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Exponential Stability for the Equations of Porous Elasticity in One-Dimensional Bounded Domains

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Abstract: This work establishes an exponential stability result for a porous-elastic system, where the dissipation mechanisms act on the porous and elastic equations. Our result completes some of the results in the literature for unbounded domains.

Keywords: porous-elastic system; exponential stability; multiplier method; dissipation mechanisms; bounded domains

MSC: 35B35; 35B40; 93D20

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1. Introduction

In this article, we investigate the following one-dimensional isothermal porous elastic problem with dissipation mechanics acting on the porous and elastic equations

$$\begin{cases} \rho u_{tt} - \alpha u_{xx} - \beta \phi_x - \gamma u_{xxt} - \varepsilon_1 \phi_{xt} = 0, & t > 0,\ x \in (0,l), \\ \kappa \phi_{tt} - \delta \phi_{xx} + \beta u_x + \eta \phi + \tau \phi_t + \varepsilon_2 u_{xt} = 0, & t > 0,\ x \in (0,l), \\ u(x,0) = u_0(x),\ u_t(x,0) = u_1(x),\ \phi(x,0) = \phi_0(x),\ \phi_t(x,0) = \phi_1(x), & x \in (0,l), \\ u_x(0,t) = u_x(l,t) = \phi(0,t) = \phi(l,t) = 0, & t > 0, \end{cases} \tag{1}$$

where the unknown scalar functions u and ϕ represent the displacement of the elastic material and the volume fraction, respectively. The coefficients satisfy

$$\rho > 0,\ \alpha > 0,\ \beta \neq 0,\ \gamma > 0,\ \varepsilon_1 \neq 0,\ \varepsilon_2 \neq 0,\ \kappa > 0,\ \delta > 0,\ \eta > 0,\ \tau > 0.$$

Furthermore, to guarantee that the internal energy is positive, we assume $\alpha\eta > \beta^2$. From a physical point of view, the system describes the interpolation of two structures: the elastic structure, which is macroscopic, and the porous structure, which can be described as microscopic. Such coupling produces internal or external forces leading to thermomechanical displacement, which are generally harmful to the system after some time. Various types of damping mechanisms are used in the literature to control the displacements. The porous-elastic materials have wide applications in petroleum engineering, material science, physics, biology, and soil mechanics. It also applies to solids characterized by tiny distributed pores such as rocks, wood, and bones, as mentioned in [1]. Júnior et al. [2] considered system (1) for $\varepsilon_1 = \varepsilon_2 = \varepsilon$ and established a lack of exponential stability result with the condition $\gamma\tau = \varepsilon^2$. Moreover, they proved an optimal polynomial stability result subject to a particular relationship between the damping parameters of the system. For the system in the whole space, that is, $t > 0,\ x \in \mathbb{R}$, we mention the work of Quintanilla and Ueda [3]. They obtained a standard decay structure with the assumption $4\gamma\tau > (\varepsilon_1 + \varepsilon_2)^2$. Furthermore, they proved that if $\varepsilon_1 = \varepsilon_2$ and either $\gamma = 0, \tau > 0$ or $\gamma > 0, \tau = 0$, then the decay structure is of regularity-loss type.

Quintanilla [4] discussed system (1) when $\gamma = \varepsilon_1 = \varepsilon_2 = 0,\ \tau > 0$ and concluded that the frictional damping ($\tau\phi_t$) was not strong enough to exponentially stabilize the system. However, Apalara [5] proved that the system was exponentially stable provided $\alpha\kappa = \delta\rho$. Similarly, when $\tau = \varepsilon_1 = \varepsilon_2 = 0,\ \gamma > 0$, Magańa and Quintanilla [6] proved that the system was not exponentially stable. On the other hand, when $\varepsilon_1 = \varepsilon_2 = 0,\ \tau > 0,\ \gamma > 0$, they obtained an exponential stability result. For some other interesting results on the porous-elastic system, we refer the reader to a non-exhaustive list of references [7–16]. We especially refer the reader to [1] for the results on some variants and a general system of (1).

Below, we mention some results concerning system (1) with other damping mechanisms. Pamplona et al. [17] considered a system of porous-thermoelasticity with microtemperatures, that is

$$\begin{cases} \rho u_{tt} - \alpha u_{xx} - \beta\phi_x - \gamma u_{xxt} - \varepsilon_1\phi_{xt} + \beta_1\theta_x - \ell_1 w_{xx} = 0, \\ \kappa\phi_{tt} - \delta\phi_{xx} + \beta u_x + \eta\phi + \tau\phi_t + \varepsilon_2 u_{xt} - m\theta + d_1 w_x - k_1\theta_{xx} - \mu\phi_{xxt} = 0, \\ c_1\theta_t - k\theta_{xx} + \beta_1 u_{xt} + m\varphi_t - \sigma_1 w_x - \sigma_2\phi_{xxt} = 0, \\ c_2 w_t - \sigma_3 w_{xx} - d_2\varphi_{xt} + k_3\theta_x + k_4 w - \ell_2 u_{xxt} = 0, \end{cases} \tag{2}$$

for $t > 0$ and $x \in (0,\pi)$, where θ and w are the temperature difference and microtemperature, respectively. They proved that when $\mu = k_1 = \sigma_2 = 0$, the semigroup generated by the solutions was not analytic, though the system was exponentially stable. However, when $\varepsilon_1 = \varepsilon_2 = \tau = 0$, they found that the semigroup, which defined the solutions was analytic. Analyticity means that the functions and the orbits are regular; hence, time derivatives can recover the solutions. In the absence of microtemperature, Casas and Quintanilla [18] investigated (2) for $\gamma = \varepsilon_1 = \ell_1 = \varepsilon_2 = d_1 = k_1 = \mu = \sigma_1 = \sigma_2 = 0$ and established an exponential stability result. However, when τ was also zero, they proved in [19] that the heat effect alone was not strong enough to bring about an exponential stability result. Contrarily, Santos et al. [20] proved that the heat effect alone was strong enough to stabilize the system exponentially, provided that $\alpha\kappa = \delta\rho$. Interestingly, when $\gamma \neq 0$, Pamplona et al. [21] showed that the system was also not exponentially stable. We refer the reader to [22–26] for some other interesting results.

