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On Certain Classes of Multivalent Analytic Functions Defined with Higher-Order Derivatives

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Abstract: This paper examines two subclasses of multivalent analytic functions defined with higher-order derivatives. These classes of functions are generalizations of several known subclasses that have been studied in separate works. Moreover, we find several interesting results for functions in these classes, including subordination results, containment relations, and integral preserving properties.

Keywords: higher-order derivatives; multivalent functions; α-convex functions; α-close to convex; n-symmetrical points

MSC: 30C45; 30C55

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1. Introduction

Let $\mathcal{A}_p$ denote the family of analytic functions $\digamma$ be defined in the open unit disc $\mathbb{U} = \{z : |z| < 1\}$ of the complex plane $\mathbb{C}$ with the following form:

$$\digamma(z) = z^p + \sum_{l=p+1}^{\infty} a_l z^l, \tag{1}$$

where $p \in \mathbb{N} = \{1, 2, \ldots\}$. Additionally, let $\mathcal{A}_1 := \mathcal{A}$. If there exists a Schwarz function ω analytic in $\mathbb{U}$ with $\omega(0) = 0$ and $|\omega(z)| < 1$, such that $\digamma(z) = g(\omega(z))$, then we say that the function $\digamma$ is subordinate to g in $\mathbb{U}$, expressed as $\digamma(z) \prec g(z)$, (or simply $\digamma \prec g$). The subordination is identical to $\digamma(0) = g(0)$ and $\digamma(\mathbb{U}) \subset g(\mathbb{U})$ if the function g is univalent in $\mathbb{U}$. A function $\digamma \in \mathcal{A}_p$ is said to be in the class $S^*_{p,k}(\alpha)$ if it satisfies the inequality

$$\Re \frac{z\digamma^{(k+1)}(z)}{(p-k)\digamma^{(k)}(z)} > \alpha \ (0 \le \alpha < 1, z \in \mathbb{U}), \tag{2}$$

where $p > k$, $p \in \mathbb{N}$, and $k \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}$. Additionally, A function $\digamma \in \mathcal{A}_p$ is said to be in the class $K_{p,k}(\alpha)$ if it satisfies the following inequality

$$\Re \frac{\left(z\digamma^{(k+1)}(z)\right)'}{(p-k)\digamma^{(k+1)}(z)} > \alpha, \ (0 \le \alpha < 1, z \in \mathbb{U}). \tag{3}$$

The classes $S^*_{p,k}(\alpha)$ and $K_{p,k}(\alpha)$ were introduced and studied by Nunokawa [1] and Srivastava et al. [2] (see also [3–9]). We note that, $S^*_{1,0}(\alpha) \cong S^*(\alpha)$ and $K_{1,0}(\alpha) \cong C(\alpha)$, where $S^*(\alpha)$ and $C(\alpha)$ are the well known families of starlike and convex functions of order $\alpha (0 \le \alpha < 1)$, respectively, introduced by Robertson [10]. It is assumed in the sequel that φ is an analytic and convex function with a positive real part in the open unit disc $\mathbb{U}$, satisfies $\varphi(0) = 1, \varphi'(0) > 0$, and $\varphi(\mathbb{U})$ is symmetrical with respect to the real axis. In [11], Ali et al.

defined the classes $S^n_{p,k}(\varphi)$ and $C^n_{p,k}(\varphi)$ consist, respectively, of Ma–Minda type starlike and convex p-valent functions $\digamma$ with higher-order derivatives given by

$$S_{p,k}(\varphi) = \left\{ \digamma \in \mathcal{A}_p : \frac{z\digamma^{(k+1)}(z)}{(p-k)\digamma^{(k)}(z)} \prec \varphi(z) \quad (z \in \mathbb{U}) \right\}$$

and

$$C_{p,k}(\varphi) = \left\{ \digamma \in \mathcal{A}_p : \frac{\left(z\digamma^{(k+1)}(z)\right)'}{(p-k)\digamma^{(k+1)}(z)} \prec \varphi(z) \quad (z \in \mathbb{U}) \right\}.$$

Here, we introduce the class $S^{\alpha}_{p,k}(\varphi)$, which unifies the classes $S_{p,k}(\varphi)$ and $C_{p,k}(\varphi)$ as follows:

