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Hadamard Product of Certain Multivalent Analytic Functions with Positive Real Parts

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Abstract: This paper aims to provide sufficient conditions for starlikeness and convexity of Hadamard product (convolution) of certain multivalent analytic functions with positive real parts. Moreover, the starlikeness conditions for a certain integral operator and other convolution results are also considered.

Keywords: analytic functions; p-valent starlike and convex functions; subordination; convolution; Bernardi integral operator

MSC: 30C45; 30C50; 30C80

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1. Introduction

Let $\mathcal{A}_p$ denote the class of functions of the form:

$$f(z) = z^p + \sum_{k=1}^{\infty} a_{k+p} z^{k+p} \qquad (p \in \mathbb{N} = \{1, 2, \ldots\}) \tag{1}$$

which are analytic in the open unit disc $\mathbb{U} = \{z : |z| < 1\}$ and let $\mathcal{A}_1 := \mathcal{A}$. A function $f \in \mathcal{A}_p$ is said to be in the class S_p^* of p-valently starlike functions in $\mathbb{U}$ if it satisfies the following inequality:

$$\Re\left(\frac{zf'(z)}{f(z)}\right) > 0 \quad (z \in \mathbb{U}). \tag{2}$$

Further, A function $f \in \mathcal{A}_p$ is said to be in the class K_p of p-valently convex in $\mathbb{U}$ if it satisfies the following inequality:

$$\Re\left(1 + \frac{zf''(z)}{f'(z)}\right) > 0 \quad (z \in \mathbb{U}).$$

The starlikeness and convexity of p-valent functions were introduced by Goodman [1] and considered recently in the works [2–12]. Let P_α be the class of functions with positive real part of order α that have the form $h(z) = 1 + \sum\limits_{k=1}^{\infty} c_k z^k$ which are analytic in $\mathbb{U}$ and satisfy the following condition

$$\Re\{h(z)\} > \alpha, \quad (0 \le \alpha < 1;\, z \in \mathbb{U}).$$

A function $f \in \mathcal{A}_p$ is said to be in the class $P(p, \alpha)$ if and only if

$$\frac{f'(z)}{pz^{p-1}} \in P_\alpha \quad (0 \le \alpha < 1; z \in \mathbb{U}).$$

For $0 \leq \alpha < 1$, we denote by $R_p(\alpha)$ the family of functions $f \in \mathcal{A}_p$ which satisfy the condition

$$\frac{f^{'}(z) + zf^{''}(z)}{p^2 z^{p-1}} \in P_\alpha \quad (z \in \mathbb{U}). \tag{3}$$

As a special case, for $p = 1$ the class $R_p(\alpha)$ reduces to the familiar class R which was studied by Chichra [13], Ali and Thomas [14], Singh and Singh [15,16], Kim and Srivastava [17], Ali et al. [18], Szasz [19] and Yang and Liu [20] . For two functions f and $g \in \mathcal{A}_p$, that is if f is given by (1) and g is given by $f(z) = z^p + \sum_{k=1}^{\infty} b_{k+p} z^{k+p}$, then their Hadamard product (convolution), $(f * g)$, is the function defined by the power series

$$(f * g)(z) = z^p + \sum_{k=1}^{\infty} a_{k+p} b_{k+p} z^{k+p} \ .$$

For a function $f \in \mathcal{A}_p$, Reddy and Padmanabhan [21] defined the following integral operator:

$$\begin{aligned} J_{p,c}(z) &= J_{p,c}(f(z)) = \frac{c+p}{z^c} \int_0^z t^{c-1} f(t) dt \ \ (p \in \mathbb{N}, c > -p) \\ &= z^p + \sum_{k=1}^{\infty} \frac{c+p}{c+p+k} a_{k+p} z^{k+p}. \end{aligned} \tag{4}$$