In the present work, we considered system (1) and proved an exponential stability result for the case $4\gamma\tau > (\varepsilon_1 + \varepsilon_2)^2$. We refer the reader to [2] for the well-posedness (existence, uniqueness, and continuous dependence on the initial data) result of the system. Meanwhile, due to the boundary conditions on u, system (1) can have solutions which do not decay. In addition, the boundary conditions prevent the use of Poincaré's inequality on u. To avoid these cases, we performed the following necessary transformation. From the first equation in (1), we have

$$\begin{aligned} \rho\int_0^l u_{tt}dx =& \alpha\int_0^l u_{xx}dx + \beta\int_0^l \phi_x dx + \gamma\int_0^l u_{xxt}dx + \varepsilon_1\int_0^l \phi_{xt}dx \\ =& \alpha\cancelto{0}{u_x\Big|_0^l} + \beta\cancelto{0}{\phi\Big|_0^l} + \gamma\cancelto{0}{u_{xt}\Big|_0^l} + \varepsilon_1\cancelto{0}{\phi_t\Big|_0^l} = 0, \end{aligned}$$

where the last equality follows from the boundary conditions $u_x(0,t) = u_x(l,t) = \phi(0,t) = \phi(l,t) = 0$. Consequently, by letting $v(t) := \displaystyle\int_0^l u(x,t)dx$ and bearing in mind the initial conditions $u(x,0) = u_0(x),\ u_t(x,0) = u_1(x)$, we obtain the following initial value problem

$$\begin{cases} v''(t) = 0, & t > 0, \\ v(0) = \displaystyle\int_0^l u_0(x)dx, & v'(0) = \displaystyle\int_0^l u_1(x)dx. \end{cases} \tag{3}$$

Solving the differential equation $v''(t) = 0$, we obtain

$$v(t) = b_1 t + b_2, \tag{4}$$

where b_1 and b_2 are some constants. Applying the initial condition $v(0) = \int_0^l u_0(x)dx$, we obtain

$$\int_0^l u_0(x)dx = b_2.$$

By differentiating (4) with respect to t and using the initial condition $v'(0) = \int_0^l u_1(x)dx$, we have

$$\int_0^l u_1(x)dx = b_1.$$

By substituting b_1 and b_2 into (4), we obtain

$$v(t) = \int_0^l u(x,t)dx = t\int_0^l u_1(x)dx + \int_0^l u_0(x)dx. \tag{5}$$

Consequently, by setting

$$\overline{u}(x,t) = u(x,t) - \frac{t}{l}\int_0^l u_1(x)dx - \frac{1}{l}\int_0^l u_0(x)dx$$

and using (5), we end up with

$$\int_0^l \overline{u}(x,t)dx = 0, \quad \forall t > 0.$$

Thus, Poincaré's inequality can be administered on $\overline{u}$. In addition, simple substitution shows that $(\overline{u}, \phi)$ is the solution to problem (1) with initial data for $\overline{u}$ given as

$$\overline{u}_0(x) = u_0(x) - \frac{1}{l}\int_0^l u_0(x)dx \text{ and } \overline{u}_1(x) = u_1(x) - \frac{1}{l}\int_0^l u_1(x)dx.$$

Henceforth, we work with $(\overline{u}, \phi)$ instead of (u, ϕ) but write (u, ϕ) for simplicity.

The breakdown of the remaining sections is as follows: We devote Section 2 to the statements and proofs of some essential technical lemmas. Our stability result is established in Section 3. The paper ends with some general comments and interesting open problems in Section 4. We use c_p throughout this paper to denote Poincaré's constant.

2. Technical Lemmas

At the beginning of this section, let us indicate that the energy functional $\mathcal{E}$ associated with system (1) is given by

$$\mathcal{E}(t) = \frac{1}{2}\int_0^l \left[\rho u_t^2 + \kappa\phi_t^2 + \delta\phi_x^2 + \alpha u_x^2 + \eta\phi^2 + 2\beta u_x\phi\right]dx, \quad \forall t > 0. \tag{6}$$

Remark 1. *Assumption $\alpha\eta > \beta^2$ guarantees that the energy functional $\mathcal{E}$, defined by (6), is nonnegative. To establish this, it is enough to show that the combination of the last three terms on the right side of (6) is nonnegative, that is*

$$\alpha u_x^2 + \eta\phi^2 + 2\beta u_x\phi > 0. \tag{7}$$

Clearly, we have

$$\alpha u_x^2 + \eta\phi^2 + 2\beta u_x\phi = \left(\alpha - \frac{\beta^2}{\eta}\right)u_x^2 + \left(\sqrt{\eta}\phi + \frac{\beta}{\sqrt{\eta}}u_x\right)^2.$$

So, the energy functional $\mathcal{E}$ becomes

$$\mathcal{E}(t) = \frac{1}{2}\int_0^l \left[\rho u_t^2 + \kappa\phi_t^2 + \delta\phi_x^2 + \left(\alpha - \frac{\beta^2}{\eta}\right)u_x^2 + \left(\sqrt{\eta}\phi + \frac{\beta}{\sqrt{\eta}}u_x\right)^2\right]dx, \quad \forall t > 0. \tag{8}$$

Consequently, by using the fact that $\alpha\eta > \beta^2$, the nonnegativity is guaranteed.

The following lemmas are designed to capture some important functionals and the estimate of their derivatives.

