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Multiple Periodic Solutions for Odd Perturbations of the Discrete Relativistic Operator

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Abstract: We obtain the existence of multiple pairs of periodic solutions for difference equations of type $-\Delta\left(\frac{\Delta u(n-1)}{\sqrt{1-|\Delta u(n-1)|^2}}\right) = \lambda g(u(n)) \quad (n \in \mathbb{Z})$, where $g : \mathbb{R} \to \mathbb{R}$ is a continuous odd function with anticoercive primitive, and $\lambda > 0$ is a real parameter. The approach is variational and relies on the critical point theory for convex, lower semicontinuous perturbations of C^1-functionals.

Keywords: discrete relativistic operator; periodic solution; critical point; genus

MSC: 39A23; 39A27; 47J20

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1. Introduction

In this note, we are concerned with the multiplicity of solutions for difference equations with relativistic operator of type

$$-\Delta[\phi(\Delta u(n-1))] = \lambda g(u(n)), \quad u(n) = u(n+T) \quad (n \in \mathbb{Z}), \tag{1}$$

where $\Delta u(n) = u(n+1) - u(n)$ is the usual forward difference operator, $\lambda > 0$ is a real parameter, $g : \mathbb{R} \to \mathbb{R}$ is a continuous odd function, and

$$\phi(y) = \frac{y}{\sqrt{1-y^2}} \quad (y \in (-1,1)).$$

In recent years, special attention has been paid to the existence and multiplicity of T-periodic solutions for problems with a discrete relativistic operator. Thus, for instance, in [1,2], variational arguments were employed to prove the solvability of systems of difference equations having the form

$$\Delta[\phi_N(\Delta u(n-1))] = \nabla_u V(n, u(n)) + h(n) \quad (n \in \mathbb{Z}), \tag{2}$$

under various hypotheses upon V and h (coerciveness, growth restriction, convexity or periodicity conditions); here, ϕ_N is the N-dimensional variant of ϕ, i.e.,

$$\phi_N(y) = \frac{y}{\sqrt{1-|y|^2}} \quad (y \in \mathbb{R}^N, \ |y| < 1).$$

The existence of at least $N+1$ geometrically distinct T-periodic solutions of (2) was proved in [3], under the assumptions that h is T-periodic, $\sum_{j=1}^{T} h(j) = 0$, and the mapping $V(n,x)$ is T-periodic in n and ω_i-periodic ($\omega_i > 0$) with respect to each x_i ($i = 1, \ldots, N$). For the proof, using an idea from the differential case [4], the singular problem (2) was reduced to an equivalent non-singular one to which classical Ljusternik–Schnirelmann category methods can be applied. In addition, under some similar assumptions on V and h, were obtained in [5] using Morse theory, conditions under which system (2) has at least 2^N geometrically distinct T-periodic solutions.

The motivation of the present study mainly comes from paper [6], where for problems involving Fisher-Kolmogorov nonlinearities of type

$$-\Delta[\phi(\Delta u(n-1))] = \lambda u(n)(1-|u(n)|^q), \quad u(n) = u(n+T) \quad (n \in \mathbb{Z}), \tag{3}$$

with $q > 0$ fixed and $\lambda > 0$ a real parameter, it was proved that *if* $\lambda > 8mT$ *for some* $m \in \mathbb{N}$ *with* $2 \leq m \leq T$, *then problem* (3) *has at least* m *distinct pairs of nontrivial solutions.* We also refer the interested reader to [6] for a discussion concerning the origin and steps in the study of this type of nonlinearity. In this respect, we shall see in Example 1 below that a sharper result holds true, namely,

(i) If $\lambda > 8\sin^2\dfrac{m\pi}{T}$ *with* $0 \leq m \leq \begin{cases} (T-1)/2 & if\ T\ is\ odd \\ (T-2)/2 & if\ T\ is\ even \end{cases}$*, then problem* (3) *has at least* $2m+1$ *distinct pairs of nontrivial solutions.*

(ii) If T *is even and* $\lambda > 8$*, then* (3) *has at least* T *distinct pairs of nontrivial solutions.*

Moreover, we prove in Theorem 2 that the above statements *(i)* and *(ii)* still remain valid for a larger class of periodic problems.

