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The ∗-Ricci Operator on Hopf Real Hypersurfaces in the Complex Quadric

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Abstract: We study the ∗-Ricci operator on Hopf real hypersurfaces in the complex quadric. We prove that for Hopf real hypersurfaces in the complex quadric, the ∗-Ricci tensor is symmetric if and only if the unit normal vector field is singular. In the following, we obtain that if the ∗-Ricci tensor of Hopf real hypersurfaces in the complex quadric is symmetric, then the ∗-Ricci operator is both Reeb-flow-invariant and Reeb-parallel. As the correspondence to the semi-symmetric Ricci tensor, we give a classification of real hypersurfaces in the complex quadric with the semi-symmetric ∗-Ricci tensor.

Keywords: Reeb-flow-invariant ∗-Ricci operator; Reeb-parallel ∗-Ricci operator; semi-symmetric ∗-Ricci tensor; singular-unit normal vector field

MSC: 53C40; 53C55

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1. Introduction

There are many Hermitian symmetric spaces of rank 2. For example, complex two-plane Grassmannians and complex hyperbolic two-plane Grassmannians, which are denoted by $G_2(\mathbb{C}^{m+2}) = SU_{m+2}/S(U_2U_m)$ and $G_2^*(\mathbb{C}^{m+2}) = SU_{m,2}/S(U_2U_m)$, respectively. They are Hermitian symmetric spaces and quaternionic Kähler symmetric spaces equipped with the Kähler structure J and quaternionic Kähler structure $\mathfrak{J}$.

The complex quadric $Q^m = SO_{m+2}/SO_mSO_2$ is another kind of compact Hermitian symmetric space different from the above ones. For $m \geq 2$, the maximal sectional curvature of Q^m is equal to 4 (see [1,2]). It is the complex hypersurface in complex projective space $\mathbb{C}P^{m+1}$ [3], and it is also a kind of real Grassmannian manifold with rank 2 [4]. So, we know that apart from the Kähler structure J, there is another distinguished geometric structure, namely, a parallel rank two vector field bundle $\mathfrak{A}$ that contains an S^1-bundle of real structures, that is, complex conjugations A on the tangent spaces of Q^m. The complex conjugation A and the Kähler structure J anti-commute with each other, that is, $AJ = -JA$.

The Kähler manifold is the subject of symplectic geometry. Contact geometry appears as the odd dimensional counterpart of symplectic geometry, in which the almost-contact manifold corresponds to the almost complex manifold. Mathematicians are interested in submanifolds or hypersurfaces with some certain structure or curvature properties (see [5–11]). The real hypersurface M in the complex quadric Q^m is naturally an almost contact metric manifold. Many mathematicians have investigated it from various aspects. For example, some classifications of M related to the parallel Ricci tensor and Reeb-parallel Ricci tensor were obtained in Suh [12,13]. Moreover, Suh studied the real hypersurface M with the commuting Ricci tensor and the Ricci soliton in [14,15]. In [16], Suh and his partner Pérez gave the classification of the real hypersurface M in Q^m with the killing shape operator, and in [17], Pérez obtained some results when the structure vector field of the almost contact structure of M was of the Jacobi type.

The real hypersurface M is a Hopf hypersurface when the integral curves of the Reeb vector field ξ are geodesic. Moreover, the integral curves of ξ are geodesic if and only if ξ is a principal curvature vector of M everywhere, that is,

$$S\xi = \alpha\xi,$$

where S is the shape operator of M, and α is Reeb function. The classification of the Hopf hypersurface M in Q^m with some other geometric properties can be found in [12].

The unit normal vector field N of the real hypersurface M in Q^m has a great impact on the geometric properties of the hypersurface M. Usually, N can be put into two classes: N is $\mathfrak{A}$-principal or $\mathfrak{A}$-isotropic. In [18], Berndt and Suh proved that if M has isometric Reeb flow, then N is $\mathfrak{A}$-isotropic, and it is locally congruent to a tube over a totally geodesic $\mathbb{C}P^k$ in Q^{2k}. When M is in contact with the $\mathfrak{A}$-principal unit normal vector field N, then the classification of M can be found in [19].

In differential geometry, the Ricci tensor Ric is very significant to the nature of a manifold. For example, in [12] Suh proved that there was no Hopf real hypersurface with a parallel Ricci tensor in the complex quadric Q^m, $m \geq 4$. Moreover, in [20], Lee, Suh, and Woo showed that there were not any Hopf real hypersurfaces in the complex quadric Q^m with the semi-symmetric Ricci tensor and the $\mathfrak{A}$-principal unit normal vector field and gave the classification when the unit normal vector field was $\mathfrak{A}$-isotropic. In [21], Suh classified the the real hypersurface in the complex quadric Q^m with the Reeb-invarient Ricci tensor, and some classification about the Reeb-parallel Ricci tensor could be found in [13]. In [22], we obtained several properties on Lorentzian generalized Sasakian space-forms, which are related to the Ricci tensor.

Apart from the Ricci tensor, there is another important curvature tensor for the almost-contact manifold, that is, the $*$-Ricci tensor Ric^*. The notion of the $*$-Ricci tensor was introduced by Tachibana in [23], and Hamada extended this notion to almost-contact manifolds in [24]. Its definition is similar to the Ricci tensor, but its properties are different from the Ricci tensor. For instance, it may be not symmetric since it is related to the structure tensor ϕ. If the $*$-Ricci tensor is symmetric, we can directly investigate it. Many authors has investigated the $*$-Ricci soliton, which replaced Ricci tensor with the $*$-Ricci tensor in the Ricci soliton (see [25,26]) . In [27], we gave the classification of the trans-Sasakian three-manifolds with the Reeb invariant $*$-Ricci opertator. In [28], we gave the notion of the semi-symmetric $*$-Ricci tensor and investigated the properties of it on the (κ, μ)-contact manifold.

In the present paper, we study the real hypersurface M in Q^m with the Reeb invariant and the Reeb-parallel $*$-Ricci operator. We also investigate the Hopf real hypersurfaces with the semi-symmetric $*$-Ricci tensor.