In particular, The operator $J_{1,c}$ was introduced by Bernardi [22] and the operator $J_{1,1}$ was studied earlier by Libera [23]. By using the Clunie-Jack Lemma [24] it was shown in [25] that if the function $f \in \mathcal{A}$ belongs to the class P_β, then $J_{1,c} \in S^*$ (S^* is the class of starllike functions) provided

$$(1 + c)\beta > \frac{\log \frac{4}{e}}{6} (c^2 \tan^2 \frac{\alpha^* \pi}{2} - 3) \tag{5}$$

where $1 = \alpha^* + \frac{2}{\pi} \tan^{-1} \alpha^*$. In their paper [14], Ali and Thomas improved the constant β in (5). In the work of Lashin [26] a criterion for convolution properties of functions of the class $P(\alpha)$ was introduced, this criterion was improved by Sokol [27] and Ponnusamy and Singh [28] . The present paper extends and improves each of these earlier results in [26–28]. Additionally, By using Miller and Mocanu Theorem [29] we will consider the starlikeness of the integral operator $J_{p,c}$ and extend the results of Ali and Thomas [14].

2. Preliminaries Lemmas

In this paper, we shall require the following lemmas.

Lemma 1 (see [15])**.** *A sequence $\{b_k\}_{k=0}^{\infty}$ of non-negative numbers is said to be a convex null sequence if $b_k \to 0$ as $k \to \infty$ and*

$$b_0 - b_1 \geq b_1 - b_2 \geq ... \geq b_k - b_{k+1} \geq 0.$$

Let the sequence $\{b_k\}_{k=0}^{\infty}$ be a convex null sequence. Then the function

$$q(z) = \frac{b_0}{2} + \sum_{k=1}^{\infty} b_k z^k \qquad (z \in \mathbb{U})$$

is analytic in $\mathbb{U}$ and $\Re\{q(z)\} > 0$.

Lemma 2 ([15])**.** *If the function $\chi(z)$ is analytic in $\mathbb{U}$ with $\chi(0) = 1$ and $\Re\{\chi(z)\} > 1/2, z \in \mathbb{U}$, then for any function F analytic in $\mathbb{U}$, the function $\chi * F$ takes its values in the convex hull of $F(\mathbb{U})$.*

Lemma 3 ([25,30])**.** *Let $\lambda > 0$ and $0 \leq \beta < 1$. If the function q is analytic in $\mathbb{U}$ with $q(0) = 1$, satisfies the inequality*

$$\Re\left\{q(z) + \lambda z q^{'}(z)\right\} > \beta \quad (z \in \mathbb{U}),$$

then

$$\Re\left\{q(z)\right\} > 1 + 2(1-\beta)\sum_{k=1}^{\infty}\frac{(-1)^k}{1+\lambda k} \quad (z \in \mathbb{U}).$$

Lemma 4 ([31])**.** *For $0 \leq \alpha < 1$ and $0 \leq \beta < 1$,*

$$P_{\alpha} * P_{\beta} \subset P_{\delta}, \quad \delta = 1 - 2(1-\alpha)(1-\beta).$$

The result is sharp.

Lemma 5 ([29])**.** *Suppose that the function $\varphi : C^2 \times U \to C$ satisfies the condition $\Re\{\varphi(ix, y; z)\} \leq \delta$ for all real $x, y \leq -\frac{(1+x^2)}{2}$ and all $z \in U$. If $q(z) = 1 + c_1 z + \cdots$ is analytic in U and*

$$\Re\{\varphi(q(z), zq^{'}(z), z)\} > \delta, \text{ for } z \in U,$$

then $\Re\{q(z)\} > 0$ in U.

Lemma 6 ([32])**.** *The nth partial sum S_n of the Alternating series $\sum_{k=1}^{\infty}(-1)^n a_n, a_n > 0$, always lies between S_{n-1} and S_{n-2}, or*

$$-1 < -a_1 < S_n < a_2 - a_1 < 0. \tag{6}$$

3. Main Results

First of all, we state and prove the following results which extend the results of Lashin [26] and Sokol [27].

