mathematics

MDPI

Article

Bifurcation of Limit Cycles and Center in 3D Cubic Systems with Z_3-Equivariant Symmetry

Ting Huang [1,2,*], **Jieping Gu** [3], **Yuting Ouyang** [2] **and Wentao Huang** [2]

[1] School of Computer Engineering, Guangzhou City University of Technology, Guangzhou 510800, China
[2] College of Mathematics and Statistics, Guangxi Normal University, Guilin 541006, China; oy97oy@163.com (Y.O.); huangwentao@163.com (W.H.)
[3] School of Education, Guangxi Vocational Normal University, Nanning 530007, China; gujieping3032@163.com
[*] Correspondence: huangting@gcu.edu.cn

Abstract: This paper focuses on investigating the bifurcation of limit cycles and centers within a specific class of three-dimensional cubic systems possessing Z_3-equivariant symmetry. By calculating the singular point values of the systems, we obtain a necessary condition for a singular point to be a center. Subsequently, the Darboux integral method is employed to demonstrate that this condition is also sufficient. Additionally, we demonstrate that the system can bifurcate 15 small amplitude limit cycles with a distribution pattern of $5-5-5$ originating from the singular points after proper perturbation. This finding represents a novel contribution to the understanding of the number of limit cycles present in three-dimensional cubic systems with Z_3-equivariant symmetry.

Keywords: three-dimensional cubic systems; Z_3-equivariant symmetry; limit cycle; center; Darboux integral method

MSC: 34C05; 34C07

check for updates

Citation: Huang, T.; Gu, J.; Ouyang, Y.; Huang, W. Bifurcation of Limit Cycles and Center in 3D Cubic Systems with Z_3-Equivariant Symmetry. *Mathematics* **2023**, *11*, 2563. https://doi.org/10.3390/math11112563

Academic Editor: Zhanybai T. Zhusubaliyev

Received: 21 April 2023
Revised: 26 May 2023
Accepted: 30 May 2023
Published: 3 June 2023

1. Introduction

The study of limit cycles in differential systems comes from practice, and many applied disciplines, such as physics, mechanics, biology, economics, ecology, electronics, and finance, may exhibit limited cycle phenomena. Limit cycles play a very important role in the study of nonlinear system dynamics. The limit cycle phenomenon was first proposed by H. Poincaré [1], and further development was promoted by the famous Hilbert's 16th problem [2]. The latter part of the sixteenth problem is as follows: what is the maximum number of limit cycles (Hilbert number: $H(n)$) that can be generated by a planar differential system of degree n? For $n=2$, the best result was obtained by Shi [3] and Chen [4] more than forty years ago, $H(2)\geq 4$. For $n=3$, scholars have conducted a lot of research work, and the best result for cubic polynomial systems is $H(3)\geq 13$ [5,6]. For more information of $n=4,5,6,\ldots$, see [7–9].

The limit cycles of a differential autonomous system are always perturbed by singular points, but it is difficult to obtain a higher number of limit cycles in terms of a single singular point. Considering this situation, it is a good direction to study from multiple singular points. In the case of two singular points, Du, Liu and Huang [10] studied a class of Kolmogorov system that is symmetrical about the line $y=x$, and showed that under certain conditions, each singularity can bifurcate into five limit cycles so there are ten limit cycles in total. Du, Huang and Zhang [11] proved that a class of systems symmetric with regard to the origin can generate 12 limit cycles of $6-6$ distribution by Hopf bifurcation. For three singular points, Yu and Han [12] used the perturbation technique to calculate the focus values and proved that a class of the Z_2-equivariant system has at most 12 limit cycles. For four singular points, Wang, Liu and Du [13] considered a class of Z_3-equivariant systems and obtained the conclusion that $H(4)\geq 16$. For more singular points, Yao and Yu [14] applied the normal form and the technique of solving coupled multivariate polynomial equations to prove that the maximal number of small limit cycles is 25 in Z_5-equivariant planar vector fields, so $H(5) \geq 25$. Li, Chan and Chung [15] gave a concrete numerical example of at least 24 limit cycles for a perturbed Hamiltonian system with Z_6 symmetry, and conjectured that $H(2k+1) \geq (2k+1)^2 - 1$ for the perturbed Hamiltonian systems. Li and Zhang [16] used the detection function method to show that $H(7) \geq 49$ by considering a Z_8-equivariant vector field. Through analyzing a Z_{10} symmetric system, Wang and Yu [17] obtained the result $H(9) \geq 80$. Recently, the fractional-order dynamic model has been widely studied. It is considered to be more valuable than the integer-order dynamic model to describe many objective natural phenomena in the real world because of its good memory characteristics and genetic function in many materials and evolutionary processes. Huang et al. [18,19] and Xu et al. [20–22] focused on the understanding of the bifurcation phenomena and dynamical behavior in fractional-order neural networks with different delays. They provide insights into the impact of fractional order and multiple delays on the stability, equilibrium points, and bifurcation patterns in neural network dynamics.

