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A New Stochastic Order of Multivariate Distributions: Application in the Study of Reliability of Bridges Affected by Earthquakes

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Abstract: In this article, we introduce and study a new stochastic order of multivariate distributions, namely, the conditional likelihood ratio order. The proposed order and other stochastic orders are analyzed in the case of a bivariate exponential distributions family. The theoretical results obtained are applied for studying the reliability of bridges affected by earthquakes. The conditional likelihood ratio order involves the multivariate stochastic ordering; it resembles the likelihood ratio order in the univariate case but is much easier to verify than the likelihood ratio order in the multivariate case. Additionally, the likelihood ratio order in the multivariate case implies this ordering. However, the conditional likelihood ratio order does not imply the weak hard rate order, and it is not an order relation on the multivariate distributions set. The new conditional likelihood ratio order, together with the likelihood ratio order and the weak hazard rate order, were studied in the case of the bivariate Marshall–Olkin exponential distributions family, which has a lack of memory type property. At the end of the paper, we also presented an application of the analyzed orderings for this bivariate distributions family to the study of the effects of earthquakes on bridges.

Keywords: stochastic order; multivariate distribution; reliability; exponential distribution; earthquake

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1. Introduction

Over time, the study of people's life expectancy and the study of the reliability of certain devices or constructions represent important research topics. There are many random phenomena that can influence the expected human life. An important factor that can impact the chance of life is the likelihood that a person will die suddenly after the age of t years. This is the hazard rate at the time t. The same problem arises in the study of reliability of a device or a construction.

It is well known (see, for example, Shaked and Shanthikumar [1]) that the hazard rate order between two random variables X and Y is equivalent to the same stochastic order between the conditional random variables $(X\,|X > t)$ and $(Y\,|Y > t)$ for all the values $t \in \mathbb{R}$.

These inequalities are applied to study the reliability of series and parallel systems. We can mention several research works that have approached this problem. Fang and Balakrishnan [2] compared the likelihood ratio order of the largest-order statistics by using a particular Weibull random variables. Fang and Balakrishnan [3] obtained results regarding some stochastic orders of the smallest and largest-order statistics in the case of an exponential-Weibull random variables family. Khaledi and Kochar [4] studied the stochastic order of extreme-order statistics in the case of the Weibull distribution. Balakrishnan and Torrado [5] and Balakrishnan et al. [6] analyzed stochastic comparisons between the largest-order statistics in the particular case of independent and identically distributed random variables and studied the usual stochastic, hazard rate, reversed hazard rate, likelihood ratio, and dispersive and star orders, concluding that all these stochastic orders are preserved in the case of parallel systems using exponentiated models for the lifetimes of components. Chen et al. [7] studied the ordering properties of extreme-order statistics arising from independent negative binomial random variables. Wang and Zhu [8] proposed and analyzed the water quality redundancy/reliability method based on information entropy technology, including Tsallis entropy and Shannon entropy in water distribution system. Triantafyllou [9] delivered a reliability study of the weighted-r-consecutive-k-out-of-n: F reliability systems consisting of independent and identically distributed components. Rykov et al. [10] studied the reliability characteristics of a k-out-of-n: F system in the case when a failure of one of the system's components leads to increasing the load on others. Montoro-Cazorla and Pérez-Ocón [11] studied an N-system with different units submitted to shock and wear.

Multivariate generalization takes place in the case of the hazard rate function and the stochastic order in the hazard rate sense. For example, we can study the stochastic order between conditional random variables $(X \,|X > t)$ and $(Y \,|Y > t)$ for all $t \in \mathbb{R}^d$. This order is the multivariate weak hazard rate order. A multivariate stochastic ordering involving the multivariate weak hazard rate order ordering is the multivariate likelihood ratio order. However, the study of the multivariate likelihood ratio order is a difficult task. The main objective of this paper is to introduce a new multivariate order relationship that involves the univariate likelihood ratio order, can be easily checked, and implies the multivariate stochastic order .

Another important research direction in this article is to prove the importance and utility of this multivariate order relationship in real applications.

One of the main objectives of seismology is to predict when and where an earthquake will occur after another earthquake. Although it does not seem to be an exact answer yet, there are probabilistic models that describe this phenomenon quite accurately. Abe and Suzuki [12] proved that the probability distribution of the time interval $\triangle t$ between successive earthquakes is approximately equal to $e_q(-\beta_t \cdot \triangle t)$, where β_t is a scale constant. Wang and Chang [13] obtained the probability function of seismic hazard and then compared it to the model prediction that uses the Poisson distribution. Kayid et al. [14] introduced the proportional reversed hazards weighted frailty model and analyzed some stochastic orders of this model. Catana [15] gave necessary and sufficient conditions of some stochastic orders in the case of the multivariate Pareto distribution family. By using the hazard function, Quintela-del-Río [16] analyzed the distribution of earthquake occurrences in two regions of Spain. Catana [17] studied the necessary and sufficient conditions for some multivariate stochastic orders of Jones–Larsen distribution and gave applications in the study of earthquakes. Dias et al. [18] studied the influence of considering partial sets of earthquake data on the temporal and spatial probability distributions of earthquakes, using data from the California region between 2003 and 2016, with different thresholds for the magnitude and depth of hypocenters. Catana and Raducan [19] gave sufficient conditions for the stochastic order of multivariate uniform distributions on closed convex sets.

If we want to calculate the probability of the next earthquake after another that occurred at a time interval $\triangle t$ in an area that may be affected more or less by other earthquakes originating in other areas, then we can use an multivariate exponential distribution.

For this distribution, we know, for example, $P(X_1 \geq t_1, X_2 \geq t_2)$. In this case, we can characterize the variation of the function $\mathbb{R}^2 \ni (t_1, t_2) \longmapsto P(X_1 \geq t_1, X_2 \geq t_2)$ when the parameters of this distribution vary.

However, how we can find $\max_{(X_1,X_2)} Eu(X_1,X_2)$ where u is an utility function? (A utility function is an increasing function).

One method that can be used in case it is difficult to calculate $Eu(X_1,X_2)$ is based-von using multivariate stochastic orders. Even so, it is still difficult to determine if $Eu(X_1,X_2) \le Eu(Y_1,Y_2)$ for any u utility function depending on the parameters of the distributions. The multivariate likelihood ratio order implies stochastic order (see Shaked and Shanthikumar [1]). It is more difficult to verify the likelihood order ratio in the multivariate case of absolute continuity according to the Lebesgue measure distributions, compared to the univariate case . It is necessary to find a new ordering that involves the stochastic ordering in the multivariate case.

The paper is structured as follows. Section 2 includes the preliminaries. In Section 3, we introduce and study the new stochatic order of multivariate distributions. Sections 4–6 discuss the new stochastic order, likelihood ratio order, and weak hazard rate order of the bivariate Marshall–Olkin exponential distributions family. Section 7 includes an application of the new stochastic order for the study of the reliability of bridges affected by earthquakes. The last section presents the conclusions.

2. Preliminaries

We present some theoretical notions related to the main distribution families used and some important definitions and results regarding multivariate stochastic orders.