As in [6], our approach to problem (1) is variational and combines a Clark-type abstract result for convex, lower semicontinuous perturbations of C^1-functionals, based on Krasnoselskii's genus. However, our technique here brings the novelty that it exploits the interference of the geometry of the energy functional with fine spectral properties of the operator $-\Delta^2$; recall that

$$\Delta^2 u(n-1) := \Delta(\Delta u(n-1)) = u(n+1) - 2u(n) + u(n-1).$$

It is worth noting that in paper [7] analogous multiplicity results are obtained in the differential case for potential systems involving parametric odd perturbations of the relativistic operator. In addition, we mention the recent paper [8], where the authors obtain the existence and multiplicity of sign-changing solutions for a slightly modified parametric problem of type (1) using bifurcation techniques.

We conclude this introductory part by briefly recalling some topics in the frame of Szulkin's critical point theory [9], which is needed in the sequel. Let $(Y, \|\cdot\|)$ be a real Banach space and $\mathcal{I} : Y \to (-\infty, +\infty]$ be a functional having the following structure:

$$\mathcal{I} = \mathcal{F} + \psi, \tag{4}$$

where $\mathcal{F} \in C^1(Y, \mathbb{R})$ and $\psi : Y \to (-\infty, +\infty]$ is proper, convex and lower semicontinuous. A point $u \in D(\psi)$ is said to be *a critical point* of $\mathcal{I}$ if it satisfies the inequality

$$\langle \mathcal{F}'(u), v-u\rangle + \psi(v) - \psi(u) \geq 0 \quad \forall\, v \in D(\psi).$$

A sequence $\{u_n\} \subset D(\psi)$ is called a (PS)-*sequence* if $\mathcal{I}(u_n) \to c \in \mathbb{R}$ and

$$\langle \mathcal{F}'(u_n), v-u_n\rangle + \psi(v) - \psi(u_n) \geq -\varepsilon_n\|v-u_n\| \quad \forall\, v \in D(\psi),$$

where $\varepsilon_n \to 0$. The functional $\mathcal{I}$ is said *to satisfy the* (PS) *condition* if any (PS)-sequence has a convergent subsequence in Y.

Let Σ be the collection of all symmetric subsets of $Y \setminus \{0\}$ which are closed in Y. The *genus* of a nonempty set $A \in \Sigma$ is defined as being the smallest integer k with the property that there exists an odd continuous mapping $h : A \to \mathbb{R}^k \setminus \{0\}$; in this case, we write $\gamma(A) = k$. If such an integer does not exist, then $\gamma(A) := +\infty$. Notice that if $A \in \Sigma$ is homeomorphic to S^{k-1} ($k-1$ dimension unit sphere in the Euclidean space $\mathbb{R}^k$) by an odd homeomorphism, then $\gamma(A) = k$ ([10], Corollary 5.5). For other properties and more details on the notion of genus, we refer the reader to [10,11]. The following theorem is an immediate consequence of ([9], Theorem 4.3).

Theorem 1. *Let $\mathcal{I}$ be of type (4) with $\mathcal{F}$ and ψ even. In addition, suppose that $\mathcal{I}$ is bounded from below, satisfies the (PS) condition and $\mathcal{I}(0) = 0$. If there exists a nonempty compact symmetric subset $A \subset Y \setminus \{0\}$ with $\gamma(A) \geq k$, such that*

$$\sup_{v \in A} \mathcal{I}(v) < 0,$$

then the functional $\mathcal{I}$ has at least k distinct pairs of nontrivial critical points.