Surprisingly, unlike the planar systems, where the number of limit cycles is finite [23], a simple three-dimensional system may have an infinite number of small amplitude limit cycles [24]. At present Huang, Gu and Wang [25] introduced some results on the bifurcation of limit cycles for three-dimensional smooth systems, such as Lotka–Volterra systems, Chen systems, Lorenz systems, Lü systems, and three-dimensional piecewise smooth systems. Sanchez and Torregrosa [26] studied several classes of three-dimensional polynomial differential systems by using the parallel computing method, and determined that there are at most 11, 31, 54, and 92 limit cycles at the origin for three-dimensional quadratic, cubic, quartic, and quintic systems, respectively. For multiple singular points, Gu et al. [27] proved the existence of seven limit cycles in a class of three-dimensional cubic Kolmogorov systems. Lu et al. [28] proved the existence of eight limit cycles in a class of Lorenz systems, which are Z_2 symmetric and quadratic three-dimensional systems. Guo, Yu and Chen [29] applied the theory of center manifold and normal forms to prove the existence of 12 limit cycles with $4-4-4$ distribution in three-dimensional vector fields with Z_3 symmetry. Then, they analyzed a $3D$ quadratic system with two symmetric singular points by using the same approach in [30] and showed that there are twelve limit cycles with $6-6$ distributed around the singular points.

Based on the above research results, numerous issues remain to be investigated in the dynamics of three-dimensional differential systems. In this study, we analyze the limit cycle bifurcation of a class of Z_3-equivariant symmetric cubic systems by using symmetry to find the focal points with the same topological structure, and the form of the systems is as follows :

$$
\begin{aligned}
\frac{dx}{dt} &= A_{010}y + (A_{010} + A_{030} - \frac{1}{3})xy + A_{020}y^2 - (2A_{010} + A_{030} + \frac{2}{3})x^2y \\
&\quad + 2A_{020}xy^2 + xu^2 + A_{030}y^3, \\
\frac{dy}{dt} &= -x - B_{030}y + x^2 + yu + B_{030}x^2y + 2xy^2 + B_{030}y^3, \\
\frac{du}{dt} &= -xu + y^2u.
\end{aligned} \tag{1}
$$

Regarding the investigations into Hopf bifurcation of ordinary differential models, several classical methods are available. These include the Poincaré normal form, Liapunov–Schmidt method [31] and the dimension reduction method of directly calculating the center manifold, see, for example, [32]. Additionally, various other research approaches have been proposed, such as the averaging theory [33,34], the technique of the inverse Jacobi multiplier [35,36], the simple normal form method [37] and the formal first integral method [38]. Recently, the authors of [39] presented a theorem on the bound of the cyclicity in terms of the Bautin ideal to determine the maximum number of limit cycles within the center manifold generated from a center.

In our research, we employ the linear and straightforward algorithm proposed in a previous work [40] to compute the quantities of singular points without the need for dimensionality reduction. This algorithm is linear in nature and avoids complex integration operations. As a result, the calculations can be easily performed using symbolic computation systems, such as Mathematica. Furthermore, the weight polynomial and characteristic curve methods introduced in [41] are utilized to search for invariant algebraic surfaces, namely, all the Darboux polynomials under certain conditions. Then, the center conditions are derived, and the maximum number of limit cycles via a Hopf bifurcation is determined. Finally, we demonstrate that the system (1) can bifurcate 15 small amplitude limit cycles with a $5-5-5$ distribution from the singular points after proper perturbation. The result is a new lower bound on the number of limit cycles in three-dimensional cubic systems with Z_3-equivariant symmetry.

