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Article

On Two-Point Boundary Value Problems and Fractional Differential Equations via New Quasi-Contractions

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Abstract: The aim of this paper is to introduce new forms of quasi-contractions in metric-like spaces and initiate more general conditions for the existence of invariant points for such operators. The proposed notions are then applied to study novel existence criteria for the existence of solutions to two-point boundary value problems in the domains of integer and fractional orders. To attract further research in this direction, important consequences are deduced and discussed to indicate the novelty and generality of our proposed concepts.

Keywords: partial metric; metric-like; quasi-contraction; fixed point; fractional differential equation

MSC: 47H10; 54H25; 46S40

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1. Introduction and Preliminaries

The origin of fixed point theory in spaces with a metric structure was announced for the first time by Banach [1]. This result has been called the contraction mapping principle (or the Banach fixed point theorem). The Banach contraction principle affirms that any contractive self-mapping on a complete metric space has one and only one fixed point (also termed the invariant point). The principle is one of the famous tests for the existence and uniqueness of solutions to various problems in science and engineering. Due to the simplicity in its applications, the contraction mapping principle has been modified in different directions. One of the extensions of the principle by replacing the contractive constant with a family of functions was established by Rakotch [2]. For some recent refinements of the Banach fixed point theorem, one can consult [3–5] and the references therein. A few of these earlier improvements that are of interest to us in this current project include the work of Ciric [6], Geraghty [7], Jaggi [8], and Dass-Gupta [9]. It is important to note that every Rakotch contraction is a special form of a Geraghty contraction (see [7] (Corollary 3.1)). The idea of quasi-contraction, defined by Ciric [6], is recognized as one of the earliest contractive mappings.

Definition 1 ([6]). *Let (X, d) be a metric. A mapping $T : X \longrightarrow X$ is called a quasi-contraction if there $\lambda \in [0, 1)$ such that for all $x, y \in X$,*

$$\begin{aligned} d(Tx, Ty) \leq\ & \lambda \max\{d(x, y), d(x, Ty), d(y, Tx), \\ & d(x, Tx), d(y, Ty)\}. \end{aligned}$$

It is now well-known that every quasi-contraction in the sense of [6] on a complete metric space has an invariant point. Along the way, one of the first rational type contractive inequalities was initiated by Jaggi [8].

Definition 2 ([8])**.** *Let (X, d) be a metric space and $T : X \longrightarrow X$ be a continuous mapping. The mapping T is called a Jaggi contraction if it satisfies the following conditions:*

$$d(Tx, Ty) \leq \frac{\alpha_1 d(x, Tx) d(y, Ty)}{d(x, y)} + \alpha_2 d(x, y), \tag{1}$$

for all $x, y \in X$, $x \neq y$ and for some $\alpha_1, \alpha_2 \in [0, 1)$ with $\sum_{i=1}^{2} \alpha_i < 1$.

Then, it was presented in [8] that every mapping T defined on a complete metric space fulfilling (1) has a unique fixed point. For some recent variants of Jaggi contraction, we direct the readers to [10,11].

Let X be a nonempty set endowed with a metric d. Consider an auxiliary function $h : X \times X \longrightarrow [0, 1)$ satisfying

$$\lim_{n \longrightarrow \infty} h(x_n, y_n) = 1 \Longrightarrow \lim_{n \longrightarrow \infty} d(x_n, y_n) = 0,$$

for all sequences $\{x_n\}_{n \in \mathbb{N}}$ and $\{y_n\}_{n \in \mathbb{N}}$ in X such that $\{d(x_n, y_n)\}_{n \in \mathbb{N}}$ is non-increasing and converges [12]. We denote the class of functions defined above by $\mathcal{H}(X)$.

Example 1 ([12])**.** *Let $\rho_1, \rho_2 : \mathbb{R}^2 \longrightarrow [0, 1)$ be given by*

(i) $\rho_1(x, y) = \eta$ *for some* $\eta \in (0, 1)$;

(ii) $\rho_2(x, y) = \frac{t}{t + x^2 + y^2}$ *for some* $t \geq 0$.

Then, $\rho_1, \rho_2 \in \mathcal{H}(\mathbb{R})$.

Geraghty [7] launched a family $\mathcal{G}$ of auxiliary functions $\zeta : \mathbb{R}_+ \longrightarrow [0, 1)$ such that if the sequence t_n is monotonic decreasing in $\mathbb{R}_+$ and

$$\zeta(t_n) \longrightarrow 1, \text{ then } t_n \longrightarrow 0.$$

After this, any function $\zeta \in \mathcal{G}$ has been called a Geraghty function. By using the members of $\mathcal{G}$, Geraghty [7] established the following result.

Theorem 1 ([7])**.** *Let (X, d) be a complete metric space. Suppose that $T : X \longrightarrow X$ is a mapping and $\zeta : \mathbb{R}_+ \longrightarrow [0, 1)$ is a function such that for all $x, y \in X$,*

$$d(Tx, Ty) \leq \zeta(d(x, y)) d(x, y), \tag{2}$$

where $\zeta \in \mathcal{G}$. Then, T has one and only one invariant point in X.

The inequality (2) is now well recognized as the Geraghty contraction in the literature. Theorem 1 has inspired many investigators (e.g., see [13–15]).

Popescu [16] proposed a variant of (triangular) τ-admissible mappings, studied in [13,17], as given below.

Definition 3 ([16])**.** *Let (X, d) be a metric space and $\tau : X \times X \longrightarrow \mathbb{R}_+$ be a mapping. The mapping $T : X \longrightarrow X$ is called τ-orbital admissible, if*

$$\tau(x, Tx) \geq 1 \text{ implies } \tau(Tx, T^2x) \geq 1.$$

If, supplementarily,

$$\tau(x, y) \geq 1 \text{ and } \tau(y, Ty) \geq 1 \text{ implies } \tau(x, Ty) \geq 1,$$

then the mapping T is called triangular τ-orbital admissible.

Observe that every τ-admissible mapping is a τ-orbital admissible mapping. For further information and some counter examples, we refer to [14,16,17].

Definition 4 ([14]). *Let (X, d) be a metric space and $\tau : X \times X \longrightarrow \mathbb{R}_+$ be a mapping. Then, X is said to be τ-regular if for every sequence $\{x_n\}_{n\in\mathbb{N}}$ such that $\tau(x_n, x_{n+1}) \geq 1$ for all $n \in \mathbb{N}$, $x_n \longrightarrow x$ as $n \longrightarrow \infty$, there exists a subsequence $\{x_{n_k}\}_{k\in\mathbb{N}}$ of $\{x_n\}_{n\in\mathbb{N}}$ such that $\tau(x_{n_k}, x) \geq 1$ for all $k \in \mathbb{N}$.*

By Φ, we depict the class of functions $\xi : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ that are continuous and non-decreasing such that $\xi(t) = 0$ if and only if $t = 0$.

Recently, by using the Geraghty contraction with the interplay of $\xi \in \Phi$, Karapinar et al. [12] defined some new contractions and examined conditions for the existence of fixed points for such operators. We recall these notions as follows.

Definition 5 ([12]). *Let (X, d) be a metric space and $T : X \longrightarrow X$, $\tau : X \times X \longrightarrow \mathbb{R}_+$ be mappings. Then, for all $x, y \in X$, define the following inequalities:*

(e_1) $\tau(x, y)\xi(d(Tx, Ty)) \leq h(x, y)\xi(E_1(x, y))$,
(e_2) $\tau(x, y)\xi(d(Tx, Ty)) \leq h(x, y)\xi(E_2(x, y))$,
(e_3) $\tau(x, y)\xi(d(Tx, Ty)) \leq h(x, y)\xi(E_3(x, y))$,

where $h \in \mathcal{H}(X)$, $\xi \in \Phi$ and

$$E_1(x, y) = \max\left\{\frac{d(x, Tx)d(y, Ty)}{d(x, y)}, d(x, y), d(x, Tx), d(y, Ty), \frac{d(x, Ty) + d(y, Tx)}{2}\right\},$$

$$E_2(x, y) = \max\left\{\frac{d(x, Tx)d(y, Ty)}{d(x, y)}, d(x, y)\right\},$$

$$E_3(x, y) = \max\{d(x, y), d(x, Tx), d(y, Ty)\}.$$

Then, the mapping T is called a Jaggi-$\tau - h - \xi$ contraction (respectively, a generalized Jaggi-type $\tau - h - \xi$ contraction) if (E_1) (respectively, (E_2)) is satisfied. We say that T is a $\tau - h - \xi$ contraction if (E_3) holds.