Definition 1. *Denote by $S^{\alpha}_{p,k}(\varphi)$ the family of functions $\digamma \in \mathcal{A}_p$ satisfying the following condition*

$$\frac{1}{p-k} \frac{\alpha z\left(z\digamma^{(k+1)}(z)\right)' + (1-\alpha)z\digamma^{(k+1)}(z)}{\alpha z\digamma^{(k+1)}(z) + (1-\alpha)\digamma^{(k)}(z)} \prec \varphi(z) \quad (\alpha \geq 0, z \in \mathbb{U}). \tag{4}$$

Remark 1. *1.* $S^0_{p,k}(\frac{1+(1-2\beta)z}{1-z}) \cong S_{p,k}(\varphi)$, *and* $S^1_{p,k}(\frac{1+(1-2\beta)z}{1-z}) \cong C_{p,k}(\varphi)$,

2. $S^{\alpha}_{p,0}(\frac{1+(1-2\beta)z}{1-z}) \cong T(p,\alpha,\beta)$ *(see Wang et al. [12]),*
3. $S^{\alpha}_{1,0}(\frac{1+(1-2\beta)z}{1-z}) \cong K(\beta)$, *where $K(\beta)$ is the class of β-starlike functions introduced by Pascu and Podaru [13].*

A function $\digamma \in \mathcal{A}$ is said to be starlike with respect to symmetrical points in $\mathbb{U}$ if it satisfies,

$$\Re\left\{ \frac{z\digamma'(z)}{\digamma(z) - \digamma(-z)} \right\} > 0 \quad (z \in \mathbb{U}).$$

Sakaguchi [14] introduced and studied this class. In addition, Shanmugam et al. [15], Lashin [16], Khan et al. [17], and Mahmood et al. [18] have studied some related classes.

For a given positive integer n, let

$$\begin{aligned} \digamma_n(z) \quad : &= \frac{1}{n}\sum_{v=0}^{n-1} \varepsilon^{-vp}\digamma(\varepsilon^v z) \\ &= z^p + a_{p+n}z^{p+n} + a_{p+2n}z^{p+2n} + \ldots \left(\varepsilon = \exp(\frac{2\pi i}{n}), z \in \mathbb{U}\right). \end{aligned} \tag{5}$$

Let $S^n_{p,k}(\varphi)$ be the class of functions $\digamma \in \mathcal{A}_p$ satisfying

$$\frac{z\digamma^{(k+1)}(z)}{(p-k)\digamma_n^{(k)}(z)} \prec \varphi(z) \quad (z \in \mathbb{U}).$$

Also, let $C^n_{p,k}(\varphi)$ be the class of functions $\digamma \in \mathcal{A}_p$ satisfying

$$\frac{\left(z\digamma^{(k+1)}(z)\right)'}{(p-k)\digamma_n^{(k+1)}(z)} \prec \varphi(z) \quad (z \in \mathbb{U}).$$

The classes $S^n_{p,0}(\varphi) \cong S^n_p(\varphi)$ of p-valent starlike functions with respect to n-symmetric points and $C^n_{p,0}(\varphi) \cong C^n_p(\varphi)$ of p-valent convex functions with respect to n-symmetric points were recently introduced and studied by Ali et al. [19]. Moreover, the classes

$S^n_{1,0}(\varphi) \cong S^n_s(\varphi)$ and $C^n_{1,0}(\varphi) \cong C^n_s(\varphi)$, which were studied by Miller and Mocanu ([20] page 314) and Wang et al. [21]. Following them, many authors discussed these classes and its subclasses (see [22–32]).
The following class $S^{n,\alpha}_{p,k}(\varphi)$ unifies the two above classes $S^n_{p,k}(\varphi)$ and $C^n_{p,k}(\varphi)$.

Definition 2. *Let F_n be the family of functions defined by (5). By $S^{n,\alpha}_{p,k}(\varphi)$, we denote the family of functions $F \in \mathcal{A}_p$ satisfying*

$$\frac{1}{p-k}\frac{\alpha z\left(zF^{(k+1)}(z)\right)' + (1-\alpha)zF^{(k+1)}(z)}{\alpha zF_n^{(k+1)}(z) + (1-\alpha)F_n^{(k)}(z)} \prec \varphi(z) \quad (\alpha \geq 0; z \in \mathbb{U}). \tag{6}$$

Remark 2. *With the appropriate selection of p, k, n, α, and φ in Definition 2, the following known subclasses are obtained.*