2. Variational Approach and Preliminaries

To introduce the variational formulation for problem (1), let H_T be the space of all *T-periodic $\mathbb{Z}$-sequences* in $\mathbb{R}$, i.e., of mappings $u : \mathbb{Z} \to \mathbb{R}$, such that $u(n) = u(n+T)$ for all $n \in \mathbb{Z}$. On H_T, we consider the following inner product and corresponding norm:

$$(u|v) := \sum_{j=1}^{T} u(j)v(j), \qquad \|u\| = \left(\sum_{j=1}^{T} |u(j)|^2 \right)^{1/2},$$

which makes it a Hilbert space. In addition, for each $u \in H_T$, we set

$$\overline{u} := \frac{1}{T} \sum_{j=1}^{T} u(j), \quad \tilde{u} := u - \overline{u}.$$

It is not difficult to check that

$$|\tilde{u}(i)| \leq T^{\frac{1}{2}} \left(\sum_{j=1}^{T} |\Delta u(j)|^2 \right)^{1/2} \quad (i \in \{1, \ldots, T\}). \tag{5}$$

Now, let the closed convex subset K of H_T be defined by

$$K := \{u \in H_T : \ |\Delta u|_\infty \leq 1\},$$

where $|\Delta u|_\infty := \max_{i=1,\ldots,T} |\Delta u(i)|$. Then, from (5), one has

$$|\overline{u}| - T \leq |u(i)| \leq |\overline{u}| + T \quad (i \in \{1, \ldots, T\}), \tag{6}$$

for all $u \in K$. We introduce the even functions

$$\Psi(u) = \begin{cases} \displaystyle\sum_{j=1}^{T} \Phi[\Delta u(j)], & \text{if } u \in K, \\ +\infty, & \text{otherwise,} \end{cases}$$

where $\Phi(y) = 1 - \sqrt{1 - y^2}$ $(y \in [-1, 1])$ and

$$\mathcal{G}_\lambda(u) = -\lambda \sum_{j=1}^{T} G(u(j)) \qquad (u \in H_T),$$

with G the primitive

$$G(x) = \int_0^x g(\tau) d\tau \qquad (x \in \mathbb{R}).$$

It is not difficult to see that Ψ is convex and lower semicontinuos, while $\mathcal{G}_\lambda$ is of class C^1, its derivative being given by

$$\langle \mathcal{G}'_\lambda(u), v \rangle = -\lambda \sum_{j=1}^{T} g(u(j))v(j) \qquad (u, v \in H_T).$$

Then, the functional $I_\lambda : H_T \to (-\infty, +\infty]$ associated to (1) is

$$I_\lambda = \Psi + \mathcal{G}_\lambda$$

and it is clear that it has the structure required by Szulkin's critical point theory. *A solution* of problem (1) is an element $u \in H_T$ such that $|\Delta u(n)| < 1$, for all $n \in \mathbb{Z}$, which satisfies the equation in (1). The following result reduces the search of solutions of problem (1) to finding critical points of I_λ.

Proposition 1. *Any critical point of I_λ is a solution of problem* (1).

Proof. Let $e \in H_T$. By virtue of Lemmas 5 and 6 in [1], the problem

$$\Delta[\phi(\Delta u(n-1))] = \overline{u} + e(n), \quad u(n) = u(n+T) \quad (n \in \mathbb{Z})$$

has a unique solution u_e, which is also the unique solution of the variational inequality

$$\sum_{j=1}^{T} \{\Phi[\Delta v(j)] - \Phi[\Delta u(j)] + \overline{u}(\overline{v} - \overline{u}) + e(j)(v(j) - u(j))\} \geq 0, \ \forall\, v \in K \tag{7}$$

([6], Proposition 3.1). Next, let $w \in K$ be a critical point of I_λ. Then, for any $v \in K$, one has

$$\sum_{j=1}^{T} \{\Phi[\Delta v(j)] - \Phi[\Delta w(j)] - \lambda g(w(j))(v(j) - w(j))\} \geq 0,$$

which can be written as

$$\sum_{j=1}^{T} \{\Phi[\Delta v(j)] - \Phi[\Delta w(j)] + \overline{w}(v(j) - w(j))\} - \sum_{j=1}^{T} [\lambda g(w(j)) + \overline{w}](v(j) - w(j)) \geq 0.$$

Hence, w is a solution of the variational inequality

$$\sum_{j=1}^{T} \{\Phi[\Delta v(j)] - \Phi[\Delta w(j)] + \overline{w}(\overline{v} - \overline{w}) + e_w(j)(v(j) - w(j))\} \geq 0, \quad \forall\, v \in K, \tag{8}$$

with $e_w \in H_T$ being given by $e_w(n) = -\lambda g(w(n)) - \overline{w}$ $(n \in \mathbb{Z})$.

