution of the order-3 ZNN model with time delay is given by the following function:

$$e_1(t) = (c_1 + c_2 t + c_3 t^2) \exp(-\lambda_1 t) + O(\tau^2) / \lambda_1 \lambda_2 \lambda_3.$$
(27)

Case II If there are exactly two equal characteristic roots of (12) (without loss of generality, we assume $p_1 = p_2 \neq p_3$), the general solution of the order-3 ZNN model without time delay is $\tilde{e}_1(t) = (c_1 + c_2 t) \exp(-\lambda_1 t) + c_3 \exp(-\lambda_3 t)$. Then, the general solution of the order-3 ZNN model with time delay is given by the following function:

$$e_1(t) = (c_1 + c_2 t) \exp(-\lambda_1 t) + c_3 \exp(-\lambda_3 t) + O(\tau^2) / \lambda_1 \lambda_2 \lambda_3.$$
(28)

Case III If the three characteristic roots of (12) mutually differ, the general solution of the order-3 ZNN model without time delay is $\tilde{e}_1(t) = c_1 \exp(-\lambda_1 t) + c_2 \exp(-\lambda_1 t) + c_3 \exp(-\lambda_3 t)$. Then, the general solution of the order-3 ZNN model with time delay is given by the following function:

$$e_{1}(t) = c_{1} \exp(-\lambda_{1} t) + c_{2} \exp(-\lambda_{1} t) + c_{3} \exp(-\lambda_{3} t) + O(\tau^{2}) / \lambda_{1} \lambda_{2} \lambda_{3}.$$
 (29)

When $t \to +\infty$, the first term $(c_1 + c_2t + c_3t^2) \exp(-\lambda_1 t)$ of function (27), the first term $(c_1 + c_2t) \exp(-\lambda_1 t) + c_3 \exp(-\lambda_3 t)$ of function (28), and the first term $c_1 \exp(-\lambda_1 t) + c_2 \exp(-\lambda_1 t) + c_3 \exp(-\lambda_3 t)$ of function (29) approach zero. Hence, as $t \to +\infty$, the first error function can be approximated as

$$e_1(t) \approx O(\tau^2) / \lambda_1 \lambda_2 \lambda_3 = O(\tau^2).$$
(30)

The proof is completed. \Box

Theorem 4. Assume that there exists a group of backward finite-divided difference formulas with truncation error $O(\tau^2)$. As the time t becomes large enough (i.e., $t \gg 0$), the relation between the solution error $e_1(t)$ of the order-4 ZNN model with time delay (18) and the truncation error $O(\tau^2)$ is formulated as $e_1(t) \approx O(\tau^2)/(\lambda_1\lambda_2\lambda_3\lambda_4) = O(\tau^2)$, where $\lambda_1, \lambda_2, \lambda_3$, and λ_4 are the ZNN design parameters. In other words, the steady-state solution error $e_1(t)$ has a positive proportional relationship with the truncation error $O(\tau^2)$.

Proof of Theorem 4. The order-4 ZNN model with time delay is a constant coefficient non-homogeneous linear differential equation (18), as follows:

$$\ddot{e}_1 + m_1 \ddot{e}_1 + m_2 \ddot{e}_1 + m_3 \dot{e}_1 + m_4 e_1 = O(\tau^2),$$

where $m_1 = \lambda_1 + \lambda_2 + \lambda_3 + \lambda_4$, $m_2 = \lambda_1\lambda_2 + \lambda_1\lambda_3 + \lambda_1\lambda_4 + \lambda_2\lambda_3 + \lambda_2\lambda_4 + \lambda_3\lambda_4$, $m_3 = \lambda_1\lambda_2\lambda_3 + \lambda_1\lambda_2\lambda_4 + \lambda_1\lambda_3\lambda_4 + \lambda_2\lambda_3\lambda_4$, and $m_4 = \lambda_1\lambda_2\lambda_3\lambda_4$. The corresponding homogeneous linear differential equation of (18), which is also the order-4 ZNN model without time delay (17), is formulated by setting the free term to zero, as follows:

$$\ddot{e}_1 + m_1 \ddot{e}_1 + m_2 \ddot{e}_1 + m_3 \dot{e}_1 + m_4 e_1 = 0,$$

and its corresponding characteristic function is the following polynomial equation with degree 4:

$$p^4 + m_1 p^3 + m_2 p^2 + m_3 p + m_4 p = 0. ag{31}$$

Then, the corresponding characteristic roots of (31) are $p_1 = -\lambda_1$, $p_2 = -\lambda_2$, $p_3 = -\lambda_3$, and $p_4 = -\lambda_4$. According to the solution structure of non-homogeneous linear ordinary differential equations [63,64], the solution of (18) is expressed as the summation of its special solution and the general solution of its corresponding homogeneous equation. Since the free term of (18) can be rewritten as $O(\tau^2) = O(\tau^2) \exp(0 \cdot t)$, and the characteristic roots of (17) satisfy $p_1 = -\lambda_1 < 0$, $p_2 = -\lambda_2 < 0$, $p_3 = -\lambda_3 < 0$, $p_4 = -\lambda_4 < 0$, the special solution of (18) is assumed to be of the form $C_4 \exp(0 \cdot t)$, which is equal to the constant C_4 . Substituting C_4 into Equation (18), one obtains the following relationship:

$$\lambda_1 \lambda_2 \lambda_3 \lambda_4 C_4 = O(\tau^2);$$

that is, $C_4 = O(\tau^2)/\lambda_1\lambda_2\lambda_3\lambda_4$. As there are four characteristic roots by solving Equation (31), the general solution of (18) is associated with the relationships between λ_1 , λ_2 , λ_3 , and λ_4 .

Case I If $\lambda_1 = \lambda_2 = \lambda_3 = \lambda_4$, the general solution of the order-4 ZNN model without time delay is $\tilde{e}_1(t) = (c_1 + c_2t + c_3t^2 + c_4t^3) \exp(-\lambda_1 t)$. Then, the general solution of the order-4 ZNN model with time delay is given by:

$$e_1(t) = (c_1 + c_2t + c_3t^2 + c_4t^3) \exp(-\lambda_1 t) + O(\tau^2) / \lambda_1 \lambda_2 \lambda_3 \lambda_4.$$
(32)

Case II If there are exactly three equal characteristic roots of (17) (without loss of generality, we assume $p_1 = p_2 = p_3 \neq p_4$), the general solution of the order-4 ZNN model without time delay is $\tilde{e}_1(t) = (c_1 + c_2t + c_3t^2) \exp(-\lambda_1 t) + c_4 \exp(-\lambda_4 t)$. Then, the general solution of the order-4 ZNN model with time delay is given by:

$$e_1(t) = (c_1 + c_2 t + c_3 t^2) \exp(-\lambda_1 t) + c_4 \exp(-\lambda_4 t) + O(\tau^2) / \lambda_1 \lambda_2 \lambda_3 \lambda_4.$$
(33)

Case III If there are exactly two equal characteristic roots of (17) (without loss of generality, we assume $p_1 = p_2 \neq p_3 \neq p_4$), the general solution of the order-4 ZNN model without time delay is $\tilde{e}_1(t) = (c_1 + c_2 t) \exp(-\lambda_1 t) + c_3 \exp(-\lambda_3 t) + c_4 \exp(-\lambda_4 t)$. Then, the general solution of the order-4 ZNN model with time delay is given by:

$$e_1(t) = (c_1 + c_2 t) \exp(-\lambda_1 t) + c_3 \exp(-\lambda_3 t) + c_4 \exp(-\lambda_4 t) + O(\tau^2) / \lambda_1 \lambda_2 \lambda_3 \lambda_4.$$
 (34)