Lemma 1. *Assume $4\gamma\tau > (\varepsilon_1 + \varepsilon_2)^2$. Then, the energy functional $\mathcal{E}$ associated with system (1) and given by (6), satisfies*

$$\frac{d}{dt}\mathcal{E}(t) \leq -\gamma_0 \int_0^l u_{xt}^2 dx, \quad \forall t > 0. \tag{9}$$

Proof. Multiplying the first equation in (1) by u_t, integrating by parts over $(0, l)$, and taking advantage of the boundary conditions, we obtain, for any $t > 0$,

$$\begin{aligned}
&\rho\int_0^l u_{tt}u_t dx + \alpha\int_0^l u_x u_{xt} dx + \beta\int_0^l \phi u_{xt} dx + \gamma\int_0^l u_{xt}^2 dx + \varepsilon_1\int_0^l \phi_t u_{xt} dx = 0\\
&\frac{\rho}{2}\frac{d}{dt}\int_0^l u_t^2 dx + \frac{\alpha}{2}\frac{d}{dt}\int_0^l u_x^2 dx + \beta\frac{d}{dt}\int_0^l \phi u_x dx - \beta\int_0^l \phi_t u_x dx + \gamma\int_0^l u_{xt}^2 dx\\
&\qquad + \varepsilon_1\int_0^l \phi_t u_{xt} dx = 0.
\end{aligned}$$

The last equation can be written as:

$$\frac{\rho}{2}\frac{d}{dt}\int_0^l u_t^2 dx + \frac{\alpha}{2}\frac{d}{dt}\int_0^l u_x^2 dx + \beta\frac{d}{dt}\int_0^l \phi u_x dx = \beta\int_0^l \phi_t u_x dx - \gamma\int_0^l u_{xt}^2 dx - \varepsilon_1\int_0^l \phi_t u_{xt} dx. \tag{10}$$

Similarly, by multiplying the second equation in (1) by ϕ_t, we obtain, for any $t > 0$,

$$\frac{\kappa}{2}\frac{d}{dt}\int_0^l \phi_t^2 dx + \frac{\delta}{2}\frac{d}{dt}\int_0^l \phi_x^2 dx + \frac{\eta}{2}\frac{d}{dt}\int_0^l \phi^2 dx = -\beta\int_0^l \phi_t u_x dx - \tau\int_0^l \phi_t^2 dx - \varepsilon_2\int_0^l \phi_t u_{xt} dx. \tag{11}$$

Summing up (10) and (11), we have, for any $t > 0$,

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int_0^l \left[\rho u_t^2 + \kappa\phi_t^2 + \delta\phi_x^2 + \alpha u_x^2 + \eta\phi^2 + 2\beta u_x\phi\right]dx\\
&\qquad\qquad = -\gamma\int_0^l u_{xt}^2 dx - \tau\int_0^l \phi_t^2 dx - (\varepsilon_1 + \varepsilon_2)\int_0^l u_{xt}\phi_t dx.
\end{aligned}$$

Thus, bearing in mind (6), we obtain

$$\frac{d}{dt}\mathcal{E}(t) = -\gamma\int_0^l u_{xt}^2 dx - \tau\int_0^l \phi_t^2 dx - (\varepsilon_1 + \varepsilon_2)\int_0^l u_{xt}\phi_t dx, \quad \forall t > 0. \tag{12}$$

Using the fact that $4\gamma\tau > (\varepsilon_1 + \varepsilon_2)^2$, we have $\gamma_0 := \gamma - \left(\frac{\varepsilon_1 + \varepsilon_2}{2\sqrt{\tau}}\right)^2 > 0$. So, from (12), we have, for any $t > 0$,

$$\begin{aligned}\frac{d}{dt}\mathcal{E}(t) &= -\left(\gamma - \left(\frac{\varepsilon_1+\varepsilon_2}{2\sqrt{\tau}}\right)^2\right)\int_0^l u_{xt}^2 dx - \int_0^l \left(\sqrt{\tau}\phi_t + \left(\frac{\varepsilon_1+\varepsilon_2}{2\sqrt{\tau}}\right)u_{xt}\right)^2 dx \\ &= -\gamma_0\int_0^l u_{xt}^2 dx - \int_0^l \left(\sqrt{\tau}\phi_t + \left(\frac{\varepsilon_1+\varepsilon_2}{2\sqrt{\tau}}\right)u_{xt}\right)^2 dx \le -\gamma_0\int_0^l u_{xt}^2 dx \le 0.\end{aligned} \tag{13}$$

□

Remark 2. *The condition* $4\gamma\tau > (\varepsilon_1+\varepsilon_2)^2$ *guarantees the dissipative nature of the system. In other words, the energy* $\mathcal{E}$ *of the system is decreasing when* $4\gamma\tau > (\varepsilon_1+\varepsilon_2)^2$*. See the proof of Lemma 1.*

Lemma 2. *The functional* $\mathcal{Q}_1$ *given by*

$$\mathcal{Q}_1(t) := \alpha\varepsilon_1\int_0^l u_x\phi dx - \varepsilon_1\rho\int_0^l \phi_t\int_0^x u_t(s)dsdx + \frac{\beta\varepsilon_1}{2}\int_0^l \phi^2 dx + \frac{\rho\beta\varepsilon_1}{2\kappa}\int_0^l u^2 dx, \quad \forall t>0,$$

satisfies, for any $\delta_1 > 0$*, the estimate*

$$\begin{aligned}\frac{d}{dt}\mathcal{Q}_1(t) \le& -\frac{\varepsilon_1^2}{2}\int_0^l \phi_t^2 dx + \delta_1\int_0^l \phi^2 dx \\ &+\left(\frac{|\varepsilon_1\varepsilon_2|\rho c_p}{\kappa} + \gamma^2 + \frac{\varepsilon_1^2}{2\delta_1}\left(\alpha - \frac{\rho\delta}{\kappa}\right)^2 + \frac{\varepsilon_1^2\rho^2\eta^2 l c_p}{2\delta_1\kappa^2} + \frac{\rho^2 l c_p}{\kappa^2}\tau^2\right)\int_0^l u_{xt}^2 dx, \quad \forall t>0.\end{aligned} \tag{14}$$

Proof. The direct derivative of $\mathcal{Q}_1$ gives, for any $t > 0$,

$$\begin{aligned}\frac{d}{dt}\mathcal{Q}_1(t) =& \alpha\varepsilon_1\int_0^l u_{xt}\phi dx + \alpha\varepsilon_1\int_0^l u_x\phi_t dx - \varepsilon_1\rho\int_0^l \phi_{tt}\int_0^x u_t(s)dsdx \\ &- \varepsilon_1\rho\int_0^l \phi_t\int_0^x u_{tt}(s)dsdx + \beta\varepsilon_1\int_0^l \phi\phi_t dx + \frac{\rho\beta\varepsilon_1}{\kappa}\int_0^l uu_t dx.\end{aligned} \tag{15}$$