Let $(\Omega,\mathcal{F},P)$ be a probability space. For a random vector $X:\Omega\to\mathbb{R}^d$ $(d\ge 2)$, we consider

$$\mu_X(B) = P(X\in B)$$

its distribution on $\left(\mathbb{R}^d,\mathcal{B}(\mathbb{R}^d)\right)$ and we denote by

$$F_X(x) = P(X\le x) = P(X_1\le x_1,\ldots,X_d\le x_d)$$

its distribution function. We also consider

$$F^*_X(x) = P(X\ge x,\ X\ne x).$$

For a function $g:\mathbb{R}^d\to\mathbb{R}$, we denote by $Supp(g) = \left\{x\in\mathbb{R}^d : g(x)\ne 0\right\}$ the support of the function g.

If μ_X is absolutely continuous according to the Lebesgue measure, we denote:

$$f_X(x) = \frac{\partial^d}{\partial x_1\ldots\partial x_d}F_X(x)$$

its density function and

$$r_X : Supp(F^*_X)\to\mathbb{R},\ r_X(x) = \nabla(-\ln(F^*_X(x)))$$

the hazard rate function.

Additionally, we denote by $pr_i(x_1,\ldots,x_d) = x_i$ and $pr_{(i_1,\ldots,i_k)}(x_1,\ldots,x_d) = \left(x_{i_1},\ldots,x_{i_k}\right)$ $(1\le i_1<i_2<\ldots<i_k\le d)$ the canonical projections.

The distribution that we will be used in this article is proposed by Marshall and Olkin (see Kotz et al. [20]), and it is given by

$$P(X\ge x,\ X\ne x) =$$

$$e^{-\sum\limits_{1\le i_1\le d}\lambda_{i_1}x_{i_1}-\sum\limits_{1\le i_1<i_2\le d}\lambda_{i_1 i_2}\max\left(x_{i_1},x_{i_2}\right)-\sum\limits_{1\le i_1\le i_2\le i_3\le d}\lambda_{i_1 i_2 i_3}\max\left(x_{i_1},x_{i_2},x_{i_3}\right)-\ldots-\lambda_{123\ldots d}\max\left(x_1,\ldots,x_d\right)}$$

where $x\in(0,\infty)^d$, $d\ge 2$ and $\lambda_{i_1\ldots i_m}\ge 0\ \forall\, m\in\{1,\ldots,d\}$.

For this distribution, it is verified that

$$P(X \geq x) = P(X \geq x,\ X \neq x)$$

where $x \in (0,\infty)^d$, $d \geq 2$.

This distribution verifies the lack of memory property (see Kotz et al. [20], p. 392):

$$P(X \geq x + (t,t,\ldots,t)) = P(X \geq x) \cdot P(X \geq (t,t,\ldots,t))\ \forall\ x \in (0,\infty)^d\ \forall\ t \in (0,\infty).$$

This distribution is very useful for the study of earthquakes. Its lack of memory property is important for the study of occurrence of earthquakes in a time interval $\triangle t$.

We now present the definition of the bivariate distribution proposed by Marshall–Olkin and an interesting property.

Definition 1 (Kotz et al. [20])**.** *We say that the positive bivariate random vector X is Marshall–Olkin Exponential distributed with parameters $\lambda_1, \lambda_2, \lambda_{12} \in (0,\infty)$ and denote this by $X \sim MOExp(\lambda_1, \lambda_2, \lambda_{12})$ if*

$$F_X^*(x_1,x_2) = e^{-\lambda_1 x_1 - (\lambda_2+\lambda_{12})x_2} \cdot 1_{\{y \in \mathbb{R}^2 : 0 < y_1 \leq y_2\}}(x_1,x_2) +$$
$$e^{-(\lambda_1+\lambda_{12})x_1 - \lambda_2 x_2} \cdot 1_{\{y \in \mathbb{R}^2 : 0 < y_2 < y_1\}}(x_1,x_2).$$

We have:

$$f_X(x_1,x_2) = \lambda_1(\lambda_2+\lambda_{12})e^{-\lambda_1 x_1 - (\lambda_2+\lambda_{12})x_2} \cdot 1_{\{y \in \mathbb{R}^2 : 0 < y_1 \leq y_2\}}(x_1,x_2) +$$
$$\lambda_2(\lambda_1+\lambda_{12})e^{-(\lambda_1+\lambda_{12})x_1 - \lambda_2 x_2} \cdot 1_{\{y \in \mathbb{R}^2 : 0 < y_2 < y_1\}}(x_1,x_2),$$
$$F_X(x_1,x_2) = \left(1 - e^{-\lambda_1 x_1}\right)\left(1 - e^{-(\lambda_2+\lambda_{12})x_2}\right) \cdot 1_{\{y \in \mathbb{R}^2 : 0 < y_1 \leq y_2\}}(x_1,x_2) +$$
$$\left(1 - e^{-(\lambda_1+\lambda_{12})x_1}\right)\left(1 - e^{-\lambda_2 x_2}\right) \cdot 1_{\{y \in \mathbb{R}^2 : 0 < y_2 < y_1\}}(x_1,x_2)$$

and

$$-\ln F_X^*(x_1,x_2) = [\lambda_1 x_1 + (\lambda_2+\lambda_{12})x_2] \cdot 1_{\{y \in \mathbb{R}^2 : 0 < y_1 \leq y_2\}}(x_1,x_2) +$$
$$[(\lambda_1+\lambda_{12})x_1 + \lambda_2 x_2] \cdot 1_{\{y \in \mathbb{R}^2 : 0 < y_2 < y_1\}}(x_1,x_2).$$

Then,

$$(r_X)_1(x) = \lambda_1 \cdot 1_{\{y \in \mathbb{R}^2 : 0 < y_1 \leq y_2\}}(x_1,x_2) + (\lambda_1+\lambda_{12}) \cdot 1_{\{y \in \mathbb{R}^2 : 0 < y_2 < y_1\}}(x_1,x_2),$$
$$(r_X)_2(x) = \lambda_2 \cdot 1_{\{y \in \mathbb{R}^2 : 0 < y_1 \leq y_2\}}(x_1,x_2) + (\lambda_2+\lambda_{12}) \cdot 1_{\{y \in \mathbb{R}^2 : 0 < y_2 < y_1\}}(x_1,x_2).$$

Proposition 1 (Kotz et al. [20])**.** *If $X \sim MOExp(\lambda_1, \lambda_2, \lambda_{12})$ then $X_i = \min(Z_i, Z_{12})$, $i \in \{1,2\}$, where $Z_1 \sim Exp(\lambda_1)$, $Z_2 \sim Exp(\lambda_2)$ and $Z_{12} \sim Exp(\lambda_{12})$ are independent random variables.*

Definition 2 (Dias et al. [18])**.** *The q-deformed Tsallis exponential function is $e_q : \mathbb{R} \to [0,\infty)$, $e_q(x) = [1 + (1-q)x]^{\frac{1}{1-q}} \cdot 1_{[0,\infty)}(1 + (1-q)x)$, where $q \in \mathbb{R}$.*

Proposition 2 describes the probability distribution of the time interval $\triangle t$ between successive earthquakes.