Therefore, by (8) and the uniqueness of the solution of (7), we obtain that, in fact, w solves problem (1). □

Proposition 2. *If G is anticoercive, i.e.,*

$$\lim_{|x| \to +\infty} G(x) = -\infty, \tag{9}$$

then I_λ is bounded from below and satisfies the (PS) condition.

Proof. From (9) we have that $-G$, hence $\mathcal{G}_\lambda$, are bounded from below on $\mathbb{R}$, respectively on H_T. This, together with the fact that Ψ is bounded from below, ensure that the same is true for I_λ.

To see that I_λ satisfies the (PS) condition, let $\{u_n\} \subset K$ be a (PS)-sequence. Assuming by contradiction that $\{|\overline{u}_n|\}$ is not bounded, we may suppose, going, if necessary, to a subsequence, that $|\overline{u}_n| \to +\infty$. Then, by virtue of (6) and (9), we deduce that $I_\lambda(u_n) \to -\infty$, contradicting the fact that $\{I_\lambda(u_n)\}$ is convergent. Consequently, $\{|\overline{u}_n|\}$ is bounded. This, together with $|\tilde{u}_n| \leq T$ shows that $\{u_n\}$ is bounded in the finite-dimensional space H_T; hence, it contains a convergent subsequence. □

Remark 1. *Notice that until here in this section, no parity assumptions on the continuous function $g : \mathbb{R} \to \mathbb{R}$ must be required.*

We end this section by reviewing some spectral properties of the operator $-\Delta^2$, which is needed in the sequel. A real number $\lambda \in \mathbb{R}$ is said to be an *eigenvalue* of $-\Delta^2$ on H_T, if there is some $u \in H_T \setminus \{0_{H_T}\}$ such that

$$-\Delta^2 u(n-1) = \lambda u(n), \qquad (n \in \mathbb{Z}) \tag{10}$$

and in this case, u is called *eigensequence* corresponding to the eigenvalue λ. On account of the periodicity of u, relation (10) is equivalent to the system

$$\begin{cases} -u(2) + 2u(1) - u(T) = \lambda u(1) \\ -u(3) + 2u(2) - u(1) = \lambda u(2) \\ \vdots \\ -u(T) + 2u(T-1) - u(T-2) = \lambda u(T-1) \\ -u(1) + 2u(T) - u(T-1) = \lambda u(T). \end{cases} \tag{11}$$

If we consider the particular circulant matrix

$$M_T := \begin{pmatrix} 2 & -1 & 0 & \cdots & 0 & 0 & -1 \\ -1 & 2 & -1 & \cdots & 0 & 0 & 0 \\ \vdots & & & & & & \\ 0 & 0 & 0 & \cdots & -1 & 2 & -1 \\ -1 & 0 & 0 & \cdots & 0 & -1 & 2 \end{pmatrix}$$

then, having in view (11), the eigenvalues of $-\Delta^2$ are precisely the characteristic roots of M_T. In addition, if $y = (y_1, \ldots, y_T) \in \mathbb{R}^T \setminus \{0_{\mathbb{R}^T}\}$ is an eigenvector corresponding to a characteristic root λ, then its *extension* $u^y \in H_T$, defined by $u^y(i) = y_i$ for $i = \overline{1,T}$, is an eigensequence corresponding to the eigenvalue λ. This means that an orthonormal basis of eigensequences $u^1, \ldots, u^T$ can be constructed from an orthonormal basis of eigenvectors $x^1, \ldots, x^T$ of M_T by extending x^i in H_T ($i = \overline{1,T}$) as above.