Case IV If the four characteristic roots of (17) mutually differ, the general solution of the order-4 ZNN model without time delay is $\tilde{e}_1(t) = c_1 \exp(-\lambda_1 t) + c_2 \exp(-\lambda_1 t) + c_3 \exp(-\lambda_3 t) + c_4 \exp(-\lambda_4 t)$. Then, the general solution of the order-4 ZNN model with time delay is given by:

$$e_{1}(t) = c_{1} \exp(-\lambda_{1} t) + c_{2} \exp(-\lambda_{1} t) + c_{3} \exp(-\lambda_{3} t) + c_{4} \exp(-\lambda_{4} t) + O(\tau^{2}) / \lambda_{1} \lambda_{2} \lambda_{3} \lambda_{4}.$$
 (35)

When $t \to +\infty$, the first term $(c_1 + c_2t + c_3t^2 + c_4t^3) \exp(-\lambda_1 t)$ of function (32), the first term $(c_1 + c_2t + c_3t^2) \exp(-\lambda_1 t) + c_4 \exp(-\lambda_4 t)$ of function (33), the first term $(c_1 + c_2t) \exp(-\lambda_1 t) + c_3 \exp(-\lambda_3 t) + c_4 \exp(-\lambda_4 t)$ of function (34) in case III, and the first term $c_1 \exp(-\lambda_1 t) + c_2 \exp(-\lambda_1 t) + c_3 \exp(-\lambda_3 t) + c_4 \exp(-\lambda_4 t)$ of function (35) approach zero. Hence, as $t \to +\infty$, the first error function can be approximated as

$$e_1(t) \approx O(\tau^2) / \lambda_1 \lambda_2 \lambda_3 \lambda_4 = O(\tau^2).$$

The proof is completed. \Box

Theorem 5. Assume that there exists a group of backward finite-divided difference formulas with truncation error $O(\tau^2)$. As the time t becomes large enough (i.e., $t \gg 0$), the solution error $e_1(t)$ of the order-N ZNN model with time delay (22) and the truncation error $O(\tau^2)$ satisfy $e_1(t) \approx O(\tau^2)/(\lambda_1\lambda_2\cdots\lambda_N) = O(\tau^2)$, where $\lambda_1, \lambda_2, \cdots$, and λ_N are the design parameters of ZNN design formulas. In other words, the steady-state solution error $e_1(t)$ has a positive proportional relationship with the truncation error $O(\tau^2)$.

Proof of Theorem 5. The order-*N* ZNN model with time delay is a constant coefficient non-homogeneous linear differential Equation (22), as follows:

$$e_1^{(N)} + n_1 e_1^{(N-1)} + \dots + n_i e_1^{(i)} + \dots + n_{N-1} \dot{e}_1 + n_N e_1 = O(\tau^2),$$

where $e_1^{(i)}$ is the *i*th-order derivative of the error function e_1 , and the coefficients of corresponding high-order derivatives are given by $n_i = \sum_{1 \le l_1 < l_2 < \cdots < l_i \le N} \lambda_{l_1} \lambda_{l_2} \cdots \lambda_{l_i}$. The corresponding homogeneous linear differential equation of (22), which is also the order-*N* ZNN model without time delay (21), is formulated by setting the free term to zero, as follows:

$$e_1^{(N)} + n_1 e_1^{(N-1)} + \dots + n_i e_1^{(i)} + \dots + n_{N-1} \dot{e}_1 + n_N e_1 = 0,$$

and its corresponding characteristic function is the following polynomial equation with degree *N*:

$$p^{N} + n_{1}p^{N-1} + \dots + n_{i}p^{i} + \dots + n_{N-1}p + n_{N} = 0.$$
 (36)

Then, the corresponding characteristic roots of (36) are $p_1 = -\lambda_1$, $p_2 = -\lambda_2$, \cdots , and $p_N = -\lambda_N$. According to the solution structure of non-homogeneous linear differential ordinary differential equations [63,64], the solution of (22) is expressed as the summation

of its special solution and the general solution of its corresponding homogeneous equation. As the free term of (22) can be rewritten as $O(\tau^2) = O(\tau^2) \exp(0 \cdot t)$, and the characteristic roots of (21) satisfy $p_1 = -\lambda_1 < 0$, $p_2 = -\lambda_2 < 0$, \cdots , $p_N = -\lambda_N < 0$, then the special solution of (22) is assumed to be of the form $C_5 \exp(0 \cdot t)$, which is equal to the constant C_5 . Substituting C_5 into Equation (22), one obtains the following relationship:

$$\lambda_1 \lambda_2 \cdots \lambda_N C_5 = O(\tau^2);$$

that is, $C_5 = O(\tau^2)/\lambda_1\lambda_2\cdots\lambda_N$. As *N* characteristic roots are obtained by solving Equation (36), the general solution of (22) is associated with the relation of $\lambda_1, \lambda_2, \cdots$, and λ_N . One divides the index set $\{1, 2, \cdots, N\}$ into *k* partitions $\bigcup_{i=1}^{k} I_i$, with the relation $\lambda_{i_1} = \lambda_{i_2}$ for any $i_1 \neq i_2 \in I_i$, $i = 1, 2, \cdots, k$. Then, the the general solution of the order-*N* ZNN model without time delay is $\tilde{e}_1(t) = \sum_{i=1}^k \sum_{j=0}^{|I_i|-1} c_{ij}t^j \exp(-\lambda_i t)$, where $|I_i|$ is the cardinality of I_i . Then, the general solution of the order-*N* ZNN model with time delay is given by the following function:

$$e_{1}(t) = \sum_{i=1}^{k} \sum_{j=0}^{|I_{i}|-1} c_{ij} t^{j} \exp(-\lambda_{i} t) + O(\tau^{2}) / \lambda_{1} \lambda_{2} \cdots \lambda_{N}.$$
(37)

When $t \to +\infty$, the first term $\sum_{i=1}^{k} \sum_{j=0}^{|I_i|-1} c_{ij} t^j \exp(-\lambda_i t)$ of function (37) approaches zero. Hence, as $t \to +\infty$, the first error function can be approximated as

$$e_1(t) \approx O(\tau^2) / \lambda_1 \lambda_2 \cdots \lambda_N = O(\tau^2).$$

The proof is completed. \Box

Remarks: One can see that Theorem 5 is the generalized situation of Theorems 1–4. We list Theorems 1 through 4 to describe the error pattern of low-order ZNN models with time delay, making the ensuing applications ready to understand. Although Theorem 5 can be extended to more generalized situations with higher-order precision, the core point is that the stable uniform BFDDFs with higher-order precision are difficult to find, which will be studied in our future work.

In summary, according to the convergence analysis of the order-*N* ZNN models with and without time delay, we find that the first solution error e_1 globally and exponentially converges to zero when the time *t* becomes large enough for the order-*N* ZNN model without time delay. Furthermore, the first solution error e_1 has a positive proportional relationship with the truncation error $O(\tau^2)$, when the time *t* becomes large enough, for the order-*N* ZNN model with time delay.