Proposition 2 (Abe and Suzuki [12]). *The probability distribution of the time interval $\triangle t$ between successive earthquakes is well adjusted by a q-exponential function and it is is approximated by*

$$e_q(-\beta_t \cdot \triangle t),$$

where β_t is a scale constant.

We now present the definitions of the basic multivariate orderings and some important properties.

Definition 3 (Shaked and Shanthikumar [1]). *We say that:*

(i) *a function $u : \mathbb{R}^d \to \mathbb{R}$ is increasing if*

$$\forall\, x, y \in \mathbb{R}^d,\, x \leq y \Longrightarrow u(x) \leq u(y).$$

(ii) *a set $C \subset \mathbb{R}^d$ is increasing if*

$$\forall\, x \in C\, \forall\, y \in \mathbb{R}^d \text{ then } x \leq y \Longrightarrow y \in C.$$

Definition 4 (Shaked and Shanthikumar [1]). *Let X and Y be two d-dimensional random vectors. We say that X is smaller than Y in the*

(i) *Stochastic order sense (written as $X \prec_{st} Y$) if*

$$P(X \in C) \leq P(Y \in C)\; \forall\, C \subset \mathbb{R}^d \text{ increasing set;}$$

(ii) *Strong stochastic order sense (written as $X \prec_{sst} Y$) if*

$$X_1 \prec_{st} Y_1$$

and

$$(X_i |\, X_1 = x_1, \ldots, X_{i-1} = x_{i-1}) \prec_{st} (Y_i |\, Y_1 = y_1, \ldots, Y_{i-1} = y_{i-1})$$
$$\forall\, i \in \{2, \ldots, d\}\; \forall\, x_i, y_i \in \mathbb{R}\; x_i \leq y_i.$$

(iii) *Weak hazard rate order sense (written as $X \prec_{whr} Y$) if*

$$r_X(x) \geq r_Y(y)\; \forall\, x \in \mathbb{R}^d;$$

(iv) *Likelihood ratio order sense (written as $X \prec_{lr} Y$) if*

$$f_X(x) \cdot f_Y(y) \leq f_X(\min(x, y)) \cdot f_Y(\max(x, y))\; \forall\, x, y \in Supp(f_X) \cup Supp(f_Y).$$

Theorem 1 (Shaked and Shanthikumar [1]). Let *X and Y be two d-dimensional random vectors. Then:*

(i) $X \prec_{sst} Y \Longrightarrow X \prec_{st} Y$;
(ii) $X \prec_{lr} Y \Longrightarrow X \prec_{whr} Y$;
(iii) $X \prec_{lr} Y \Longrightarrow X \prec_{st} Y$.

Theorem 2 (Shaked and Shanthikumar [1]). *Let X and Y be two d-dimensional random vectors. Then,*

$$X \prec_{st} Y \Longleftrightarrow Eu(X) \leq Eu(Y)\; \forall\, u : \mathbb{R}^d \to \mathbb{R} \text{ increasing.}$$

Definition 5 (Shaked and Shanthikumar [1], p. 290). *A function $K : \mathbb{R}^d \to (0, \infty)$ is said to be multivariate totally positive of order 2 (MTP_2) if*

$$K(x) \cdot K(y) \leq K(\min(x, y)) \cdot K(\max(x, y))\; \forall\, x, y \in \mathbb{R}^d.$$

Lemma 1 (Ruggeri et al. [21]). *A function* $K : \mathbb{R}^d \to (0,\infty)$ *with* 2 *continuous derivatives is* MTP_2 *if and only if*

$$\frac{\partial^2}{\partial x_i \partial x_j} \ln K(x) \geq 0 \; \forall\, x \in \mathbb{R}^d \; \forall\, i,j \in \{1,\ldots,d\} \; i \neq j.$$

Remark 1. *Let* X *and* Y *be two d-dimensional random vectors with* $Supp(X) = Supp(Y)$ *and* $Supp(X)$ *is lattice. Then* $X \prec_{lr} Y$ *if and only if the function* $x \longmapsto \frac{f_Y(x)}{f_X(x)}$ *is increasing on* $Supp(f_X)$ *and* f_X *or* f_Y *is* MTP_2*.*

3. The New Stochastic Order of Multivariate Distributions and Some Properties

We introduce the following order given by the definition:

Definition 6. *Let the d-dimensional random vector* X *and* Y*. We say that* X *is smaller in the conditional likelihood ratio order sense than* Y *(and we denote* $X \prec_{clr} Y$*) if*

$$X_1 \prec_{lr} Y_1$$

and

$$(X_i |\, X_1 = x_1, \ldots, X_{i-1} = x_{i-1}) \prec_{lr} (Y_i |\, Y_1 = y_1, \ldots, Y_{i-1} = y_{i-1})$$
$$\forall\, i \in \{2,\ldots,d\} \; \forall\, x_i, y_i \in \mathbb{R} \; x_i \leq y_i.$$

Proposition 3 describes an implication between the conditional likelihood ratio order and the multivariate stochastic order. Theorem 3 describes a a characterization of the conditional likelihood ratio. The conditions from this order are similar to those in the case of ordering the distributions on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$. In Proposition 4, a relationship between the multivariate likelihood ratio order and the conditional likelihood ratio order is established. Propositions 5, 6 and 7 describe some of the properties of the conditional likelihood ratio order.

Proposition 3. *Let the d-dimensional random vector* X *and* Y*. Then,* $X \prec_{clr} Y \Longrightarrow X \prec_{st} Y$*.*

Proof. Thus from Theorem 1. □

Theorem 3. *Let the d-dimensional random vectors* X *and* Y *with* μ_X *and* μ_Y *be absolutely continuous with respect to the Lebesgue measure. Then,* $X \prec_{clr} Y$ *if and only if*

$$t \longmapsto \frac{f_{Y_1}(t)}{f_{X_1}(t)} \text{ is increasing on } Supp(f_{X_1}) \cup Supp(f_{Y_1})$$

and

$$t \longmapsto \frac{f_{(Y_1,Y_2,\ldots,Y_{i-1},Y_i)}(y_1,y_2,\ldots,y_{i-1},t)}{f_{(X_1,X_2,\ldots,X_{i-1},X_i)}(x_1,x_2,\ldots,x_{i-1},t)} \text{ is increasing on}$$
$$pr_i\Big(Supp\Big(f_{(X_1,\ldots,X_i)}\Big) \cup Supp\Big(f_{(Y_1,\ldots,Y_i)}\Big)\Big)$$
$$\forall\, x,y \in pr_{(1,2,\ldots,i-1)}\Big(Supp\Big(f_{(X_1,\ldots,X_i)}\Big) \cup Supp\Big(f_{(Y_1,\ldots,Y_i)}\Big)\Big) \; x \leq y \; \forall\, i \in \{2,\ldots,d\}.$$

Proof. $X_1 \prec_{lr} Y_1 \Longleftrightarrow t \longmapsto \frac{f_{Y_1}(t)}{f_{X_1}(t)}$ is increasing on $Supp(f_{X_1}) \cup Supp(f_{Y_1})$.