Definition 6 ([12]). *Let (X, d) be a metric space and $T : X \longrightarrow X$ be a self-mapping. Suppose that there exist $\xi \in \Phi$, $h \in \mathcal{H}(X)$ and $\tau : X \times X \longrightarrow \mathbb{R}_+$ such that for all $x, y \in X$,*

(E_4) $\tau(x, y)\xi(d(Tx, Ty)) \leq h(x, y)\xi(E_4(x, y))$,

where

$$E_4(x, y) = \max\left\{ \frac{d(x,Tx)(1+d(y,Ty))}{1+d(x,y)}, d(x, y), \frac{d(y,Ty)(1+d(x,Tx))}{1+d(x,y)} \right\}.$$

Then, T is said to be a generalized Dass-Gupta type τ-h-ξ-contraction.

The following is the principal result of [12].

Theorem 2 ([12]). *Let (X, d) be a complete metric space and $\tau : X \times X \longrightarrow \mathbb{R}_+$, $T : X \longrightarrow X$ be mappings. Assume that the following conditions are satisfied:*

(C1) T is a Jaggi-$\tau - h - \xi$ contraction;
(C2) T is continuous and forms a triangular τ-orbital admissible;
(C3) There exists $x_0 \in X$ such that $\tau(x_0, Tx_0) \geq 1$.

Then, T has an invariant point in X.

On the other hand, the study of new spaces and the corresponding invariant point results has been a very vigorous activity in mathematical research groups. In this way, Matthews [18] initiated the idea of a partial metric space as an aspect of the denotational semantics of data flow networks. In this space, the Euclidean metric is replaced with the partial metric (PM) with the property that the self-distance of any point in the space may be nonzero. It was established in [18] that the Banach contraction principle hold good in PM spaces and can be utilized in program verification. Neil [19] improved the idea of a partial metric space by allowing negative distances. The idea of partial metric space due to [19] is called the dualistic PM. Heckman [20] extended the PM concept by removing the small self-distance axiom. The PM proposed by Heckman [20] is termed weak PM.

Not long ago, Amini-Harandi [21] extended the PM spaces by launching the notion of metric-like (ML) space and discussed some invariant point results that subsume some related ones in the literature. Shortly after, Shukla [22] initiated the concept of $0-\sigma$-complete ML space and extended the idea of Amini-Harandi [21]. Following [21,22], several invariant point results in ML spaces have been examined; a few of these can be found in [23–25].

Hereafter, specific fundamentals of PM and ML spaces are gathered.

Definition 7 ([18])**.** *A PM on a nonempty set X is a function $\rho : X \times X \longrightarrow \mathbb{R}_+$ such that for all $x, y, z \in X$,*

(ρ1) $x = y$ *if and only if* $\rho(x,x) = \rho(x,y) = \rho(y,y)$;
(ρ2) $\rho(x,x) \leq \rho(x,y)$;
(ρ3) $\rho(x,y) = \rho(y,x)$;
(ρ4) $\rho(x,y) \leq \rho(x,z) + \rho(z,y) - \rho(z,z)$.

The pair (X,ρ) is called a partial metric space.

Remark 1. *It is obvious that if $\rho(x,y) = 0$, then $(\rho1)$ and $(\rho2)$ yield $x = y$. However, if $x = y$, $\rho(x,y)$ may not be zero. One of the cardinal examples of a partial metric space is the pair $(\mathbb{R},\rho)$, where $\rho(x,y) = \max\{x,y\}$ for all $x,y \in \mathbb{R}$.*

Definition 8 ([21])**.** *An ML on a nonempty set X is a function $\varrho : X \times X \longrightarrow \mathbb{R}_+$ such that for all $x, y, z \in X$,*

(ϱ_1) $\varrho(x,y) = 0$ *implies* $x = y$;
(ϱ_2) $\varrho(x,y) = \varrho(y,x)$;
(ϱ_3) $\varrho(x,y) \leq \varrho(x,z) + \varrho(z,y)$.

The pair (X,ϱ) is called an ML space.

Observe that an ML satisfies all the axioms of a metric, except that $\varrho(x,x)$ is positive for $x \in X$. Every ML on X generates a topology μ_ϱ on X whose base is the family of open ϱ-balls, given as

$$\mathcal{B}_\varrho(x,\epsilon) = \{y \in X : |\varrho(x,y) - \varrho(x,x)| < \epsilon\},$$

for all $x \in X$ and $\epsilon > 0$.

A sequence $\{x_n\}_{n\in\mathbb{N}}$ in X converges to a point $x \in X$ if and only if $\lim\limits_{n\longrightarrow\infty} \varrho(x_n,x) = \varrho(x,x)$. The sequence $\{x_n\}_{n\in\mathbb{N}}$ in X is said to be ϱ-Cauchy if $\lim\limits_{n,m\longrightarrow\infty} \varrho(x_n,x_m)$ exists and is finite. The ML space (X,ϱ) is ϱ-complete if for each Cauchy sequence $\{x_n\}_{n\in\mathbb{N}}$ there exists $u \in X$ such that

$$\lim_{n\longrightarrow\infty} \varrho(x_n,u) = \varrho(u,u) = \lim_{n,m\longrightarrow\infty} \varrho(x_n,x_m).$$

It is interesting to know that every partial metric space is an ML space, but the converse is not always true, as the following example shows.

Example 2 ([21]). *Let $X = \{0, 1\}$ and $\varrho : X \times X \longrightarrow \mathbb{R}_+$ be defined by*

$$\varrho(x, y) = \begin{cases} 2, & \text{if } x = y = 0, \\ 1, & \text{otherwise.} \end{cases}$$

Then, (X, ϱ) is an ML space. However, (X, ϱ) is not a PM, since $\varrho(0, 0) > \varrho(0, 1)$.

The following Lemma is needed in the sequel. Its proof can be adapted from its metric space version established in [26].

Lemma 1. *Let (X, ϱ) be an ML space and $\{x_n\}_{n \in \mathbb{N}}$ be a sequence in X such that*

$$\lim_{n \longrightarrow \infty} \varrho(x_n, x_{n+1}) = 0. \tag{3}$$

If $\{x_n\}_{n \in \mathbb{N}}$ is not Cauchy in (X, ϱ), then there exist $\epsilon > 0$ and two sequences $\{y_k\}_{k \in \mathbb{N}}$ and $\{n_k\}_{k \in \mathbb{N}}$ of positive integers such that $n_k > y_k > k$, and the following hold:

(i) $\lim_{k \longrightarrow \infty} \varrho(x_{y_k}, x_{n_k}) = \epsilon$;

(ii) $\lim_{k \longrightarrow \infty} \varrho(x_{y_k}, x_{n_{k+1}}) = \epsilon$;

(iii) $\lim_{k \longrightarrow \infty} \varrho(x_{y_{k-1}}, x_{n_k}) = \epsilon$;

(iv) $\lim_{k \longrightarrow \infty} \varrho(x_{y_{k-1}}, x_{n_{k+1}}) = \epsilon$.

Following our survey of the existing literature, it is observed that invariant point results of Jaggi type and Dass-Gupta type extensions in the sense of Karapinar et al. [12] have not been exhaustively examined in ML or dislocated metric spaces. Hence, following the above chain of developments, and particularly motivated by the ideas presented in [12,21,27,28], the principal objectives of this manuscript are twofold. The first is to introduce new classes of contractions in ML spaces, viz. Jaggi type (τ, h, ξ)-quasi-contraction, Dass-Gupta type (τ, h, ξ)-quasi-contraction, and to analyze criteria for the existence of fixed points for such operators. The second focus is to apply the obtained results to examine novel conditions for the existence of solutions to ordinary boundary value problems and fractional boundary value problems with integral boundary conditions. A few consequences are discussed to indicate that the ideas initiated herein improve and subsume a significant number of existing concepts in the related literature. In particular, our results extend the notions studied in [7–9,12,29] from complete metric spaces to ϱ-complete ML spaces.

The rest of the paper is structured in the following style: in Section 2, new classes of contractions are defined, and the conditions under which their fixed points exist are investigated. Section 2.1 discusses a few particular cases of our principal ideas. Some roles of the obtained results are considered in Sections 2.2 and 2.3, concerning differential equations of integer and non-integer orders, respectively. Section 3 provides an overview of the proposed concepts in this paper.

2. Main Results

We begin this section by introducing the following concept.

Definition 9. *Let (X, ϱ) be an ML space and $\tau : X \times X \longrightarrow \mathbb{R}_+$, $T : X \longrightarrow X$ be mappings. Then, T is called a Jaggi-type (τ, h, ξ)-quasi-contraction if for all $x, y \in X$,*

$$\tau(x, y)\xi(\varrho(Tx, Ty)) \le h(x, y)\xi(\mathcal{J}(x, y)), \tag{4}$$

where $h \in \mathcal{H}(X)$, $\xi \in \Phi$, and

$$\begin{aligned}\mathcal{J}(x,y) = \ &\max\left\{\frac{\varrho(x,Tx)\varrho(y,Ty)}{\varrho(x,y)}, \varrho(x,y), \varrho(x,Tx), \varrho(y,Ty),\right. \\ &\left.\frac{\varrho(x,Ty)+\varrho(y,Tx)}{4}, \frac{\varrho(x,x)+\varrho(y,y)}{4}\right\}.\end{aligned} \tag{5}$$

Theorem 3. *Let (X,ϱ) be a ϱ-complete ML space and $\tau : X \times X \longrightarrow \mathbb{R}_+$, $T : X \longrightarrow X$ be mappings. Suppose that the following conditions are satisfied:*

(i) T is a Jaggi-type (τ, h, ξ)-quasi-contraction;
(ii) T is continuous and forms a triangular τ-orbital admissible;
(iii) There exists $x_0 \in X$ such that $\tau(x_0, Tx_0) \geq 1$.