Theorem 1. *Let $p \in \mathbb{N}, 0 \leq \alpha, \beta < 1$, and let $\psi(p) = \sum_{k=1}^{\infty}\frac{(-1)^k}{p+k}$. If $f, g \in \mathcal{A}_p$ satisfy $f \in P(p, \alpha)$ and $g \in P(p, \beta)$, then $\xi = (f * g) \in S_p^*$, provided that*

$$(1-\alpha)(1-\beta) < \min\{\frac{2p+1}{8p^2\psi^2 + 4p}, \frac{p+1}{4p^2(1-\ln\frac{4}{e})}\}. \tag{7}$$

Proof. It is easy to see that,

$$\frac{f^{'}(z)}{pz^{p-1}} * \frac{g^{'}(z)}{pz^{p-1}} = \frac{\xi^{'}(z) + z\xi^{''}(z)}{p^2 z^{p-1}}. \tag{8}$$

By the hypothesis on f and g, it follows from (8) and Lemma 4 that

$$\Re\left(\frac{\xi^{'}(z) + z\xi^{''}(z)}{p^2 z^{p-1}}\right) > 1 - 2(1-\alpha)(1-\beta). \tag{9}$$

Let

$$\phi(z) = \frac{\xi^{'}(z)}{pz^{p-1}}, \tag{10}$$

then $\phi(z) = 1 + b_1 z + b_2 z^2 + \ldots$ is analytic in $\mathbb{U}$. Using (9) and (10) we obtain

$$\Re\left(\frac{\xi^{'}(z) + z\xi^{''}(z)}{p^2 z^{p-1}}\right) = \phi(z) + \frac{1}{p}z\phi^{'}(z) > 1 - 2(1-\alpha)(1-\beta).$$

If we apply Lemma 3, then we have

$$\Re\left(\frac{\xi'(z)}{pz^{p-1}}\right) > 1 + 4(1-\alpha)(1-\beta)p\sum_{k=1}^{\infty}\frac{(-1)^k}{p+k} =: \lambda, \ \ (z \in \mathbb{U}). \tag{11}$$

Since $\psi(p) > \psi(1), p \geq 1$, it follows that $\lambda > 1 - 2(1-\alpha)(1-\beta)p(1 - \ln\frac{4}{e})$. If

$$(1-\alpha)(1-\beta) < \frac{p+1}{4p^2(1-\ln\frac{4}{e})}, \tag{12}$$

then

$$\lambda > \frac{p-1}{2p} > 0. \tag{13}$$

Applying Lemma 3 again, (11) gives

$$\Re\left\{\frac{\xi(z)}{z^p}\right\} > 1 + 2p(1-\lambda)\psi. \tag{14}$$

If we apply Lemma 6, then we have

$$\psi > -\frac{1}{p+1}. \tag{15}$$

Inequality (15) together with (13) implies $1 + 2p(1-\lambda)\psi > 0$. Let $q(z) = \frac{z\xi'(z)}{p\xi(z)}$ and $\tau(z) = \frac{\xi(z)}{z^p}$, then $q(z)$ is analytic in U with $q(0) = 1$ and

$$\Re\{\tau(z)\} > 1 - 8(1-\alpha)(1-\beta)p^2\psi^2. \tag{16}$$

By simple calculation, we find that

$$\frac{\xi'(z) + z\xi''(z)}{p^2 z^{p-1}} = \tau(z)\left[q^2(z) + \frac{1}{p}zq'(z)\right] = \varphi\Big(q(z), zq'(z), z\Big),$$

where $\varphi(u,v;z) = \tau(z)(u^2 + \frac{1}{p}v)$. By (9) we get

$$\Re\Big[\varphi\Big(q(z), zq'(z), z\Big)\Big] > 1 - 2(1-\alpha)(1-\beta) \ \ (z \in U).$$

Moreover $\Re\{\varphi(ix,y,z)\} = \Re\{\tau(z)(\frac{1}{p}y - x^2)\}$, and for real $x, y \leq -\frac{1}{2}(1+x^2)$, we have

$$\Re\{\varphi(ix,y,z)\} \leq -\frac{1}{2p}\{1 + (1+2p)x^2\}\Re\{\tau(z)\} \leq -\frac{1}{2p}\Re\{\tau(z)\} \quad (z \in U). \tag{17}$$

Thus by (16) and (17) we get

$$\Re\{\varphi(ix,y,z)\} \leq 1 - 2(1-\alpha)(1-\beta),$$

for all $z \in U$. Thus by Lemma 5, $\Re\{q(z)\} > 0$. Thus, $\Re\left\{\frac{z\xi'(z)}{p\xi(z)}\right\} > 0$, that is, $\xi \in S^*_p$. $\square$