3. Tracking Control of Triple-Integrator System with NOF

In this section, we solve the problem of tracking control for a TI system with NOF using the order-*N* ZNN methods with and without time delay, and validate the corresponding error theory through simulation experiments.

3.1. Controllers with and without Time Delay for TI System with NOF

In this subsection, we present the problem of tracking control for a TI system with NOF using the order-*N* ZNN method without time delay. We also give the corresponding order-*N* ZNN method with time delay approximated using the BFDDFs with truncation errors $O(\tau^2)$ in Table 1 and the BFDDFs with truncation errors $O(\tau)$ in [64]; the latter of which are adopted as the benchmark for comparison.

Let us consider the following problem of tracking control for the TI system [65] with NOF:

$$\dot{x}_1 = x_2,$$

 $\dot{x}_2 = x_3,$
 $\dot{x}_3 = u_1,$
(38)

where x_1 , x_2 , and x_3 are the states of the TI system and u_1 is the input of the system. Corresponding to the universal form of the tracking control problem for nonlinear systems (1) with single input, the tracking control problem for the TI system [65] with NOF is defined as $\mathbf{x}(t) = [x_1(t), x_2(t), x_3(t)]^T$, $\mathbf{f}(\mathbf{x}, u, t) = [x_2(t), x_3(t), u_1(t)]^T$, and $h(\mathbf{x}) = \sin(x_1(t))$. The output of the dynamic system (38) is set as $y_1 = \sin(x_1)$. The controller u_1 was designed based on the state system (38) using the ZNN method as follows. We define the first error function as

$$\hat{e}_1 = y_1 - y_{1d} = \sin(x_1) - y_{1d}, \tag{39}$$

where y_{1d} is the desired tracking function of y_1 , and it is order-3 differentiable. The ZNN design formula is adopted for the first error function \hat{e}_1 ,

$$\dot{\hat{e}}_1 = -\hat{\lambda}_1 \hat{e}_1,\tag{40}$$

where $\hat{\lambda}_1 > 0 \in \mathbb{R}$ is the ZNN design parameter. Substituting Equation (39) into Equation (40), we obtain $\dot{x}_1 \cos(x_1) - \dot{y}_{1d} = -\hat{\lambda}_1(\sin(x_1) - y_{1d})$. Using the first equation of the QI system (50), the differential equation of the state x_1 is formulated as

$$x_2 \cos(x_1) - \dot{y}_{1d} = -\dot{\lambda}_1 (\sin(x_1) - y_{1d}).$$
(41)

In the following, the ZNN method is adopted three times to obtain the controller u_1 step-by-step, based on the state functions of the TI system (38). From Equation (41), we define the second error function, \hat{e}_2 , as

$$\hat{e}_2 = x_2 \cos(x_1) - \dot{y}_{1d} + \hat{\lambda}_1 (\sin(x_1) - y_{1d}).$$
(42)

The ZNN design formula is adopted for the second error function \hat{e}_2 ,

$$\hat{\theta}_2 = -\hat{\lambda}_2 \hat{e}_2,\tag{43}$$

where $\hat{\lambda}_2 > 0 \in \mathbb{R}$ is the ZNN design parameter. Substituting Equation (42) into Equation (43), and reformulating it using the state functions of the TI system (38), we have $x_3 \cos(x_1) - x_2^2 \sin x_1 - \ddot{y}_{1d} + (\hat{\lambda}_1 + \hat{\lambda}_2)(x_2 \cos(x_1) - \dot{y}_{1d}) + \hat{\lambda}_1 \hat{\lambda}_2(\sin(x_1) - y_{1d}) = 0$. Define the third error function \hat{e}_3 as

$$\hat{e}_3 = x_3 \cos(x_1) - x_2^2 \sin x_1 - \ddot{y}_{1d} + (\hat{\lambda}_1 + \hat{\lambda}_2)(x_2 \cos(x_1) - \dot{y}_{1d}) + \hat{\lambda}_1 \hat{\lambda}_2 (\sin(x_1) - y_{1d}).$$
(44)

The ZNN design formula is adopted for the third error function \hat{e}_3 ,

$$\dot{\hat{e}}_3 = -\hat{\lambda}_3 \hat{e}_3,\tag{45}$$

where $\hat{\lambda}_3 > 0 \in \mathbb{R}$ is the ZNN design parameter. Substituting Equation (44) into Equation (45), and reformulating it by using the state functions of the TI system (38), we obtain the following equation, which includes the input u_1 :

$$(u_1 - x_2^2)\cos(x_1) - 3x_2x_3\sin(x_1) - \ddot{y}_{1d} + (\hat{\lambda}_1 + \hat{\lambda}_2 + \hat{\lambda}_3)(x_3\cos(x_1) - x_2^2\sin(x_1) - \ddot{y}_{1d}) + (\hat{\lambda}_1\hat{\lambda}_2 + \hat{\lambda}_1\hat{\lambda}_3 + \hat{\lambda}_2\hat{\lambda}_3)(x_2\cos(x_1) - \dot{y}_{1d}) + (\hat{\lambda}_1\hat{\lambda}_2\hat{\lambda}_3)(\sin(x_1) - y_{1d}) = 0.$$

$$(46)$$

Finally, we define a function of a nonlinear state variable and different-order derivatives of the desired path as $f_1 = \tilde{y}_{1d} + \alpha_1 \tilde{y}_{1d} + \alpha_2 \dot{y}_{1d} + \alpha_3 (y_{1d} - \sin(x_1))$, where $\alpha_1 = \hat{\lambda}_1 + \hat{\lambda}_2 + \hat{\lambda}_3$, $\alpha_2 = \hat{\lambda}_1 \hat{\lambda}_2 + \hat{\lambda}_1 \hat{\lambda}_3 + \hat{\lambda}_2 \hat{\lambda}_3$, and $\alpha_3 = \hat{\lambda}_1 \hat{\lambda}_2 \hat{\lambda}_3$. Then, rewriting Equation (46), the controller without time delay for the TI system with NOF (38), in the form of u_1 , is formulated as follows:

$$\begin{cases} f_1 = y_{1d} + \alpha_1 \ddot{y}_{1d} + \alpha_2 \dot{y}_{1d} + \alpha_3 (y_{1d} - \sin(x_1)), \\ f_2 = (3x_1x_2 + \alpha_1 x_2^2) \sin(x_1) + (x_2^2 - \alpha_1 x_3 - \alpha_2 x_2) \cos(x_1), \\ u_1 = \frac{f_1 + f_2}{\cos(x_1)}. \end{cases}$$
(47)