From ([12], p. 38), we know that the characteristic roots of M_T, hence the eigenvalues of $-\Delta^2$, are $4\sin^2 i\pi/T$ ($i = \overline{0, T-1}$). We can label them according to the parity of T as follows:

<u>Todd</u> :

$$\lambda_0 = 0, \qquad \lambda_{2k-1} = \lambda_{2k} = 4\sin^2 \frac{k\pi}{T}, \quad k = 1, \ldots, \frac{T-1}{2};$$

<u>Teven</u> :

$$\lambda_0 = 0, \qquad \lambda_{2k-1} = \lambda_{2k} = 4\sin^2 \frac{k\pi}{T}, \quad k = 1, \ldots, \frac{T-2}{2}, \qquad \lambda_{T-1} = 4.$$

In both cases, we consider an orthonormal basis $e^0, \ldots, e^{T-1}$ in H_T, such that e^i is an eigensequence corresponding to λ_i ($i = \overline{0, T-1}$). Observe that, by multiplying equality (10) by arbitrary $v \in H_T$ and using summation by parts formula, one obtains that if $u \in H_T$ and $\lambda \in \mathbb{R}$ satisfy (10), then

$$\sum_{j=1}^{T} \Delta u(j) \Delta v(j) = \lambda (u|v).$$

This yields

$$\sum_{j=1}^{T} \Delta e^i(j) \Delta e^k(j) = \lambda_k \delta_{ik} \qquad (i, k \in \{0, \ldots, T-1\}), \tag{12}$$

where δ_{ik} stands for the Kronecker delta function.

3. Main Result

Our main result is given in the following.

Theorem 2. *Assume that $g : \mathbb{R} \to \mathbb{R}$ is a continuous odd function and that G satisfies (9) together with*

$$\liminf_{x \to 0} \frac{2G(x)}{x^2} \geq 1. \tag{13}$$

Then, the following hold true:

(i) If

$$\lambda > 8\sin^2 \frac{m\pi}{T} \; (= 2\lambda_{2m}) \text{ with } 0 \leq m \leq \begin{cases} (T-1)/2 & \text{if } T \text{ is odd} \\ (T-2)/2 & \text{if } T \text{ is even} \end{cases}, \tag{14}$$

then problem (1) has at least $2m+1$ distinct pairs of nontrivial solutions.

(ii) If T is even and

$$\lambda > 8 \; (= 2\lambda_{T-1}), \tag{15}$$

then (1) has at least T distinct pairs of nontrivial solutions.

Proof. We show (i) in the odd case because the even case follows by exactly the same arguments, and under assumption (15), a quite similar strategy works by simply replacing "$2m$" with "$T-1$".

Thus, let $0 \leq m \leq (T-1)/2$. On account of Theorem 1 and Propositions 1 and 2, we have to prove that there exists a nonempty compact symmetric subset $A_m \subset H_T \setminus \{0\}$ with $\gamma(A_m) \geq 2m+1$, such that

$$\sup_{v \in A_m} I_\lambda(v) < 0. \tag{16}$$

Since $\lambda > 2\lambda_{2m}$, we can choose $\varepsilon \in (0,1)$, so that $\lambda > 2\lambda_{2m}/(1-\varepsilon)$. Then, by virtue of (13), there exists $\delta > 0$ such that

$$2G(x) \geq (1-\varepsilon)x^2 \quad \text{as } |x| \leq \delta. \tag{17}$$

Next, we introduce the set

$$A_m := \left\{ \sum_{k=0}^{2m} \alpha_k e^k \, : \, \alpha_0^2 + \cdots + \alpha_{2m}^2 = \rho^2 \right\},$$

where ρ is a positive number, which is chosen $\leq \min\left\{ \dfrac{1}{2\sqrt{2m+1}}, \delta \right\}$.

Then, it is not difficult to see that the odd mapping $H : A_m \to S^{2m}$ defined by

$$H\left(\sum_{k=0}^{2m} \alpha_k e^k \right) = \left(\frac{\alpha_0}{\rho}, \frac{\alpha_1}{\rho} \ldots, \frac{\alpha_{2m}}{\rho} \right)$$

is a homeomorphism between A_m and S^{2m}; therefore, $\gamma(A_m) = 2m+1$.