Let $i \in \{2, \ldots, d\}$. For $x = (x_1, \ldots, x_{i-1}) \in pr_{(1,2,\ldots,i-1)} Supp\Big(f_{(X_1,\ldots,X_i)}\Big)$, the probability density funcion of $(X_i | \ X_1 = x_1, \ldots, X_{i-1} = x_{i-1})$ is

$$f_{(X_i | \ X_1 = x_1,\ldots,X_{i-1}=x_{i-1})}(t) = \frac{f_{(X_1,\ldots,X_{i-1},X_i)}(x_1, \ldots, x_{i-1}, t)}{f_{(X_1,\ldots,X_{i-1})}(x_1, \ldots, x_{i-1})}$$

Now, let $x, y \in pr_{(1,2,\ldots,i-1)}\Big(Supp\Big(f_{(X_1,\ldots,X_i)}\Big) \cup Supp\Big(f_{(Y_1,\ldots,Y_i)}\Big)\Big)$ with $x \leq y$. Then,

$$(X_i | \ X_1 = x_1, \ldots, X_{i-1} = x_{i-1}) \prec_{lr} (Y_i | \ Y_1 = y_1, \ldots, Y_{i-1} = y_{i-1}) \iff$$

$$t \longmapsto \frac{f_{(Y_i | \ X_1 = x_1,\ldots,X_{i-1}=x_{i-1})}(t)}{f_{(X_i | \ X_1 = x_1,\ldots,X_{i-1}=x_{i-1})}(t)} \text{ is increasing on}$$

$$pr_i\Big(Supp\Big(f_{(X_1,\ldots,X_i)}\Big) \cup Supp\Big(f_{(Y_1,\ldots,Y_i)}\Big)\Big) \iff$$

$$t \longmapsto \frac{f_{(Y_1,\ldots,Y_{i-1},Y_i)}(y_1, \ldots, y_{i-1}, t)}{f_{(X_1,\ldots,X_{i-1},X_i)}(x_1, \ldots, x_{i-1}, t)} \cdot \frac{f_{(X_1,\ldots,X_{i-1})}(x_1, \ldots, x_{i-1})}{f_{(Y_1,\ldots,Y_{i-1})}(y_1, \ldots, y_{i-1})}$$

$$\text{is increasing on } pr_i\Big(Supp\Big(f_{(X_1,\ldots,X_i)}\Big) \cup Supp\Big(f_{(Y_1,\ldots,Y_i)}\Big)\Big) \iff$$

$$t \longmapsto \frac{f_{(Y_1,\ldots,Y_{i-1},Y_i)}(y_1, \ldots, y_{i-1}, t)}{f_{(X_1,\ldots,X_{i-1},X_i)}(x_1, \ldots, x_{i-1}, t)} \text{ is increasing on}$$

$$pr_i\Big(Supp\Big(f_{(X_1,\ldots,X_i)}\Big) \cup Supp\Big(f_{(Y_1,\ldots,Y_i)}\Big)\Big).$$

□

Proposition 4. *Let the d-dimensional random vectors X and Y with μ_X and μ_Y be absolutely continuous with respect to the Lebesgue measure. Then, $X \prec_{lr} Y \Longrightarrow X \prec_{clr} Y$.*

Proof. $X \prec_{lr} Y \Longrightarrow X_1 \prec_{lr} Y_1$ and $X \prec_{lr} Y \Longrightarrow f_X(x) \cdot f_Y(y) \leq f_X(y) \cdot f_Y(x) \ \forall \ x, y \in Supp(f_X) \cup Supp(f_Y)$ with $y \leq x$

$$\Longrightarrow t \longmapsto \frac{f_Y(t)}{f_X(t)} \text{ is increasing on } Supp(f_X) \cup Supp(f_Y)$$

$$\Longrightarrow f_{(X_1,\ldots,X_{i-1},X_i)}(x_1, \ldots, x_{i-1}, s) f_{(Y_1,\ldots,Y_{i-1},Y_i)}(y_1, \ldots, y_{i-1}, t) \leq$$

$$f_{(X_1,\ldots,X_{i-1},X_i)}(x_1, \ldots, x_{i-1}, t) f_{(Y_1,\ldots,Y_{i-1},Y_i)}(y_1, \ldots, y_{i-1}, s)$$

$$i \in \{2, \ldots, d\} \ x_i \leq y_i \ t \leq s$$

$$\Longrightarrow \frac{f_{(Y_1,\ldots,Y_{i-1},Y_i)}(y_1, \ldots, y_{i-1}, t)}{f_{(X_1,\ldots,X_{i-1},X_i)}(x_1, \ldots, x_{i-1}, t)} \leq \frac{f_{(Y_1,\ldots,Y_{i-1},Y_i)}(y_1, \ldots, y_{i-1}, s)}{f_{(X_1,\ldots,X_{i-1},X_i)}(x_1, \ldots, x_{i-1}, s)}$$

$$\Longrightarrow t \longmapsto \frac{f_{(Y_1,\ldots,Y_{i-1},Y_i)}(y_1, \ldots, y_{i-1}, t)}{f_{(X_1,\ldots,X_{i-1},X_i)}(x_1, \ldots, x_{i-1}, t)} \text{ is increasing on}$$

$$pr_i\Big(Supp\Big(f_{(X_1,\ldots,X_i)}\Big) \cup Supp\Big(f_{(Y_1,\ldots,Y_i)}\Big)\Big)$$

$$\forall \ x, y \in pr_{(1,2,\ldots,i-1)} Supp\Big(f_{(X_1,\ldots,X_i)}\Big) \cup Supp\Big(f_{(Y_1,\ldots,Y_i)}\Big) \ x \leq y \ \forall \ i \in \{2, \ldots, d\}.$$

□

Proposition 5. *$\prec_{clr}$ is an antisymmetric order relationship on the multivariate random vectors set with their measure absolutely continuous according to the Lebesgue measure.*

Proof. Let the d-dimensional random vectors X and Y with $X \prec_{clr} Y$ and $Y \prec_{clr} X$.

$X_1 \prec_{lr} Y_1$ and $Y_1 \prec_{lr} X_1$. $\prec_{lr}$ is an antisymmetric order relationship on the random variables set; therefore,

$$f_{X_1} = f_{Y_1}.$$

Now, let $i \in \{2, \ldots, d\}$.

$t \longmapsto \frac{f_{(Y_1,\ldots,Y_{i-1},Y_i)}(x_1,\ldots,x_{i-1},t)}{f_{(X_1,\ldots,X_{i-1},X_i)}(x_1,\ldots,x_{i-1},t)}$ and $t \longmapsto \frac{f_{(X_1,\ldots,X_{i-1},X_i)}(x_1,\ldots,x_{i-1},t)}{f_{(Y_1,\ldots,Y_{i-1},Y_i)}(x_1,\ldots,x_{i-1},t)}$ are increasing on $pr_i Supp\Big(f_{(X_1,\ldots,X_i)}\Big) \cup Supp\Big(f_{(Y_1,\ldots,Y_i)}\Big)$.