In the time delay situation, order-*n* BFDDFs with truncation errors $O(\tau^2)$ are adopted to approximate the order-*n* (with $n \leq 3$) derivatives: $\dot{y}_{1d} \approx (3y_{1d}(t) - 4y_{1d}(t - \tau) + y_{1d}(t - 2\tau))/2\tau$, $\ddot{y}_{1d} \approx (2y_{1d}(t) - 5y_{1d}(t - \tau) + 4y_{1d}(t - 2\tau) - y_{1d}(t - 3\tau))/\tau^2$, and $\ddot{y}_{1d} \approx (5y_{1d}(t) - 18y_{1d}(t - \tau) + 24y_{1d}(t - 2\tau) - 14y_{1d}(t - 3\tau) + 3y_{1d}(t - 4\tau))/2\tau^3$. Substituting the above-approximated order-*n* derivatives into the controller without time delay (47), the controller with time delay \tilde{u}_1 for the TI system with NOF is formulated as follows:

$$\begin{cases} \tilde{f}_{1}(t) = \frac{1}{2\tau^{3}} (\beta_{1}y_{1d}(t) - \beta_{2}y_{1d}(t-\tau) + \beta_{3}y_{1d}(t-2\tau) - \beta_{4}y_{1d}(t-3\tau) + 3y_{1d}(t-4\tau) \\ -2\alpha_{3}\tau^{3}\sin(x_{1}(t))), \\ f_{2}(t) = (3x_{1}(t)x_{2}(t) + \alpha_{1}x_{2}^{2}(t))\sin(x_{1}(t)) + (x_{2}^{2}(t) - \alpha_{1}x_{3}(t) - \alpha_{2}x_{2}(t))\cos(x_{1}(t)), \\ \tilde{u}_{1}(t) = \frac{\tilde{f}_{1}(t) + f_{2}(t)}{\cos(x_{1}(t))}, \end{cases}$$

$$(48)$$

where $\beta_1 = 5 + 4\alpha_1\tau + 3\alpha_2\tau^2 + 2\alpha_3\tau^3$, $\beta_2 = 18 + 10\alpha_1\tau + 4\alpha_2\tau^2$, $\beta_3 = 24 + 8\alpha_1\tau + \alpha_2\tau^2$, and $\beta_4 = 14 + 2\alpha_1\tau$.

For comparison, the controller with time delay for the TI system (38) with NOF was approximated by the BFDDFs with truncation errors $O(\tau)$. The order-*n* derivatives (with $n \leq 3$) in the model (47) were approximated as follows: $\dot{y}_{1d}(t) \approx (y_{1d}(t) - y_{1d}(t-\tau))/\tau$, $\ddot{y}_{1d}(t) \approx (y_{1d}(t) - 2y_{1d}(t-\tau) + y_{1d}(t-2\tau))/\tau^2$, and $\ddot{y}_{1d}(t) \approx (y_{1d}(t) - 3y_{1d}(t-\tau) + 3y_{1d}(t-2\tau) - y_{1d}(t-3\tau))/\tau^3$. Then, the controller with time delay for the TI system with NOF (38) in the form of \bar{u}_1 was formulated as

$$\begin{cases} \bar{f}_{1}(t) = \frac{1}{\tau^{3}} \left(\bar{\beta}_{1} y_{1d}(t) - \bar{\beta}_{2} y_{1d}(t-\tau) + \bar{\beta}_{3} y_{1d}(t-2\tau) - y_{1d}(t-3\tau) - \alpha_{3} \tau^{3} \sin(x_{1}(t)) \right), \\ f_{2}(t) = (3x_{1}(t)x_{2}(t) + \alpha_{1}x_{2}^{2}(t)) \sin(x_{1}(t)) + (x_{2}^{2}(t) - \alpha_{1}x_{3}(t) - \alpha_{2}x_{2}(t)) \cos(x_{1}(t)), \\ \bar{u}_{1}(t) = \frac{\bar{f}_{1}(t) + f_{2}(t)}{\cos(x_{1}(t))}, \end{cases}$$

$$\tag{49}$$

where $\bar{\beta}_1 = 1 + \alpha_1 \tau + \alpha_2 \tau^2 + \alpha_3 \tau^3$, $\bar{\beta}_2 = 3 + 2\alpha_1 \tau + \alpha_2 \tau^2$, and $\bar{\beta}_3 = 3 + \alpha_1 \tau$.

Having presented the construction of the controllers for the TI system with NOF using the ZNN method, the corresponding numerical experiments considering the controllers with and without time delay for the TI system with NOF are detailed in the following subsection.

3.2. TI System Tests with NOF

In this subsection, we describe the TI (with NOF) system tests, which were conducted using the MATLAB R2021a simulation platform on a PC, in order to demonstrate the validity of the proposed ZNN model with time delay. The hardware environment for the TI system tests was a laptop with an Intel^(R) Core^(TM) i7-7700HQ CPU (2.80 GHz) and 8.00 GB RAM. The numerical solution of the TI system (38) was obtained using the ode15s function (i.e., ode15s('TI-system', [0,T],[x_1 , x_2 , x_3], odeset('Reltol', 10⁻⁹, 'Abstol', 10⁻⁹))) in the Matlab ODE toolbox. In the tests, we set the simulation duration as 40 s, and chose the desired path as $y_{1d} = \sin(2t)\cos(t)$. The initial states of the TI system were set as $x_1(0) = 0.02$, $x_2(0) = 0$, and $x_3(0) = 0$. In addition, we set the ZNN design parameters as $\hat{\lambda}_1 = \hat{\lambda}_2 = \hat{\lambda}_3 = 1000$, and one can see that these model parameters decided the parameters of the controllers with and without time delay. In Table 2, we give a list of the parameters for the controller without time delay depicted in (47), the controller with time delay constructed using BFDDFs with truncation error $O(\tau^2)$ depicted in (48), and the controller with time delay constructed using BFDDFs with truncation error $O(\tau)$ depicted in (49), when the time delay τ was equal to 0.01 s. The test results for the controller without time delay for the TI system (47) with NOF are illustrated in Figure 3. The actual trajectory and the desired path are illustrated in Figure 3a, where the blue dashed line represents the desired path, while the red solid line represents the corresponding actual trajectory. This result indicates that the corresponding actual trajectory y_1 tracked the desired path y_{1d} well, which implies that the proposed tracking controller designed using the order-3 ZNN method without time delay showed good performance. Furthermore, Figure 3c illustrates the tracking error \hat{e}_1 of the controller without time delay for the TI system (47) with NOF, which converged to zero in a short time. From Figure 3d, one can see that the maximal steady-state tracking error (MSSTE) of the output y_1 of the TI system with NOF was on the order of 10^{-9} , which implies that the accuracy was good. Moreover, Figure 3b shows that the smooth and continuous control input u_1 of the TI system with NOF was within a suitable and reasonable range. Therefore, the proposed tracking controller (47) developed using the order-3 ZNN method without time delay is applicable in hardware implementations for real-world integrator systems.



Figure 3. Tracking performance of TI system (38) equipped with controller without time delay (47) for the desired path $y_{1d} = \sin(2t)\cos(t)$.

Table 2. Parameters for controller without time delay depicted in (47), controller with time delay constructed using BFDDFs with truncation error $O(\tau^2)$ depicted in (48), and controller with time delay constructed using BFDDFs with truncation error $O(\tau)$ depicted in (49), with time delay τ equal to 0.01 s.