Remark 1. *Putting $p = 1$ in Theorem 1 we get the result obtained by Lashin ([26], Theorem 1).*

Theorem 2. *Let* $p \in \mathbb{N}$ *and* $0 \le \alpha, \beta, \gamma < 1$. *If* $f, g, h \in \mathcal{A}_p$ *satisfy* $f \in P(p, \alpha)$, $g \in P(p, \beta)$ *and* $h \in P(p, \gamma)$, *then* $\zeta = (f * g * h) \in K_p$, *where*

$$(1-\alpha)(1-\beta)(1-\gamma) < \min\{\frac{2p+1}{16p^2\left(\sum_{k=1}^{\infty}\frac{(-1)^k}{p+k}\right)^2 + 8p}, \frac{p+1}{8p^2(1-\ln\frac{4}{e})}\}.$$

Proof. It is sufficient to show that $\eta(z) = \frac{z\zeta'(z)}{p} \in S_p^*$. Note that,

$$\frac{f'(z)}{pz^{p-1}} * \frac{g'(z)}{pz^{p-1}} * \frac{h'(z)}{pz^{p-1}} = \frac{\eta'(z) + z\eta''(z)}{p^2 z^{p-1}}. \tag{18}$$

By the hypothesis of Theorem 2, it follows from (18) and Lemma 4 that

$$\Re\left[\frac{\eta'(z) + z\eta''(z)}{p^2 z^{p-1}}\right] > 1 - 4(1-\alpha)(1-\beta)(1-\gamma),$$

and the proof is completed similar to the proof of Theorem 1. □

Remark 2. *Putting* $p = 1$ *in Theorem 2 we get the result obtained by Lashin ([26], Theorem 2).*

Theorem 3. *Let* $p \in \mathbb{N}, c > -p$ *and* $0 \le \alpha < 1$. *If* $f \in \mathcal{A}_p$ *given by (1) be in the class* $P(p, \alpha)$, *then the function* $J_{p,c}$ *defined by (4) belongs to the class* $P(p, \beta)$, *where*

$$\beta = 1 + 2(1-\alpha)(p+c)\sum_{k=1}^{\infty}\frac{(-1)^k}{p+c+k}.$$

Proof. From (4) we have

$$zJ''_{p,c}(z) + (c+1)J'_{p,c}(z) = (c+p)f'(z). \tag{19}$$

Let

$$q(z) = \frac{J'_{p,c}(z)}{pz^{p-1}} \tag{20}$$

so that $q(z) = 1 + c_1 z + c_2 z^2 + \ldots$ is analytic in $\mathbb{U}$. Therefore (19) and (20) leads us to

$$\Re\left\{q(z) + \frac{1}{p+c}zq'(z)\right\} = \Re\left\{\frac{f'(z)}{pz^{p-1}}\right\} > \alpha \quad (c > -p, p \in \mathbb{N}).$$

Now by applying Lemma 3 with $\lambda = \frac{1}{c+p}, c > -p$ and $\beta = \alpha$, we deduce that

$$\Re\left\{\frac{J'_{p,c}(f(z))}{pz^{p-1}}\right\} > 1 + 2(1-\alpha)(p+c)\sum_{k=1}^{\infty}\frac{(-1)^k}{p+c+k}.$$

This evidently ends the proof of Theorem 3. □

Remark 3. *The result (asserted by Theorem 3 above) was also obtained, by means of a markedly different technique, by Aouf and Ling ([33], Theorem 1).*

Remark 4. *The result presented in Theorem 4 below generalizes the results shown by Ali and Thomas [14], by employing a different technique*

Theorem 4. *Let $f \in \mathcal{A}_p$ and $J_{p,c}$ given by (4). If $f \in P(p,\alpha)$, then $J_{p,c} \in S_p^*$ $(-p < c \leq 0)$, where*

$$1-\alpha < \min\{\frac{2p+1}{2(p+c)[1+2\delta(c+p\psi)]}, \frac{p+1}{2p(p+c)\ln 4}\},$$

$\delta(c+p) = \sum_{k=1}^{\infty} \frac{(-1)^k}{p+c+k}$ and $\psi(p) = \sum_{k=1}^{\infty} \frac{(-1)^k}{p+k}$.