1. $S^{n,0}_{p,k}(\varphi) \cong S^n_{p,k}(\varphi)$ and $S^{n,1}_{p,k}(\varphi) \cong C^n_{p,k}(\varphi)$,
2. $S^{n,0}_{1,0}(\varphi) \equiv S^n_s(\varphi)$ and $S^{n,1}_{1,0}(\varphi) \equiv C^n_s(\varphi)$,
3. The class $S^{n,\alpha}_{1,0}(\varphi)$ is equivalent to the class $K_n(\alpha,\varphi)$ of α-starlike functions with respect to n-symmetric points introduced by Paravatham and Radha [33],
4. If we put $n = 2, \alpha = 0$, and $\varphi = \frac{1+z}{1-z}$ then $S^{2,0}_{1,0}(\frac{1+z}{1-z})$ is equivalent to the class S_s of starlike functions with respect to the symmetrical points introduced by Sakaguchi [14].

Definition 3. *A function $F \in \mathcal{A}_p$ is said to be α-close to convex of higher order with respect to n-symmetric points if it satisfies,*

$$\frac{1}{p-k}\frac{\alpha z\left(zF^{(k+1)}(z)\right)' + (1-\alpha)zF^{(k+1)}(z)}{\alpha z\varsigma_n^{(k+1)}(z) + (1-\alpha)\varsigma_n^{(k)}(z)} \prec \varphi(z) \quad (\alpha \geq 0; z \in \mathbb{U}),$$

where $\varsigma_n(z) = \frac{1}{n}\sum_{v=0}^{n-1}\omega^{-pv}\varsigma(\omega^v z)$ with $\varsigma(z) \in S^{n,\alpha}_{p,k}(\varphi)$. We denote this class by $K^{n,\alpha}_{p,k}(\varphi)$.

Using techniques involving differential subordination, we examine some interesting subordination criteria and inclusion relations, as well as integral operators for functions belonging to the class $S^{n,\alpha}_{p,k}(\varphi)$. In addition, we discuss some properties of functions belonging to the class $K^{n,\alpha}_{p,k}(\varphi)$.

2. Preliminary Lemmas and Some Properties of $S^{\alpha}_{p,k}(\varphi)$ and $S_{p,k}(\varphi)$

The two lemmas below are often used in our subsequent investigations.

Lemma 1 ([33])**.** *Let α, σ any two complex numbers, and φ be convex and univalent in $\mathbb{U}$ with $\varphi(0) = 1$ and $\Re[\alpha\varphi(z) + \sigma] > 0$. Also let Q analytic in $\mathbb{U}$ with $Q(0) = 1$ and $Q(z) \prec \varphi(z)$. If $\rho = 1 + \rho_1 z + \ldots$, analytic in $\mathbb{U}$, then*

$$\rho(z) + \frac{z\rho'(z)}{\alpha Q(z) + \sigma} \prec \varphi(z) \quad \Rightarrow \rho(z) \prec \varphi(z). \; (z \in \mathbb{U}).$$

Lemma 2 ([34])**.** *Let α, σ any two complex numbers, and φ be convex and univalent in $\mathbb{U}$ with $\varphi(0) = 1$ and $\Re[\alpha\varphi(z) + \sigma] > 0$. If $\rho = 1 + \rho_1 z + \ldots$, analytic in $\mathbb{U}$, then*

$$\rho(z) + \frac{z\rho'(z)}{\alpha\rho(z) + \sigma} \prec \varphi(z) \quad \Rightarrow \rho(z) \prec \varphi(z) \; (z \in \mathbb{U}).$$

Proposition 1. *Let $F \in \mathcal{A}_p$ and $\Re\frac{1}{\alpha}[(p-k)\alpha\varphi(z)+1-\alpha] > 0$. If $F \in S^{\alpha}_{p,k}(\varphi)$, then $F \in S_{p,k}(\varphi)$.*

Proof. Let

$$h(z) := \frac{zF^{(k+1)}(z)}{(p-k)zF^{(k)}(z)}, \tag{7}$$

then the function h is of the form $h(z) = 1 + h_1 z + \dots$. By taking the derivatives in the both sides of (7), we obtain

$$\frac{1}{p-k}\frac{\alpha z\left(zF^{(k+1)}(z)\right)' + (1-\alpha)zF^{(k+1)}(z)}{\alpha zF^{(k+1)}(z) + (1-\alpha)F^{(k)}(z)} = h(z) + \frac{\alpha z h'(z)}{(p-k)\alpha h(z) + 1 - \alpha}.$$

If we apply Lemma 2 with $\beta = p-k$ and $\delta = \frac{1-\alpha}{\alpha}$, then

$$\frac{zF^{(k+1)}(z)}{(p-k)zF^{(k)}(z)} = h(z) \prec \varphi(z).$$

□

Proposition 2. *Let $\Re(c) > -p$ and $\Re[(p-k)\varphi(z)+k+c] > 0$. Also let $F \in \mathcal{A}_p$ and*

$$\xi(z) = L[F(z)] = \frac{c+p}{z^c}\int_0^z F(t)t^{c-1}dt. \tag{8}$$

If $F \in S_{p,k}(\varphi)$ then $\xi \in S_{p,k}(\varphi)$.