(47)			(48)	(49)	
$\begin{array}{c} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{array}$	$\begin{array}{c} 3.0000 \times 10^{3} \\ 3.0000 \times 10^{6} \\ 1.0000 \times 10^{9} \end{array}$	$egin{array}{c} eta_1 \ eta_2 \ eta_3 \end{array}$	$\begin{array}{c} 3.0250 \times 10^{3} \\ 1.5180 \times 10^{3} \\ 5.6400 \times 10^{2} \end{array}$	$\begin{array}{c}\bar{\beta}_1\\\bar{\beta}_2\\\bar{\beta}_3\end{array}$	$\begin{array}{c} 1.3310 \times 10^{3} \\ 3.6300 \times 10^{2} \\ 3.3000 \times 10^{1} \end{array}$
-	-	eta_4	$7.4000 imes 10^1$	-	-

With a time delay $\tau = 0.01$ s, the tracking control performance for the TI system (38) when equipped with the controller with time delay approximated by the BFDDFs with truncation errors $O(\tau^2)$ depicted in (48), for the desired path $y_{1d} = \sin(2t)\cos(t)$ is illustrated in Figure 4. As is displayed in Figure 4a, the corresponding actual trajectory y_1 converged to the desired path y_{1d} within a short time. Figure 4c illustrates that the tracking error \hat{e}_1 converged to zero within a small time interval. One can also see that the MSSTE of the corresponding actual trajectory y_1 was on the order of 10^{-6} from Figure 4d. Moreover, Figure 4b shows that the continuous and smooth control input \tilde{u}_1 was within a suitable and reasonable range. From the above, one sees that the tracking performance approximated by the BFDDFs with truncation error $O(\tau^2)$ was still good in the presence of a time delay. Due to the similarity of the trajectories for different values of time delay τ , we only displayed the tracking control results with a time delay of $\tau = 0.01$ s.



Figure 4. Tracking performance of TI system (38) equipped with controller with time delay (48) for the desired path $y_{1d} = \sin(2t)\cos(t)$.

In Figure 5, the tracking errors for the TI system (38) equipped with the controller with time delay approximated using BFDDFs with truncation errors $O(\tau^2)$ depicted in (48), are illustrated with the different time delay values: 0.1, 0.05, 0.01, or 0.005 s. In the case of $\tau = 0.1$ s, the order of the MSSTE was 10^{-4} . In the case of $\tau = 0.01$ s, the order of the MSSTE was 10^{-6} . In the case of $\tau = 0.05$ s, the order of the MSSTE was 10^{-5} . In the case of $\tau = 0.005$ s, the order of the MSSTE was 10^{-7} . Therefore, from the test results, it can be seen that, if the time delay τ decreases by a factor of 10^{-1} , then the order of the MSSTE decreases by a factor of 10^{-2} . That is, the tracking errors showed a positive proportional relationship with the time delay τ^2 ; or, in other words, with the truncation errors $O(\tau^2)$. These results are consistent with Theorems 1, 2, and 4 in Section 2.



Figure 5. Orders of absolute tracking error $|\hat{e}_1|$ for TI system (38) equipped with controller with time delay (48) for the desired path $y_{1d} = \sin(2t)\cos(t)$, with τ equal to 0.1, 0.05, 0.01, or 0.005 s.

In Figure 6, the tracking errors are illustrated for the controllers with time delay constructed using BFDDFs with truncation error $O(\tau)$ depicted in (49), and using BFDDFs with truncation error $O(\tau^2)$ depicted in (48), with time delay of $\tau = 0.01$ s. From Figure 6, one can see that the MSSTE was 7.1296×10^{-5} for the controller with time delay constructed using the BFDDFs with truncation error $O(\tau)$ depicted in (49), while the MSSTE was 1.4×10^{-6} for the controller constructed using the BFDDFs with truncation error $O(\tau)$ depicted in (48). In order to compare more details of these two models, more simulation results are provided in Table 3. The tracking errors with different time delay values for the controllers with time delay constructed using the BFDDFs with truncation error $O(\tau)$ depicted in (49), and BFDDFs with truncation error $O(\tau^2)$ depicted in (49), and BFDDFs with truncation error $O(\tau^2)$ depicted in (49), and BFDDFs with truncation error $O(\tau^2)$ depicted in (49), has a positive proportional relationship with the truncation error $O(\tau)$ [62], while the tracking error of (48) has a positive proportional relationship with the truncation error $O(\tau^2)$.



Figure 6. Order of absolute tracking error (ATE) $|\hat{e}_1|$ for TI system (38) equipped with controllers constructed using BFDDFs with truncation error $O(\tau)$ depicted in (49), and BFDDFs with truncation error $O(\tau^2)$ depicted in (48), with the time delay of $\tau = 0.01$ s for the desired path $y_{1d} = \sin(2t)\cos(t)$.

Table 3. Performance of controllers with time delay constructed using BFDDFs with truncation error $O(\tau)$ depicted in (49), and controller with time delay constructed using BFDDFs with truncation error $O(\tau^2)$ depicted in (48), with respect to different time delay values.

	Criterion	$ au_0 = 0.2~\mathrm{s}$	$ au_1 = 0.1 { m s}$	$ au_2 = 0.05~\mathrm{s}$	$ au_3 = 0.01 \mathrm{s}$	$ au_4=0.005~{ m s}$
	$rac{ au_{k-1}}{ au_k}$	-	2	2	5	2
(40)	MSSTE	1.4000×10^{-3}	7.1146×10^{-4}	3.5629×10^{-4}	7.1296×10^{-5}	3.5653×10^{-5}
(49)	$\frac{\text{MSSTE}(\tau_{k-1})}{\text{MSSTE}(\tau_k)} \\ (\approx)$	-	2	2	5	2
(49)	MSSTE	5.4759×10^{-4}	1.3928×10^{-4}	3.4962×10^{-5}	1.4000×10^{-6}	3.5001×10^{-7}
(48)	$\frac{\text{MSSTE}(\tau_{k-1})}{\text{MSSTE}(\tau_k)}$ (\approx)	-	4	4	25	4

4. Tracking Control of Quintuple-Integrator System with LOF

In this section, we solve the problem of tracking control for the QI system with LOF using the order-*N* ZNN methods with and without time delay, and validate the corresponding error theory through simulation experiments.

4.1. Controllers with and without Time Delay for QI System with LOF

In this subsection, we present the problem of tracking control for the QI system with LOF using the order-*N* ZNN method without time delay. We also give the corresponding order-*N* ZNN methods with time delay, using the BFDDFs with truncation errors $O(\tau^2)$ in Table 1 and the BFDDFs with truncation errors $O(\tau)$ in [64]; again, the latter are adopted as the benchmark for comparison.

Consider the following problem of tracking control for the QI system [65] with LOF:

$$\begin{aligned}
\dot{x}_1 &= x_2, \\
\dot{x}_2 &= x_3, \\
\dot{x}_3 &= x_4, \\
\dot{x}_4 &= u_2,
\end{aligned}$$
(50)

where x_1, x_2, x_3 , and x_4 are the states of the QI system, and u_2 is the input of the system. The output of the dynamic system (50) is set as $y_2 = x_1$. Corresponding to the universal tracking control problem form for nonlinear systems (1) with single input, the tracking control problem for the QI system [65] with LOF is defined as $\mathbf{x}(t) = [x_1(t), x_2(t), x_3(t), x_4(t)]^T$, $\mathbf{f}(\mathbf{x}, u, t) = [x_2(t), x_3(t), x_4(t), u_2(t)]^T$, and $h(\mathbf{x}) = x_1(t)$. The controller u_2 is designed based on the state system (50) using the ZNN method as follows. We define the first error function of y_2 as \hat{e}_4 , formulated as

$$\hat{z}_4 = y_2 - y_{2d} = x_1 - y_{2d}, \tag{51}$$

where y_{2d} is the desired tracking function of y_2 , which is order-4 differentiable. The ZNN design formula is adopted for the error function \hat{e}_4 ,