Generally, the conditions of the Reeb invariant $*$-Ricci operator and the Reeb-parallel $*$-Ricci operator are not the same since the Reeb invariant $*$-Ricci operator is defined by $L_\xi Q^* = 0$ and the Reeb-parallel $*$-Ricci operator is $\nabla_\xi Q^* = 0$; in other words, one is a Lie derivative and the other is a connection derivative. However, we can see from the following theorem that they are the same for the Hopf real hypersurface in the complex quadric with the singular-unit normal vector field.

Theorem 1. *Let M be a Hopf real hypersurface in the complex quadric Q^m, $m \geq 3$, with the $\mathfrak{A}$-principal or $\mathfrak{A}$-isotropic unit normal vector field N; then,*

$$L_\xi Q^* = \nabla_\xi Q^* = 0,$$

*where Q^* is the $*$-*Ricci *operator, ξ is Reeb vector field, L is Lie derivative, and ∇ is Riemannian connection of M. That is, the $*$-*Ricci *operator on a Hopf real hypersurface in the complex quadric Q^m, $m \geq 3$, with a singular-unit normal vector field that is both Reeb-flow-invariant and Reeb-parallel.*

Aa an analogue to the notion of the semi-symmetric Ricci tensor, we consider the notion of the semi-symmetric $*$-Ricci tensor defined by

$$0 = (R(X,Y)\text{Ric}^*)(Z,W) = -\text{Ric}^*(R(X,Y)Z,W) - \text{Ric}^*(Z,R(X,Y)W),$$

for any vector field X, Y, Z, and W on the manifold. It has been proved that there are no Hopf hypersurfaces in the complex quadric with the semi-symmetric Ricci tensor and the $\mathfrak{A}$-principal unit normal vector field in [20]. For the $*$-Ricci tensor, we draw the conclusion that:

Theorem 2. *Hopf real hypersurfaces with the semi-symmetric $*$-Ricci tensor and $\mathfrak{A}$-principal unit normal vector field do not exist in the complex quadric Q^m, $m \geq 3$.*

2. Some General Equations and Key Lemmas

As we have mentioned above, the complex quadric Q^m is the complex hypersurface in the complex projective space $\mathbb{C}P^{m+1}$. If $z_0, \ldots, z_{m+1}$ are the homogeneous coordinates of $\mathbb{C}P^{m+1}$, then Q^m is the image of the equation $z_0^2 + \ldots + z_{m+1}^2 = 0$. Now, we denote the Kähler structure of $\mathbb{C}P^{m+1}$ by $(J, \bar{g})$, where $\bar{g}$ is the Fubini–Study metric on $\mathbb{C}P^{m+1}$, which has constant holomorphic sectional curvature 4. We know that the complex hypersurface of a Kähler manifold has an induced Kähler structure; in other words, it is a Kähler manifold. Then, the complex quadric Q^m has a canonical induced Kähler structure (J, g), where g is the Riemannian metric on Q^m induced from the Fubini–Study metric $\bar{g}$. Now, we explain why Q^m is SO_{m+2}/SO_mSO_2. Firstly, it is known that the complex projective space $\mathbb{C}P^{m+1} = SU_{m+2}/S(U_{m+1}U_1)$ because it is a Hermitian symmetric space of the special unitary group SU_{m+2}. As the subgroup of SU_{m+2}, SO_{m+2} acts on $\mathbb{C}P^{m+1}$ with cohomogeneity one. If the orbit of SO_{m+2} contains the fixed point of the action of the stabilizer $S(U_{m+1}U_1)$, namely, $o = [0, \ldots, 0, 1] \in \mathbb{C}P^{m+1}$, then this orbit is a totally geodesic real projective space $\mathbb{R}P^{m+1} \subset \mathbb{C}P^{m+1}$. The complex quadric $Q^m = SO_{m+2}/SO_mSO_2$ is just the second singular orbit of this action. It also gives the geometric interpretation of why Q^m is the Grassmann manifold $G_2^+(\mathbb{R}^{m+2})$ of oriented 2-planes in $\mathbb{R}^{m+2}$. In this paper, we focus on the condition of $m \geq 3$ because Q^1 is just S^1 and Q^2 is $S^1 \times S^1$.

Let us denote the unit normal vector field of Q^m by $\bar{N}$, and $A_{\bar{N}}$ is the shape operator of Q^m respect to $\bar{N}$. $A_{\bar{N}}$ is anti-commuting with the Kähler structure J, and it is involution. Then, the shape operator $A_{\bar{N}}$ is one of the complex conjugations A restricted to TQ^m. In some sense, we can consider the set of all shape operators of Q^m as the complex conjugations on TQ^m. Then, the tangent space of Q^m can be decomposed as

$$TQ^m = V(A_{\bar{N}}) \oplus JV(A_{\bar{N}}),$$

where $V(A_{\bar{N}})$ and $JV(A_{\bar{N}})$ are the $(+1)$-eigenspace and (-1)-eigenspace, respectively. So, $A_{\bar{N}}$ defines a real structure, and since the real codimension of Q^m in $\mathbb{C}P^{m+1}$ is 2, there is an S^1-subbundle $\mathfrak{A}$ of the endomorphism bundle $\text{End}(TQ^m)$ consisting of complex conjugations.

In terms of the complex conjugations $A \in \mathfrak{A}$ and the Kähler structure J, we can obtain the curvature tensor $\bar{R}$ of Q^m from the Gauss equation for $Q^m \subset \mathbb{C}P^{m+1}$

$$\begin{aligned}\bar{R}(X,Y)Z \quad = \quad & g(Y,Z)X - g(X,Z)Y + g(JY,Z)JX - g(JX,Z)JY - 2g(JX,Y)JZ \\ & + g(AY,Z)AX - g(AX,Z)AY + g(JAY,Z)JAX - g(JAX,Z)JAY.\end{aligned}$$

A nonzero vector field $Z \in TQ^m$ is singular if it is $\mathfrak{A}$-principal or $\mathfrak{A}$-isotropic. For these two types of singular vector fields, we have

1. If there is a conjugation $A \in \mathfrak{A}$ so that $Z \in V(A)$, then Z is $\mathfrak{A}$-principal.
2. If there is a conjugation $A \in \mathfrak{A}$ and two orthonormal vector fields $X, Y \in V(A)$ so that $Z/||Z|| = (X + JY)/\sqrt{2}$, then Z is $\mathfrak{A}$-isotropic.