Proof. From (8) we obtain

$$z\xi'(z) + c\xi(z) = (c+p)F(z). \tag{9}$$

Differentiating (9) k times we have

$$z\xi^{(k+1)}(z) + (k+c)\xi^{(k)}(z) = (c+p)F^{(k)}(z). \tag{10}$$

If we set $h(z) := \frac{z\xi^{(k+1)}(z)}{(p-k)\xi^{(k)}(z)}$, then

$$h(z) + \frac{zh'(z)}{(p-k)h(z)+k+c} = \frac{zF^{(k+1)}(z)}{(p-k)F^{(k)}(z)}.$$

Applying Lemma 2 with $\beta = p-k$ and $\delta = k+c$ now yields $h(z) = \frac{z\xi^{(k+1)}(z)}{(p-k)\xi^{(k)}(z)} \prec \varphi(z)$. □

3. Main Results

Throughout this section, inclusion relations for functions in classes $S^{n,\alpha}_{p,k}(\varphi)$ and $K^{n,\alpha}_{p,k}(\varphi)$ are obtained. Bernardi integral operator is also discussed.

Theorem 1. *Let the function $F \in \mathcal{A}_p$ belongs to the class $S^{n,\alpha}_{p,k}(\varphi)$, then the function F_n defined by equality (5) belongs to the class $S^{\alpha}_{p,k}(\varphi)$. Further, if $\Re\frac{1}{\alpha}[(p-k)\alpha\varphi(z)+1-\alpha] > 0$ and $\alpha \neq 0$, then $F_n \in S_{p,k}(\varphi)$.*

Proof. From the definition of $S^{n,\alpha}_{p,k}(\varphi)$, we have

$$\frac{1}{p-k}\frac{\alpha z\left(zF^{(k+1)}(z)\right)' + (1-\alpha)zF^{(k+1)}(z)}{\alpha zF_n^{(k+1)}(z) + (1-\alpha)F_n^{(k)}(z)} \prec \varphi(z) \quad (\alpha \geq 0; z \in \mathbb{U}). \tag{11}$$

Since $\varepsilon^n = 1$, if we change z by $\varepsilon^v z$ in (11) where $v = 0, 1, \ldots, n-1$, then the left side of (11) is equivalent to

$$\frac{1}{p-k}\frac{\alpha\varepsilon^v z[F^{(k+1)}(\varepsilon^v z) + \varepsilon^v zF^{(k+2)}(\varepsilon^v z)] + (1-\alpha)\varepsilon^v zF^{(k+1)}(\varepsilon^v z)}{\alpha z\varepsilon^v F_n^{(k+1)}(\varepsilon^v z) + (1-\alpha)F_n^{(k)}(\varepsilon^v z)}, \tag{12}$$

where $z \in \mathbb{U}$. The following identities are immediately derived from the definition (5) of F_n.

$$\begin{cases} F_n(\varepsilon^v z) = \varepsilon^{vp}F_n(z), \\ \varepsilon^{v(k-p)}F_n^{(k)}(\varepsilon^v z) = F_n^{(k)}(z) = \frac{1}{n}\sum\limits_{\nu=0}^{n-1}\varepsilon^{v(k-p)}F^{(k)}(\varepsilon^\nu z). \end{cases} \tag{13}$$

Using (13), (12) becomes

$$\frac{1}{p-k}\frac{\alpha z[\varepsilon^{v(1+k-p)}F^{(k+1)}(\varepsilon^v z) + z\varepsilon^{v(2+k-p)}F^{(k+2)}(\varepsilon^v z)] + (1-\alpha)z\varepsilon^{v(1+k-p)}F^{(k+1)}(\varepsilon^v z)}{\alpha zF_n^{(k+1)}(z) + (1-\alpha)F_n^{(k)}(z)}.$$