$$\dot{\hat{e}}_4 = -\hat{\lambda}_4 \hat{e}_4,\tag{52}$$

where $\hat{\lambda}_4 > 0 \in \mathbb{R}$ is the ZNN design parameter. Substituting Equation (51) into Equation (52), we obtain $\dot{x}_1 - \dot{y}_{2d} = -\hat{\lambda}_4(x_1 - y_{2d})$. Using the first equation of the QI system (50), the differential equation of the state x_1 can be formulated as

$$x_2 - \dot{y}_{2d} = -\hat{\lambda}_4(x_1 - y_{2d}).$$

In order to construct the controller u_2 , the second error function \hat{e}_5 of y_2 is defined as

$$\hat{e}_5 = x_2 - \dot{y}_{2d} + \hat{\lambda}_4 (x_1 - y_{2d}), \tag{53}$$

and the ZNN design formula is adopted for the second error function \hat{e}_5 ,

$$\dot{\hat{e}}_5 = -\hat{\lambda}_5 \hat{e}_5,\tag{54}$$

where $\hat{\lambda}_5 > 0 \in \mathbb{R}$ is the ZNN design parameter. Substituting Equation (53) into Equation (54), we have $\dot{x}_2 - \ddot{y}_{2d} + \hat{\lambda}_4(\dot{x}_1 - \dot{y}_{2d}) = -\hat{\lambda}_5(x_2 - \dot{y}_{2d} + \hat{\lambda}_4(x_1 - y_{2d}))$. Using the first and second equations of the QI system (50), the differential equation considering the states x_1 and x_2 is formulated as

$$x_3 - \ddot{y}_{2d} + \hat{\lambda}_4(x_2 - \dot{y}_{2d}) = -\hat{\lambda}_5(x_2 - \dot{y}_{2d} + \hat{\lambda}_4(x_1 - y_{2d})).$$

The third error function \hat{e}_6 of y_2 is defined as

$$\hat{z}_6 = x_3 - \ddot{y}_{2d} + (\hat{\lambda}_4 + \hat{\lambda}_5)(x_2 - \dot{y}_{2d}) + \hat{\lambda}_4 \hat{\lambda}_5(x_1 - y_{2d}), \tag{55}$$

and the ZNN design formula is adopted for the third error function \hat{e}_6 ,

$$\dot{\hat{e}}_6 = -\hat{\lambda}_6 \hat{e}_6,\tag{56}$$

where $\hat{\lambda}_6 > 0 \in \mathbb{R}$ is the ZNN design parameter. Substituting Equation (55) into Equation (56), we obtain $\dot{x}_3 - \ddot{y}_{2d} + (\hat{\lambda}_4 + \hat{\lambda}_5)(\dot{x}_2 - \ddot{y}_{2d}) + \hat{\lambda}_4 \hat{\lambda}_5(\dot{x}_1 - \dot{y}_{2d}) = -\hat{\lambda}_6(x_3 - \ddot{y}_{2d} + (\hat{\lambda}_4 + \hat{\lambda}_5)(x_2 - \dot{y}_{2d}))$

 \dot{y}_{2d}) + $\hat{\lambda}_4 \hat{\lambda}_5(x_1 - y_{2d})$). Using the first, second, and third equations of the QI system (50), the differential equation of the states x_1 , x_2 , and x_3 is formulated as

$$\begin{aligned} x_4 &- \ddot{y}_{2d} + (\hat{\lambda}_4 + \hat{\lambda}_5 + \hat{\lambda}_6)(x_3 - \ddot{y}_{2d}) + (\hat{\lambda}_4 \hat{\lambda}_5 + \hat{\lambda}_4 \hat{\lambda}_6 + \hat{\lambda}_5 \hat{\lambda}_6)(x_2 - \dot{y}_{2d}) \\ &+ \hat{\lambda}_4 \hat{\lambda}_5 \hat{\lambda}_6(x_1 - y_{2d}) = 0. \end{aligned}$$

From the state functions of the QI system, the controller u_2 is defined using the derivative of x_4 . Thus, the forth error function \hat{e}_7 is formulated as

$$\hat{e}_7 = x_4 - \ddot{y}_{2d} + (\hat{\lambda}_4 + \hat{\lambda}_5 + \hat{\lambda}_6)(x_3 - \ddot{y}_{2d}) + (\hat{\lambda}_4 \hat{\lambda}_5 + \hat{\lambda}_4 \hat{\lambda}_6 + \hat{\lambda}_5 \hat{\lambda}_6)(x_2 - \dot{y}_{2d}) + \hat{\lambda}_4 \hat{\lambda}_5 \hat{\lambda}_6(x_1 - y_{2d}),$$
(57)

and the ZNN design formula is adopted for the forth error function \hat{e}_7 ,

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$$\dot{\hat{c}}_7 = -\hat{\lambda}_7 \hat{e}_7,\tag{58}$$

where $\hat{\lambda}_7 > 0 \in \mathbb{R}$ is the ZNN design parameter. Substituting Equation (57) into Equation (58), we have $\dot{x}_4 - \ddot{y}_{2d} + (\hat{\lambda}_4 + \hat{\lambda}_5 + \hat{\lambda}_6)(\dot{x}_3 - \ddot{y}_{2d}) + (\hat{\lambda}_4\hat{\lambda}_5 + \hat{\lambda}_4\hat{\lambda}_6 + \hat{\lambda}_5\hat{\lambda}_6)(\dot{x}_2 - \ddot{y}_{2d}) + \hat{\lambda}_4\hat{\lambda}_5\hat{\lambda}_6$ $(\dot{x}_1 - \dot{y}_{2d}) = -\hat{\lambda}_7(x_4 - \ddot{y}_{2d} + (\hat{\lambda}_4 + \hat{\lambda}_5 + \hat{\lambda}_6)(x_3 - \ddot{y}_{2d}) + (\hat{\lambda}_4\hat{\lambda}_5 + \hat{\lambda}_4\hat{\lambda}_6 + \hat{\lambda}_5\hat{\lambda}_6)(x_2 - \dot{y}_{2d}) + \hat{\lambda}_4\hat{\lambda}_5\hat{\lambda}_6(x_1 - y_{2d}))$. Substituting the state equations of the QI system (50) into the above equation, the controller u_2 for the QI system with LOF is formulated by the following equation in the situation without time delay:

$$u_{2} = \ddot{y}_{2d} + q_{1}(\ddot{y}_{2d} - x_{4}) + q_{2}(\ddot{y}_{2d} - x_{3}) + q_{3}(\dot{y}_{2d} - x_{2}) + q_{4}(y_{2d} - x_{1}),$$
(59)

where the model parameters are $q_1 = \hat{\lambda}_4 + \hat{\lambda}_5 + \hat{\lambda}_6 + \hat{\lambda}_7$, $q_2 = \hat{\lambda}_4 \hat{\lambda}_5 + \hat{\lambda}_4 \hat{\lambda}_6 + \hat{\lambda}_4 \hat{\lambda}_7 + \hat{\lambda}_5 \hat{\lambda}_6 + \hat{\lambda}_5 \hat{\lambda}_7 + \hat{\lambda}_6 \hat{\lambda}_7$, $q_3 = \hat{\lambda}_4 \hat{\lambda}_5 \hat{\lambda}_6 + \hat{\lambda}_4 \hat{\lambda}_5 \hat{\lambda}_7 + \hat{\lambda}_4 \hat{\lambda}_6 \hat{\lambda}_7 + \hat{\lambda}_5 \hat{\lambda}_6 \hat{\lambda}_7$, and $q_4 = \hat{\lambda}_4 \hat{\lambda}_5 \hat{\lambda}_6 \hat{\lambda}_7$.