Thus, $t \longmapsto \frac{f_{(Y_1,\ldots,Y_{i-1},Y_i)}(x_1,x_2,\ldots,x_{i-1},t)}{f_{(X_1,\ldots,X_{i-1},X_i)}(x_1,x_2,\ldots,x_{i-1},t)}$ is constant.

$$\frac{f_{(Y_1,\ldots,Y_{i-1},Y_i)}(x)}{f_{(X_1,\ldots,X_{i-1},X_i)}(x)} = k \ \forall\, x \in Supp\Big(f_{(X_1,\ldots,X_i)}\Big) \cup Supp\Big(f_{(Y_1,\ldots,Y_i)}\Big)$$

$$\Longrightarrow f_{(Y_1,\ldots,Y_{i-1},Y_i)}(x) = k f_{(X_1,\ldots,X_{i-1},X_i)}(x) \ \forall\, x \in Supp\Big(f_{(X_1,\ldots,X_i)}\Big) \cup Supp\Big(f_{(Y_1,\ldots,Y_i)}\Big)$$

$$\Longrightarrow \int_{\mathbb{R}^i} f_{(Y_1,\ldots,Y_{i-1},Y_i)} d\lambda^i = k \int_{\mathbb{R}^i} f_{(X_1,\ldots,X_{i-1},X_i)} d\lambda^i \Longrightarrow k = 1$$

$$\Longrightarrow f_{(Y_1,\ldots,Y_{i-1},Y_i)} = f_{(X_1,\ldots,X_{i-1},X_i)}$$

Therefore $\prec_{clr}$ is an antisymmetric order relationship on the multivariate random vectors set. □

Proposition 6. *$\prec_{clr}$ is a transitive order relationship on the multivariate random vectors set with their measure absolutely continuous according to the Lebesgue measure.*

Proof. $\prec_{lr}$ is a transitive-order relationship on the random variables set.

Let the d-dimensional random vectors X, Y and Z with $X \prec_{clr} Y$ and $Y \prec_{clr} Z$.

Then, $X_1 \prec_{lr} Y_1$ and $Y_1 \prec_{lr} Z_1$. Thus, $X_1 \prec_{lr} Z_1$.

Now, let $i \in \{2, \ldots, d\}$ and

$x, y, z \in pr_{(1,2,\ldots,i-1)}\Big(Supp\Big(f_{(X_1,\ldots,X_i)}\Big) \cup Supp\Big(f_{(Y_1,\ldots,Y_i)}\Big) \cup Supp\Big(f_{(Z_1,\ldots,Z_i)}\Big)\Big)$ with $x \leq y \leq z$.

$$(X_i|\, X_1 = x_1, \ldots, X_{i-1} = x_{i-1}) \prec_{lr} (Y_i|\, Y_1 = y_1, \ldots, Y_{i-1} = y_{i-1})$$

and

$$(Y_i|\, Y_1 = y_1, \ldots, Y_{i-1} = y_{i-1}) \prec_{lr} (Z_i|\, Z_1 = z_1, \ldots, Z_{i-1} = z_{i-1}).$$

Then,

$$(X_i|\, X_1 = x_1, \ldots, X_{i-1} = x_{i-1}) \prec_{lr} (Z_i|\, Z_1 = z_1, \ldots, Z_{i-1} = z_{i-1}).$$

Therefore, $\prec_{clr}$ is a transitive-order relationship on the multivariate random vectors set. □

Proposition 7. *Let the bivariate random vector X with μ_X be absolutely continuous with respect to the Lebesgue measure. Then, $X \prec_{clr} X$ if and only if f_X is MTP_2.*

Proof. $t \longmapsto \frac{f_{(Y_1,Y_2)}(y_1,t)}{f_{(X_1,X_2)}(x_1,t)}$ is increasing on $pr_2\Big(Supp\Big(f_{(X_1,X_2)}\Big) \cup Supp\Big(f_{(Y_1,Y_2)}\Big)\Big) \ \forall\, x, y \in \mathbb{R}$ $x \leq y$.

Thus,

$$\frac{f_{(Y_1,Y_2)}(y_1,t)}{f_{(X_1,X_2)}(x_1,t)} \leq \frac{f_{(Y_1,Y_2)}(y_1,s)}{f_{(X_1,X_2)}(x_1,s)} \; \forall\, x,y,t,s \in \mathbb{R}\; x \leq y \; t \leq s$$

$$\Longleftrightarrow f_{(X_1,X_2)}(x_1,s) f_{(Y_1,Y_2)}(y_1,t) \leq f_{(X_1,X_2)}(x_1,t) f_{(Y_1,Y_2)}(y_1,s) \; \forall\, x,y,t,s \in \mathbb{R}\; x \leq y \; t \leq s$$

$$\Longleftrightarrow f_{(X_1,X_2)}(x_1,s) f_{(Y_1,Y_2)}(y_1,t) \leq f_{(X_1,X_2)}(\min((x_1,s),(y_1,t))) f_{(Y_1,Y_2)}(\max((x_1,s),(y_1,t)))$$

$$\forall\, (x_1,s),(y_1,t) \in \mathbb{R}^2$$

Thus, $X \prec_{clr} X$ if and only if f_X is MTP_2. □

Remark 2. *$\prec_{clr}$ is not a reflexive relationship on the multivariate random vectors set.*

4. Conditional Likelihood Ratio Order of the Marshall–Olkin Exponential Distributions Family

Theorem 4 gives necessary and sufficient conditions for the conditional likelihood ratio order of the bivariate Marshall–Olkin exponential distributions family.

Theorem 4. *Let, $X \sim MOExp(\alpha_1, \alpha_2, \alpha_{12})$ and $Y \sim MOExp(\beta_1, \beta_2, \beta_{12})$. Then, $X \prec_{clr} Y$ if and only if $\alpha_i + \alpha_{12} \geq \beta_i + \beta_{12}\; \forall\, i \in \{1,2\}$ and $\alpha_2 \geq \beta_2$.*

Proof. $\frac{f_{Y_1}(t)}{f_{X_1}(t)} = e^{[(\alpha_1+\alpha_{12})-(\beta_1+\beta_{12})]t}$, $t \in (0,\infty)$.

Then, $X_1 \prec_{lr} Y_1$ if and only if $\alpha_1 + \alpha_{12} \geq \beta_1 + \beta_{12}$.