By taking the summation relation to v from 0 to $n-1$, we find

$$\begin{aligned} &\frac{1}{n(p-k)} \\ &\times \sum_{v=0}^{n-1}\frac{\alpha z[\varepsilon^{v(1+k-p)}F^{(k+1)}(\varepsilon^v z) + z\varepsilon^{v(2+k-p)}F^{(k+2)}(\varepsilon^v z)] + (1-\alpha)z\varepsilon^{v(1+k-p)}F^{(k+1)}(\varepsilon^v z)}{\alpha zF_n^{(k+1)}(z) + (1-\alpha)F_n^{(k)}(z)} \\ = &\frac{1}{p-k}\frac{\alpha z\left(zF_n^{(k+1)}(z)\right)' + (1-\alpha)zF_n^{(k+1)}(z)}{\alpha zF_n^{(k+1)}(z) + (1-\alpha)F_n^{(k)}(z)}. \end{aligned} \tag{14}$$

Since $F \in S^{n,\alpha}_{p,k}(\varphi)$, then each the terms in the left-hand side of Equation (14) is subordinate to $\varphi(z)$. Therefore, there exists $\mu_v's$ in $\mathbb{U}$ such that

$$\frac{1}{p-k}\frac{\alpha z\left(zF_n^{(k+1)}(z)\right)' + (1-\alpha)zF_n^{(k+1)}(z)}{\alpha zF_n^{(k+1)}(z) + (1-\alpha)F_n^{(k)}(z)} \prec \frac{1}{n}\sum_{v=0}^{n-1}\varphi(\mu_v) = \varphi(\mu_o).$$

Since $\varphi(\mathbb{U})$ is convex, we have $\mu_o \in \mathbb{U}$, and so $F_n \in S^{\alpha}_{p,k}(\varphi)$. Since $\Re\frac{1}{\alpha}[(p-k)\alpha\varphi(z) + 1 - \alpha] > 0$, it follows from Proposition 1 that $F_n \in S_{p,k}(\varphi)$. □

Theorem 2 below shows that $S^{n,\alpha}_{p,k}(\varphi) \subset S^{n,0}_{p,k}(\varphi)$.

Theorem 2. *Let* $\Re\frac{1}{\alpha}[(p-k)\alpha\varphi(z) + 1 - \alpha] > 0$ *with* $\alpha \neq 0$. *If* $F \in \mathcal{A}_p$ *belongs to the class* $S^{n,\alpha}_{p,k}(\varphi)$, *then we have* $F \in S^{n}_{p,k}(\varphi)$.

Proof. Let $F \in \mathcal{A}_p$ belongs to the class $S^{n,\alpha}_{p,k}(\varphi)$, and let

$$h(z) := \frac{zF^{(k+1)}(z)}{(p-k)F_n^{(k)}(z)}, Q(z) =: \frac{zF_n^{(k+1)}(z)}{(p-k)F_n^{(k)}(z)},$$

then Theorem 1 gives $Q(z) \prec \varphi(z)$. It is easy to obtain that

$$\frac{1}{p-k}\frac{\alpha z\left(zF^{(k+1)}(z)\right)' + (1-\alpha)zF^{(k+1)}(z)}{\alpha zF_n^{(k+1)}(z) + (1-\alpha)F_n^{(k)}(z)} = h(z) + \frac{\alpha zh'(z)}{(p-k)\alpha Q(z) + 1 - \alpha}.$$

Since $F \in S_{p,k}^{n,\alpha}(\varphi)$, then we have

$$h(z) + \frac{\alpha zh'(z)}{(p-k)\alpha Q(z) + 1 - \alpha} \prec \varphi(z).$$

If we apply Lemma 1, we get

$$h(z) \prec \varphi(z),$$

which implies $F \in S_{p,k}^{n}(\varphi)$, and this completes the proof. □

Remark 3. *1. Putting $p = 1, k = 0$, and $\varphi = \frac{1+z}{1-z}$ in Theorem 2, we get the result obtained by Wang et al. [12].*
2. Putting $p = 1, k = 0, \alpha = 0$, and $\varphi = \frac{1+z}{1-z}$ in Theorem 2, we get the result obtained by Wang et al. [21].
3. Putting $p = 1$ and $k = 0$ in Theorem 2, we obtain the result obtained by Parvatham and Radha [33].