Let M be the real hypersurface of Q^m and (ϕ, ξ, η, g) be its induced almost contact structure. Then, we have the following basic equations [29]:

$$\phi\xi = 0, \quad \eta \circ \phi = 0,$$

$$\phi^2 X = -X + \eta(X)\xi, \quad \eta(\xi) = 1,$$

$$\eta(X) = g(\xi, X),$$

$$g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y),$$

where ϕ is the structure tensor, ξ is Reeb vector field, and η is the dual 1-form of ξ, for any vector fields X and Y. Moreover, $\xi = -JN$ where J is the Kähler structure of Q^m and N is the unit normal vector field of M. The structure tensor ϕ and the Kähler structure J are related by

$$JX = \phi X + \eta(X)N.$$

Thus, ϕ and J coincide with each other when restricted to the kernel of η.

For any complex conjugation $A \in \mathfrak{A}$, we can choose two orthonormal vectors $Z_1, Z_2 \in V(A)$, such that

$$\begin{aligned} N &= \cos(t)Z_1 + \sin(t)JZ_2, \\ AN &= \cos(t)Z_1 - \sin(t)JZ_2, \\ \xi &= \sin(t)Z_2 - \cos(t)JZ_1, \\ A\xi &= \sin(t)Z_2 + \cos(t)JZ_1, \end{aligned}$$

where $0 \le t \le \frac{\pi}{4}$ (see [12]). The $\mathfrak{A}$-principal unit normal vector field N corresponds to the value $t = 0$; thus, we have $g(AN, N) = -g(\xi, A\xi) = 1$, $g(N, AY) = g(AN, Y) = 0$. The $\mathfrak{A}$-isotropic unit normal vector field N corresponds to the value $t = \frac{\pi}{4}$, so we have $g(AN, N) = g(\xi, A\xi) = 0$. Thus, $AN \in TM$.

In particular, we see that $A\xi$ is always the tangent on M (because it holds

$$\begin{aligned} g(A\xi, N) &= g(\sin(t)Z_2 + \cos(t)JZ_1, \cos(t)Z_1 + \sin(t)JZ_2) \\ &= \sin(t)\cos(t)g(Z_2, Z_1) + \sin 2(t)g(Z_2, JZ_2) \\ &\quad + \cos 2(t)g(JZ_1, Z_1) + \cos(t)\sin(t)g(JZ_1, JZ_2) \\ &= 0, \end{aligned}$$

for two orthonormal vectors $Z_1z, Z_2 \in V(A)$). So, from this and the property of $JA = -AJ$, we obtain

$$AN = AJ\xi = -JA\xi = -\phi A\xi - g(A\xi, \xi)N.$$

In fact, on a real hypersurface M in the complex quadric Q^m, for any vector field X on M, we can put

$$AX = BX + g(AX, N)N = BX + \rho(X)N,$$

here, BX denotes the tangential part of AX and 1-form ρ is given by

$$\begin{aligned} \rho(X) &= g(X, AN) = g(AX, N) \\ &= g(X, -\phi A\xi - g(A\xi, \xi)N) \\ &= -g(X, \phi A\xi), \end{aligned}$$

so

$$\begin{aligned} JAX &= JBX + g(X, AN)JN \\ &= JBX - g(X, \phi A\xi)JN \\ &= JBX + g(X, \phi A\xi)\xi \\ &= \phi BX + \eta(BX)N + g(X, \phi A\xi)\xi \\ &= \phi BX + \eta(BX)N - \rho(X)\xi, \end{aligned}$$

and

$$(JAX)^T = \phi BX - \rho(X)\xi,$$

where $(\cdots)^T$ denotes the tangential component of the vector $(\cdots)$ in Q^m.

Denote the induced Riemannian connection and the shape operator on M by ∇, S, respectively. Then, the Gauss–Weingarten equations are

$$\bar{\nabla}_X Y = \nabla_X Y + g(SX, Y)N, \quad \bar{\nabla}_X N = -SX,$$

where $\bar{\nabla}$ is the Riemannian connection on Q^m with respect to $\bar{g}$. Moreover, we have the following two equations:

$$(\nabla_X \phi)Y = \eta(Y)SX - g(SX, Y)\xi, \quad \nabla_X \xi = \phi SX.$$

Additionally, from the Gauss–Weingarten equation, in terms of the Kähler structure J and the complex conjugation $A \in \mathfrak{A}$, the curvature tensor R of M induced from $\bar{R}$ of Q^m is

$$\begin{aligned}
R(X,Y)Z &= g(Y,Z)X - g(X,Z)Y + g(\phi Y, Z)\phi X - g(\phi X, Z)\phi Y - 2g(\phi X, Y)\phi Z \\
&\quad + g(AY, Z)(AX)^T - g(AX, Z)(AY)^T + g(JAY, Z)(JAX)^T \\
&\quad - g(JAX, Z)(JAY)^T + g(SY, Z)SX - g(SX, Z)SY \\
&= g(Y,Z)X - g(X,Z)Y + g(\phi Y, Z)\phi X - g(\phi X, Z)\phi Y - 2g(\phi X, Y)\phi Z \\
&\quad + g(BY, Z)BX - g(BX, Z)BY \\
&\quad + g(\phi BY, Z)\phi BX - g(\phi BY, Z)\rho(X)\xi - \rho(Y)\eta(Z)\phi BX \\
&\quad - g(\phi BX, Z)\phi BY + g(\phi BY, Z)\rho(Y)\xi + \rho(X)\eta(Z)\phi BY \\
&\quad + g(SY, Z)SX - g(SX, Z)SY.
\end{aligned}$$

