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On Fuzzy C-Paracompact Topological Spaces

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Abstract: The aim of this paper is to study fuzzy extensions of some covering properties defined by A. V. Arhangel'skii and studied by other authors. Indeed, in 2016, A. V. Arhangel'skii defined other paracompact-type properties: C-paracompactness and C_2-paracompactness. Later, M. M. Saeed, L. Kalantan and H. Alzumi investigated these two properties. In this paper, we define fuzzy extensions of these notions and obtain results about them, and in particular, prove that these are good extensions of those defined by Arhangel'skii.

Keywords: topology; fuzzy sets; covering properties; fuzzy paracompactness

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1. Introduction

In 2016, A. V. Arhangel'skii defined other paracompact-type properties: C-paracompactness and C_2-paracompactness. A topological space X is called C-paracompact if there is a paracompact space Y and a bijective function *f*: $X \to Y$ such that the restriction $f_{|A} : A \to f(A)$ is a homeomorphism for each compact subspace $A \subset X$. A topological space X is called C_2—paracompact if there is a Hausdorff paracompact space Y and a bijective function *f*: $X \to Y$ *f* such that the restriction $f_{|A} : A \to f(A)$ is a homeomorphism for each compact subspace $A \subset X$. Later, M. M. Saeed, L. Kalantan and H. Alzumi [1] investigated these two properties and gave some examples that illustrate relations between them. In this paper, we define fuzzy extensions of these notions and obtain results about them.

2. Definitions

First, we give some previous definitions:

Definition 1. *Let (X,τ) be a fuzzy topological space. We will say that (X,τ) is fuzzy C-paracompact if there exists a fuzzy paracompact space (Y,ζ) (in various senses, which is will specify) and a bijection map f: $X \to Y$ such that restriction $f_{|K} : K \to f(K)$ is a fuzzy homeomorphism for each $K \subset X$ such that its characteristic map χ_K is a Lowen's fuzzy compact subset.*

Definition 2. *Let (X,τ) be a fuzzy topological space. We will say that (X,τ) is fuzzy C_2-paracompact if there exists a fuzzy paracompact (in various senses, which is will specify in next definitions) Hausdorff space (Y,ζ) and a bijection map f: $X \to Y$, such that restriction $f_{|K} : K \to f(K)$ is a fuzzy homeomorphism for each $K \subset X$ such that its characteristic map χ_K is a Lowen's fuzzy compact subset.*

Remark 1. *The kinds of fuzzy paracompact, fuzzy compact and fuzzy Hausdorff spaces cited in the above definitions should be good extensions of paracompactness, compactness and Hausdorff topological spaces. We list these definitions:*

Definition 3 ([2])**.** *Let $r \in (0, 1]$, μ be a set in a fuzzy topological space (X,τ). We say that μ is r-paracompact (resp. r*-paracompact) if for each r-open Q-cover of μ there exists an open refinement of it which is both locally finite (resp. $\star$-locally finite) in μ and a r-Q-cover of μ. Additionally,*

*μ is called S-paracompact (resp. S *-paracompact) if for every $r \in (0, 1]$, μ is r-paracompact (resp. r*-paracompact). We say that (X,τ) is r-paracompact (resp. r*-paracompact, S-paracompact, S*-paracompact) if set X verifies this property.*

Definition 4 ([3]). *Let μ be a fuzzy set in a fuzzy topological space (X,τ). We say that μ is fuzzy paracompact (resp. r*-paracompact) if for each open L-cover $\mathscr{U}$ of μ and for each $r \in (0, 1]$, there exists an open refinement $\mathscr{V}$ of $\mathscr{U}$ which is both locally finite (resp. $\star$-locally finite) in μ and L-cover of $\mu - r$. We say that a fuzzy topological space (X,τ) is fuzzy paracompact (resp. $\star$-fuzzy paracompact) if each constant fuzzy set in X is fuzzy paracompact (resp. $\star$-fuzzy paracompact).*

Definition 5 ([4]). *A fuzzy topological space (X,τ) is called fuzzy paracompact if for each $\mathscr{U} \subset \tau$ and for each $r \in (0, 1]$ such that $\sup\{\mu \mid \mu \in \mathscr{U}\} \geq r$, and for all ε $(0 < \varepsilon \leq r)$, there exists a locally finite open refinement $\mathscr{V}$ of $\mathscr{U}$ such that $\sup\{\mu \mid \mu \in \mathscr{V}\} \geq r - \varepsilon$.*

Definition 6 ([5]). *A fuzzy set μ in a fuzzy topological space (X,τ) is called fuzzy compact (in the Lowen's sense) if for all family $\mathscr{U} \subset \tau$ such that $\sup\{v \mid v \in \mathscr{U}\} \geq \mu$ and for all $\varepsilon>0$ there exists a finite subfamily $\mathscr{U}_0 \subset \mathscr{U}$ such that $\sup\{v \mid v \in \mathscr{U}_0\} \geq \mu - \varepsilon$. The fuzzy topological space (X,τ) is fuzzy compact if each constant fuzzy set in (X,τ) is fuzzy compact.*