Then, T has an invariant point in X.

Proof. Define an iterative sequence $\{x_n\}_{n\in\mathbb{N}}$ by $x_n = Tx_{n-1}$ for all $n \in \mathbb{N}$. On account of (iii), and using the τ-orbital admissibility of T, we can verify that $\tau(x_n, x_{n+1}) \geq 1$ for each $n \in \mathbb{N}$. Then, using the fact that T is triangular τ-orbital admissible, we have $\tau(x_n, x_{n+1}) \geq$ 1and $\tau(x_{n+1}, x_{n+2}) \geq 1$ implies $\tau(x_n, x_{n+2}) \geq 1$. Again, using the same argument, we have $\tau(x_n, x_{n+2}) \geq 1$ and $\tau(x_{n+2}, x_{n+3}) \geq 1$ implies $\tau(x_n, x_{n+3}) \geq 1$. Continuing in the same manner, we get $\tau(x_n, x_{n+1}) \geq 1$. Assume that for some positive integer p, $x_p = x_{p+1}$, then $x_p = Tx_p$, and hence x_p is an invariant point of T. For this purpose, we presume that $x_n \neq x_{n+1}$ for all $n = 0, 1, 2, \ldots$. Since T is a Jaggi-type (τ, h, ξ)-quasi-contraction, then for all $n \in \mathbb{N}$,

$$\begin{aligned}\xi(\varrho(x_n, x_{n+1})) \leq\ & \tau(x_{n-1}, x_n)\xi(\varrho(x_n, x_{n+1})) \\ =\ & \tau(x_{n-1}, x_n)\xi(\varrho(Tx_{n-1}, Tx_n)) \\ \leq\ & h(x_{n-1}, x_n)\xi(\mathcal{J}(x_{n-1}, x_n)) \\ <\ & \xi(\mathcal{J}(x_{n-1}, x_n)).\end{aligned} \tag{6}$$

Using (5), we have

$$\begin{aligned}&\mathcal{J}(x_{n-1}, x_n) \\ &= \max\left\{\frac{\varrho(x_{n-1}, Tx_{n-1})\varrho(x_n, Tx_n)}{\varrho(x_{n-1}, x_n)}, \varrho(x_{n-1}, x_n), \varrho(x_{n-1}, Tx_{n-1}),\right. \\ &\left.\varrho(x_n, Tx_n), \frac{\varrho(x_{n-1}, Tx_n)+\varrho(x_n, Tx_{n-1})}{4}, \frac{\varrho(x_{n-1}, x_{n-1})+\varrho(x_n, x_n)}{4}\right\} \\ &= \max\left\{\frac{\varrho(x_{n-1}, x_n)\varrho(x_n, x_{n+1})}{\varrho(x_{n-1}, x_n)}, \varrho(x_{n-1}, x_n), \varrho(x_{n-1}, x_n), \varrho(x_n, x_{n+1}),\right. \\ &\left.\frac{\varrho(x_{n-1}, x_{n+1})+\varrho(x_n, x_n)}{4}, \frac{\varrho(x_{n-1}, x_{n-1})+\varrho(x_n, x_n)}{4}\right\} \\ &= \max\left\{\varrho(x_n, x_{n+1}), \varrho(x_{n-1}, x_n),\right. \\ &\left.\frac{\varrho(x_{n-1}, x_{n+1})+\varrho(x_n, x_n)}{4}, \frac{\varrho(x_{n-1}, x_{n-1})+\varrho(x_n, x_n)}{4}\right\} \\ &\leq \max\left\{\varrho(x_n, x_{n+1}), \varrho(x_{n-1}, x_n),\right. \\ &\frac{\varrho(x_{n-1}, x_n)+\varrho(x_n, x_{n+1})+\varrho(x_n, x_{n-1})+\varrho(x_{n-1}, x_n)}{4}, \\ &\left.\frac{\varrho(x_{n-1}, x_n)+\varrho(x_n, x_{n-1})+\varrho(x_{n-1}, x_n)+\varrho(x_n, x_{n-1})}{4}\right\} \\ &= \max\left\{\varrho(x_n, x_{n+1}), \varrho(x_{n-1}, x_n), \frac{3}{4}\varrho(x_{n-1}, x_n)+\frac{1}{4}\varrho(x_n, x_{n+1})\right\}.\end{aligned} \tag{7}$$

Suppose that $\varrho(x_{n-1}, x_n) < \varrho(x_n, x_{n+1})$; then, (7) becomes

$$\begin{aligned}\mathcal{J}(x_{n-1}, x_n) &\leq \max\left\{\varrho(x_n, x_{n+1}), \frac{3}{4}\varrho(x_n, x_{n+1}) + \frac{1}{4}\varrho(x_n, x_{n+1})\right\} \\ &= \max\{\varrho(x_n, x_{n+1}), \varrho(x_n, x_{n+1})\} = \varrho(x_n, x_{n+1}).\end{aligned}$$

Therefore, (6) yields

$$\xi(\varrho(x_n, x_{n+1})) < \xi(\mathcal{J}(x_{n-1}, x_n)) \leq \xi(\varrho(x_n, x_{n+1})),$$

a contradiction. Thus, we infer that for all $n \in \mathbb{N}$, $\varrho(x_n, x_{n+1}) \leq \varrho(x_{n-1}, x_n)$, and

$$\mathcal{J}(x_{n-1}, x_n) \leq \varrho(x_n, x_{n+1}). \tag{8}$$

Therefore, the sequence $\{\varrho(x_n, x_{n+1})\}_{n\in\mathbb{N}}$ is non-increasing. Consequently, there exists $\eta \geq 0$ such that $\lim_{n\longrightarrow\infty} \varrho(x_n, x_{n+1}) = \eta$. Now, we show that $\eta = 0$. Assume on the contrary that $\eta > 0$. Then, from (6) and (8), we get

$$0 < \frac{\xi(\varrho(x_n, x_{n+1}))}{\xi(\varrho(x_{n-1}, x_n))} \leq h(x_{n-1}, x_n),$$

from which it follows that $\lim_{n\longrightarrow\infty} h(x_{n-1}, x_n) = 1$. Given that $h \in \mathcal{H}(X)$ yields $\lim_{n\longrightarrow\infty} \varrho(x_{n-1}, x_n) = 0$—proving that $\eta = 0$, a contradiction—hence,

$$\lim_{n\longrightarrow\infty} \varrho(x_n, x_{n+1}) = 0. \tag{9}$$

We now show that the sequence $\{x_n\}_{\in\mathbb{N}}$ is Cauchy. Suppose on the contrary that $\{x_n\}_{n\in\mathbb{N}}$ is not a Cauchy sequence; then, all the conclusions of Lemma 1 hold. As we have already noted in the beginning, using τ-orbital admissibility, we have

$$\tau\left(x_{y_p}, x_{y_p+1}\right) \geq 1 \text{ for all } y_p.$$

Then , using the fact that T is triangular τ-orbital admissible, we have

$$\tau\left(x_{y_p}, x_{y_p+1}\right) \geq 1 \text{ and } \tau\left(x_{y_p+1}, x_{y_p+2}\right) \geq 1 \Longrightarrow \tau\left(x_{y_p}, x_{y_p+2}\right) \geq 1.$$

Again, using the same argument, we have

$$\tau\left(x_{y_p}, x_{y_p+2}\right) \geq 1 \text{ and } \tau\left(x_{y_p+2}, x_{y_p+3}\right) \geq 1 \Longrightarrow \tau\left(x_{y_p}, x_{y_p+3}\right) \geq 1.$$

Continuing the same manner, we get

$$\tau\left(x_{y_p}, x_{n_p}\right) \geq 1.$$

Using the point that $\tau(x_{n_p}, x_{y_p}) \geq 1$ for all $p \in \mathbb{N}$, we note that for each $p \in \mathbb{N}$,

$$\begin{aligned}\xi(\varrho(x_{n_p+1}, x_{y_p+1})) &\leq \tau(x_{n_p}, x_{y_p})\xi(\varrho(x_{n_p+1}, x_{y_p+1})) \\ &= \tau(x_{n_p}, x_{y_p})\xi(\varrho(Tx_{n_p}, Tx_{y_p})) \\ &\leq h(x_{n_p}, x_{y_p})\xi(\mathcal{J}(x_{n_p}, x_{y_p})).\end{aligned} \tag{10}$$