We have that $A_m \subset K$. Indeed, let $v = \sum_{k=0}^{2m} \alpha_k e^k \in A_m$. Then, for all $j \in \{1, \ldots, T\}$, we obtain

$$\begin{aligned} |\Delta v(j)| &\leq \sum_{k=0}^{2m} \left| \alpha_k e^k(j+1) \right| + \sum_{k=0}^{2m} \left| \alpha_k e^k(j) \right| \leq 2 \sum_{k=0}^{2m} |\alpha_k| \\ &\leq 2\sqrt{2m+1} \left(\sum_{k=0}^{2m} \alpha_k^2 \right)^{1/2} = 2\rho\sqrt{2m+1} \end{aligned} \tag{18}$$

and since $\rho \leq 1/(2\sqrt{2m+1})$, one has $|\Delta v|_\infty \leq 1$, which shows that $v \in K$. On the other hand, using (12), we obtain

$$\begin{aligned}
\sum_{j=1}^{T} |\Delta v(j)|^2 &= \sum_{j=1}^{T} \left| \Delta \left(\sum_{k=0}^{2m} \alpha_k e^k(j) \right) \right|^2 = \sum_{j=1}^{T} \left(\sum_{k=0}^{2m} \alpha_k \Delta e^k(j) \right)^2 \\
&= \sum_{j=1}^{T} \left(\sum_{k=0}^{2m} \alpha_k^2 (\Delta e^k(j))^2 + \sum_{\substack{i,k=0 \\ i \neq k}}^{2m} \alpha_i \alpha_k \Delta e^k(j) \Delta e^i(j) \right) \\
&= \sum_{k=0}^{2m} \alpha_k^2 \sum_{j=1}^{T} (\Delta e^k(j))^2 + \sum_{\substack{i,k=0 \\ i \neq k}}^{2m} \alpha_i \alpha_k \sum_{j=1}^{T} \Delta e^k(j) \Delta e^i(j) \\
&= \sum_{k=0}^{2m} \lambda_k \alpha_k^2 \leq \lambda_{2m} \sum_{k=0}^{2m} \alpha_k^2 = \lambda_{2m} \rho^2.
\end{aligned} \tag{19}$$

In addition, it is clear that

$$\sum_{j=1}^{T} |v(j)|^2 = \|v\|^2 = (v|v) = \sum_{k=0}^{2m} \alpha_k^2 = \rho^2. \tag{20}$$

Then, from (17), (19), (20) and $|v(j)| \leq \rho \leq \delta$ ($j \in \{1, \ldots, T\}$), it follows that

$$\begin{aligned}
I_\lambda(v) &= \Psi(v) + \mathcal{G}_\lambda(v) \leq \sum_{j=1}^{T} |\Delta v(j)|^2 - \frac{\lambda}{2}(1-\varepsilon) \sum_{j=1}^{T} |v(j)|^2 \\
&\leq \rho^2 \lambda_{2m} - \frac{\lambda}{2}(1-\varepsilon)\rho^2 = \rho^2 \frac{2\lambda_{2m} - \lambda(1-\varepsilon)}{2} < 0.
\end{aligned}$$

Therefore, (16) holds true and the proof of (i) is complete. □

Example 1. *If* (14) *holds true, then problem* (3) *has at least* $2m+1$ *distinct pairs of nontrivial solutions. In addition, if T is even, under assumption* (15)*, problem* (3) *has at least T distinct pairs of nontrivial solutions. Notice that besides the trivial solution, problem* (3) *always has the pair of constant solutions* $u \equiv \pm 1$*, and these are the only constant nontrivial solutions of* (3)*. Therefore, problem* (3) *has at least* $2m$ *(resp.* $T-1$*) distinct pairs of nonconstant solutions if hypothesis* (14) *is satisfied (resp.* (15) *holds true).*

Consider the eigenvalue type problem

$$-\Delta[\phi(\Delta u(n-1))] = \lambda u(n) + h(u(n)), \quad u(n) = u(n+T) \quad (n \in \mathbb{Z}) \tag{21}$$

and set $H(x) = \int_0^x h(\tau) d\tau$ ($x \in \mathbb{R}$).