In the time delay situation, the order-*n* BFDDFs with truncation errors $O(\tau^2)$ are adopted to approximate the order-*n* (with n = 1, 2, 3, 4) derivatives: $\dot{y}_{2d} \approx (3y_{2d}(t) - 4y_{2d}(t - \tau) + y_{2d}(t - 2\tau))/2\tau$, $\ddot{y}_{2d} \approx (2y_{2d}(t) - 5y_{2d}(t - \tau) + 4y_{2d}(t - 2\tau) - y_{2d}(t - 3\tau))/\tau^2$, $\ddot{y}_{2d} \approx (5y_{2d}(t) - 18y_{2d}(t - \tau) + 24y_{2d}(t - 2\tau) - 14y_{2d}(t - 3\tau) + 3y_{2d}(t - 4\tau))/2\tau^3$ and $\ddot{y}_{2d} \approx (3y_{2d}(t) - 14y_{2d}(t - \tau) + 26y_{2d}(t - 2\tau) - 24y_{2d}(t - 3\tau) + 11y_{2d}(t - 4\tau) - 2y_{2d}(t - 5\tau))/\tau^4$. Then, the controller with time delay \tilde{u}_2 for the QI system with LOF is formulated as follows:

$$\tilde{u}_{2}(t) = \frac{1}{2\tau^{4}} (\phi_{1}y_{2d}(t) - \phi_{2}y_{2d}(t-\tau) + \phi_{3}y_{2d}(t-2\tau) - \phi_{4}y_{2d}(t-3\tau) + \phi_{5}y_{2d}(t-4\tau) - 4y_{2d}(t-5\tau) - 2\tau^{4}(q_{4}x_{1}(t) + q_{3}x_{2}(t) + q_{2}x_{3}(t) + q_{1}x_{4}(t))),$$
(60)

where $\phi_1 = 6 + 5q_1\tau + 4q_2\tau^2 + 3q_3\tau^3 + 2q_4\tau^4$, $\phi_2 = 28 + 18q_1\tau + 10q_2\tau^2 + 4q_3\tau^3$, $\phi_3 = 52 + 24q_1\tau + 8q_2\tau^2 + q_3\tau^3$, $\phi_4 = 48 + 14q_1\tau + 2q_2\tau^2$, and $\phi_5 = 22 + 3q_1\tau$.

For comparison, another controller with time delay for the QI system (50) with LOF was approximated using the BFDDFs with truncation errors $O(\tau)$. The order-*n* derivatives (with n = 1, 2, 3, 4) in the model (59) were approximated by: $\dot{y}_{2d}(t) \approx (y_{2d}(t) - y_{2d}(t - \tau))/\tau$, $\ddot{y}_{2d}(t) \approx (y_{2d}(t) - 2y_{2d}(t - \tau) + y_{2d}(t - 2\tau))/\tau^2$, $\ddot{y}_{2d}(t) \approx (y_{2d}(t) - 3y_{2d}(t - \tau) + 3y_{2d}(t - 2\tau) - y_{2d}(t - 3\tau))/\tau^3$, and $\ddot{y}_{2d}(t) \approx (y_{2d}(t) - 4y_{2d}(t - \tau) + 6y_{2d}(t - 2\tau) - 4y_{2d}(t - 3\tau) + y_{2d}(t - 4\tau))/\tau^4$. Then, the controller with time delay for QI system with LOF (50), in the form of \bar{u}_2 , was formulated as

$$\bar{u}_{2}(t) = \frac{1}{\tau^{4}} \big(\gamma_{1} y_{2d}(t) - \gamma_{2} y_{2d}(t-\tau) + \gamma_{3} y_{2d}(t-2\tau) - \gamma_{4} y_{2d}(t-3\tau) + y_{2d}(t-4\tau) - \tau^{4} (q_{4} x_{1}(t) + q_{3} x_{2}(t) + q_{2} x_{3}(t) + q_{1} x_{4}(t)) \big),$$
(61)

where $\gamma_1 = 1 + q_1\tau + q_2\tau^2 + q_3\tau^3 + q_4\tau^4$, $\gamma_2 = 4 + 3q_1\tau + 2q_2\tau^2 + q_3\tau^3$, $\gamma_3 = 6 + 3q_1\tau + q_2\tau^2 + q_3\tau^3$, and $\gamma_4 = 4 + q_1\tau$.

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Having presented the construction of the controllers for the QI system with LOF using the ZNN method, the corresponding numerical experiments of the controllers with and without time delay for the QI system with LOF are detailed in the following subsection.

4.2. QI with LOF System Tests

Next, the QI (with LOF) system tests were conducted on the MATLAB R2021a simulation platform on a PC, in order to demonstrate the validity of the proposed ZNN model with time delay. The hardware environment for the QI with LOF system tests was a laptop with an Intel^(R) Core^(TM) i7-7700HQ CPU (2.80 GHz) and 8.00 GB RAM. The numerical solution of the QI system (50) was obtained using the ode15s function (i.e., ode15s('QIsystem', [0,T], $[x_1, x_2, x_3, x_4]$, odeset('Reltol', 10^{-9} , 'Abstol', 10^{-9}))) in the Matlab ODE toolbox. In the tests, we set the simulation duration as 40 s, and chose the desired path as $y_{2d} = \sin(0.1\pi t) \cos(0.2\pi t)$. The initial states of the QI system were set as $x_1(0) = 0.1$, $x_2(0) = 0$, $x_3(0) = 0$, and $x_4(0) = 0$. In addition, we set the ZNN design parameters as $\hat{\lambda}_4 = \hat{\lambda}_5 = \hat{\lambda}_6 = \hat{\lambda}_7 = 100$, and one can see that these model parameters decided the parameters of controllers with and without time delay. We also list the parameters of the controller without time delay in (59), the controller with time delay constructed using BFDDFs with truncation error $O(\tau^2)$ depicted in (60), and the controller with time delay constructed using BFDDFs with truncation error $O(\tau)$ depicted in (61), when the time delay τ was equal to 0.01 s, in Table 4. The test results for the controller without time delay for the QI system (59) with LOF is illustrated in Figure 7. The actual trajectory and the desired path are illustrated in Figure 7a, where the blue dashed line represents the desired path, while the red solid line represents the corresponding actual trajectory. This result indicates that the corresponding actual trajectory y_2 tracked the desired path y_{4d} well, which implies that this tracking controller designed using the order-4 ZNN method without time delay had good performance. Furthermore, Figure 7c illustrates the tracking error \hat{e}_4 of the controller without time delay for the QI system (59) with LOF, which converged to zero in a short time. From Figure 7d, one can see that the MSSTE of the outputs y_2 of the QI system with LOF was on the order of 10^{-10} , which implies that the accuracy was good. Moreover, Figure 7b shows that the continuous and smooth control input u_2 was within a suitable and reasonable range. Therefore, the proposed tracking controller developed using the order-4 ZNN method without time delay is applicable in hardware implementations for real-world integrator systems.

Table 4. Parameters of controller without time delay depicted in (59), controller with time delay constructed using BFDDFs with truncation error $O(\tau^2)$ depicted in (60), and controller with time delay constructed using BFDDFs with truncation error $O(\tau)$ depicted in (61), with time delay τ equal to 0.01 s.