Using the second equation in (1), we see that the third term on the right-hand side of Equation (15) can be written as

$$-\varepsilon_1\rho\int_0^l \phi_{tt}\int_0^x u_t(s)dsdx = \frac{\varepsilon_1\rho}{\kappa}\int_0^l [-\delta\phi_{xx} + \beta u_x + \eta\phi + \tau\phi_t + \varepsilon_2 u_{xt}]\int_0^x u_t(s)dsdx.$$

Using integration by parts and the boundary conditions, we end up with

$$\begin{aligned}-\varepsilon_1\rho\int_0^l \phi_{tt}\int_0^x u_t(s)dsdx =& -\frac{\rho\delta\varepsilon_1}{\kappa}\int_0^l u_{xt}\phi dx - \frac{\rho\beta\varepsilon_1}{\kappa}\int_0^l uu_t dx - \frac{\varepsilon_1\varepsilon_2\rho}{\kappa}\int_0^l u_t^2 dx \\ &+ \frac{\varepsilon_1\rho\eta}{\kappa}\int_0^l \phi\int_0^x u_t(s)dsdx + \frac{\tau\varepsilon_1\rho}{\kappa}\int_0^l \phi_t\int_0^x u_t(s)dsdx.\end{aligned} \tag{16}$$

Similarly, using the first equation in (1), we obtain that the fourth term on the right-hand side of Equation (15) equals

$$-\varepsilon_1\rho\int_0^l \phi_t\int_0^x u_{tt}(s)dsdx = -\alpha\varepsilon_1\int_0^l u_x\phi_t dx - \beta\varepsilon_1\int_0^l \phi\phi_t dx - \varepsilon_1\gamma\int_0^l u_{xt}\phi_t dx - \varepsilon_1^2\int_0^l \phi_t^2 dx. \tag{17}$$

The combination of (15)–(17) gives, for any $t > 0$,

$$\begin{aligned}\frac{d}{dt}\mathcal{Q}_1(t) = &-\varepsilon_1^2\int_0^l \phi_t^2 dx \underbrace{-\frac{\varepsilon_1\varepsilon_2\rho}{\kappa}\int_0^l u_t^2 dx}_{f_1} \underbrace{-\varepsilon_1\gamma\int_0^l u_{xt}\phi_t dx}_{f_2} + \underbrace{\frac{\varepsilon_1\rho\tau}{\kappa}\int_0^l \phi_t \int_0^x u_t(s)dsdx}_{f_3} \\ &+ \underbrace{\varepsilon_1\left(\alpha - \frac{\rho\delta}{\kappa}\right)\int_0^l u_{xt}\phi dx}_{f_4} + \underbrace{\frac{\varepsilon_1\rho\eta}{\kappa}\int_0^l \phi\int_0^x u_t(s)dsdx}_{f_5}.\end{aligned} \tag{18}$$

The estimation of $f_i, i = 1\cdots 5$, using Young's, Cauchy-Schwarz, and Poincaré's inequalities gives, for any $t > 0$,

$$\begin{aligned}f_1(t) &\le \frac{|\varepsilon_1\varepsilon_2|\rho c_p}{\kappa}\int_0^l u_{xt}^2 dx \\ f_2(t) &\le \frac{\varepsilon_1^2}{4}\int_0^l \phi_t^2 dx + \gamma^2\int_0^l u_{xt}^2 dx \\ f_3(t) &\le \frac{\varepsilon_1^2}{4}\int_0^l \phi_t^2 dx + \frac{\rho^2}{\kappa^2}\tau^2\int_0^l\left(\int_0^x u_t(s)ds\right)^2 dx \\ &\le \frac{\varepsilon_1^2}{4}\int_0^l \phi_t^2 dx + \frac{\rho^2 l}{\kappa^2}\tau^2\int_0^l u_t^2 dx \\ &\le \frac{\varepsilon_1^2}{4}\int_0^l \phi_t^2 dx + \frac{\rho^2 l c_p}{\kappa^2}\tau^2\int_0^l u_{xt}^2 dx \\ f_4(t) &\le \frac{\delta_1}{2}\int_0^l \phi^2 dx + \frac{\varepsilon_1^2}{2\delta_1}\left(\alpha - \frac{\rho\delta}{\kappa}\right)^2\int_0^l u_{xt}^2 dx \\ f_5(t) &\le \frac{\delta_1}{2}\int_0^l \phi^2 dx + \frac{\varepsilon_1^2\rho^2\eta^2 l c_p}{2\delta_1\kappa^2}\int_0^l u_{xt}^2 dx.\end{aligned}$$

Replacing $f_i, i = 1\cdots 5$, in (18) with their respective estimates yields (14). □

Lemma 3. *Suppose that $\alpha\eta > \beta^2$. The functional $\mathcal{Q}_2$ given by*

$$\mathcal{Q}_2(t) := \kappa\int_0^l \phi_t\phi dx + \frac{\beta\rho}{\alpha}\int_0^l \phi\int_0^x u_t(s)dsdx + \frac{1}{2}\left(\tau - \frac{\beta\varepsilon_1}{\alpha}\right)\int_0^l \phi^2 dx, \quad \forall t > 0,$$

can be estimated, for some positive constant η_0, by the the following expression:

$$\begin{aligned}\frac{d}{dt}\mathcal{Q}_2(t) \le &-\delta\int_0^l \phi_x^2 dx - \frac{\eta_0}{2}\int_0^l \phi^2 dx + \left(\kappa + \frac{\beta^2\rho}{2\alpha}\right)\int_0^l \phi_t^2 dx \\ &+ \left(\frac{1}{2\eta_0}\left(\frac{\beta\gamma}{\alpha} - \varepsilon_2\right)^2 + \frac{\rho c_p}{2\alpha}\right)\int_0^l u_{tx}^2 dx, \quad \forall t > 0.\end{aligned} \tag{19}$$