For an almost contact metric manifold, the $*$-Ricci tensor Ric^* is (see [24,25])

$$\mathrm{Ric}^*(X,Y) = \frac{1}{2}\, trace\{Z \to R(X, \phi Y)\phi Z\}.$$

So, we can calculate the $*$-Ricci tensor Ric^* of M

$$\begin{aligned}
\mathrm{Ric}^*(X,Y) &= \frac{1}{2}\sum_{i=1}^{2m-1} g(R(X, \phi Y)\phi e_i, e_i) \\
&= \frac{1}{2}\{g(\phi X, \phi Y) + g(\phi X, \phi Y) + g(\phi X, \phi Y) \\
&\quad + g(\phi X, \phi Y) + 4(m-1)g(\phi X, \phi Y) - g(\phi B\phi Y, BX) \\
&\quad + g(\phi BX, B\phi Y) - g(\phi^2 B\phi Y, \phi BX) + g(\phi^2 B\phi Y, \xi)\rho(X) \\
&\quad + g(\phi^2 B\phi X, \phi B\phi Y) + g(\phi^2 BX, \xi)\rho(\phi Y) \\
&\quad - g(SX, \phi S\phi Y) + g(\phi SX, S\phi Y)\} \\
&= 2mg(\phi X, \phi Y) + 2g(\phi BX, B\phi Y) + g(\phi SX, S\phi Y),
\end{aligned}$$

where $\{e_i\}$ is a local orthonormal basis of M.

Generally, Ric^* is not symmetric because it has an asymmetric part $g(\phi BX, B\phi Y)$ and $g(\phi SX, S\phi Y)$. So, it is not a geometric invariant. The asymmetric $*$-Ricci tensor is just a tensor on a manifold; it makes little sense of geometry or physics . Hence, when we investigate the $*$-Ricci tensor, we only focus on the symmetric $*$-Ricci tensor or the symmetric part of the $*$-Ricci tensor. The following theorem tells us when the $*$-Ricci tensor is symmetric on a Hopf hypersurface in the complex quadric.

Theorem 3. *Let M be a Hopf hypersurface in the complex quadric Q^m, $m \geq 3$. Then, the $*$-*Ricci *tensor* Ric^* *of M is symmetric if and only if the unit normal vector field N of M is singular, that is, N is either $\mathfrak{A}$-principal or $\mathfrak{A}$-isotropic.*

In particular, if N is $\mathfrak{A}$-principal, then

$$\mathrm{Ric}^*(X,Y) = 2(m-1)g(\phi X, \phi Y) - g((\phi S)^2 X, Y),$$

if N is $\mathfrak{A}$-isotropic, then

$$\begin{aligned}\mathrm{Ric}^*(X,Y) &= 2(m-1)g(\phi X,\phi Y) - g((\phi S)^2X,Y)\\ &\quad +2g(X,A\xi)g(Y,A\xi) + 2g(X,AN)g(Y,AN),\end{aligned}$$

for any vector fields X, Y on M.

Proof. In [25], it has been proved that if M is Hopf, then $(\phi S)^2 = (S\phi)^2$. So, we have

$$g(\phi SX, S\phi Y) = -g((\phi S)^2X,Y) = -g((S\phi)^2X,Y) = g(\phi SY, S\phi X).$$

Now, we calculate $g(\phi BX, B\phi Y)$:

$$\begin{aligned}g(\phi BX, B\phi Y) &= g(JBX - \eta(BX)N, B\phi Y) = g(JBX, B\phi Y)\\ &= g(JBX, A\phi Y - g(A\phi Y, N)N)\\ &= -g(BX, JA\phi Y - g(A\phi Y, N)JN)\\ &= -g(AX - g(AX,N)N, JA\phi Y - g(A\phi Y,N)JN)\\ &= -g(AX, JA\phi Y) + g(A\phi Y, N)g(AX, JN) + g(AX,N)g(N, JA\phi Y)\\ &= g(X,\phi^2Y) + g(JY,AN)g(AX,JN) - \eta(Y)g(N,AN)g(AX,JN)\\ &\quad +g(X,AN)g(Y,AN)\\ &= g(X,\phi^2Y) + g(Y,A\xi)g(X,A\xi) + \eta(Y)g(N,AN)g(X,A\xi)\\ &\quad +g(X,AN)g(Y,AN).\end{aligned}$$

First, we assume the $*$-Ricci tensor is symmetric, that is, $\mathrm{Ric}^*(X,Y) = \mathrm{Ric}^*(Y,X)$. From the above equation, there must be

$$\eta(Y)g(N,AN)g(X,A\xi) = \eta(X)g(N,AN)g(Y,A\xi),$$

If $g(N,AN) = 0$, that is, N is $\mathfrak{A}$-isotropic. If $g(N,AN) \neq 0$, putting $X = \xi, Y = A\xi$, we have $g(A\xi,\xi)^2 = \eta(\xi)g(A\xi,A\xi) = 1$. We know

$$\begin{aligned}g(A\xi,\xi) &= g(\sin(t)Z_2 + \cos(t)JZ_1, \sin(t)Z_2 - \cos(t)JZ_1)\\ &= -\cos(2t),\end{aligned}$$

where $0 \leq t \leq \frac{\pi}{4}$. According to these facts, $g(A\xi,\xi) = -1$, that is, $t = 0$. It implies that the normal vector field N is $\mathfrak{A}$-principal.