(59)		(60)	(61)	
91 92 93 94	$\begin{array}{c} 4.0000 \times 10^2 \\ 6.0000 \times 10^4 \\ 4.0000 \times 10^6 \\ 1.0000 \times 10^8 \end{array}$	$\begin{array}{c} 6.4000 \times 10^{1} \\ 1.7600 \times 10^{2} \\ 2.0000 \times 10^{2} \\ 1.1600 \times 10^{2} \\ 2.4000 \times 10^{1} \end{array}$	γ1 γ2 γ3 γ4	$\begin{array}{c} 1.6000 \times 10^{1} \\ 3.2000 \times 10^{1} \\ 2.8000 \times 10^{1} \\ 8.0000 \times 10^{0} \end{array}$



Figure 7. Tracking performance for QI system (50) equipped with controller without time delay (59) for the desired path $y_{2d} = \sin(0.1\pi t)\cos(0.2\pi t)$.

With a time delay $\tau = 0.01$ s, the tracking control performance of the QI system (50) equipped with the controller with time delay approximated by the BFDDFs with truncation errors $O(\tau^2)$ depicted in (59) for the desired path $y_{2d} = \sin(0.1\pi t) \cos(0.2\pi t)$ is illustrated in Figure 8. As displayed in Figure 8a, the corresponding actual trajectory y_2 converged to the desired path y_{2d} within a very short time. Figure 8c illustrates the tracking error \hat{e}_4 , which also converged to zero within a small time interval. One can see, from Figure 8d, that the MSSTE of output y_2 was on the order of 10^{-6} . Moreover, Figure 8b shows the continuous smooth control of input \tilde{u}_2 within a suitable and reasonable range. From the above evidence, one can see that the tracking performance is still good under the time delay situation. Due to the similarity of the trajectories under different values of τ , we only display the tracking control results with time delay $\tau = 0.01$ s.

In Figure 9, the tracking errors of the QI system (50) equipped with the controller with time delay approximated by the BFDDFs with truncation errors $O(\tau^2)$ depicted in (60) are illustrated with different values of the time delay, where $\tau = 0.1, 0.01, 0.05$, or 0.005 s. In the situation of $\tau = 0.1$ s, the order of the MSSTE was 10^{-5} . In the situation of $\tau = 0.01$ s, the order of the MSSTE was 10^{-7} . In the situation of $\tau = 0.05$ s, the order of the MSSTE was 10^{-5} . In the situation of $\tau = 0.005$ s, the order of MSSTE was 10^{-7} . Therefore, it can be concluded that, when the value of the time delay τ decreases tenfold, the order of the MSSTE decreases by a factor of 10^{-2} . That is to say, the tracking errors had a positive proportional relationship with the time delay τ^2 ; or, in other words, with the truncation errors $O(\tau^2)$. These results are consistent with Theorems 1–4 in Section 2.



Figure 8. Tracking performance of QI system (50) equipped with controller with time delay (60) for the desired path $y_{2d} = \sin(0.1\pi t) \cos(0.2\pi t)$.



Figure 9. Orders of absolute tracking error $|\hat{e}_4|$ for QI system (50) equipped with controller with time delay (60) for the desired path $y_{2d} = \sin(0.1\pi t) \cos(0.2\pi t)$ with time delay τ equal to 0.1, 0.05, 0.01, or 0.005 s.

In Figure 10, the tracking errors for the controllers with time delay constructed using the BFDDFs with truncation error $O(\tau)$ depicted in (61), and using the BFDDFs with truncation error $O(\tau^2)$ depicted in (60) with the time delay $\tau = 0.01$ s are illustrated. From Figure 10, one can see that the order of the MSSTE was 10^{-5} for the controller with time delay constructed using the BFDDFs with truncation error $O(\tau)$ depicted in (61), while the order of the MSSTE was 10^{-6} with truncation error $O(\tau)$ depicted in (61), while the order of the MSSTE was 10^{-6} with that constructed using the BFDDFs with truncation error $O(\tau^2)$ depicted in (60). In order to compare more details of the above two models, more simulation results are provided in Table 5. The tracking errors under different time delay values for the controllers with time delay constructed using the BFDDFs with truncation error $O(\tau)$ depicted in (61), and using the BFDDFs with truncation error $O(\tau^2)$ depicted in (61), and using the BFDDFs with truncation error $O(\tau^2)$ depicted in (61), while the tracking error of (61) has a positive proportional relationship with the truncation error $O(\tau^2)$.



Figure 10. Orders of absolute tracking errors (ATE) $|\hat{e}_4|$ of QI system (50) equipped with controllers constructed using BFDDFs with truncation error $O(\tau)$ depicted in (61), and using BFDDFs with truncation error $O(\tau^2)$ depicted in (60), with time delay $\tau = 0.01$ s, for the desired path $y_{2d} = \sin(0.1\pi t)\cos(0.2\pi t)$.

Table 5. Performance of controller with time delay constructed using BFDDFs with truncation error $O(\tau)$ depicted in (61), and controller with time delay constructed using BFDDFs with truncation error $O(\tau^2)$ depicted in (60), with different time delay values.

	Criterion	$ au_0=0.5~{ m s}$	$ au_1 = 0.1 \ \mathrm{s}$	$ au_2 = 0.05~\mathrm{s}$	$ au_3 = 0.01~{ m s}$	$ au_4=0.005~{ m s}$
	$\frac{\tau_{k-1}}{\tau_{k-1}}$	-	5	2	5	2
(61)	MSSTE	4.9000×10^{-3}	$9.873 imes10^{-4}$	4.9358×10^{-4}	9.8698×10^{-5}	4.9347×10^{-5}
	$\underline{\text{MSSTE}(\tau_{k-1})}$					
	$MSSTE(\tau_k)$	-	5	2	5	2
	(\approx)	4	-	-	7	7
(60)	MSSTE	1.4000×10^{-4}	5.6890×10^{-5}	1.4220×10^{-5}	5.8650×10^{-7}	1.4352×10^{-7}
(00)	$\frac{\text{MSSTE}(\tau_{k-1})}{\text{MSSTE}(\tau_{k-1})}$		25	4	25	4
	$MSSTE(\tau_k)$	-	25	4	25	4
	(≈)					

5. Conclusions

In order to adapt to the requirements of real-world hardware implementations for nonlinear systems, a ZNN-based method was proposed, whose multiple-order derivatives were estimated using backward finite-divided difference formulas (BFDDFs) with quadratic-order precision, thus producing time delays. As such, we named the developed method the ZNN method with time delay. The theoretical results presented with systematical proofs in Section 2 described the quadratic-order error pattern of the ZNN method with time delay derived using backward finite-divided difference formulas with quadratic-order precision, which specifically indicate the effect of the time delay. The results of numerical experiments implied that the problems related to tracking control for a TI system with NOF and a QI system with LOF were effectively solved by the ZNN method without time delay. In addition, we also validated the effectiveness of the order-N ZNN method with time delay for solving the above tracking control problems, and the corresponding numerical results substantiated the theoretical results for the order-N ZNN method with time delay. Furthermore, the controller with time delay constructed using the BFDDFs with $O(\tau^2)$ truncation errors showed better performance than that constructed using the BFDDFs with $O(\tau)$ truncation errors. Studying the effects of various time delays is a potential future research direction, while the application of higher accuracy approximation formulas to the ZNN method is the another potential research direction in our future work.

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