Proof. Let $f \in \mathcal{A}_p$ be in the class $P(p,\alpha)$, by using Theorem 3, we have

$$\Re\left\{\frac{J'_{p,c}(f(z))}{pz^{p-1}}\right\} > 1+2(1-\alpha)(p+c)\sum_{k=1}^{\infty}\frac{(-1)^k}{p+c+k} := \mu, say.$$

Since $\delta(c+p) > \delta(0)$ for $-p < c \leq 0$, then $\mu > 1-(1-\alpha)(p+c)\ln 4$. If

$$(1-\alpha) < \frac{p+1}{2p(p+c)\ln 4}, \tag{21}$$

then $\mu > \frac{p-1}{2p} > 0$. Let us define the function φ by

$$\varphi(z) = \frac{J_{p,c}(z)}{z^p},$$

so that $\varphi(z) = 1 + c_1 z + c_2 z^2 + \ldots$ is analytic in $\mathbb{U}$ and

$$\Re\left\{\varphi(z) + \frac{1}{p} z\varphi'(z)\right\} = \Re\left\{\frac{J'_{p,c}(f(z))}{pz^{p-1}}\right\} > \mu.$$

If we apply Lemma 3 with $\lambda = \frac{1}{p}$ and $\beta = \mu$, then we have

$$\Re\left\{\frac{J_{p,c}(z)}{z^p}\right\} > 1+2p(1-\mu)\psi. \tag{22}$$

Since $2p(1-\mu) < p+1$, (15) gives $1+2p(1-\mu)\psi > 0$. Note also that from (19), we have

$$zJ''_{p,c}(z) + J'_{p,c}(z) = (c+p)f'(z) - cJ'_{p,c}(z).$$

Since $c \leq 0$, the above equation and Theorem 3 give

$$\begin{aligned} \Re\left\{\frac{zJ''_{p,c}(z) + J'_{p,c}(z)}{p^2 z^{p-1}}\right\} &= \frac{(c+p)}{p}\Re\left\{\frac{f'(z)}{pz^{p-1}}\right\} - \frac{c}{p}\Re\left\{\frac{J'_{p,c}(z)}{pz^{p-1}}\right\} \\ &> \frac{(c+p)}{p}\alpha - \frac{c}{p}\mu. \end{aligned} \tag{23}$$

Let $q(z) = \frac{zJ'_{p,c}(z)}{pJ_{p,c}(z)}$ and $\rho(z) = \frac{J_{p,c}(z)}{z^p}$, then $q(z)$ is analytic in U with $q(0) = 1$ and

$$\Re\{\rho(z)\} > 1+2p(1-\mu)\psi. \tag{24}$$

Applying the same method and technique as in our proof of Theorem 1, we get

$$\frac{zJ''_{p,c}(z) + J'_{p,c}(z)}{p^2 z^{p-1}} = \rho(z)\left[q^2(z) + \frac{1}{p}zq'(z)\right] = \varphi\Big(q(z), zq'(z), z\Big),$$

where $\varphi(u,v;z) = \rho(z)(u^2 + \frac{1}{p}v)$. By (23) we get

$$\Re\Big[\varphi\Big(q(z), zq'(z), z\Big)\Big] > \frac{(c+p)}{p}\alpha - \frac{c}{p}\mu \ \ (z \in U).$$

Moreover $\Re\{\varphi(ix,y,z)\} = \Re\{\rho(z)(\frac{1}{p}y - x^2)\}$, and for real $x, y \leq -\frac{1}{2}(1+x^2)$, we have

$$\Re\{\varphi(ix,y,z)\} \leq -\frac{1}{2p}\{1+(1+2p)x^2\}\Re\{\rho(z)\} \leq -\frac{1}{2p}\Re\{\rho(z)\} \quad (z \in U). \tag{25}$$

Thus by (24) and (25) we get

$$\begin{aligned}\Re\{\varphi(ix,y,z)\} &\leq -\frac{1}{2p}\{1+2p(1-\mu)\psi\} \\ &< \frac{(c+p)}{p}\alpha - \frac{c}{p}\mu.\end{aligned}$$

for all $z \in U$. Thus by Lemma 5, $\Re\{q(z)\} > 0$. Thus , $\Re\left\{\frac{zJ'_{p,c}(z)}{pJ_{p,c}(z)}\right\} > 0$, that is, $J_{p,c} \in S^*_p$ and this ends the proof. □