In addition, for each $p \in \mathbb{N}$, we get

$$\begin{aligned}
\mathcal{J}(x_{n_p}, x_{y_p}) = \; & \max\left\{ \frac{\varrho(x_{n_p}, Tx_{n_p})\varrho(x_{y_p}, Tx_{y_p})}{\varrho(x_{n_p}, x_{y_p})}, \varrho(x_{n_p}, x_{y_p}), \right. \\
& \varrho(x_{n_p}, Tx_{n_p}), \varrho(x_{y_p}, Tx_{y_p}), \\
& \left. \frac{\varrho(x_{n_p}, Tx_{y_p}) + \varrho(x_{y_p}, Tx_{n_p})}{4}, \frac{\varrho(x_{n_p}, x_{n_p}) + \varrho(x_{y_p}, x_{y_p})}{4} \right\} \\
= \; & \max\left\{ \frac{\varrho(x_{n_p}, x_{n_{p+1}})\varrho(x_{y_p}, x_{y_{p+1}})}{\varrho(x_{n_p}, x_{y_p})}, \varrho(x_{n_p}, x_{y_p}), \right. \\
& \varrho(x_{n_p}, x_{n_{p+1}}), \varrho(x_{y_p}, x_{y_{p+1}}), \\
& \left. \frac{\varrho(x_{n_p}, x_{y_{p+1}}) + \varrho(x_{y_p}, x_{n_{p+1}})}{4}, \frac{\varrho(x_{n_p}, x_{n_p}) + \varrho(x_{y_p}, x_{y_p})}{4} \right\} \\
\leq \; & \max\left\{ \frac{\varrho(x_{n_p}, x_{n_{p+1}})\varrho(x_{y_p}, x_{y_{p+1}})}{\varrho(x_{n_p}, x_{y_p})}, \varrho(x_{n_p}, x_{y_p}), \right. \\
& \varrho(x_{n_p}, x_{n_{p+1}}), \varrho(x_{y_p}, x_{y_{p+1}}), \\
& \frac{\varrho(x_{n_p}, x_{y_p}) + \varrho(x_{y_p}, x_{y_{p+1}}) + \varrho(x_{y_p}, x_{n_p}) + \varrho(x_{n_p}, x_{n_{p+1}})}{4}, \\
& \left. \frac{\varrho(x_{n_p}, x_{y_p}) + \varrho(x_{y_p}, x_{n_p}) + \varrho(x_{y_p}, x_{n_p}) + \varrho(x_{n_p}, x_{y_p})}{4} \right\}.
\end{aligned} \tag{11}$$

Keeping note of $\lim_{p\to\infty} \varrho\left(x_{n_p}, x_{n_p+1}\right) = 0$, and $\lim_{p\to\infty} \varrho\left(x_{y_p}, x_{y_p+1}\right) = 0$, together with the results of Lemma 1, the limit of the right hand side of (11) is $\lim_{p\to\infty} \varrho\left(x_{y_p}, x_{n_p}\right) = \varepsilon$. Note that, by the definition of $\mathcal{J}$, we have

$$\varrho\left(x_{y_p}, x_{n_p}\right) \leq \mathcal{J}\left(x_{y_p}, x_{n_p}\right).$$

So, using squeeze theorem, we have

$$\lim_{p\to\infty} \mathcal{J}\left(x_{y_p}, x_{n_p}\right) = \varepsilon. \tag{12}$$

By utilizing the continuity of ξ and using (12), we have

$$\lim_{p\to\infty} \xi\left(\varrho\left(x_{y_p}, x_{n_p}\right)\right) \leq \lim_{p\to\infty} h\left(x_{y_p}, x_{n_p}\right) \lim_{p\to\infty} \xi\left(\mathcal{J}\left(x_{y_p}, x_{n_p}\right)\right).$$

Since we have $\lim_{p\to\infty} \varrho\left(x_{y_p}, x_{n_p}\right) = \lim_{p\to\infty} \mathcal{J}\left(x_{y_p}, x_{n_p}\right) = \varepsilon$, we conclude that $\lim_{p\longrightarrow\infty} h(x_{n_p}, x_{y_p}) = 1$. Given that $h \in \mathcal{H}(X)$, we get $\lim_{p\longrightarrow\infty} \varrho(x_{n_p}, x_{y_p}) = 0$, which is a contradiction. Hence, $\{x_n\}_{n\in\mathbb{N}}$ a Cauchy sequence. Thus, we can find $u \in X$ such that $\lim_{n\longrightarrow\infty} x_n = u$. Since T is continuous, we obtain $\lim_{n\longrightarrow\infty} x_{n+1} = \lim_{n\longrightarrow\infty} Tx_n = Tu$; from which it follows that, $Tu = u$. $\square$

Definition 10. *Let (X, ϱ) be an ML space and $\tau : X \times X \longrightarrow \mathbb{R}_+, T : X \longrightarrow X$ be mappings. Then, T is called a Dass-Gupta type (τ, h, ξ)-quasi-contraction if, for all $x, y \in X$,*

$$\tau(x, y)\xi(\varrho(Tx, Ty)) \leq h(x, y)\xi(\mathcal{D}(x, y)), \tag{13}$$

where $h \in \mathcal{H}(X), \xi \in \Phi$, *and*

$$\begin{aligned}\mathcal{D}(x,y) = \ & \max\left\{\frac{\varrho(x,Tx)[1+\varrho(y,Ty)]}{1+\varrho(x,y)}, \frac{\varrho(y,Ty)[1+\varrho(x,Tx)]}{1+\varrho(x,y)},\right. \\ & \left.\varrho(x,y), \frac{\varrho(x,x)+\varrho(y,y)}{4}\right\}.\end{aligned} \tag{14}$$

Theorem 4. *Let* (X,ϱ) *be a* ϱ*-complete ML space and* $\tau : X \times X \longrightarrow \mathbb{R}_+$, $T : X \longrightarrow X$ *be mappings. Assume that*

(i) T *is a Dass-Gupta type* (τ, h, ξ)*-quasi-contraction;*
(ii) T *is continuous and forms triangular* τ*-orbital admissible;*
(iii) *There exists* $x_0 \in X$ *such that* $\tau(x_0, Tx_0) \geq 1$.

Then, T *has an invariant point in* X.

Proof. From condition (iii), there exists $x_0 \in X$ such that $\tau(x_0, Tx_0) \geq 1$. Let the sequence $\{x_n\}$ be constructed as $x_n = Tx_{n-1}$, for all $n \in \mathbb{N}$. Assume that $x_p = x_{p+1}$, for some $p \in \mathbb{N}$, then $Tx_p = x_{p+1} = x_p$, that is, x_p is an invariant point of T. So, we presume that $x_n \neq x_{n+1}$, $n = 0,1,2,...n$. As in the previous proof, we can easily see that $\tau(x_{n-1}, x_n) \geq 1$. Hence,

$$\begin{aligned}\xi(\varrho(x_n,x_{n+1})) &\leq \tau(x_{n-1},x_n)\xi(\varrho(x_n,x_{n+1})) \\ &= \tau(x_{n-1},x_n)\xi(\varrho(Tx_{n-1},Tx_n)) \\ &\leq h(x_{n-1},x_n)\xi(\mathcal{D}(x_{n-1},x_n)) \\ &< \xi(\mathcal{D}(x_{n-1},x_n))\end{aligned} \tag{15}$$

On the other hand,

$$\begin{aligned}\mathcal{D}(x_{n-1},x_n) &= \max\left\{\begin{array}{c}\frac{\varrho(x_{n-1},Tx_{n-1})[1+\varrho(x_n,Tx_n)]}{1+\varrho(x_{n-1},x_n)}, \\ \frac{\varrho(x_n,Tx_n)[1+\varrho(x_{n-1},Tx_{n-1})]}{1+\varrho(x_{n-1},x_n)}, \varrho(x_{n-1},x_n), \\ \frac{\varrho(x_{n-1},x_{n-1})+\varrho(x_n,x_n)}{4}\end{array}\right\} \\ &= \max\left\{\begin{array}{c}\frac{\varrho(x_{n-1},x_n)[1+\varrho(x_n,x_{n+1})]}{1+\varrho(x_{n-1},x_n)}, \\ \frac{\varrho(x_n,x_{n+1})[1+\varrho(x_{n-1},x_n)]}{1+\varrho(x_{n-1},x_n)}, \varrho(x_{n-1},x_n), \\ \frac{\varrho(x_{n-1},x_{n-1})+\varrho(x_n,x_n)}{4}\end{array}\right\} \\ &\leq \max\left\{\begin{array}{c}\frac{\varrho(x_{n-1},x_n)[1+\varrho(x_n,x_{n+1})]}{1+\varrho(x_{n-1},x_n)}, \\ \varrho(x_n,x_{n+1}), \varrho(x_{n-1},x_n), \\ \frac{\varrho(x_{n-1},x_n)+\varrho(x_n,x_{n-1})+\varrho(x_n,x_{n-1})+\varrho(x_{n-1},x_n)}{4}\end{array}\right\} \\ &= \max\left\{\frac{\varrho(x_{n-1},x_n)[1+\varrho(x_n,x_{n+1})]}{1+\varrho(x_{n-1},x_n)}, \varrho(x_n,x_{n+1}), \varrho(x_{n-1},x_n)\right\}.\end{aligned}$$