Corollary 1. *If the continuous function* $h : \mathbb{R} \to \mathbb{R}$ *is odd and*

$$\liminf_{x \to 0} \frac{H(x)}{x^2} \geq 0, \quad \lim_{x \to +\infty} \frac{H(x)}{x^2} = -\infty,$$

then the conclusions (i) *and* (ii) *of Theorem 2 remain valid with* (21) *instead of* (1)*.*

Proof. Theorem 2 applies to the problem

$$-\Delta[\phi(\Delta u(n-1))] = \lambda \left(u(n) + \frac{h(u(n))}{\lambda} \right), \quad u(n) = u(n+T) \quad (n \in \mathbb{Z}).$$

□

Theorem 2 can be employed to derive the multiplicity of nontrivial solutions of autonomous non-parametric problems having the form

$$-\Delta[\phi(\Delta u(n-1))] = f(u(n)), \quad u(n) = u(n+T) \quad (n \in \mathbb{Z}). \tag{22}$$

Setting $F(x) = \int_0^x f(\tau)d\tau$ $(x \in \mathbb{R})$, we have the following.

Corollary 2. *Assume that $f : \mathbb{R} \to \mathbb{R}$ is a continuous odd function and that*

$$\lim_{x \to +\infty} F(x) = -\infty. \tag{23}$$

Then, the following hold true:
(i) If

$$\liminf_{x \to 0} \frac{F(x)}{x^2} > 4 \sin^2 \frac{m\pi}{T} \text{ with } 0 \leq m \leq \begin{cases} (T-1)/2 & \text{if } T \text{ is odd} \\ (T-2)/2 & \text{if } T \text{ is even} \end{cases}, \tag{24}$$

then problem (22) has at least $2m + 1$ distinct pairs of nontrivial solutions.

(ii) If T is even and

$$\liminf_{x \to 0} \frac{F(x)}{x^2} > 4, \tag{25}$$

then (22) has at least T distinct pairs of nontrivial solutions.

Proof. From (24), there exists $\overline{\lambda} > 0$ such that

$$\liminf_{x \to 0} \frac{2F(x)}{x^2} \geq \overline{\lambda} > 8 \sin^2 \frac{m\pi}{T}$$

and the result follows from Theorem 2 with $g(x) = f(x)/\overline{\lambda}$; a similar argument works when (25) is fulfilled. □

Example 2. *Let $f_a : \mathbb{R} \to \mathbb{R}$ be given by*

$$f_a(x) = 2x \sin |x|^{-\frac{1}{2}} - \frac{x|x|^{-\frac{1}{2}} \cos |x|^{-\frac{1}{2}}}{2} + 2ax - 4x^3 \quad (x \in \mathbb{R}).$$

Then,

$$F_a(x) = x^2 \left(\sin |x|^{-\frac{1}{2}} + a - x^2 \right) \quad (x \in \mathbb{R})$$

and by Corollary 2, we obtain that, if

$$a > 1 + 4 \sin^2 \frac{m\pi}{T} \text{ with } 0 \leq m \leq \begin{cases} (T-1)/2 & \text{if } T \text{ is odd} \\ (T-2)/2 & \text{if } T \text{ is even} \end{cases},$$

then the equation

$$-\Delta[\phi(\Delta u(n-1))] = f_a(u(n)) \quad (n \in \mathbb{Z}) \tag{26}$$

has at least $2m + 1$ distinct pairs of nontrivial T-periodic solutions, while if T is even and $a > 5$, then (26) has at least T distinct pairs of nontrivial T-periodic solutions.

Remark 2. *A multiplicity result for odd perturbations of the discrete p-Laplacian operator is obtained in [13] using a Clark-type result in the frame of the classical critical point theory.*

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