Conversely, if N is $\mathfrak{A}$-principal, from $g(AN,N) = -g(\xi,A\xi) = 1$, $g(N,AY) = g(AN,Y) = 0$, we have

$$\begin{aligned}\mathrm{Ric}^*(X,Y) &= 2mg(\phi X,\phi Y) + 2g(\phi BX, B\phi Y) + g(\phi SX, S\phi Y)\\ &= 2mg(\phi X,\phi Y) + 2g(X,\phi^2Y) + g(X,\xi)g(Y,\xi) - \eta(Y)g(X,\xi)\\ &\quad +g(\phi SX, S\phi Y)\\ &= 2(m-1)g(\phi X,\phi Y) - g((\phi S)^2X,Y).\end{aligned}$$

If N is $\mathfrak{A}$-isotropic, from $g(AN,N) = g(\xi,A\xi) = 0$, we have

$$\begin{aligned}\mathrm{Ric}^*(X,Y) &= 2mg(\phi X,\phi Y) + 2g(\phi BX, B\phi Y) + g(\phi SX, S\phi Y)\\ &= 2mg(\phi X,\phi Y) + 2(g(X,\phi^2Y) + g(Y,A\xi)g(X,A\xi)\\ &\quad +g(X,AN)g(Y,AN)) + g(\phi SX, S\phi Y)\\ &= 2(m-1)g(\phi X,\phi Y) - g((\phi S)^2X,Y)\\ &\quad +2g(X,A\xi)g(Y,A\xi) + 2g(X,AN)g(Y,AN),\end{aligned}$$

From the above two equations, we know that when the condition of N is singular, the $*$-Ricci tensor is symmetric. □

When the $*$-Ricci tensor is symmetric, we can define the $*$-Ricci operator by

$$\mathrm{Ric}^*(X,Y) = g(Q^*X,Y).$$

The following are some important theorems that will be used in the proof of our main theorems.

Theorem 4 ([30])**.** *Let M be a real hypersurface in the complex quadric Q^m, $M \geq 3$, with $\mathfrak{A}$-principal normal vector field N. Then,*

(a) $A\phi X = -\phi AX$,

(b) $A\phi SX = -\phi SX$,

(c) $ASX = SX - 2g(SX,\xi)\xi$ and $SAX = SX - 2\eta(X)S\xi$,

for any $X \in TM$.

In particular, if M is Hopf, then we obtain $ASX = SAX$ for any tangent vector field X on M.

Theorem 5 ([12])**.** *Let M be a Hopf real hypersurface in the complex quadric Q^m, $M \geq 3$. Then, M has an $\mathfrak{A}$-principal singular normal vector field N if and only if M is a contact real hypersurface with constant mean curvature and non-vanishing Reeb function in Q^m.*

Moreover, for a contact manifold, we have

Theorem 6 ([29])**.** *Let M be a hypersurface of a Kähler manifold, (ϕ,ξ,η,g) its induced almost contact metric structure, and S its shape operator. Then, (ϕ,ξ,η,g) is a contact metric structure if and only if $S\phi + \phi S = -2\phi$.*

Theorem 7 ([31])**.** *Let M be a Hopf hypersurface in the complex quadric Q^m with the singular unite normal vector field; then, the Reeb function α is the constant function.*

3. Proof of Theorem 1 with $\mathfrak{A}$-Principal unit Normal VECTOR field

Firstly, let us calculate the derivative and Lie derivative of Q^* along ξ. Now

$$L_\xi(g(Q^*X,Y)) = \xi(g(X,Y)) = \nabla_\xi(g(Q^*X,Y)).$$

So, we have

$$\begin{aligned}(L_\xi g)(Q^*X,Y) + g((L_\xi Q^*)X,Y) + g(Q^*(L_\xi X),Y) + g(Q^*X,L_\xi Y)\\ = g((\nabla_\xi Q^*)X,Y) + g(Q^*(\nabla_\xi X),Y) + g(Q^*X,\nabla_\xi Y).\end{aligned} \tag{1}$$

From $\nabla_X\xi = \phi SX$, we have

$$\begin{aligned}(L_\xi g)(X,Y)) &= g(\nabla_X\xi,Y) + g(X,\nabla_Y\xi)\\ &= g(\phi SX,Y) + g(X,\phi SY)\\ &= g((\phi S - S\phi)X,Y).\end{aligned}$$

Then, Equation (1) becomes

$$\begin{aligned}&g((\nabla_\xi Q^*)X,Y) + g(Q^*(\nabla_\xi X),Y) + g(Q^*X,\nabla_\xi Y)\\ =\ &g((\phi S - S\phi)Q^*X,Y) + g((L_\xi Q^*)X,Y)\\ &+g(Q^*(\nabla_\xi X - \nabla_X\xi),Y) + g(Q^*X,\nabla_\xi Y - \nabla_Y\xi).\end{aligned}$$

From the above equation, we have

$$\begin{aligned}g((L_\xi Q^*)X,Y) = g((\nabla_\xi Q^*)X,Y) - g(\phi SQ^*X,Y) + g(Q^*\phi SX,Y)\\ = g((\nabla_\xi Q^*)X,Y) + g(Q^*X,S\phi Y) + g(Q^*\phi SX,Y)\end{aligned} \tag{2}$$

In this section, we assume the real hypersurface M in Q^m is Hopf and the unit normal vector field is $\mathfrak{A}$-principal. From Theorem 3, we have

$$\begin{aligned} g((L_\xi Q^*)X, Y) &= g((\nabla_\xi Q^*)X, Y) + g(Q^*X, S\phi Y) + g(Q^*\phi SX, Y) \\ &= g((\nabla_\xi Q^*)X, Y) \\ &\quad +2(m-1)g(\phi X, \phi S\phi Y) - g((\phi S)^2 X, S\phi Y) \\ &\quad +2(m-1)g(\phi^2 SX, \phi Y) - g((\phi S)^2\phi SX, Y) \\ &= g((\nabla_\xi Q^*)X, Y), \end{aligned}$$

we have $(L_\xi Q^*)X = (\nabla_\xi Q^*)X$.