Now, let $x_1, y_1 \in pr_1\Big(Supp\Big(f_{(X_1,X_2)}\Big) \cup Supp\Big(f_{(Y_1,Y_2)}\Big)\Big)$ with $x_1 \leq y_1$. Then,

$$\frac{f_{(Y_1,Y_2)}(y_1,t)}{f_{(X_1,X_2)}(x_1,t)} =$$

$$\frac{\beta_1(\beta_2+\beta_{12})e^{-\beta_1 y_1-(\beta_2+\beta_{12})t}\cdot 1_{\{s\in\mathbb{R}^2:0<s_1\leq s_2\}}(y_1,t) + \beta_2(\beta_1+\beta_{12})e^{-(\beta_1+\beta_{12})y_1-\beta_2 t}\cdot 1_{\{s\in\mathbb{R}^2:0<s_2<s_1\}}(y_1,t)}{\alpha_1(\alpha_2+\alpha_{12})e^{-\alpha_1 x_1-(\alpha_2+\alpha_{12})t}\cdot 1_{\{s\in\mathbb{R}^2:0<s_1\leq s_2\}}(x_1,t) + \alpha_2(\alpha_1+\alpha_{12})e^{-(\alpha_1+\alpha_{12})x_1-\alpha_2 t}\cdot 1_{\{y\in\mathbb{R}^2:0<s\in\mathbb{R}^2:0<s_2<s_1\}}(x_1,t)}$$

If $(y_1,t) \in \{s \in \mathbb{R}^2 : 0 < s_1 \leq s_2\}$ then $x_1 \leq y_1 \leq t$.

Thus, $(x_1,t) \in \{s \in \mathbb{R}^2 : 0 < s_1 \leq s_2\}$.

Thus,

$$\frac{f_{(Y_1,Y_2)}(y_1,t)}{f_{(X_1,X_2)}(x_1,t)} = \frac{\beta_1(\beta_2+\beta_{12})e^{-\beta_1 y_1-(\beta_2+\beta_{12})t}}{\alpha_1(\alpha_2+\alpha_{12})e^{-\alpha_1 x_1-(\alpha_2+\alpha_{12})t}} = \frac{\beta_1(\beta_2+\beta_{12})}{\alpha_1(\alpha_2+\alpha_{12})}\cdot e^{(\alpha_1 x_1-\beta_1 y_1)+[(\alpha_2+\alpha_{12})-(\beta_2+\beta_{12})]t}$$

and in this case, $t \longmapsto \frac{f_{(Y_1,Y_2)}(y_1,t)}{f_{(X_1,X_2)}(x_1,t)}$ is increasing on $[y_1,\infty)\; \forall\, 0 < x_1 \leq y_1$ if and only if $\alpha_2 + \alpha_{12} \geq \beta_2 + \beta_{12}$.

If $(y_1,t) \notin \{s \in \mathbb{R}^2 : 0 < s_1 \leq s_2\}$ then $t < y_1$.

Case 1: $t < x_1 \leq y_1$

Then, $(x_1,t) \in \{s \in \mathbb{R}^2 : 0 < s_2 < s_1\}$

Thus,

$$\frac{f_{(Y_1,Y_2)}(y_1,t)}{f_{(X_1,X_2)}(x_1,t)} = \frac{\beta_2(\beta_1+\beta_{12})e^{-(\beta_1+\beta_{12})y_1-\beta_2 t}}{\alpha_2(\alpha_1+\alpha_{12})e^{-(\alpha_1+\alpha_{12})x_1-\alpha_2 t}} = \frac{\beta_2(\beta_1+\beta_{12})}{\alpha_2(\alpha_1+\alpha_{12})}\cdot e^{[(\alpha_1+\alpha_{12})x_1-(\beta_1+\beta_{12})y_1]+(\alpha_2-\beta_2)t}$$

and in this case $t \longmapsto \frac{f_{(Y_1,Y_2)}(y_1,t)}{f_{(X_1,X_2)}(x_1,t)}$ is increasing on $(-\infty,x_1)\; \forall\, 0 < x_1 \leq y_1$ if and only if $\alpha_2 \geq \beta_2$.

Case 2: $x_1 \leq t < y_1$

Then, $(x_1,t) \in \{s \in \mathbb{R}^2 : 0 < s_1 \leq s_2\}$

Thus,

$$\frac{f_{(Y_1,Y_2)}(y_1,t)}{f_{(X_1,X_2)}(x_1,t)} = \frac{\beta_2(\beta_1+\beta_{12})e^{-(\beta_1+\beta_{12})y_1-\beta_2 t}}{\alpha_1(\alpha_2+\alpha_{12})e^{-\alpha_1 x_1-(\alpha_2+\alpha_{12})t}} = \frac{\beta_2(\beta_1+\beta_{12})}{\alpha_1(\alpha_2+\alpha_{12})} \cdot e^{[\alpha_1 x_1-(\beta_1+\beta_{12})y_1]+[(\alpha_2+\alpha_{12})-\beta_2]t}$$

and in this case $t \longmapsto \frac{f_{(Y_1,Y_2)}(y_1,t)}{f_{(X_1,X_2)}(x_1,t)}$ is increasing on $[x_1,y_1)\ \forall\, 0 < x_1 \leq y_1$ because $\alpha_2+\alpha_{12} > \alpha_2 \geq \beta_2$.

Thus, $t \longmapsto \frac{f_{(Y_1,Y_2)}(y_1,t)}{f_{(X_1,X_2)}(x_1,t)}$ is increasing on $pr_2\Big(Supp\Big(f_{(X_1,X_2)}\Big) \cup Supp\Big(f_{(Y_1,Y_2)}\Big)\Big)$ if and only if $\alpha_2 \geq \beta_2$ and $\alpha_2+\alpha_{12} \geq \beta_2+\beta_{12}$. □

5. Likelihood Ratio Order of the Marshall–Olkin Exponential Distributions Family

Theorem 5 describes a property of the probability density function of a Marshall–Olkin Exponential distribution and Theorem 6 gives necessary and sufficient conditions for multivariate likelihood ratio order of the bivariate Marshall–Olkin exponential distributions family.

Theorem 5. *Let $X \sim MOExp(\alpha_1,\alpha_2,\alpha_{12})$. Then, f_X is MTP_2.*

Proof. We have $\frac{\partial^2}{\partial x_1 \partial x_2} \ln f_X(x) = 0\ \forall\ x \in \mathbb{R}^2$. From Lemma 1, it follows that f_X is MTP_2. □

Theorem 6. *Let $X \sim MOExp(\alpha_1,\alpha_2,\alpha_{12})$ and $Y \sim MOExp(\beta_1,\beta_2,\beta_{12})$. Then $X \prec_{lr} Y$ if and only if $\alpha_i \geq \beta_i$ and $\alpha_i+\alpha_{12} \geq \beta_i+\beta_{12}\ \forall\, i \in \{1,2\}$.*

Proof. f_X and f_Y are MTP_2. From Remark 1, it follows that

$$X \prec_{lr} Y \Longleftrightarrow (x_1,x_2) \longmapsto \frac{f_Y(x_1,x_2)}{f_X(x_1,x_2)} \text{ is increasing on } (0,\infty)^2.$$