This paper is structured as follows. Section 2 introduces fundamental theories and methods of differential Hopf bifurcation systems, which are essential for a comprehensive discussion of system (1). In Section 3, we present a recursive formula for singular point values, which is linear and can simplify complex integral operations. As a result, we can conveniently use computer symbolic operation systems, such as Mathematica or Maple, to calculate the singular point values of system (1). In Section 4, we establish a necessary condition for the singular point to be a center, and then we use the Darboux integral method to demonstrate that this condition is also sufficient. Finally, in Section 5, we show that system (1) can bifurcate 15 small amplitude limit cycles with a $5-5-5$ distribution from the singular points after proper perturbation.

2. Preliminary Theory and Method

For the Z_q-equivariant system, we only present the results we need in the next section in this section. For more details, the reader can refer to [42,43]

Lemma 1 (see [43])**.** *If a system is a Z_q-equivariant system and it has a real singular point except for the origin, then the system must have q real singular points, which are q-symmetric about the origin.*

To discuss the limit cycle of a system, it is usually necessary to calculate all the focal values at the singular point. The consequent difficulty arises that the calculation of all the focus values of the singular point of the system is too complicated for us to calculate all the focus values. However, according to Hilbert's finite basis principle, there must be a finite number of focus values that can express all the focus values of the singular point of the system (the finite number of focus values in front is called the focus basis). Wang [44] proved that the first non-zero focal values v_{2m+1} at the origin of system (2) are algebraically equivalent to the first non-zero singular values μ_m at the origin of its complex system (4) (and the Lyapunov constant V_{2m} at the origin of the center manifold of system (2)). Therefore, the study of the focal values of the differential system can be simplified to the investigation of its singular point values. The focus basis issue, namely the highest-order fine focus problem, can be effectively solved in this process, and then the maximum number of limit cycle branches is also analyzed. In this paper, we apply the formal series method proposed by Wang [40] to study system (1).

Next, we show (for details see [40]) some preparations for the calculation of the singular point values of the system. Consider the following three-dimensional system:

$$\begin{aligned}
\frac{dx}{dt} &= -y + \sum_{k+j+l=2}^{\infty} A_{kjl} x^k y^j u^l = X(x, y, u), \\
\frac{dy}{dt} &= x + \sum_{k+j+l=2}^{\infty} B_{kjl} x^k y^j u^l = Y(x, y, u), \\
\frac{du}{dt} &= -du + \sum_{k+j+l=2}^{\infty} d_{kjl} x^k y^j u^l = U(x, y, u),
\end{aligned} \tag{2}$$

where $x, y, u, t, d, A_{kjl}, B_{kjl}, d_{kjl} \in \mathbb{R}(k, j, l \in \mathbb{N})$ and $d > 0$.

The linear matrix of system (2) has a pair of purely imaginary eigenvalues $\pm i$ and a negative eigenvalue $-d$ at the origin of the singular point. Reviewing the related concepts and results, it is obvious that Hopf bifurcation occurs in system (2). The qualitative analysis of the high-dimensional systems is generally utilized to reduce their dimensions to plane systems by using the center manifold theorem, and then the limit cycles, centers, and isochronous centers are studied by means of the plane bifurcation theory. For system (2), there exists an approximation to the center manifold $u = u(x, y)$, which can be represented as the polynomial series in x and y formally as follows:

$$u = u(x, y) = x^2 + y^2 + h.o.t.,$$

where $h.o.t.$ denotes a higher-order term with an order greater than or equal to 3.