Now, we prove that when N is $\mathfrak{A}$-principal, then $(L_\xi Q^*)X = (\nabla_\xi Q^*)X = 0$. The Codazzi equation (see [12]) is

$$\begin{aligned} g((\nabla_X S)Y - (\nabla_Y S)X, Z) &= \eta(X)g(\phi Y, Z) - \eta(Y)g(\phi X, Z) - 2\eta(Z)g(\phi X, Y) \\ &\quad +g(X, AN)g(AY, Z) - g(Y, AN)g(AX, Z) \\ &\quad +g(X, A\xi)g(JAY, Z) - g(Y, A\xi)g(JAX, Z). \end{aligned} \tag{3}$$

Putting $X = \xi$ in (3) and in considerationation of $g(AN, N) = -g(\xi, A\xi) = 1$, we have

$$g((\nabla_\xi S)Y - (\nabla_Y S)\xi, Z) = g(\phi Y, Z) - g(JAY, Z). \tag{4}$$

Since M is Hopf, $S\xi = \alpha\xi$ and α are constant from Lemma 7,

$$(\nabla_Y S)\xi = \nabla_Y(S\xi) - S(\nabla_Y \xi) = \alpha\nabla_Y \xi - S\phi SY = \alpha\phi SY - S\phi SY. \tag{5}$$

From Equations (4) and (5), we have

$$\begin{aligned} g((\nabla_\xi S)Y, Z) &= g(\phi Y, Z) - g(JAY, Z) + g((\nabla_Y S)\xi, Z) \\ &= g(\phi Y, Z) - g(JAY, Z) + g(\alpha\phi SY - S\phi SY, Z). \end{aligned} \tag{6}$$

In [12], Suh proved that for a Hopf hypersurface M in Q^m, the following equation:

$$\begin{aligned} 0 &= 2g(S\phi SY, Z) - \alpha g((\phi S + S\phi)Y, Z) - 2g(\phi Y, Z) \\ &\quad +2g(Y, AN)g(Z, A\xi) - 2g(Z, AN)g(Y, A\xi) \\ &\quad +2g(\xi, A\xi)\{g(Z, AN)\eta(Y) - g(Y, AN)\eta(Z)\}, \end{aligned} \tag{7}$$

holds for all vector fields Y, Z on M. From Equations (6) and (7), in consideration of $g(X, AN) = 0$, we have

$$\begin{aligned} g((\nabla_\xi S)Y, Z) &= -g(JAY, Z) + \alpha g(\phi SY, Z) - \frac{\alpha}{2}g((\phi S + S\phi)Y, Z) \\ &= g(AJY, Z) + \frac{\alpha}{2}g((\phi S - S\phi)Y, Z). \end{aligned} \tag{8}$$

When the unit normal vector field N is $\mathfrak{A}$-principal, we have that the $*$-Ricci tensor Ric^* on M is

$$g(Q^*Y, Z) = \mathrm{Ric}^*(Y, Z) = 2(m-1)g(\phi Y, \phi Z) - g((\phi S)^2 Y, Z),$$

from Theorem 3. Applying ∇_ξ to both side of this equation, we have

$$g((\nabla_\xi Q^*)Y, Z) = g((\nabla_\xi S)\phi SY, \phi Z) - g((\nabla_\xi S)Y, \phi S\phi Z), \tag{9}$$

by $(\nabla_\xi \phi)Y = \eta(Y)S\xi - g(S\xi, Y)\xi = 0$. Putting Equation (8) in Equation (9), we have

$$\begin{aligned} g((\nabla_\xi Q^*)Y,Z) &= g(AJ\phi SY,\phi Z)+\frac{\alpha}{2}g((\phi S-S\phi)\phi SY,\phi Z) \\ &\quad -g(AJY,\phi S\phi Z)-\frac{\alpha}{2}g((\phi S-S\phi)Y,\phi S\phi Z) \\ &= g(AJ\phi SY,\phi Z)+g(JAY,\phi S\phi Z) \\ &= g(\phi^2 SY,A\phi Z)-g(AY,\phi^2 S\phi Z) \\ &= -g(SY,A\phi Z)+g(AY,S\phi Z) \\ &= g((\phi AS-\phi SA)Y,Z), \end{aligned} \tag{10}$$

by $JX=\phi X+\eta(X)N$ and $A\xi=-\xi$ if N is $\mathfrak{A}$-principal.

From Lemma 4 and Equation (10), we have

$$g((\nabla_\xi Q^*)Y,Z)=g((\phi AS-\phi SA)Y,Z)=0.$$

That is

$$(\nabla_\xi Q^*)X=0.$$

4. Proof of Theorem 1 with $\mathfrak{A}$-Isotropic unit Normal Vector Field

In this section, we assume the real hypersurface M in Q^m is Hopf and the unit normal vector field is $\mathfrak{A}$-isotropic. We have $g(AN,N)=g(\xi,A\xi)=0$ and $AN\in TM$.

In [12], the authors have proved that for a Hopf hypersurface M in Q^m, $m\geq 3$, with $\mathfrak{A}$-isotropic unit normal vector field N, the following two equations are satisfied:

$$SAN=0,\quad and \quad SA\xi=0.$$

Thus, we have

$$\begin{aligned} g(X,AN)g(S\phi Y,AN) &= g(X,AN)g(\phi Y,SAN)=0, \\ g(X,A\xi)g(S\phi Y,A\xi) &= g(X,A\xi)g(\phi Y,SA\xi)=0, \\ g(Y,AN)g(\phi SX,AN) &= g(Y,AN)g(AN,JSX-\eta(SX)N) \\ &= g(Y,AN)g(AJN,SX) \\ &= -g(Y,AN)g(SA\xi,X)=0 \\ g(Y,A\xi)g(\phi SX,A\xi) &= g(Y,A\xi)g(A\xi,JSX-\eta(SX)N) \\ &= -g(Y,A\xi)g(JA\xi,SX) \\ &= g(Y,A\xi)g(AJ\xi,SX) \\ &= g(Y,A\xi)g(SAN,X)=0. \end{aligned}$$