If $\varrho(x_{n-1},x_n) < \varrho(x_n,x_{n+1})$, then

$$\begin{aligned}\mathcal{D}(x_{n-1},x_n) &< \max\left\{\frac{\varrho(x_n,x_{n+1})[1+\varrho(x_n,x_{n+1})]}{1+\varrho(x_n,x_{n+1})}, \varrho(x_n,x_{n+1})\right\} \\ &= \max\{\varrho(x_n,x_{n+1}), \varrho(x_n,x_{n+1})\} = \varrho(x_n,x_{n+1}).\end{aligned}$$

So, using (15), we have

$$\xi(\varrho(x_n,x_{n+1})) < \xi(\mathcal{D}(x_{n-1},x_n)) < \xi(\varrho(x_n,x_{n+1})),$$

a contradiction. Hence, $\varrho(x_n, x_{n+1}) \leq \varrho(x_{n-1}, x_n)$ for all $n \in \mathbb{N}$, and so,

$$\begin{aligned}\mathcal{D}(x_{n-1}, x_n) &\leq \max\left\{\frac{\varrho(x_{n-1}, x_n)[1+\varrho(x_{n-1}, x_n)]}{1+\varrho(x_{n-1}, x_n)}, \varrho(x_{n-1}, x_n)\right\} \\ &= \max\{\varrho(x_{n-1}, x_n), \varrho(x_{n-1}, x_n)\} = \varrho(x_{n-1}, x_n).\end{aligned}$$

That is,

$$\mathcal{D}(x_{n-1}, x_n) \leq \varrho(x_{n-1}, x_n). \tag{16}$$

This shows that $\{\varrho(x_n, x_{n+1})\}_{n\in\mathbb{N}}$ is a non-increasing sequence. Therefore, there exists $\eta \geq 0$ such that $\lim_{n\to\infty} \varrho(x_n, x_{n+1}) = \eta$. Now, we claim that $\eta = 0$. Assume on the contrary that $\eta > 0$. Then, from (15) and (16) , we get

$$0 < \frac{\xi(\varrho(x_n, x_{n+1}))}{\xi(\varrho(x_{n-1}, x_n))} \leq h(x_{n-1}, x_n),$$

from which it follows $\lim_{n\to\infty} h(x_{n-1}, x_n) = 1$. Since $h \in \mathcal{H}(X)$, so $\lim_{n\to\infty} \varrho(x_{n-1}, x_n) = 0$, gives a contradiction. Hence, $\eta = 0$. Next, we shall demonstrate that $\{x_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence. Assume on the contrary that $\{x_n\}_{n\in\mathbb{N}}$ is not a Cauchy sequence. Then, all the conclusions of Lemma 1 hold. Consistent with the previous proof, we can show that $\tau(x_{n_p}, x_{y_p}) \geq 1$. Therefore, for $p \in \mathbb{N}$

$$\begin{aligned}\xi(\varrho(x_{n_p+1}, x_{y_p+1})) &\leq \tau(x_{n_p}, x_{y_p})\xi(\varrho(x_{n_p+1}, x_{y_p+1})) \\ &= \tau(\varrho(x_{n_p}, x_{y_p}))\xi(\varrho(Tx_{n_p}, Tx_{y_p})) \\ &\leq h(x_{n_p}, x_{y_p})\xi(\mathcal{D}((x_{n_p}, x_{y_p})))\end{aligned} \tag{17}$$

Then, we consider

$$\mathcal{D}(x_{n_p}, x_{y_p}) = \max\left\{\begin{array}{c}\frac{\varrho(x_{n_p},Tx_{n_p})[1+\varrho(x_{y_p},Tx_{y_p})]}{1+\varrho(x_{n_p},x_{y_p})}, \\ \frac{\varrho(x_{y_p},Tx_{y_p})[1+\varrho(x_{n_p},Tx_{n_p})]}{1+\varrho(x_{n_p},x_{y_p})}, \varrho(x_{n_p}, x_{y_p}), \\ \frac{\varrho(x_{n_p},x_{n_p})+\varrho(x_{y_p},x_{y_p})}{4}\end{array}\right\}$$

$$\leq \max\left\{\begin{array}{c}\frac{\varrho(x_{n_p},x_{n_p+1})[1+\varrho(x_{y_p},x_{y_p+1})]}{1+\varrho(x_{n_p},x_{y_p})}, \\ \frac{\varrho(x_{y_p},x_{y_p+1})[1+\varrho(x_{n_p},x_{n_p+1})]}{1+\varrho(x_{n_p},x_{y_p})}, \varrho(x_{n_p}, x_{y_p}), \\ \frac{\varrho(x_{n_p},x_{n_p+1})+\varrho(x_{n_p+1},x_{n_p})+\varrho(x_{y_p},x_{y_p+1})+\varrho(x_{y_p+1},x_{y_p})}{4}\end{array}\right\}.$$

Keeping note of $\lim_{p\to\infty} \varrho\left(x_{n_p}, x_{n_p+1}\right) = 0$, and $\lim_{p\to\infty} \varrho\left(x_{y_p}, x_{y_p+1}\right) = 0$, together with the results of Lemma 1, the limit of the right-hand side of the above equation will be equal to $\lim_{p\to\infty} \varrho\left(x_{y_p}, x_{n_p}\right) = \varepsilon$.

Note that, by the definition of $\mathcal{D}$, we have

$$\varrho(x_{y_p}, x_{n_p}) \leq \mathcal{D}(x_{y_p}, x_{n_p}).$$

So, using the squeeze theorem, we have

$$\lim_{p\to\infty} \mathcal{D}\left(x_{y_p}, x_{n_p}\right) = \varepsilon. \tag{18}$$

Using the continuity of ξ and together with (17) and (18), we have

$$\lim_{p\to\infty}\xi\Big(\varrho\Big(x_{y_p},x_{n_p}\Big)\Big)\le\lim_{p\to\infty}h\Big(x_{y_p},x_{n_p}\Big)\lim_{p\to\infty}\xi\Big(\mathcal{D}\Big(x_{y_p},x_{n_p}\Big)\Big).$$

Since we have $\lim_{p\to\infty}\varrho\Big(x_{y_p},x_{n_p}\Big)=\lim_{p\to\infty}\mathcal{D}\Big(x_{y_p},x_{n_p}\Big)=\varepsilon$, and so $\lim_{p\to\infty}h(x_{n_p},x_{y_p})=1$. Given that $h\in\mathcal{H}(X)$, we obtain $\lim_{p\to\infty}\varrho(x_{n_p},x_{y_p})=0$, a contradiction. Consequently, $\{x_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence. By the completeness of X, there exists $u\in X$ such that $\lim_{n\to\infty}x_n=u$. Since T is continuous, then $\lim_{n\to\infty}x_{n+1}=\lim_{n\to\infty}Tx_n=Tu$, proving that $Tu=u$. $\square$

Example 3. *Let $X=[0,1]$ and*

$$\varrho(x,y)=|x-y|+|x|+|y| \tag{19}$$

for all $x,y\in X$; then, (X,ϱ) is a complete ML space ([22]). Define the mapping $T:X\longrightarrow X$ by $T(x)=\ln(1+\frac{x}{3})$, for all $x\in X$; then, T is not a contraction mapping. Let $\tau:X\times X\longrightarrow\mathbb{R}_+$ be defined as

$$\tau(x,y)=\begin{cases}1, & \text{if } x,y\in[0,1),\\ 0, & \text{otherwise.}\end{cases}$$

Let $\xi(t)=t$ for all $t\ge 0$, and $h:X\times X\longrightarrow[0,1)$ be set as

$$h(x,y)=\begin{cases}\frac{\arctan(|x-y|+|x|+a)}{|x-y|+|x|+a}, & \text{if } x\neq y, 0\le a<1,\\ 0, & \text{if } x=y.\end{cases}$$

Clearly, $h\in\mathcal{H}(X)$ and $\xi\in\Phi$. Now, using the fact that $\ln(1+t)\le\arctan(t)$, for all $t\in[0,1]$, then for all $x,y\in[0,1)$, we have

$$\begin{aligned}\tau(x,y)\varrho(Tx,Ty) &= \Big|\ln\Big(1+\frac{x}{3}\Big)-\ln\Big(1+\frac{y}{3}\Big)\Big|=\Big|\ln\Big(\frac{1+x}{1+y}\Big)\Big|\\ &\le \ln(1+|x|)\le\ln(1+|x|+a)\\ &\le \ln(1+|x-y|+|y|+a)\le\arctan(|x-y|+|y|+a)\\ &= \frac{\arctan(|x-y|+|y|+a)}{|x-y|+|y|+a}\cdot(|x-y|+|y|+a)\\ &= h(x,y)\xi(\varrho(x,y))\\ &\le h(x,y)\xi(\mathcal{J}(x,y)).\end{aligned}$$

Note that if $x=1$ or $y=1$, then $\tau(x,y)=0$, and hence

$$\tau(x,y)\xi(\varrho(Tx,Ty))\le h(x,y)\xi(\mathcal{J}(x,y)).$$

Obviously, other hypotheses of Theorem 3 are satisfied. Consequently, we see that $u=0$ is the invariant point of T.