Proof. Multiplying the second equation in (1) by ϕ, then integrating it by parts over $(0, l)$, and taking into account the boundary conditions $\phi(0, t) = \phi(l, t) = 0$, we obtain, for any $t > 0$,

$$\begin{aligned}\kappa\frac{d}{dt}\int_0^l \phi_t\phi dx - \kappa\int_0^l \phi_t^2 dx + \delta\int_0^l \phi_x^2 dx + \beta\int_0^l u_x\phi dx + \eta\int_0^l \phi^2 dx + \frac{\tau}{2}\frac{d}{dt}\int_0^l \phi^2 dx \\ + \varepsilon_2\int_0^l u_{xt}\phi dx = 0,\end{aligned}$$

which implies

$$\kappa\frac{d}{dt}\int_0^l \phi_t\phi dx+\frac{\tau}{2}\frac{d}{dt}\int_0^l \phi^2 dx=-\delta\int_0^l \phi_x^2 dx-\eta\int_0^l \phi^2 dx+\kappa\int_0^l \phi_t^2 dx \\ -\beta\int_0^l u_x\phi dx-\varepsilon_2\int_0^l u_{xt}\phi dx. \tag{20}$$

Integrating the first equation in (1) over $(0,x)$ and using the boundary conditions $u_x(0,t)=\phi(0,t)=0$, we obtain, for any $t>0$,

$$\rho\int_0^x u_{tt}(s)ds-\alpha u_x-\beta\phi-\gamma u_{xt}-\varepsilon_1\phi_t=0. \tag{21}$$

Multiplying (21) by ϕ and integrating over $(0,l)$, we obtain, for any $t>0$,

$$\rho\frac{d}{dt}\int_0^l \phi\int_0^x u_t(s)dsdx-\rho\int_0^l \phi_t\int_0^x u_t(s)dsdx-\alpha\int_0^l u_x\phi dx-\beta\int_0^l \phi^2 dx \\ -\gamma\int_0^l u_{xt}\phi dx-\frac{\varepsilon_1}{2}\frac{d}{dt}\int_0^l \phi^2 dx=0. \tag{22}$$

Multiplying (22) by $\frac{\beta}{\alpha}$, we end up with

$$\frac{\beta\rho}{\alpha}\frac{d}{dt}\int_0^l \phi\int_0^x u_t(s)dsdx-\frac{\beta\varepsilon_1}{2\alpha}\frac{d}{dt}\int_0^l \phi^2 dx=\frac{\beta\rho}{\alpha}\int_0^l \phi_t\int_0^x u_t(s)dsdx+\beta\int_0^l u_x\phi dx \\ +\frac{\beta^2}{\alpha}\int_0^l \phi^2 dx+\frac{\beta\gamma}{\alpha}\int_0^l u_{xt}\phi dx. \tag{23}$$

The addition of (20) and (23) gives, for any $t>0$,

$$\frac{d}{dt}\underbrace{\left(\kappa\int_0^l \phi_t\phi dx+\frac{\beta\rho}{\alpha}\int_0^l \phi\int_0^x u_t(s)dsdx+\frac{1}{2}\left(\tau-\frac{\beta\varepsilon_1}{\alpha}\right)\int_0^l \phi^2 dx\right)}_{=\mathcal{Q}_2(t)} \\ =-\delta\int_0^l \phi_x^2 dx-\left(\eta-\frac{\beta^2}{\alpha}\right)\int_0^l \phi^2 dx+\kappa\int_0^l \phi_t^2 dx \\ +\underbrace{\left(\frac{\beta\gamma}{\alpha}-\varepsilon_2\right)\int_0^l u_{xt}\phi dx}_{f_6}+\underbrace{\frac{\beta\rho}{\alpha}\int_0^l \phi_t\int_0^x u_t(s)dsdx}_{f_7}. \tag{24}$$

Using the fact that $\alpha\eta>\beta^2$, we have $\eta_0=\eta-\frac{\beta^2}{\alpha}>0$. Therefore, proceeding similarly as in the proof of Lemma 2, while estimating the functions f_2 and f_3, we obtain, for any $t>0$,

$$f_6(t)\le\frac{\eta_0}{2}\int_0^l \phi^2 dx+\frac{1}{2\eta_0}\left(\frac{\beta\gamma}{\alpha}-\varepsilon_2\right)^2\int_0^l u_{xt}^2 dx$$
$$f_7(t)\le\frac{\beta^2\rho}{2\alpha}\int_0^l \phi_t^2 dx+\frac{\rho c_p}{2\alpha}\int_0^l u_{xt}^2 dx.$$

By replacing f_6 and f_7 in (24) with the above estimates, we obtain (19). □

Lemma 4. *The functional $\mathcal{Q}_3$ given by*

$$\mathcal{Q}_3(t):=\rho\int_0^l u_t u dx+\frac{\gamma}{2}\int_0^l u_x^2 dx+\varepsilon_1\int_0^l u_x\phi dx,\quad \forall t>0,$$

satisfies

$$\frac{d}{dt}\mathcal{Q}_3(t) \leq -\frac{\alpha}{2}\int_0^l u_x^2 dx + \left(\rho c_p + \frac{\varepsilon_1}{2}\right)\int_0^l u_{xt}^2 dx + \left(\frac{\beta^2}{2\alpha} + \frac{\varepsilon_1}{2}\right)\int_0^l \phi^2 dx, \quad \forall t > 0. \tag{25}$$

Proof. Multiplying the first equation in (1) by u, we have

$$\rho u_{tt}u - \alpha u_{xx}u - \beta\phi_x u - \gamma u_{xxt}u - \varepsilon_1\phi_{xt}u = 0, \qquad \forall t > 0.$$