By transformation

$$x = \frac{z+w}{2}, \quad y = \frac{(w-z)i}{2}, \quad u = u, \quad t = -Ti, \quad i = \sqrt{-1}, \tag{3}$$

the system (2) can be transformed into the following complex system:

$$\begin{aligned}
\frac{dz}{dT} &= z + \sum_{k+j+l=2}^{\infty} a_{kjl} z^k w^j u^l = Z(z, w, u), \\
\frac{dw}{dT} &= -w - \sum_{k+j+l=2}^{\infty} b_{kjl} w^k z^j u^l = -W(z, w, u), \\
\frac{du}{dT} &= idu + \sum_{k+j+l=2}^{\infty} \tilde{d}_{kjl} z^k w^j u^l = \tilde{U}(z, w, u),
\end{aligned} \tag{4}$$

in which $z, w, T, a_{kjl}, b_{kjl}, \tilde{d}_{kjl} \in \mathbb{C}(k, j, l \in \mathbb{N})$. It is worth noting that the coefficients a_{kjl} and b_{kjl} of system (4) satisfy a conjugate relationship, namely, $b_{kjl} = \overline{a_{kjl}}, k \geq 0, j \geq 0, l \geq 0, k + j + l \geq 2$. System (2) and system (4) are referred to as being concomitant. To make it easier to write $\tilde{d}_{kjl}$ and $\tilde{U}$, we still use d_{kjl}, U.

For system (4), we have the following preliminary results:

Lemma 2 (see [40]). *For system* (4)*, when taking* $c_{110} = 1, c_{101} = c_{011} = c_{200} = c_{020} = 0, c_{kk0} = 0, k = 2, 3, \ldots$, *we can derive successively and uniquely the following series:*

$$F(z, w, u) = zw + \sum_{\alpha+\beta+\gamma=3}^{\infty} c_{\alpha\beta\gamma} z^\alpha w^\beta u^\gamma,$$

such that

$$\frac{dF}{dT} = \frac{\partial F}{\partial z} Z - \frac{\partial F}{\partial w} W + \frac{\partial F}{\partial u} U = \sum_{m=1}^{\infty} \mu_m (zw)^{m+1}.$$

The expression μ_m given in (5) is called the m-th singular point value at the origin of system (4), and if $\alpha \neq \beta$ or $\alpha = \beta, \gamma \neq 0$, $c_{\alpha\beta\gamma}$ is determined by the following recursive formula:

$$c_{\alpha\beta\gamma} = \frac{1}{\beta - \alpha - id\gamma} \sum_{k+j+l=3}^{\alpha+\beta+\gamma+2} \Big[(\alpha - k + 1)a_{k,j-1,l} - (\beta - j + 1)b_{j,k-1,l} \\ +(\gamma - l)d_{k-1,j-1,l+1}\Big] c_{\alpha-k+1,\beta-j+1,\gamma-l},$$

and for any positive integer m, μ_m is determined by the following recursive formula:

$$\mu_m = \sum_{k+j+l=3}^{2(m+1)} \Big[(m - k + 2)a_{k,j-1,l} - (m - j + 2)b_{j,k-1,l} - ld_{k-1,j-1,l+1}\Big] c_{m-k+2,m-j+2,-l}, \quad (5)$$

and if $\alpha < 0$ or $\beta < 0$ or $\gamma < 0$ or $\gamma = 0, \alpha = \beta$, we put $c_{\alpha\beta\gamma} = 0$.

A singular point of a system becomes a center if and only if all its focal values are zero. So we have the following results.

Definition 1 (see [45])**.** *For system (2), if $v_1 \neq 1$, then the origin is called the rough focus (strong focus); if $v_1 = 1$, and $v_2 = v_3 = \cdots = v_{2k} = 0, v_{2k+1} \neq 0$, then the origin is called the fine focus (weak focus) of order k, and the quantity of v_{2k+1} is called the k-th focal value at the origin $(k = 1, 2, \ldots)$; if $v_1 = 1$, and for any positive integer k, $v_{2k+1} = 0$, then the origin is called a center.*

Lemma 3 (see [44])**.** *Let μ_k be the k-th singular point value at the origin of system (4), and v_{2k+1} be the k-th focal value at the origin of system (2) $(k = 1, 2, \ldots)$. If $\mu_1 = \mu_2 = \cdots = \mu_{k-1} = 0$, then*

$$v_{2k+1} = i\pi\mu_k. \quad (6)$$

The maximum number of limit cycles bifurcated from the fine focus of the system is closely related to the highest order of the fine focus. So solving the highest-order fine focus question of the system is equivalent to analyzing the issue of the maximum number of limit cycle bifurcations as follows:

Lemma 4 (see [46])**.** *If the origin of system (2) is a fine focus of order k, then the origin of system (2) can bifurcate at most k small-amplitude limit cycles under a suitable perturbation.*

Lemma 5 (see [45])**.** *Suppose that the focal values v_i of system (2) associated with the origin are determined by k independent parameters $\lambda_1, \lambda_2, \ldots, \lambda_k$, i.e., $v_i = v_i(\lambda_1, \lambda_2, \ldots, \lambda_k), i = 0, 1, \ldots, k$. If there exists a point $\lambda^* = (\lambda_1^*, \lambda_2^*, \ldots, \lambda_k^*)$ such that $v_3(\lambda^*) = v_5(\lambda^*) = \cdots = v_{2k-1}(\lambda^*) = 0, v_{2k+1}(\lambda^*) \neq 0$, and*

$$\det\left[\frac{\partial(v_3, v_5, \ldots, v_{2k-1})}{\partial(\lambda_1, \lambda_2, \ldots, \lambda_k)}\right]_{\lambda^*} \neq 0,$$

then system (2) has exactly k small-amplitude limit cycles bifurcating from the origin under a suitable perturbation on λ^*.

3. Singular Point Values

For the Z_q-equivariant symmetric system, it can be seen from Lemma 1 that its singular points appear as multiples of q. Moreover, these q singular points have the same topological structure, so their limit cycles are also generated in multiples of q, and the properties around the corresponding limit cycles are similar. Therefore, when we study the center-focus problem and the limit cycle bifurcation of the Z_q-equivariant symmetric system, we only need to select one of the q singular points with the same topological structure for research. We can easily determine that the points $(1,0,0)$, $(-\frac{1}{2}, \frac{\sqrt{3}}{2}, 0)$ and $(-\frac{1}{2}, -\frac{\sqrt{3}}{2}, 0)$ are three singular points of the system (1), which are symmetric about the Z-axis. So we only need to choose one of them, and at the same time for the convenience of calculation, we only calculate the singular point values of the singular point $(1,0,0)$.

To calculate the values of a single non-origin singular point, the following steps are typically involved: First, the singular point is moved to the origin by a transformation. Second, the real system is changed into a complex system by complex transformation. Third, calculate the singular point value in the complex system.

Firstly, let

$$x = x_1 + 1, \quad y = y_1, \quad u = u_1,$$

and system (1) can be rewritten in the following form:

$$\begin{aligned}
\frac{dx_1}{dt} &= -y_1 + u_1^2 + u_1^2 x_1 - \frac{5x_1y_1}{3} - 3A_{010}x_1y_1 - A_{030}x_1y_1 - \frac{2x_1^2y_1}{3} - 2A_{010}x_1^2y_1 \\
&\quad - A_{030}x_1^2y_1 + 3A_{020}y_1^2 + 2A_{020}x_1y_1^2 + A_{030}y_1^3, \\
\frac{dy_1}{dt} &= x_1 + x_1^2 + u_1y_1 + 2B_{030}x_1y_1 + B_{030}x_1^2y_1 + 2y_1^2 + 2x_1y_1^2 + B_{030}y_1^3, \\
\frac{du_1}{dt} &= -u_1 - u_1x_1 + u_1y_1^2.
\end{aligned} \tag{7}$$

Then, let

$$x = \frac{z+w}{2}, \quad y = \frac{(w-z)i}{2}, \quad u = u, \quad t = -Ti, \quad i = \sqrt{-1},$$

and system (7) can be rewritten in the following form:

$$\begin{aligned}
\frac{dz}{dT} &= z - iu^2 + \frac{iuw}{2} - \frac{iu^2w}{2} - \frac{2w^2}{3} - \frac{3A_{010}w^2}{4} + \frac{3iA_{020}w^2}{4} - \frac{A_{030}w^2}{4} + \frac{iB_{030}w^2}{2} \\
&\quad - \frac{w^3}{3} - \frac{A_{010}w^3}{4} + \frac{iA_{020}w^3}{4} - \frac{A_{030}w^3}{4} - \frac{iuz}{2} - \frac{iu^2z}{2} + \frac{3wz}{2} - \frac{3iA_{020}wz}{2} \\
&\quad + \frac{w^2z}{6} - \frac{A_{010}w^2z}{4} - \frac{iA020w^2z}{4} + \frac{A_{030}w^2z}{4} + \frac{iB_{030}w^2z}{2} + \frac{z^2}{6} + \frac{3A_{010}z^2}{4} \\
&\quad + \frac{3iA_{020}z^2}{4} + \frac{A_{030}z^2}{4} - \frac{iB_{030}z^2}{2} + \frac{wz^2}{3} + \frac{A_{010}wz^2}{4} - \frac{iA_{020}wz^2}{4} - \frac{A_{030}wz^2}{4} \\
&\quad - \frac{iB_{030}wz^2}{2} - \frac{z^3}{6} + \frac{A_{010}z^3}{4} + \frac{iA_{020}z^3}{4} + \frac{A_{030}z^3}{4},
\end{aligned}$$

$$\begin{aligned}
\frac{dw}{dT} &= -w - (iu^2 - \frac{iuw}{2} + \frac{iu^2w}{2} - \frac{2w^2}{3} - \frac{3A_{010}w^2}{4} - \frac{3iA_{020}w^2}{4} - \frac{A_{030}w^2}{4} - \frac{iB_{030}w^2}{2} \\
&\quad - \frac{w^3}{3} - \frac{A_{010}w^3}{4} - \frac{iA_{020}w^3}{4} - \frac{A_{030}w^3}{4} + \frac{iuz}{2} + \frac{iu^2z}{2} + \frac{3wz}{2} + \frac{3iA_{020}wz}{2} \\
&\quad + \frac{w^2z}{6} - \frac{A_{010}w^2z}{4} + \frac{iA020w^2z}{4} + \frac{A_{030}w^2z}{4} - \frac{iB_{030}w^2z}{2} + \frac{z^2}{6} + \frac{3A_{010}z^2}{4} \\
&\quad - \frac{3iA_{020}z^2}{4} + \frac{A_{030}z^2}{4} + \frac{iB_{030}z^2}{2} + \frac{wz^2}{3} + \frac{A_{010}wz^2}{4} + \frac{iA_{020}wz^2}{4} - \frac{A_{030}wz^2}{4} \\
&\quad + \frac{iB_{030}wz^2}{2} - \frac{z^3}{6} + \frac{A_{010}z^3}{4} - \frac{iA_{020}z^3}{4} + \frac{A_{030}z^3}{4}),
\end{aligned}$$

$$\frac{du}{dT} = iu + \frac{iuw}{2} + \frac{iuw^2}{4} + \frac{iuz}{2} - \frac{iuwz}{2} + \frac{iuz^2}{4}. \tag{8}$$

Then, by using Lemma 2, we can calculate the singular point values of the singular point $(0,0,0)$ of the complex system (8). With the aid of ycomputer algebra software Mathematica 12, the recurrence formula can be obtained as follows:

Lemma 6. *For system (8), the singular point values $\mu_m (m = 1, 2, \ldots)$ at the origin are determined by the following recursive formulas:*

$$\begin{aligned}\mu_m = \frac{1}{12}\{&(4 + 3A_{010} + 3iA_{020} + 3A_{030})(2 + m)c_{-2+m,2+m,0}\\ &+ 2(-3iA_{020} - 3A_{030} + 3iB_{030} - 2m + 3A_{010}m + 3iB_{030}m)c_{-1+m,1+m,0}\\ &+ (8 + 9A_{010} + 9iA_{020} + 3A_{030} + 6iB_{030})(2 + m)c_{-1+m,2+m,0}\\ &+ [-18 + (-16 + 9A_{010} + 3A_{030} - 6iB_{030})m - 9iA_{020}(2 + m)]c_{m,1+m,0}\\ &+ 2(-3iA_{020} + 3A_{030} + 3iB_{030} + 2m - 3A_{010}m + 3iB_{030}m)c_{1+m,-1+m,0}\\ &- [-18 + (-16 + 9A_{010} + 3A_{030} + 6iB_{030})m + 9iA_{020}(2 + m)]c_{1+m,m,0}\\ &- (4 + 3A_{010} - 3iA_{020} + 3A - 030)(2 + m)c_{2+m,-2+m,0}\\ &- (8 + 9A_{010} - 9iA_{020} + 3A_{030} - 6iB_{030})(2 + m)c_{2+m,-1+m,0}\\ &- 6i(A_{020} + 2B_{030})mc_{m,m,0}\},\end{aligned}$$