Definition 7 ([6]). *A fuzzy topological space (X,τ) is said to be fuzzy Hausdorff if for any two distinct fuzzy points $p,q \in X$, there are disjoint $\mathscr{U},\mathscr{V} \in \tau$ with $p \in \mathscr{U}$ and $q \in \mathscr{V}$.*

3. Results

Lemma 1. *Let (X,T) be a topological space and A be a subset of X. Then, A is compact in (X,T) if and only if its characteristic map χ_A is a Lowen's fuzzy compact subset in $(X,\omega(T))$.*

Proof. ($\Leftarrow$) For each $\mathscr{U} \subset$ T, such that $A \subset \bigcup_{U \in \mathscr{U}} U$ is $\sup\{\chi_U \mid U \in \mathscr{U}\} \geq \chi_A$.

For each $\varepsilon \in (0, 1)$, from the hypothesis there exists a finite subfamily $\mathscr{U}_0 \subset \mathscr{U}$ such that $\sup\{\chi_U \mid U \in \mathscr{U}\} \geq \chi_A - \varepsilon$. Then, $\mathscr{U}_0$ is a finite subcovering of $\mathscr{U}$.

($\Rightarrow$) Let $\mathscr{F} \subset \omega(T)$ such that $\sup\{\mu \in \mathscr{F}\} \geq \chi_A$. For each $\varepsilon > 0$ and for each $\mu \in \mathscr{F}$ if $\mu^\varepsilon = \mu + \varepsilon$, and $\mathscr{L}(\mu^\varepsilon) = \{(x,r) \mid \mu^\varepsilon(x) > r\}$ is open in $X \times R$. Additionally, $\bigcup_{\mu \in \mathscr{F}} L(\mu^\varepsilon) \supset \bigcap A \times I$ (which is compact), because for each $(x,r) \in A \times I$ (where $\varepsilon < r$), is $\sup\{\mu(x) \mid \mu \in \mathrm{F}\} = 1 \geq r > \varepsilon$, then, there exists $\mu_0 \in \mathscr{F}$ such that $\varepsilon < r \leq \mu_0(x)$.

Thus, $\mu_0(x) + \varepsilon > r > \varepsilon$, then $(x,r) \in \mathscr{L}(\mu^\varepsilon)$.

Finally, there exists a finite subfamily $\mathscr{F}_0 \subset \mathscr{F}$ such that $\bigcup_{\mu \in \mathscr{F}} \mathscr{L}(\mu^\varepsilon) \supset A \times I$ and $\sup\{\mu \mid \mu \in \mathscr{F}_0\} \geq \chi_A - \varepsilon$ because for each $(a,1) \in A \times I$ there is $\mu \in \mathscr{F}_0$ such that $(a,1) \in \mathscr{L}(\mu^\varepsilon)$, then $\mu_0(a) + \varepsilon > 1$, and $\sup\{\mu \mid \mu \in \mathscr{F}_0\} \geq \chi_A - \varepsilon$. □

Proposition 1. *Let (X,τ) be a fuzzy topological space. Then, (X,τ) is fuzzy C-paracompact if and only if $(X, \iota(\tau))$ is C-paracompact, i.e., fuzzy C-paracompactness is a good extension of C-paracompactness.*

Proof. (X,τ) is fuzzy C-paracompact, i.e., there exists a fuzzy paracompact space (Y,ζ) (in the sense of some good extension of paracompactness [2–4,7]) and a bijection map $f\colon X \to Y$ such that restriction $f_{|K} : K \to f(K)$ is a fuzzy homeomorphism for each $K \subset X$ such that its characteristic map χ_K is a Lowen's fuzzy compact subset [5]. That is, there is a paracompact space $(Y, \iota(\varsigma))$ and a bijection map $f\colon X \to Y$ such that the restriction $f_{|K} : K \to f(K)$ is a homeomorphism for each compact subspace $K \subset X$, i.e., $(X, \iota(\tau))$ is C-paracompact. □

Corollary 1. *Fuzzy C_2-paracompactness is a good extension of C_2-paracompactness.*

Proposition 2. *Every fuzzy Hausdorff topological space (in the Srivastava, Lal and Srivastava or in the Wagner and McLean sense) which is fuzzy locally compact (in the Kudri and Wagner sense) is fuzzy C_2-paracompact.*

Proof. It follows from ([6], Prop.3.2), ([8], Prop.3.1), ([9], Th.3.3) and above Corollary. (For undefined concepts and previous results see [10]). □

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