It is worthy of note that the mapping ϱ in (19) is not a metric, since for $x=y=1$, $\varrho(1,1)=2$. Similarly, ϱ is not a PM, since for $x=0$ and $y=1$, $\varrho(0,1)=3>0=\varrho(0,0)$. Hence, our result does not coincide with the main ideas of [13] and some citations therein.

2.1. Some Consequences

In this section, we discuss a few particular cases of our principal results. First, we use the following auxiliary function, launched in [13].

Let Ψ be the class of all upper semi-continuous from the right functions $\psi : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ such that $\psi^{-1}(0) = 0$ and $\psi(t) < t$ for all $t > 0$.

Corollary 1. *Let (X, ϱ) be a ϱ-complete ML space and $\tau : X \times X \longrightarrow \mathbb{R}_+$, $T : X \longrightarrow X$ be mappings. Assume that the following conditions are satisfied:*

(i) $\tau(x,y)\xi(\varrho(Tx,Ty)) \leq \zeta(\xi(\mathcal{J}(x,y)))\xi(\mathcal{J}(x,y))$ *for each $x, y \in X$, where $\xi \in \Phi$, $\zeta \in \mathcal{G}$ and $\mathcal{J}(x,y)$ is as given in (5);*
(ii) *T is triangular τ-orbital admissible and either T is continuous or X is τ-regular;*
(iii) *There exists $x_0 \in X$ such that $\tau(x_0, Tx_0) \geq 1$.*

Then, T has an invariant point in X.

Proof. Define the mapping $h : X \times X \longrightarrow \mathbb{R}_+$ by

$$h(x,y) = \zeta(\xi(\mathcal{J}(x,y))), \text{ for all } x, y \in X.$$

Suppose that the sequences $\{x_n\}_{n\in\mathbb{N}}$, $\{y_n\}_{n\in\mathbb{N}}$ in X are such that $h(x_n, y_n) = 1$. Then, $\xi(\mathcal{J}(x_n, y_n)) = 0$. Since ξ is continuous and $\xi^{-1}(0) = 0$, then $\mathcal{J}(x_n, y_n) = 0$. It follows that

$$\lim_{n\to\infty} \varrho(x_n, y_n) = \lim_{n\to\infty} \varrho(x_n, x_{n+1}) = \lim_{n\to\infty} \varrho(y_n, y_{n+1}) = 0. \tag{20}$$

Thus, $h \in \mathcal{H}(X)$, and by Condition (i), we obtain

$$\tau(x,y)\xi(\varrho(Tx,Ty)) \leq h(x,y)\xi(\mathcal{J}(x,y)), \text{ for each } x, y \in X.$$

Hence, T is a Jaggi-type (τ, h, ξ)-quasi-contraction. By applying (20) and the triangle inequality, we get

$$\lim_{n\to\infty} \varrho(Tx_n, Ty_n) = \lim_{n\to\infty} \varrho(x_{n+1}, y_{n+1}) = 0.$$

Consequently, all the hypotheses of Theorem 3 are satisfied. Therefore, T has an invariant point $u \in X$. $\square$

Motivated by Definition 1, we deduce the next concept, which unifies and extends Theorem 1 due to Geraghty [7] and the results of [6,12,13].

Definition 11. *Let (X, ϱ) be an ML space and $\tau : X \times X \longrightarrow \mathbb{R}_+, T : X \longrightarrow X$ be mappings. Then, T is called a (τ, ζ, ξ)-quasi-contraction (or simply generalized quasi-contraction) if there exist $\zeta \in \mathcal{G}$ and $\xi \in \Phi$ such that for all $x, y \in X$,*

$$\tau(x,y)\varrho(Tx,Ty) \leq \zeta(\xi(\mathcal{M}(x,y)))\xi(\mathcal{M}(x,y)), \tag{21}$$

where

$$\mathcal{M}(x,y) = \max\{\varrho(x,y), \varrho(x,Tx), \varrho(y,Ty), \varrho(x,x), \varrho(y,y)\}. \tag{22}$$

Corollary 2. *Let (X, ϱ) be a ϱ-complete ML space and $\tau : X \times X \longrightarrow \mathbb{R}_+$, $T : X \longrightarrow X$ be mappings. Assume that the following conditions are satisfied:*

(i) *T is a (τ, ζ, ξ)-quasi-contraction;*
(ii) *T is triangular τ-orbital admissible and either T is continuous or X is τ-regular;*
(iii) *There exists $x_0 \in X$ such that $\tau(x_0, Tx_0) \geq 1$.*

Then, T has an invariant point in X.

Proof. First, observe that Theorem 3 is still valid if (5) is replaced with (22). Now, define the mapping $h : X \times X \longrightarrow \mathbb{R}_+$ by

$$h(x,y) = \zeta(\xi(\mathcal{M}(x,y))), \text{ for all } x,y \in X.$$

Following Corollary 1, we can deduce that $h \in \mathcal{H}(X)$ and for any sequences $\{x_n\}_{n\in\mathbb{N}}$, $\{y_n\}_{n\in\mathbb{N}}$ in X,

$$\lim_{n\to\infty} h(x_n,y_n) = 1 \;\Rightarrow\; \lim_{n\to\infty} \varrho(Tx_n,Ty_n) = \lim_{n\to\infty} \varrho(x_{n+1},y_{n+1}) = 0.$$

Since ξ is a non-decreasing function, then for all $x,y \in X$,

$$\tau(x,y)\xi(\varrho(Tx,Ty)) \le h(x,y)\xi(\mathcal{M}(x,y)) \le h(x,y)\xi(\mathcal{J}(x,y)), \text{ for all } x,y \in X.$$

Hence, T is a Jaggi-type (τ,h,ξ)-quasi-contraction. Therefore, all the hypotheses of Theorem 3 are satisfied, from which it follows that T has an invariant point in X. □

Corollary 3. *(Amini-Harandi [21] (Theorem 2.7)). Let (X,ϱ) be a ϱ-complete ML space and $\tau : X \times X \longrightarrow \mathbb{R}_+$, $T : X \longrightarrow X$ be mappings such that*

$$\varrho(Tx,Ty)) \le \varrho(x,y) - \varphi(\varrho(x,y)),$$

for all $x,y \in X$, where $\varphi \in \Phi$. Then T has an invariant point in X.

Proof. Define the mapping $h : X \times X \longrightarrow [0,1)$ by

$$h(x,y) = \begin{cases} \frac{\varrho(x,y)-\varphi(\varrho(x,y))}{\varrho(x,y)}, & \text{if } x \neq y; \\ 0, & \text{if } x = y. \end{cases}$$

Let $\{x_n\}_{n\in\mathbb{N}}$, $\{y_n\}_{n\in\mathbb{N}}$ be sequences in X such that $\{\varrho(x_n,y_n)\}_{n\in\mathbb{N}}$ is non-increasing and $\lim_{n\to\infty} \varrho(x_n,y_n) = \eta$. Assume that $\lim_{n\to\infty} h(x_n,y_n) = 1$. Then, we need to prove that $\lim_{n\to\infty} \varrho(x_n,y_n) = 0$. Suppose on the contrary that $\lim_{n\to\infty} \varrho(x_n,y_n) = \eta > 0$. Since φ is continuous, then

$$\begin{aligned} \lim_{n\to\infty} h(x_n,y_n) &= \lim_{n\to\infty} \frac{\varrho(x_n,y_n) - \varphi(\varrho(x_n,y_n))}{\varrho(x_n,y_n)} \\ &= \frac{\eta - \varphi(\eta)}{\eta} = 1, \end{aligned}$$

from which it follows that $\varphi(\eta) = 0$, and hence $\eta = 0$, a contradiction. Consequently, $\lim_{n\to\infty} \varrho(x_n,y_n) = 0$, that is, $h \in \mathcal{H}(X)$.