Now, integrating by parts over $(0,l)$ and using the boundary conditions $u_x(0,t) = u_x(l,t) = \phi(0,t) = \phi(l,t) = 0$, we obtain, for any $t > 0$,

$$\begin{aligned}
&\rho\int_0^l u_{tt}u dx + \alpha\int_0^l u_x^2 dx + \beta\int_0^l \phi u_x dx + \gamma\int_0^l u_x u_{xt} dx + \varepsilon_1\int_0^l u_x\phi_t dx = 0\\
&\rho\frac{d}{dt}\int_0^l u_t u dx - \rho\int_0^l u_t^2 dx + \alpha\int_0^l u_x^2 dx + \beta\int_0^l \phi u_x dx + \frac{\gamma}{2}\frac{d}{dt}\int_0^l u_x^2 dx\\
&\qquad + \varepsilon_1\frac{d}{dt}\int_0^l u_x\phi dx - \varepsilon_1\int_0^l u_{xt}\phi dx = 0,
\end{aligned}$$

which implies

$$\begin{aligned}
\frac{d}{dt}\underbrace{\left(\rho\int_0^l u_t u dx + \frac{\gamma}{2}\int_0^l u_x^2 dx + \varepsilon_1\int_0^l u_x\phi dx\right)}_{\mathcal{Q}_3(t)} &= -\alpha\int_0^l u_x^2 dx + \rho\int_0^l u_t^2 dx\\
&\quad \underbrace{-\beta\int_0^l u_x\phi dx}_{f_8} + \underbrace{\varepsilon_1\int_0^l u_{xt}\phi dx}_{f_9}.
\end{aligned} \tag{26}$$

Using Young's inequality, we have, for any $t > 0$,

$$\begin{aligned}
f_8(t) &\leq \frac{\alpha}{2}\int_0^l u_x^2 dx + \frac{\beta^2}{2\alpha}\int_0^l \phi^2 dx\\
f_9(t) &\leq \frac{\varepsilon_1}{2}\int_0^l u_{xt}^2 dx + \frac{\varepsilon_1}{2}\int_0^l \phi^2 dx.
\end{aligned}$$

Using the above estimates as well as Poincaré's inequality, we establish (25), and so the proof is complete. □

3. Exponential Stability

The following is our exponential stability result.

Theorem 1. *Suppose that $4\gamma\tau > (\varepsilon_1 + \varepsilon_2)^2$ and $\alpha\eta > \beta^2$. Then, the energy $\mathcal{E}(t)$ of the system (1) decays exponentially. In other words, there exist two positive constants k_0 and k_1 such that the energy functional given by (6) satisfies*

$$\mathcal{E}(t) \leq k_0\exp(-k_1 t), \quad \forall t > 0. \tag{27}$$

Proof. We define a Lyapunov functional (which is a linear combination of the functionals defined in the previous section)

$$\mathcal{L}(t) := \mathcal{N}\mathcal{E}(t) + \sum_{i=1}^{3}\mathcal{N}_i\mathcal{Q}_i(t), \quad \forall t > 0, \tag{28}$$

where $\mathcal{N}$ and $\mathcal{N}_i$, $i = 1\cdots 3$, are positive constants to be appropriately chosen later on in the proof. By differentiating (28) and using (9), (14), (19) and (25), we have, for any $t > 0$,

$$\begin{aligned}
\frac{d}{dt}\mathcal{L}(t) \leq & -\delta\mathcal{N}_2\int_0^l \phi_x^2 dx - \frac{\alpha}{2}\mathcal{N}_3\int_0^l u_x^2 dx \\
& -\left[\frac{\varepsilon_1^2}{2}\mathcal{N}_1 - \left(\kappa + \frac{\beta^2\rho}{2\alpha}\right)\mathcal{N}_2\right]\int_0^l \phi_t^2 dx - \left[\frac{\eta_0}{2}\mathcal{N}_2 - \delta_1\mathcal{N}_1 - \left(\frac{\beta^2}{2\alpha} + \frac{\varepsilon_1}{2}\right)\mathcal{N}_3\right]\int_0^l \phi^2 dx \\
& -\Bigg[\gamma_0\mathcal{N} - \left(\frac{1}{2\eta_0}\left(\frac{\beta\gamma}{\alpha} - \varepsilon_2\right)^2 + \frac{\rho c_p}{2\alpha}\right)\mathcal{N}_2 - \left(\rho c_p + \frac{\varepsilon_1}{2}\right)\mathcal{N}_3 \\
& \quad - \left(\gamma^2 + \frac{|\varepsilon_1\varepsilon_2|\rho c_p}{\kappa} + \frac{\varepsilon_1^2}{2\delta_1}\left(\alpha - \frac{\rho\delta}{\kappa}\right)^2 + \frac{\varepsilon_1^2\rho^2\eta^2 l c_p}{2\delta_1\kappa^2} + \frac{\rho^2 l c_p}{\kappa^2}\tau^2\right)\mathcal{N}_1\Bigg]\int_0^l u_{xt}^2 dx.
\end{aligned} \tag{29}$$

We let $\mathcal{N}_3 = 1$, and take $\mathcal{N}_2$ large enough so that

$$c_0 := \frac{\eta_0}{2}\mathcal{N}_2 - \left(\frac{\beta^2}{2\alpha} + \frac{\varepsilon_1}{2}\right)\mathcal{N}_3 > 0.$$

Next, we choose $\mathcal{N}_1$ large enough so that

$$c_1 := \frac{\varepsilon_1^2}{2}\mathcal{N}_1 - \left(\kappa + \frac{\beta^2\rho}{2\alpha}\right)\mathcal{N}_2 > 0$$

and, then, let

$$\delta_1 = \frac{c_0}{2\mathcal{N}_1}.$$

Thus, we have, for any $t > 0$,

$$\frac{d}{dt}\mathcal{L}(t) \leq -c_2\int_0^l \phi_x^2 dx - c_3\int_0^l u_x^2 dx - c_1\int_0^l \phi_t^2 dx - \frac{c_0}{2}\int_0^l \phi^2 dx - \left[\gamma_0\mathcal{N} - c_4\right]\int_0^l u_{xt}^2 dx, \tag{30}$$

where

$$\begin{aligned}
c_2 :=& \delta\mathcal{N}_2 > 0, \\
c_3 :=& \frac{\alpha}{2}\mathcal{N}_3 > 0, \\
c_4 :=& \left(\frac{1}{2\eta_0}\left(\frac{\beta\gamma}{\alpha} - \varepsilon_2\right)^2 + \frac{\rho c_p}{2\alpha}\right)\mathcal{N}_2 + \left(\rho c_p + \frac{\varepsilon_1}{2}\right)\mathcal{N}_3 \\
& + \left(\gamma^2 + \frac{|\varepsilon_1\varepsilon_2|\rho c_p}{\kappa} + \frac{\varepsilon_1^2}{2\delta_1}\left(\alpha - \frac{\rho\delta}{\kappa}\right)^2 + \frac{\varepsilon_1^2\rho^2\eta^2 l c_p}{2\delta_1\kappa^2} + \frac{\rho^2 l c_p}{\kappa^2}\tau^2\right)\mathcal{N}_1 > 0.
\end{aligned}$$