Now, we have

$$\frac{f_Y(x_1,x_2)}{f_X(x_1,x_2)} = \frac{\beta_1(\beta_2+\beta_{12})}{\alpha_1(\alpha_2+\alpha_{12})} \cdot e^{(\alpha_1-\beta_1)x_1-(\alpha_2+\alpha_{12}-\beta_2-\beta_{12})x_2} \cdot 1_{\{y\in\mathbb{R}^2:0<y_1\leq y_2\}}(x_1,x_2)+$$
$$\frac{\beta_2(\beta_1+\beta_{12})}{\alpha_2(\alpha_1+\alpha_{12})} \cdot e^{(\alpha_1+\alpha_{12}-\beta_1-\beta_{12})x_1-(\alpha_2-\beta_2)x_2} \cdot 1_{\{y\in\mathbb{R}^2:0<y_2<y_1\}}(x_1,x_2).$$

However,
$(x_1,x_2) \longmapsto \frac{f_Y(x_1,x_2)}{f_X(x_1,x_2)}$ is increasing on $\{y \in \mathbb{R}^2 : 0 < y_1 \leq y_2\}$ and on $\{y \in \mathbb{R}^2 : 0 < y_2 < y_1\}$
is equivalent to

$$\alpha_i \geq \beta_i \text{ and } \alpha_i+\alpha_{12} \geq \beta_i+\beta_{12}\ \forall\, i \in \{1,2\}.$$

Obviously, if $(x_1,x_2) \longmapsto \frac{f_Y(x_1,x_2)}{f_X(x_1,x_2)}$ is increasing on $\{y \in \mathbb{R}^2 : 0 < y_2 < y_1\}$ then, from the continuity of this function on $(0,\infty)^2$, it is increasing on $\{y \in \mathbb{R}^2 : 0 < y_2 \leq y_1\}$.

If $(x_1,x_2) \in \{y \in \mathbb{R}^2 : 0 < y_1 \leq y_2\}$ and $(t_1,t_2) \in \{y \in \mathbb{R}^2 : 0 < y_2 < y_1\}$ with $(x_1,x_2) \leq (t_1,t_2)$, then there exists the point

$$(s_1,s_2) \in \Big\{y \in \mathbb{R}^2 : 0 < y_1 = y_2\Big\}$$

with

$$(x_1,x_2) \leq (s_1,s_2) \leq (t_1,t_2).$$

Thus,
$$\frac{f_Y(x_1,x_2)}{f_X(x_1,x_2)} \le \frac{f_Y(s_1,s_2)}{f_X(s_1,s_2)} \le \frac{f_Y(t_1,t_2)}{f_X(t_1,t_2)}.$$
Similarly, we prove that if $(x_1,x_2) \in \{y \in \mathbb{R}^2 : 0 < y_2 < y_1\}$ and $(t_1,t_2) \in \{y \in \mathbb{R}^2 : 0 < y_1 \le y_2\}$ with $(x_1,x_2) \le (t_1,t_2)$ then $\frac{f_Y(x_1,x_2)}{f_X(x_1,x_2)} \le \frac{f_Y(t_1,t_2)}{f_X(t_1,t_2)}$.

Therefore, $(x_1,x_2) \longmapsto \frac{f_Y(x_1,x_2)}{f_X(x_1,x_2)}$ is increasing on $(0,\infty)^2$ if and only if $\alpha_i \ge \beta_i$ and $\alpha_i + \alpha_{12} \ge \beta_i + \beta_{12}\ \forall\, i \in \{1,2\}$.

It follows that $X \prec_{lr} Y \Longleftrightarrow \alpha_i \ge \beta_i$ and $\alpha_i + \alpha_{12} \ge \beta_i + \beta_{12}\ \forall\, i \in \{1,2\}$. □

Remark 3. *From Proposition 4, Theorem 4, and Theorem 6 we have that* $X \prec_{clr} Y \not\Leftrightarrow X \prec_{lr} Y$.

6. Weak Hazard Rate Order of the Marshall–Olkin Exponential Distributions Family

Theorem 7 describes the multivariate weak hazard rate order of the bivariate Marshall–Olkin exponential distributions family.

Theorem 7. *Let* $X \sim MOExp(\alpha_1,\alpha_2,\alpha_{12})$ *and* $Y \sim MOExp(\beta_1,\beta_2,\beta_{12})$. *Then,* $X \prec_{whr} Y \Longleftrightarrow \alpha_i \ge \beta_i$ *and* $\alpha_i + \alpha_{12} \ge \beta_i + \beta_{12}\ \forall\, i \in \{1,2\}$.

Proof. Let us suppose that $X \prec_{whr} Y$.

We have
$$\alpha_1 \cdot 1_{\{y\in\mathbb{R}^2:0<y_1\le y_2\}}\left(n,n^2\right) + (\alpha_1+\alpha_{12}) \cdot 1_{\{y\in\mathbb{R}^2:0<y_2<y_1\}}\left(n,n^2\right) \ge$$
$$\beta_1 \cdot 1_{\{y\in\mathbb{R}^2:0<y_1\le y_2\}}\left(n,n^2\right) + (\beta_1+\beta_{12}) \cdot 1_{\{y\in\mathbb{R}^2:0<y_2<y_1\}}\left(n,n^2\right),$$
$$\alpha_1 \cdot 1_{\{y\in\mathbb{R}^2:0<y_1\le y_2\}}\left(n^2,n\right) + (\alpha_1+\alpha_{12}) \cdot 1_{\{y\in\mathbb{R}^2:0<y_2<y_1\}}\left(n^2,n\right) \ge$$
$$\beta_1 \cdot 1_{\{y\in\mathbb{R}^2:0<y_1\le y_2\}}\left(n^2,n\right) + (\beta_1+\beta_{12}) \cdot 1_{\{y\in\mathbb{R}^2:0<y_2<y_1\}}\left(n^2,n\right),$$
$$\alpha_2 \cdot 1_{\{y\in\mathbb{R}^2:0<y_1\le y_2\}}\left(n,n^2\right) + (\alpha_2+\alpha_{12}) \cdot 1_{\{y\in\mathbb{R}^2:0<y_2<y_1\}}\left(n,n^2\right) \ge$$
$$\beta_2 \cdot 1_{\{y\in\mathbb{R}^2:0<y_1\le y_2\}}\left(n,n^2\right) + (\beta_2+\beta_{12}) \cdot 1_{\{y\in\mathbb{R}^2:0<y_2<y_1\}}\left(n,n^2\right)$$
and
$$\alpha_2 \cdot 1_{\{y\in\mathbb{R}^2:0<y_1\le y_2\}}\left(n^2,n\right) + (\alpha_2+\alpha_{12}) \cdot 1_{\{y\in\mathbb{R}^2:0<y_2<y_1\}}\left(n^2,n\right) \ge$$
$$\beta_2 \cdot 1_{\{y\in\mathbb{R}^2:0<y_1\le y_2\}}\left(n^2,n\right) + (\beta_2+\beta_{12}) \cdot 1_{\{y\in\mathbb{R}^2:0<y_2<y_1\}}\left(n^2,n\right), \text{ where } n \in \mathbb{Z},\ n \ge 2.$$
Thus, $\alpha_i \ge \beta_i$ and $\alpha_i + \alpha_{12} \ge \beta_i + \beta_{12}\ \forall\, i \in \{1,2\}$.