Let $\xi(t) = t$ for all $t \in \mathbb{R}_+$. Then, using (22), we infer that

$$\xi(\varrho(Tx,Ty)) \le h(x,y)\xi(\varrho(x,y)) \le h(x,y)\xi(\mathcal{J}(x,y)), \text{ for all } x,y \in X.$$

Hence, with $\tau(x,y) = 1$ for all $x,y \in X$, Theorem 3 can be applied to conclude that T has an invariant point in X. □

Corollary 4. *(Amini-Harandi [21] (Theorem 2.11)). Let (X,ϱ) be a ϱ-complete ML space and $\tau : X \times X \longrightarrow \mathbb{R}_+$, $T : X \longrightarrow X$ be mappings such that the following conditions are satisfied:*

(i) $\varrho(Tx,Ty) \le T(\varrho(x,y))\varrho(x,y)$ for all $x,y \in X$, where $T : \mathbb{R}_+ \longrightarrow [0,1)$ is a non-increasing continuous function with $T^{-1}(0) = 0$;

(ii) T is triangular τ-orbital admissible and either T is continuous or X is τ-regular;

(iii) There exists $x_0 \in X$ such that $\tau(x_0,Tx_0) \ge 1$.

Then, T has an invariant point in X.

Proof. Consider the mapping $h : X \times X \longrightarrow [0, 1)$ given by

$$h(x, y) = \zeta(\varrho(x, y)), \text{ for all } x, y \in X, \ \zeta \in \mathcal{G}.$$

Let $\xi(t) = t$ for all $t \in \mathbb{R}_+$. Then, from Condition (i), we have

$$\xi(\varrho(Tx, Ty)) \leq h(x, y)\xi(\varrho(x, y)) \leq h(x, y)\xi(\mathcal{J}(x, y)), \text{ for all } x, y \in X.$$

Hence, all the hypotheses of Theorem 3 are satisfied with $\tau(x, y) = 1$ for all $x, y \in X$. Consequently, T has an invariant point in X. □

2.2. An Application to Ordinary Differential Equations

In this section, a role of one of our obtained results is examined in the domain of ordinary boundary value problems (Bvp). For related studies, the reader can consult [12,30,31] and some references therein. In particular, we adopt the method of [12]. Let $X = C([0, 1], \mathbb{R}_+)$ be the space of all continuous real-valued functions defined on $\mathcal{I} = [0, 1]$ and let $u \in X$. Consider the two-point Bvp of order two:

$$\begin{cases} -u''(t) - g(t, u(t)) = 0, & t \in \mathcal{I} \\ u(0) = u(1) = 0, \end{cases} \tag{23}$$

where $g : \mathcal{I} \times \mathbb{R}$ is a continuous function. The integral reformulation of (23) is given by

$$u(t) = \int_0^1 L(t, s)g(s, u(s))\, ds, \tag{24}$$

where $L(t, s)$ is the Green's function obtained as

$$L(t, s) = \begin{cases} t(1 - s), & \text{if } 0 \leq t \leq s \leq 1 \\ s(1 - t), & \text{if } 0 \leq s \leq t \leq 1. \end{cases}$$

Consider an operator $T : X \longrightarrow X$ defined by

$$Tu(t) = \int_0^1 L(t, s)g(s, u(s))\, ds, \ t \in \mathcal{I} \tag{25}$$

for all $t \in \mathcal{D}$. We recall that $u \in X$ is a solution of (23) if and only if $u \in X$ is an invariant point of T defined in (25). Define the mapping $\varrho : X \times X \longrightarrow \mathbb{R}$ by $\varrho(x, y) = \sup_{t \in \mathcal{I}}(|x(t)| + |y(t)|)$, for all $x, y \in X$. Then, (X, ϱ) is a ϱ-complete ML space.

Theorem 5. *Assume that the following conditions are satisfied:*

(C1) There exist $T : \mathbb{R}^2 \longrightarrow \mathbb{R}$ and $\psi \in \Phi$ such that for all $t \in \mathcal{I}$ and $p, q \in \mathbb{R}$ with $T(p, q) \geq 0$,

$$|g(t, p) - g(t, q)| \leq 8\psi(|p| + |q|);$$

(C2) there exists $u_1 \in X$ such that for all $t \in \mathcal{I}$,

$$T\left(u_1(t), \int_0^1 L(t, s)g(s, u_1(s))\right) \geq 0;$$

(C3) for all $t \in \mathcal{I}$ and $u, v \in X$,

$$T(u(t), v(t)) \geq 0 \text{ implies } T\left(\int_0^1 L(t, s)g(s, u(s))\, ds, \int_0^1 L(t, s)g(s, v(s))\, ds\right) \geq 0;$$

(C4) *let* $\{u_n\}_{n\in\mathbb{N}}$ *be a sequence in* X *such that* $u_n \longrightarrow u \in X$*, and for all* $t \in \mathcal{I}$ *with* $n \in \mathbb{N}$*,*

$$T(u_n(t), u_{n+1}(t)) \geq 0 \text{ implies } T(u_n(t), u(t)) \geq 0.$$

Then, the Bvp (23) *has a solution in* X*.*

Proof. We show that the operator T given in (25) is a Jaggi-type (τ, h, ξ)-quasi-contraction. Let $u, v \in X$ such that $T(u(t), v(t)) \geq 0$ for all $t \in \mathcal{I}$. Employing (C1), we have

$$\begin{aligned} |Tu(t) - Tv(t)| &= \left| \int_0^1 L(t,s)(g(s,u(s)) - g(s,v(s)))\, ds \right| \\ &\leq \int_0^1 L(t,s)|g(s,u(s)) - g(s,v(s))|\, ds \\ &\leq \int_0^1 L(t,s)(8\psi(|u(s)| + |v(s)|))\, ds \\ &\leq 8\psi(\varrho(u,v)) \sup_{t\in\mathcal{I}} \int_0^1 L(t,s)\, ds = \psi(\varrho(u,v)), \end{aligned}$$

where we recognized the fact that for each $t \in \mathcal{I}, \int_0^t L(t,s)\, ds = \frac{t}{2} - \frac{t^2}{2}$, and hence $\sup_{t\in\mathcal{I}} \int_0^1 L(t,s)\, ds = \dfrac{1}{8}$. Now, define $\tau : X \times X \longrightarrow \mathbb{R}_+$ by

$$\tau(u,v) = \begin{cases} 1, & \text{if } T(u(t), v(t)) \geq 0 \\ 0, & \text{otherwise,} \end{cases}$$

and

$$h(u,v) = \begin{cases} \frac{\psi(\varrho(u,v))}{\varrho(u,v)}, & \text{if } u \neq v \\ 0, & \text{elsewhere.} \end{cases}$$

Then, for all $u, v \in X$, we get

$$\begin{aligned} \tau(u,v)\varrho(Tu, Tv) &\leq \psi(\varrho(u,v)) = \frac{\psi(\varrho(u,v))}{\varrho(u,v)}\varrho(u,v) \\ &= h(u,v)\varrho(u,v) \leq h(u,v)\mathcal{J}(u,v). \end{aligned}$$

Then, taking $\xi(t) = t$ for all $t \in \mathbb{R}_+$, we find that T is a Jaggi-type (τ, h, ξ)-quasi-contraction. Moreover, let $\{u_n\}_{n\in\mathbb{N}}$ be a sequence in X such that $x_n \longrightarrow u \in X$ and $\lim_{n\longrightarrow\infty} h(u_n, u) = 1$. By the definition of τ, for all $n \in \mathbb{N}$ and $t \in \mathcal{I}$, $T(u_n(t), u(t)) \geq 0$. So,

$$\varrho(Tu_n(t), Tu(t)) \leq \psi(\varrho(u_n(t), u(t))),$$

from which it follows that for all $n \in \mathbb{N}$, $\varrho(Tu_n, Tu) \leq \psi(\varrho(u_n, u))$. By assumption, $\lim_{n\longrightarrow\infty} \varrho(u_n, u) = 0$. Hence, $\lim_{n\longrightarrow\infty} \varrho(u_{n+1}, u) = 0$. Utilizing hypotheses $(C2) - (C4)$, it is clear that all the assumptions of Theorem 3 are fulfilled. Consequently, there exists $u^* \in X$ such that $Tu^* = u^*$. □

2.3. Application to Nonlinear Differential Equations in the Setting of Fractional Derivatives with Singular Kernel

In this section, one of our main results is applied to examine new conditions for the existence of a solution to the Caputo-type fractional boundary value problem of order $\varsigma \in (1, 2]$ possessing an integral boundary condition.

Let ς be a positive real number, and Γ denotes the gamma function. The Caputo derivative of fractional order ς is given as

$$({}_0^C\mathcal{D}^{\varsigma} f)(t) = \frac{1}{\Gamma(n-\varsigma)} \int_0^t ((t-s)^{n-\varsigma-1} f^{(n)}(s))\, ds,\ n = [\varsigma] + 1,$$

where $f : \mathbb{R}_+ \longrightarrow \mathbb{R}$ is a continuous function.