On the other hand, from (28), we have, for any $t > 0$,

$$\begin{aligned}
|\mathcal{L}(t) - \mathcal{N}\mathcal{E}(t)| \leq & \mathcal{N}_1\left|\alpha\varepsilon_1\int_0^l u_x\phi dx - \rho\varepsilon_1\int_0^l \phi_t\int_0^x u_t(s)ds dx + \frac{\beta\varepsilon_1}{2}\int_0^l \phi^2 dx\right. \\
& \left. + \frac{\rho\beta\varepsilon_1}{2\kappa}\int_0^l u^2 dx\right| + \mathcal{N}_3\left|\rho\int_0^l u_t u dx + \frac{\gamma}{2}\int_0^l u_x^2 dx + \varepsilon_1\int_0^l u_x\phi dx\right| \\
& + \mathcal{N}_2\left|\kappa\int_0^l \phi_t\phi dx + \frac{\beta\rho}{\alpha}\int_0^l \phi\int_0^x u_t(s)ds dx + \frac{1}{2}\left(\tau - \frac{\beta\varepsilon_1}{\alpha}\right)\int_0^l \phi^2 dx\right|.
\end{aligned}$$

Using Young's, Cauchy-Schwarz, and Poincaré's inequalities, we obtain, for any $t > 0$,

$$
\begin{aligned}
|\mathcal{L}(t)-\mathcal{N}\mathcal{E}(t)| \leq & \left[\frac{\alpha|\varepsilon_1|}{2}\mathcal{N}_1+\frac{\rho|\beta\varepsilon_1|c_p}{2\kappa}\mathcal{N}_1+\frac{\rho c_p}{2}\mathcal{N}_3+\frac{\gamma}{2}\mathcal{N}_3+\frac{|\varepsilon_1|}{2}\mathcal{N}_3\right]\int_0^l u_x^2 dx \\
& +\left[\frac{\alpha|\varepsilon_1|}{2}\mathcal{N}_1+\frac{|\beta\varepsilon_1|}{2}\mathcal{N}_1+\frac{|\varepsilon_1|}{2}\mathcal{N}_3+\frac{\kappa}{2}\mathcal{N}_2+\frac{|\beta|\rho}{2\alpha}\mathcal{N}_2+\frac{1}{2}\left|\tau-\frac{\beta\varepsilon_1}{\alpha}\right|\right]c_p\int_0^l \phi_x^2 dx \\
& +\left[\frac{\rho|\varepsilon_1|}{2}\mathcal{N}_1+\frac{\kappa}{2}\mathcal{N}_2\right]\int_0^l \phi_t^2 dx+\left[\frac{\rho|\varepsilon_1|}{2}\mathcal{N}_1+\frac{\rho}{2}\mathcal{N}_3+\frac{|\beta|\rho}{2\alpha}\mathcal{N}_2\right]\int_0^l u_t^2 dx \\
\leq & k\int_0^l\left(u_x^2+\phi_x^2+\phi_t^2+u_t^2\right)dx,
\end{aligned}
$$

where the constant $k>0$ can be taken as

$$
\begin{aligned}
k= & \frac{\alpha|\varepsilon_1|}{2}\mathcal{N}_1+\frac{\rho|\beta\varepsilon_1|c_p}{2\kappa}\mathcal{N}_1+\frac{\rho c_p}{2}\mathcal{N}_3+\frac{\gamma}{2}\mathcal{N}_3+\frac{|\varepsilon_1|}{2}\mathcal{N}_3+\frac{\alpha|\varepsilon_1|c_p}{2}\mathcal{N}_1+\frac{|\beta\varepsilon_1|c_p}{2}\mathcal{N}_1+\frac{|\varepsilon_1|c_p}{2}\mathcal{N}_3 \\
& +\frac{\kappa c_p}{2}\mathcal{N}_2+\frac{|\beta|\rho c_p}{2\alpha}\mathcal{N}_2+\frac{c_p}{2}\left|\tau-\frac{\beta\varepsilon_1}{\alpha}\right|+\frac{\rho|\varepsilon_1|}{2}\mathcal{N}_1+\frac{\kappa}{2}\mathcal{N}_2+\frac{\rho|\varepsilon_1|}{2}\mathcal{N}_1+\frac{\rho}{2}\mathcal{N}_3+\frac{|\beta|\rho}{2\alpha}\mathcal{N}_2.
\end{aligned}
$$

Using (8), it is obvious that

$$
\int_0^l u_t^2 dx \leq \frac{2}{\rho}\mathcal{E}(t), \quad \int_0^l \phi_x^2 dx \leq \frac{2}{\delta}\mathcal{E}(t), \quad \int_0^l \phi_t^2 dx \leq \frac{2}{\kappa}\mathcal{E}(t), \quad \int_0^l u_x^2 dx \leq \frac{2}{\alpha-\dfrac{\beta^2}{\eta}}\mathcal{E}(t).
$$

Consequently, we have, for any $t>0$,

$$
|\mathcal{L}(t)-\mathcal{N}\mathcal{E}(t)| \leq a_0\mathcal{E}(t), \qquad a_0>0,
$$

which implies

$$
(\mathcal{N}-a_0)\mathcal{E}(t) \leq \mathcal{L}(t) \leq (\mathcal{N}+a_0)\mathcal{E}(t), \qquad \forall t>0. \tag{31}
$$

Finally, we choose $\mathcal{N}$ large enough so that

$$
c_5 := \gamma_0\mathcal{N}-c_4>0 \quad \text{and} \quad c_6 := \mathcal{N}-a_0>0.
$$