where $c_{k,k,0} = 0, k = 2, 3, \ldots$, and if $k < 0$ or $j < 0$ or $l < 0$, we let $c_{k,j,l} = 0$, else

$$\begin{aligned}c_{\alpha\beta\gamma} = \frac{1}{12(-\alpha + \beta - i\gamma)}\{&(4 + 3A_{010} + 3iA_{020} + 3A_{030})(1 + \beta)c_{-3+\alpha,1+\beta,\gamma} + [4 - 6A_{030}\\ &- 2\alpha + 3A_{030}\alpha + 3iA_{020}(-2 + \alpha - \beta) - 2\beta - 3A_{030}\beta + 6iB_{030}\beta + 3i\gamma\\ &+ 3A_{010}(-2 + \alpha + \beta)]c_{-2+\alpha,\beta,\gamma} + (8 + 9A_{010} + 9iA_{020} + 3A_{030} + 6iB_{030})\\ &\times (1 + \beta)c_{-2+\alpha,1+\beta,\gamma} + [4\alpha + 3A_{010}\alpha - 3A_{030}\alpha - 4\beta - 3A010\beta + 3A_{030}\beta\\ &- 3iA_{020}(-2 + \alpha + \beta) - 6iB_{030}(-2 + \alpha + \beta) - 6i\gamma]c_{-1+\alpha,-1+\beta,\gamma} + [(2\\ &+ 9iA_{020} + 3A_{030} - 6iB_{030})(-1 + \alpha) - 18i(-i + A_{020})\beta + 6i\gamma]c_{-1+\alpha,\beta,\gamma}\\ &- 6i(1 + \beta)c_{-1+\alpha,1+\beta,-2+\gamma} + 6i(1 + \beta)c_{-1+\alpha,1+\beta,-1+\gamma} + [-4 + 6A_{030} + 2\alpha\\ &+ 3A_{030}\alpha + 6iB_{030}\alpha - 3iA_{020}(2 + \alpha - \beta) + 2\beta - 3A_{030}\beta - 3A_{010}(-2 + \alpha\\ &+ \beta) + 3i\gamma]c_{\alpha,-2+\beta,\gamma} + [18(1 - iA_{020})\alpha - (2 + 9A_{010} - 9iA_{020} + 3A_{030}\\ &+ 6iB_{030})(-1 + \beta) + 6i\gamma]c_{\alpha,-1+\beta,\gamma} - 6i(\alpha + \beta)c_{\alpha,\beta,-2+\gamma}\\ &- 6i(\alpha + \beta)c_{\alpha,\beta,-1+\gamma} - 12i(1 + \beta)c_{\alpha,1+\beta,-2+\gamma}\\ &- (4 + 3A_{010} - 3iA_{020} + 3A_{030})(1 + \alpha)c_{1+\alpha,-3+\beta,\gamma}\\ &- (8 + 9A_{010} - 9iA_{020} + 3A_{030} - 6iB_{030})(1 + \alpha)c_{1+\alpha,-2+\beta,\gamma}\\ &- 6i(1 + \alpha)c_{1+\alpha,-1+\beta,-2+\gamma} + 6i(1 + \alpha)c_{1+\alpha,-1+\beta,-1+\gamma}\\ &- 12i(1 + \alpha)c_{1+\alpha,\beta,-2+\gamma}\}.\end{aligned}$$

Next, using the computer algebra system Mathematica, the recursive formula is programmed, and the singular point values of system (8) corresponding to the Hopf critical point located at the origin are calculated, and the following theorem is obtained.