Remark 5. *For $p = 1$ Theorem 4 gives the result obtained by Ali and Thomas [14].*

Theorem 5. *If $f \in R_p(\alpha)$, then $f \in P(p,\alpha)$.*

Proof. Let $f \in \mathcal{A}_p$ defined by (1) satisfies the condition (3), then

$$\Re\left\{\frac{f'(z) + zf''(z)}{p^2 z^{p-1}}\right\} = \Re\left\{1 + \sum_{k=1}^{\infty}\left(\frac{p+k}{p}\right)^2 a_{p+k}z^k\right\} > \alpha.$$

Hence, we have

$$\Re\left\{1 + \frac{1}{2(1-\alpha)}\sum_{k=1}^{\infty}\left(\frac{p+k}{p}\right)^2 a_{p+k}z^k\right\} > \frac{1}{2}.$$

Note that

$$\begin{aligned}\frac{f'(z)}{pz^{p-1}} &= 1 + \sum_{k=1}^{\infty}\frac{p+k}{p}a_{p+k}z^k \\ &= \left\{1 + \frac{1}{2(1-\alpha)}\sum_{k=1}^{\infty}\left(\frac{p+k}{p}\right)^2 a_{p+k}z^k\right\} * \left\{1 + 2(1-\alpha)\sum_{k=1}^{\infty}\frac{p}{p+k}z^k\right\}.\end{aligned}$$

Applying Lemma 1, with $c_0 = 1$ and $c_k = \frac{p}{p+k}, k = 1, 2, ...,$ we get

$$\Re\left\{1 + 2(1-\alpha)\sum_{k=1}^{\infty}\frac{p}{p+k}z^k\right\} > \alpha,$$

which implies that $Re\left\{\frac{f'(z)}{pz^{p-1}}\right\} > \alpha$, by using Lemma 2. □

Remark 6. *Theorem 5 is immediate from Hallenbeck-Ruscheweyh theorem [34]. Indeed, define $\phi(z)$ by (10) with f in place of ξ. Then $f \in R_p(\alpha)$ means $\phi(z) + \frac{1}{p}z\phi'(z) \prec \frac{1+(1-2\alpha)z}{1-z} = L(z)$. Now Hallenbeck-Ruscheweyh theorem (see also Miller-Mocanu ([29], P.71, Theorem 3.1b)) implies $\phi(z) \prec L(z)$,*

Remark 7. *Putting $p = 1$ in Theorem 5 we get the result obtained by Al-Oboudi ([35], Theorem 2.3, when $\lambda = n = 1$).*

Theorem 6. *Let $f \in R_p(\alpha)$. Then $f \in P(p, \xi)$, where*

$$\xi = \frac{2 + (p^2 + 3p)\alpha}{(1+p)(p+2)} \geq \alpha.$$

Proof. It is shown in [36] that, if $\gamma \geq 0$ and if $g(z) = z + \sum_{k=1}^{\infty} \frac{1}{1+\gamma k} z^{k+1}$ then

$$\Re\left\{\frac{g(z)}{z}\right\} \geq \frac{2\gamma^2 + 3\gamma + 1}{2(1+\gamma)(1+2\gamma)}.$$

Hence

$$\Re\left\{1 + 2(1-\alpha)\sum_{k=1}^{\infty} \frac{p}{p+k} z^k\right\} \geq \frac{(2 + p^2 + 3p)\alpha}{(1+p)(p+2)}. \tag{26}$$

Using (26) in the Theorem 5 we get the result. □

Remark 8. *Putting $p = 1$ in Theorem 6 we get the result obtained by Al-Oboudi ([35], Remark 2.5, when $\lambda = 1$).*

4. Conclusions

The convolution method has recently been used to study many interesting subclasses of analytical functions. An interesting criterion was given by Lashin [26] to be starlike for convolution of functions with positive real parts, which was improved by Sokol [27]. Each of these earlier results has been extended and improved in this paper. Additionally, by using Miller and Mocanu Theorem [29] , Ali and Thomas' results [14] for the starlikeness of the Bernardi integral operator have been extended.

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