Consider the following fractional differential equation:

$$({}_0^C\mathcal{D}^{\varsigma} u)(t) = g(t, u(t)),\ t \in I = (0, 1], \tag{26}$$

with the boundary condition

$$u(0) = 0, u(1) = \int_0^r u(s)\, ds,\ r \in (0, 1),$$

where $u \in C([0,1], \mathbb{R}) = X$ and $g : I \times \mathbb{R} \longrightarrow \mathbb{R}$ is a continuous function. The integral reformulation of (26) is given by

$$\begin{aligned} u(t) \quad = \quad & \frac{1}{\Gamma(\eta)} \int_0^t (t-s)^{\eta-1} g(s, u(s)) ds - \frac{2t}{(2-r^2)\Gamma(\varsigma)} \int_0^1 (1-s)^{\eta-1} g(s, u(s))\, ds \\ & + \frac{2t}{(2-r^2)\Gamma(\eta)} \int_0^r \left(\int_0^s (s-z)^{\eta-1} g(z, u(z))\, dz \right) ds,\ t \in I. \end{aligned}$$

Define an operator $T : X \longrightarrow X$ by

$$\begin{aligned} Tu(t) \quad = \quad & \frac{1}{\Gamma(\eta)} \int_0^t (t-s)^{\eta-1} g(s, u(s)) ds - \frac{2t}{(2-r^2)\Gamma(\varsigma)} \int_0^1 (1-s)^{\eta-1} g(s, u(s))\, ds \\ & + \frac{2t}{(2-r^2)\Gamma(\eta)} \int_0^r \left(\int_0^s (s-z)^{\eta-1} g(z, u(z))\, dz \right) ds,\ t \in I. \end{aligned} \tag{27}$$

It is a fact that $u \in X$ solves (26) if and only if $u \in X$ is an invariant point of T in (27). Consider the mapping $\varrho : X \times X \longrightarrow \mathbb{R}$ given as

$$\varrho(x, y) = \sup_{t \in I} (|x(t) - y(t)| + |x(t)| + |y(t)|) = \|x - y\|_\infty + \|x\|_\infty + \|y\|_\infty,$$

for all $x, y \in X$. Then, (X, ϱ) is a ϱ-complete ML space. However, the mapping ϱ is not a metric.

Now, we investigate the solvability conditions of Problem (26) via the following assumptions.

(A_1) There exist $\omega : \mathbb{R}^2 \longrightarrow \mathbb{R}$ and $\psi \in \Phi$ such that for all $t \in I$ and $p, q \in \mathbb{R}$ with $\omega(p, q) \geq 0$, $|g(t, p) - g(t, q)| \leq M_0 \psi(|p - q| + |p| + |q|)$, $M_0 = \frac{\Gamma(\varsigma+2)}{5+3\varsigma}$;

(A_2) There exists $u_0 \in X$ such that $\omega(u_0, Tu_0) \geq 0$ for all $t \in I$;

(A_3) For all $t \in I$ and $u, v \in X$, $\omega(u(t), v(t)) \geq 0$ implies $\omega(Tu(t), Tv(t)) \geq 0$;

(A_4) Let $\{u_n\}_{n \in \mathbb{N}}$ be a sequence in X such that $u_n \longrightarrow u \in X$, and for all $t \in I$, $\omega(u_n(t), u_{n+1}(t)) \geq 0$ for all $n \in \mathbb{N}$ implies $\omega(u_n(t), u(t)) \geq 0$.

Theorem 6. *Under the assumptions (A_1)–(A_4), Problem (26) has at least one solution u^* in X.*

Proof. We show that the mapping T is a Jaggi-type (τ, h, ξ)-quasi-contraction. Accordingly, let $u, v \in X$ such that for all $t \in I$, $\omega(u(t), v(t)) \geq 0$. Then, using (A_1),

$$
\begin{aligned}
& |Tu(t) - Tv(t)| \\
= & \left| \frac{1}{\Gamma(\varsigma)} \int_0^t (t-s)^{\varsigma-1} g(s, u(s)) \right. \\
& - \frac{2t}{(2-r^2)\Gamma(\varsigma)} \int_0^1 (1-s)^{\varsigma-1} g(s, u(s))\, ds \\
& + \frac{2t}{(2-r^2)\Gamma(\varsigma)} \int_0^r \left(\int_0^s (s-z)^{\varsigma-1} g(z, u(s))\, dz \right) ds \\
& - \frac{1}{\Gamma(\varsigma)} \int_0^t (t-s)^{\varsigma-1} g(s, v(s))\, ds + \frac{2t}{(2-r^2)\Gamma(\varsigma)} \int_0^1 (1-s)^{\varsigma-1} g(s, v(s))\, ds \\
& - \frac{2t}{(2-r^2)\Gamma(\varsigma)} \int_0^r \left(\int_0^s (s-z)^{\varsigma-1} g(z, v(z))\, dz \right) ds \\
\leq & \ \frac{1}{\Gamma(\varsigma)} \int_0^t |t-s|^{\varsigma-1} |g(s, u(s)) - g(s, v(s))|\, ds \\
& + \frac{2t}{(2-r^2)\Gamma(\varsigma)} \int_0^1 (1-s)^{\varsigma-1} |g(s, u(s)) - g(s, v(s))|\, ds \\
& + \frac{2t}{(2-r^2)\Gamma(\varsigma)} \int_0^r \left| \int_0^s (s-z)^{\varsigma-1} |g(s, u(s)) - g(s, v(s))|\, dz \right| ds \\
\leq & \ \frac{1}{\Gamma(\varsigma)} \int_0^t |t-s|^{\varsigma-1} M_0 \psi(|u(s) - v(s)| + |u(s)| + |v(s)|)\, ds \\
& + \frac{2t}{(2-r^2)\Gamma(\varsigma)} \int_0^1 |1-s|^{\varsigma-1} M_0 \psi(|u(s) - v(s)| + |u(s)| + |v(s)|)\, ds \\
& + \frac{2t}{(2-r^2)\Gamma(\varsigma)} \int_0^r \left(|s-z|^{\varsigma-1} M_0 \psi(|u(s) - v(s)| + |u(z)| + |v(z)|)\, dz \right) ds \\
\leq & \ M_0 \psi(\|u-v\|_\infty + \|u\|_\infty + \|v\|_\infty) \cdot \sup_{t \in (0,1)} \left(\frac{1}{\Gamma(\varsigma)} \int_0^t |t-s|^{\varsigma-1}\, ds \right. \\
& \left. + \frac{2t}{(2-r^2)} \Gamma(\varsigma) \int_0^1 |1-s|^{\varsigma-1}\, ds + \frac{2t}{(2-r^2)\Gamma(\varsigma)} \int_0^r \int_0^s |s-z|^{\varsigma-1}\, dz\, ds \right) \\
\leq & \ \psi(\|u-v\|_\infty + \|u\|_\infty + \|v\|_\infty) = \psi(\varrho(u, v)).
\end{aligned}
$$

That is, $\varrho(Tu, Tv) \leq \psi(\varrho(u, v))$. Now, define the mapping $\tau : X \times X \longrightarrow \mathbb{R}_+$ as follows:

$$
\tau(x, y) = \begin{cases} 1, & \text{if } \omega(u(t), v(t)) \geq 0, \text{ for all } t \in I \\ 0, & \text{elsewhere,} \end{cases}
$$

and let

$$
h(u, v) = \begin{cases} \frac{\psi(\varrho(u,v))}{\varrho(u,v)}, & \text{if } u \neq v \\ 0, & \text{otherwise.} \end{cases}
$$

Then, for all $u, v \in X$, we get

$$
\begin{aligned}
\tau(u, v)\varrho(Tu, Tv) & \leq \psi(\varrho(u, v)) = \frac{\psi(\varrho(u, v))}{\varrho(u, v)} \varrho(u, v) \\
& = h(u, v)\varrho(u, v) \leq h(u, v)\mathcal{J}(u, v).
\end{aligned}
$$

By setting $\xi(t) = t$ for all $t \in \mathbb{R}_+$, we see that T is a Jaggi-type (τ, h, ξ)-quasi-contraction. Clearly, all the assumptions of Theorem 3 are satisfied. Consequently, there exists $u^* \in X$ such that $u^* = Tu^*$. $\square$

3. Conclusions

This manuscript proposed new forms of quasi-contractions, under the names Jaggi type (τ, h, ξ)-quasi-contraction and Dass-Gupta type (τ, h, ξ)-quasi-contraction in ML spaces, and analyzed novel conditions for the existence of invariant points for such operators. The launched ideas are then utilized to examine the existence criteria for the solutions of boundary value problems in the bodywork of integer and non-integer orders. It is apposite to state that the present concepts in this article, being investigated in an ML setup, are fundamental. Hence, our approach can be moved further to some realms such as b-ML spaces, fuzzy metric spaces, and related quasi or pseudo-metric spaces.

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