Let us prove the converse.

From $\alpha_i \ge \beta_i$ and $\alpha_i + \alpha_{12} \ge \beta_i + \beta_{12}\ \forall\, i \in \{1,2\}$, it follows that
$$\alpha_1 \cdot 1_{\{y\in\mathbb{R}^2:0<y_1\le y_2\}}(x_1,x_2) + (\alpha_1+\alpha_{12}) \cdot 1_{\{y\in\mathbb{R}^2:0<y_2<y_1\}}(x_1,x_2) \ge$$
$$\beta_1 \cdot 1_{\{y\in\mathbb{R}^2:0<y_1\le y_2\}}(x_1,x_2) + (\beta_1+\beta_{12}) \cdot 1_{\{y\in\mathbb{R}^2:0<y_2<y_1\}}(x_1,x_2)\ \forall\, x \in \mathbb{R}^2$$
and
$$\alpha_2 \cdot 1_{\{y\in\mathbb{R}^2:0<y_1\le y_2\}}(x_1,x_2) + (\alpha_2+\alpha_{12}) \cdot 1_{\{y\in\mathbb{R}^2:0<y_2<y_1\}}(x_1,x_2) \ge$$
$$\beta_2 \cdot 1_{\{y\in\mathbb{R}^2:0<y_1\le y_2\}}(x_1,x_2) + (\beta_2+\beta_{12}) \cdot 1_{\{y\in\mathbb{R}^2:0<y_2<y_1\}}(x_1,x_2)\ \forall\, x \in \mathbb{R}^2$$
Thus,
$$r_X(x) \ge r_Y(y)\ \forall\, x \in Supp(F_X^*) \cap Supp(F_Y^*).$$
□

Remark 4. *From Theorem 4 and Theorem 7 we have that* $X \prec_{clr} Y \nRightarrow X \prec_{whr} Y$.

7. Application in the Study of Reliability of Bridges Affected by Earthquakes

Suppose that we analyze the reliability of two bridges that cross each of the two areas with significant seismic risk. It is obvious that there is an area of each bridge that can be affected by earthquakes in both areas of the bridge, but there are also areas that can be affected only by earthquakes in that area. The risk of the first (respectively, the second) bridge collapsing suddenly due to significant damage caused by an earthquake after a period $t \geq 0$ from the construction of the bridge is represented by the bivariate vector $X \sim MOExp(\alpha_1, \alpha_2, \alpha_{12})$ (respectively, $Y \sim MOExp(\beta_1, \beta_2, \beta_{12})$).

$Eu(X_1, X_2)$ is increasing when $\alpha_i + \alpha_{12}$ is decreasing for all $i \in \{1,2\}$, and only α_2 is decreasing. Thus, from Theorem 4 we have that if $\alpha_i + \alpha_{12} \geq \beta_i + \beta_{12}$ for all $i \in \{1,2\}$ and $\alpha_2 \geq \beta_2$, then $X \prec_{clr} Y$, thus $X \prec_{st} Y$. We can interpret that the second bridge will last longer.

From Theorem 7, we have that the risk of the bridge collapsing is increasing when $\alpha_i + \alpha_{12}$ and α_i are decreasing for all $i \in \{1,2\}$.

Now, we have the parameters of the bivariate Marshall–Olkin exponential distribution in Table 1, which describes the lifetime of a bridge. The parameters were randomly generated in the interval $(0,1)$ and will be used to illustrate the application of multivariate parametric inequalities for estimating the probability of an earthquake after another earthquake.

Table 1. The lifetime of a bridge.

Name of the Bridge	α_1	α_2	α_3
Golden Gate bridge	0.32	0.26	0.21
Brooklyn bridge	0.25	0.28	0.22
London bridge	0.40	0.32	0.28
Sunshine Skyway bridge	0.21	0.23	0.22
Williamsburg bridge	0.24	0.45	0.31
Bixby Creek bridge	0.23	0.40	0.39
New River Gorge bridge	0.22	0.21	0.24

Let us consider $X_i \sim MOExp(\alpha_1, \alpha_2, \alpha_{12})$, where $\alpha_1, \alpha_2, \alpha_{12}$ are the values from the $(i+1)$-th row of the table, $i \in \{1, 2, \ldots, 7\}$.

Then, we have:

$X_3 \prec_{whr} X_1 \prec_{whr} X_2 \prec_{whr} X_4$; $X_5 \prec_{whr} X_6 \prec_{whr} X_4$; $X_6 \prec_{whr} X_7$

but $X_1 \nprec_{whr} X_7$; $X_4 \nprec_{whr} X_7$;

Therefore, among the Golden Gate, Brooklyn, London, and Sunshine Skyway bridges and after a $t > 0$ time interval, the London bridge has the highest probability of collapsing suddenly due to significant damage caused by an earthquake and the Sunshine Skyway bridge has the the lowest probability.

Additionally, among the Bixby Creek and New River Gorge bridges and after a $t > 0$ period, the Bixby Creek bridge has the higher probability of collapsing suddenly due to significant damage caused by an earthquake than the New River Gorge bridge.

Between the Golden Gate and New River Gorge bridges, it cannot be determined which probability of collapsing suddenly due to significant damage caused by an earthquake after a $t > 0$ period is greater;

$X_4 \prec_{lr} X_2 \prec_{lr} X_1 \prec_{lr} X_3$; $X_4 \prec_{lr} X_6 \prec_{lr} X_5$; $X_7 \prec_{lr} X_6$

but $X_1 \nprec_{lr} X_7$; $X_4 \nprec_{lr} X_7$;

These order relations imply that

$X_4 \prec_{clr} X_2 \prec_{clr} X_1 \prec_{clr} X_3$; $X_4 \prec_{clr} X_6 \prec_{clr} X_5$; $X_7 \prec_{clr} X_6$; $X_1 \nprec_{lr} X_7$; $X_4 \nprec_{lr} X_7$.

Additionally, $X_4 \nprec_{clr} X_7$ and $X_7 \nprec_{clr} X_4$.

8. Conclusions

In this article, we proposed and studied a new stochastic order for multivariate distributions. The study was focused on absolutely continuous distributions according to the Lebesgue measure. This new order, namely, the conditional likelihood ratio order, involves the multivariate stochastic ordering; it resembles the likelihood ratio order in the univariate case but is much easier to verify than the likelihood ratio order in the multivariate case. Additionally, the likelihood ratio order in the multivariate case implies this ordering. However, the conditional likelihood ratio order does not imply the weak hard rate order, and it is not an order relation on the multivariate distributions set.

The new conditional likelihood ratio order, together with the likelihood ratio order and the weak hazard rate order, were studied in the case of the bivariate Marshall–Olkin exponential distributions family, which has a lack of memory type property. At the end of the article, we also presented an application of the analyzed orderings for this bivariate distributions family to the study of the effects of earthquakes on bridges. New research directions opened by the present approach include the study of the conditional likelihood ratio order for other distributions families, together with new applications, including economics and finance domains.

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