Thus, for some positive constants σ_1 and σ_2, the following equivalence relation holds

$$
\sigma_1\mathcal{E}(t) \leq \mathcal{L}(t) \leq \sigma_2\mathcal{E}(t), \quad \forall t>0. \tag{32}
$$

Moreover, referring to (30), we obtain, for any $t>0$,

$$
\frac{d}{dt}\mathcal{L}(t) \leq -c_2\int_0^l \phi_x^2 dx - c_3\int_0^l u_x^2 dx - c_1\int_0^l \phi_t^2 dx - \frac{c_0}{2}\int_0^l \phi^2 dx - c_5\int_0^l u_{xt}^2 dx. \tag{33}
$$

Using Poincaré's inequality, we have

$$
-c_5\int_0^l u_{xt}^2 dx \leq -\frac{c_5}{c_p}\int_0^l u_t^2 dx.
$$

Accordingly, we end up with

$$
\frac{d}{dt}\mathcal{L}(t) \leq -\nu\int_0^l\left(u_t^2+u_x^2+\phi_t^2+\phi_x^2+\phi^2\right)dx, \quad \forall t>0, \tag{34}
$$

for some positive constant ν. Meanwhile, by considering (6) and using Young's inequality, we obtain

$$\mathcal{E}(t) \leq \frac{1}{2}\int_0^l \left[\rho u_t^2 + \left(\alpha + \frac{\beta^2}{2}\right)u_x^2 + \kappa\phi_t^2 + \delta\phi_x^2 + \left(\eta + \frac{1}{2}\right)\phi^2\right]dx, \quad \forall t > 0.$$

Letting $c_7 := \rho + \left(\alpha + \frac{\beta^2}{2}\right) + \kappa + \delta + \left(\eta + \frac{1}{2}\right) > 0$, we have

$$\mathcal{E}(t) \leq c_7 \int_0^l \left[u_t^2 + u_x^2 + \phi_t^2 + \phi_x^2 + \phi^2\right]dx, \quad \forall t > 0. \tag{35}$$

Consequently, from (34) and (35), we have, for some $a_1 > 0$

$$\frac{d}{dt}\mathcal{L}(t) \leq -a_1\mathcal{E}(t), \quad \forall t > 0.$$

Using the equivalence relation (32), we obtain

$$\frac{d}{dt}\mathcal{L}(t) \leq -k_1\mathcal{L}(t), \quad \forall t > 0. \tag{36}$$

Simple integration of (36), as well as the application of (32), yields the desired exponential stability result (27). □

Remark 3. *If the condition $4\gamma\tau > (\varepsilon_1 + \varepsilon_2)^2$ is replaced with $2\gamma\tau \geq (\varepsilon_1 + \varepsilon_2)^2$, then the energy functional $\mathcal{E}$ statisfies*

$$\frac{d}{dt}\mathcal{E}(t) \leq -\gamma_0 \int_0^l u_{xt}^2 dx, \quad \forall t > 0.$$

Consequently, the exponential stability result (27) also holds when $2\gamma\tau = (\varepsilon_1 + \varepsilon_2)^2$.

4. General Comments and Open Problems

This last section gives some comments and highlights some open problems. The result in this paper completes the result obtained by Quintanilla and Ueda [3] for unbounded domains. Our exponential decay result also holds when $2\gamma\tau = (\varepsilon_1 + \varepsilon_2)^2$; however, we do not know whether or not it is valid for the case of $4\gamma\tau = (\varepsilon_1 + \varepsilon_2)^2$. Júnior et al. [2] established an optimal polynomial stability result for the case $\varepsilon_1 = \varepsilon_2$, which is equivalent to the result obtained by Quintanilla and Ueda [3] for the same case but unbounded domains. An interesting open problem is establishing an optimal polynomial stability result for the general case $4\gamma\tau = (\varepsilon_1 + \varepsilon_2)^2$. More importantly, it is interesting to investigate the system for multidimensional cases, perhaps with nonlinear terms corresponding to the Navier–Stokes equations. Some numerical analysis could also be carried out to illustrate some of the results. Other open problems include:

(a) The case when $\gamma = 0, \tau > 0$ is an interesting problem to investigate. It would not be easy to obtain an exponential stability result; perhaps setting $\varepsilon_1 = \varepsilon_2$ might help. This is the same for the case $\gamma > 0, \tau = 0$.

(b) The case when the term $\tau\phi_t$ is nonlinear, that is, $\tau g(\phi_t)$, is also an interesting problem to consider.

(c) Another interesting problem is to consider the more general system proposed by Munoz et al. [1]

$$\begin{cases} \rho u_{tt} - \alpha u_{xx} - \beta\phi_x - \gamma u_{xxt} - \varepsilon_1\phi_{xt} - d_1\phi_{xx} - b_1\phi_{xxt} = 0, \\ \kappa\phi_{tt} - \delta\phi_{xx} + \beta u_x + \eta\phi + \tau\phi_t + \varepsilon_2 u_{xt} - d_1 u_{xx} - k\phi_{xxt} - b_2 u_{xxt} - \mu\phi_{xt} = 0, \\ u(x,0) = u_0(x),\ u_t(x,0) = u_1(x),\ \phi(x,0) = \phi_0(x),\ \phi_t(x,0) = \phi_1(x), \\ u(0,t) = u(l,t) = \phi(0,t) = \phi(l,t) = 0. \end{cases} \tag{37}$$

The necessary assumption to guarantee the positivity of the internal energy of system (37) is

$$\alpha > \frac{\beta^2}{\eta} + \frac{d_1^2}{\delta}. \tag{38}$$

In addition, the following assumption

$$4\gamma > \frac{(\varepsilon_1 + \varepsilon_2)^2}{\tau} + \frac{(b_1 + b_2)^2}{k} \tag{39}$$

assures the dissipation of the energy. The inequality $2\gamma \geq \frac{(\varepsilon_1 + \varepsilon_2)^2}{\tau} + \frac{(b_1 + b_2)^2}{k}$ could also be considered instead of (39), see Remark 3.

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