Theorem 3. *Let $F \in \mathcal{A}_p$ and $\Re[(p-k)\varphi(z) + k + c] > 0$, also, let ξ given by Equation (8) . If $F \in S_{p,k}^{n}(\varphi)$, then $\xi \in S_{p,k}^{n}(\varphi)$.*

Proof. Take into account the definition of $F_n(z)$ given by Equation (5) with $\xi(z)$ instead of $F(z)$. Hence, $\xi_n(z) = \frac{1}{n}\sum_{v=0}^{n-1}\varepsilon^{-vp}\xi(\omega^v z)$, it is obvious that $\xi_n(z) = \frac{c+p}{z^c}\int_0^z F_n(t)t^{c-1}dt$. Differentiating this equation with respect to z, we have

$$c\xi_n(z) + z\xi_n'(z) = (c+p)F_n(z). \tag{15}$$

Differentiating (15) k times we get

$$z\xi_n^{(k+1)}(z) + (k+c)\xi_n^{(k)}(z) = (c+p)F_n^{(k)}(z). \tag{16}$$

Also, from (8) we get

$$z\xi^{(k+1)}(z) + (k+c)\xi^{(k)}(z) = (c+p)F^{(k)}(z). \tag{17}$$

Considering that $F \in S_{p,k}^{n}(\varphi)$ then, by applying the first section of Theorem 1 when $\alpha = 0$ we obt ain$F_n \in S_{p,k}(\varphi)$. Then, Proposition 2 gives $\xi_n \in S_{p,k}(\varphi)$, or equivalently,

$$\frac{z\xi_n^{(k+1)}(z)}{(p-k)\xi_n^{(k)}(z)} \prec \varphi(z). \tag{18}$$

If we let

$$h(z) := \frac{z\xi^{(k+1)}(z)}{(p-k)\xi_n^{(k)}(z)}, \tag{19}$$

and

$$Q(z) := \frac{z\xi_n^{(k+1)}(z)}{(p-k)\xi_n^{(k)}(z)}, \tag{20}$$

then, both the functions h and Q are analytic in $\mathbb{U}$ such that $p(0) = Q(0) = 1$ and $Q(z) \prec \varphi(z)$. Differentiating (17) and using (19), we obtain

$$h(z) + \frac{zh'(z)}{(p-k)Q + k + c} = \frac{(c+p)zF^{(k+1)}(z)}{(p-k)[(p-k)Q + k + c]\xi_n^{(k)}(z)}.$$

On using (16), we see that

$$h(z) + \frac{zh'(z)}{(p-k)Q + k + c} = \frac{zF^{(k+1)}(z)}{(p-k)F_n^{(k)}(z)} \prec \varphi(z).$$

Based on Lemma 1, we have $h(z) \prec \varphi(z)$, which ends the proof. □

Theorem 4. *Let* $\Re\frac{1}{\alpha}[(p-k)\alpha\varphi(z) + 1 - \alpha] > 0$. *Then, we have* $K_{p,k}^{n,\alpha}(\varphi) \subset K_{p,k}^{n,0}(\varphi)$.

Proof. Let $F \in K_{p,k}^{n,\alpha}(\varphi)$. Setting

$$h(z) =: \frac{zF^{(k+1)}(z)}{(p-k)\varsigma_n^{(k)}(z)},\ Q(z) =: \frac{z\varsigma_n^{(k+1)}(z)}{(p-k)\varsigma_n^{(k)}(z)},$$

we obtain

$$\frac{1}{p-k}\frac{\alpha z\left(zF^{(k+1)}(z)\right)' + (1-\alpha)zF^{(k+1)}(z)}{\alpha z\varsigma_n^{(k+1)}(z) + (1-\alpha)\varsigma_n^{(k)}(z)} = h(z) + \frac{\alpha zh'(z)}{\alpha(p-k)Q(z) + (1-\alpha)}.$$

Theorem 1 gives us $Q(z) \prec \varphi(z)$ because $\varsigma(z) \in S_{p,k}^{n,\alpha}(\varphi)$. Lemma 1 is once more applied to produce $h(z) \prec \varphi(z)$, proving Theorem 4. □

Remark 4. *Putting* $p = 1$ *and* $k = 0$ *in Theorem 4, we reach the result obtained by Parvatham and Radha [33].*

4. Conclusions

Analytic p-valent functions were recently studied using higher-order derivatives. With the higher-order derivatives, we defined two subclasses of analytic p-valent functions with n-symmetric points, which unify the previously introduced and studied subclasses. This paper aims to present several exciting subordination results, containment relations, and integral preserving properties for functions in these classes. Some of our results extend previously known results and some of our results are new. This work can be extended to the classes of harmonic multivalent n-symmetric and meromorphic multivalent n-symmetric type functions involving quantum calculus, as discussed in [35–39].

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