Then, from Equation (2) and Theorem 3, we have

$$\begin{aligned} g((L_\xi Q^*)X,Y) &= g((\nabla_\xi Q^*)X,Y)+g(Q^*X,S\phi Y)+g(Q^*\phi SX,Y) \\ &= g((\nabla_\xi Q^*)X,Y) \\ &\quad +2(m-1)g(\phi X,\phi S\phi Y)-g((\phi S)^2X,S\phi Y) \\ &\quad +g(X,A\xi)g(S\phi Y,A\xi)+g(X,AN)g(S\phi Y,AN) \\ &\quad +2(m-1)g(\phi^2 SX,\phi Y)-g((\phi S)^2\phi SX,Y) \\ &\quad +g(Y,A\xi)g(\phi SX,A\xi)+g(Y,AN)g(\phi SX,AN) \\ &= g((\nabla_\xi Q^*)X,Y), \end{aligned}$$

we obtain $(L_\xi Q^*)X=(\nabla_\xi Q^*)X$. From

$$\begin{aligned} g(Q^*X,Y) &= 2(m-1)g(\phi X,\phi Y)-g((\phi S)^2X,Y) \\ &\quad +2g(X,A\xi)g(Y,A\xi)+2g(X,AN)g(Y,AN), \end{aligned}$$

we can calculate that

$$
\begin{aligned}
g((\nabla_\xi Q^*)X, Y) \quad = \quad & 2g(\nabla_\xi(AN), X)g(AN, Y) + 2g(\nabla_\xi(AN), Y)g(AN, X) \\
& + 2g(\nabla_\xi(A\xi), X)g(A\xi, Y) + 2g(\nabla_\xi(A\xi), Y)g(A\xi, X) \\
& - g(\phi(\nabla_\xi S)\phi SX, Y) - g(\phi S\phi(\nabla_\xi S)X, Y), \qquad (11)
\end{aligned}
$$

by $AN \in TM$ and $(\nabla_\xi \phi)X = 0$.

In the following, we give the proof of

$$g(\phi(\nabla_\xi S)\phi SX, Y) + g(\phi S\phi(\nabla_\xi S)X, Y) = 0. \qquad (12)$$

From Equation (7) and $g(\xi, A\xi) = 0$, we have

$$
\begin{aligned}
0 \quad = \quad & 2g(S\phi SX, Y) - \alpha g((\phi S + S\phi)X, Y) - 2g(\phi X, Y) \\
& + 2g(X, AN)g(Y, A\xi) - 2g(Y, AN)g(X, A\xi).
\end{aligned}
$$

Then, we have

$$S\phi SX = \frac{1}{2}\alpha(\phi S + S\phi)X + \phi X - g(X, AN)A\xi + g(X, A\xi)AN. \qquad (13)$$

From $\phi AN = JAN = A\xi$ and $\phi A\xi = JA\xi = -AN$, we have

$$
\begin{aligned}
S\phi SX + \phi S\phi S\phi X \quad = \quad & \frac{1}{2}\alpha(\phi S + S\phi)X + \phi X - g(X, AN)A\xi + g(X, A\xi)AN \\
& \frac{1}{2}\alpha\phi(\phi S + S\phi)\phi X + \phi^3 X - g(\phi X, AN)\phi A\xi \\
& + g(\phi X, A\xi)\phi AN \\
= \quad & 0 \qquad (14)
\end{aligned}
$$

Putting $X = \xi$ in Codazzi Equation (3) and in consideration of

$$g(AN, N) = g(\xi, A\xi) = 0,$$

we have

$$g((\nabla_\xi S)Y - (\nabla_Y S)\xi, Z) = g(\phi Y, Z) - g(Y, AN)g(A\xi, Z) - g(Y, A\xi)g(JA\xi, Z),$$

thus,

$$
\begin{aligned}
g((\nabla_\xi S)Y, Z) \quad = \quad & g(\phi Y, Z) - g(Y, AN)g(A\xi, Z) \\
& - g(Y, A\xi)g(JA\xi, Z) + g(\alpha\phi SY - S\phi SY, Z),
\end{aligned}
$$

by Equation (5). So, we have

$$(\nabla_\xi S)Y = \phi Y - g(Y, AN)A\xi + g(Y, A\xi)AN + \alpha\phi SY - S\phi SY. \qquad (15)$$

Then, from Equations (13) and (15), we have

$$(\nabla_\xi S)Y = \alpha\phi SY - \frac{1}{2}\alpha(\phi S + S\phi)Y = \frac{\alpha}{2}(\phi S - S\phi)Y.$$

From Equation (14), we have

$$
\begin{aligned}
g(\phi(\nabla_\xi S)\phi SX, Y) \quad + \quad & g(\phi S\phi(\nabla_\xi S)X, Y) \\
= \quad & \frac{\alpha}{2}g(\phi(\phi S - S\phi)\phi SX, Y) + \frac{\alpha}{2}g(\phi S\phi(\phi S - S\phi)X, Y) \\
= \quad & 0.
\end{aligned}
$$

Thus, we prove Equation (12).

The derivative of AN and $A\xi$ is

$$\begin{aligned}
\nabla_X(AN) &= \bar{\nabla}_X(AN) - g(SX, AN)N \\
&= (\bar{\nabla}_X A)N + A(\bar{\nabla}_X N) \\
&= q(X)JAN - ASX \\
&= q(X)A\xi - ASX, \\
\nabla_X(A\xi) &= \bar{\nabla}_X(A\xi) - g(SX, A\xi)N \\
&= (\bar{\nabla}_X A)\xi + A(\bar{\nabla}_X \xi) \\
&= (\bar{\nabla}_X A)\xi + A(\bar{\nabla}_X(-JN)) \\
&= q(X)JA\xi - A((\bar{\nabla}_X J)N + J(\bar{\nabla}_X N)) \\
&= q(X)JA\xi + AJSX \\
&= q(X)JA\xi - JASX,
\end{aligned}$$

by $(\bar{\nabla}_U A)V = q(U)JAV$ for all $U, V \in TQ^m$, so $\nabla_\xi(AN) = q(\xi)A\xi - \alpha A\xi$ and $\nabla_\xi(A\xi) = q(\xi)JA\xi - \alpha JA\xi$, to obtain Equation (11) , we have

$$\begin{aligned}
g((\nabla_\xi Q^*)X, Y) &= 2g(q(\xi)A\xi - \alpha A\xi, X)g(AN, Y) \\
&\quad +2g(q(\xi)A\xi - \alpha A\xi, Y)g(AN, X) \\
&\quad +2g(q(\xi)JA\xi - \alpha JA\xi, X)g(A\xi, Y) \\
&\quad +2g(q(\xi)JA\xi - \alpha JA\xi, Y)g(A\xi, X) \\
&= 2(q(\xi) - \alpha)(g(A\xi, X)g(AN, Y) + g(A\xi, Y)g(AN, X)) \\
&\quad +2(q(\xi) - \alpha)(g(JA\xi, X)g(A\xi, Y) + g(JA\xi, Y)g(A\xi, X)) \\
&= 0
\end{aligned}$$

So, there must be $(\nabla_\xi Q^*)X = 0$. So $(L_\xi Q^*)X = (\nabla_\xi Q^*)X = 0$.

5. Proof of Theorem 2

First, we assume that the $*$-Ricci tensor of the Hopf real hypersurface M^{2m-1} of the complex quadric Q^m is semi-symmetric, that is,

$$0 = (R(X,Y)\mathrm{Ric}^*)(Z, W) = -\mathrm{Ric}^*(R(X,Y)Z, W) - \mathrm{Ric}^*(Z, R(X,Y)W).$$

Putting $W = Y = \xi$ and from the fact that

$$\mathrm{Ric}^*(R(X,\xi)Z, \xi) = 0,$$

and

$$\begin{aligned}
R(X,\xi)\xi &= X - \eta(X)\xi + g(A\xi, \xi)(AX)^T - g(AX, \xi)(A\xi)^T \\
&\quad +g(JA\xi, \xi)(JAX)^T - g(JAX, \xi)(JA\xi)^T \\
&\quad +\alpha SX - \alpha^2\eta(X)\xi,
\end{aligned}$$

since the unit normal vector filed N is $\mathfrak{A}$-principal, we have $AN = N$ and $A\xi = -\xi$, $(AX)^T = BX = AX$; then, the above equation becomes

$$\begin{aligned}
R(X,\xi)\xi &= X - \eta(X)\xi - BX - \eta(X)\xi + \alpha SX - \alpha^2\eta(X)\xi \\
&= X - 2\eta(X)\xi - AX + \alpha SX - \alpha^2\eta(X)\xi.
\end{aligned}$$

Then, from Theorem 3, we have

$$\begin{aligned} 0 &= \mathrm{Ric}^*(R(X,\xi)\xi, Z) \\ &= 2(m-1)g(\phi R(X,\xi)\xi, \phi Z) - g((\phi S)^2 R(X,\xi)\xi, Z) \\ &= 2(m-1)g(\phi X - \phi AX + \alpha\phi SX, \phi Z) \\ &\quad -g((\phi S)^2 X - (\phi S)^2 AX + \alpha(\phi S)^2 SX, Z) \\ &= 2(m-1)g(AX - X - \alpha SX, \phi^2 Z) \\ &\quad -g((X - AX + \alpha SX, (\phi S)^2 Z) \\ &= g(AX - X - \alpha SX, 2(m-1)\phi^2 Z + (\phi S)^2 Z) \end{aligned} \tag{16}$$

where we have used the fact that $(\phi S)^2 = (S\phi)^2$ since M is Hopf.

By replacing X with AX in Equation (16) and from Lemma 4, we have

$$\begin{aligned} 0 &= \mathrm{Ric}^*(R(AX,\xi)\xi, Z) \\ &= g(A^2X - AX - \alpha SAX, 2(m-1)\phi^2 Z + (\phi S)^2 Z), \\ &= g(X - AX - \alpha SX + 2\alpha^2\eta(X)\xi, 2(m-1)\phi^2 Z + (\phi S)^2 Z), \\ &= g(X - AX - \alpha SX, 2(m-1)\phi^2 Z + (\phi S)^2 Z). \end{aligned} \tag{17}$$

From Equations (16) and (17), we have

$$0 = \alpha g(SX, 2(m-1)\phi^2 Z + (\phi S)^2 Z).$$

By replacing Z by ϕZ in the above equation, we have

$$\begin{aligned} 0 &= \alpha g(SX, 2(m-1)\phi^3 Z + (\phi S)^2 \phi Z) \\ &= \alpha g(SX, -2(m-1)\phi Z + \phi^2 S\phi SZ) \\ &= \alpha g(SX, -2(m-1)\phi Z - S\phi SZ) \\ &= \alpha g(X, -2(m-1)S\phi Z - S^2\phi SZ). \end{aligned}$$

So, we have

$$2(m-1)S\phi Z + S^2\phi SZ = 0, \tag{18}$$

since α is a nonzero constant from Lemma 5 and the arbitrariness of vector field X.

Applying A to both sides of Equation (18), and the fact that $A\phi SZ = -\phi SZ$, $ASZ = SAZ$ from Lemma 4, we have

$$\begin{aligned} 0 &= 2(m-1)AS\phi Z + AS^2\phi SZ \\ &= 2(m-1)AS\phi Z + S^2 A\phi SZ \\ &= 2(m-1)AS\phi Z - S^2\phi SZ \end{aligned} \tag{19}$$

From Equations (18) and (19), we have

$$AS\phi Z + S\phi Z = 0. \tag{20}$$

From Lemma 4, we have

$$AS\phi Z = S\phi Z - 2g(S\phi Z, \xi)\xi = S\phi Z,$$

to obatain Equation (20) , we have

$$S\phi Z = 0.$$

From Lemmas 5 and 6, we know the Hopf hypersurface M is in contact and $S\phi Z + \phi SZ = -2\phi Z$. So,

$$\phi SZ = -2\phi Z.$$

Then, we will have

$$0 = (S\phi)^2 Z = (\phi S)^2 Z = 4\phi^2 Z.$$

That is, $\phi^2 = 0$, which cannot happen. Thus, we complete the proof of Theorem 2.

6. Conclusions

In our paper, we study the Hopf real hypersurface M in the complex quadric Q^m, $m \geq 3$, with some certain $*$-Ricci operator properties. We give the necessary and sufficient condition that the $*$-Ricci tensor on the Hopf real hypersurface in the complex quadric is symetric. We know that the $*$-Ricci operator on the Hopf real hypersurface M with the singular-unit normal vector field N is Reeb-invariant and Reeb-parallel. Moveover, we prove that the $*$-Ricci tensor on the Hopf real hypersurface M in the complex quadric with the $\mathfrak{A}$-principal unit normal vector field cannot be semi-symmetric.

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