Theorem 1. *For the flow on the center manifold of system (8), the first five singular point values at the origin are given as follows:*

$$\mu_1 = \frac{1}{4}i(-9A_{020} + 9A_{010}A_{020} + 3A_{020}A_{030} + 2B_{030}).$$

If $B_{030} = -\frac{3}{2}(-3A_{020} + 3A_{010}A_{020} + A_{020}A_{030})$, then $\mu_2 = -\frac{1}{144}iA_{020}F_1$, where

$$\begin{aligned}F_1 = 258 &+ 1503A_{010} - 1539A_{010}^2 - 486A_{010}^3 + 6318A_{020}^2 - 2673A_{010}A_{020}^2 - 12393A_{010}^2A_{020}^2\\ &+ 8748A_{010}^3A_{020}^2 + 1117A_{030} - 1890A_{010}A_{030} - 162A_{010}^2A_{030} - 891A_{020}^2A_{030}\\ &- 8262A_{010}A_{020}^2A_{030} + 8748A_{010}^2A_{020}^2A_{030} - 459A_{030}^2 + 54A_{010}A_{030}^2 - 1377A_{020}^2A_{030}^2\\ &+ 2916A_{010}A_{020}^2A_{030}^2 + 18A_{030}^3 + 324A_{020}^2A_{030}^3.\end{aligned}$$

Case 1 : *if* $A_{020} = 0$, *then* $B_{030} = 0$ *and*

$$\mu_2 = \mu_3 = \mu_4 = \mu_5 = 0.$$

Case 2 : *if* $A_{020} \neq 0$, *then*

$$\mu_3 = \frac{1}{124416} i A_{020} H_3,$$
$$\mu_4 == -\frac{1}{134369280} i A_{020} H_4,$$
$$\mu_5 = \frac{1}{348285173760} i A_{020} H_5,$$

where

$$\begin{aligned}H_3 = {} & 6246384 + 27244530A_{010} - 68699907A_{010}^2 + 32937192A_{010}^3 + 4279959A_{010}^4 - 2401326A_{010}^5 \\ & + 157002786A_{020}^2 - 132687234A_{010}A_{020}^2 - 429523884A_{010}^2A_{020}^2 + 648917892A_{010}^3A_{020}^2 \\ & - 270719982A_{010}^4A_{020}^2 + 25863462A_{010}^5A_{020}^2 + 505616904A_{020}^4 - 811136430A_{010}A_{020}^4 \\ & - 872685171A_{010}^2A_{020}^4 + 2549853918A_{010}^3A_{020}^4 - 1760664033A_{010}^4A_{020}^4 + 22767910A_{030} \\ & + 389014812A_{010}^5A_{020}^4 - 82142550A_{010}A_{030} + 66573900A_{010}^2A_{030} - 6467688A_{010}^3A_{030} \\ & - 2401326A_{010}^4A_{030} - 15339942A_{020}^2A_{030} - 389539692A_{010}A_{020}^2A_{030} \\ & + 789431184A_{010}^2A_{020}^2A_{030} - 444022236A_{010}^3A_{020}^2A_{030} + 60348078A_{010}^4A_{020}^2A_{030} \\ & - 270378810A_{020}^4A_{030} - 581790114A_{010}A_{020}^4A_{030} + 2549853918A_{010}^2A_{020}^4A_{030} \\ & - 2347552044A_{010}^3A_{020}^4A_{030} + 648358020A_{010}^4A_{020}^4A_{030} - 20509215A_{030}^2 \\ & + 34447464A_{010}A_{030}^2 - 9670914A_{010}^2A_{030}^2 - 533628A_{010}^3A_{030}^2 - 82121688A_{020}^2A_{030}^2 \\ & + 309981492A_{010}A_{020}^2A_{030}^2 - 263542248A_{010}^2A_{020}^2A_{030}^2 + 51726924A_{010}^3A_{020}^2A_{030}^2 \\ & - 96965019A_{020}^4A_{030}^2 + 849951306A_{010}A_{020}^4A_{030}^2 - 1173776022A_{010}^2A_{020}^4A_{030}^2 \\ & + 432238680A_{010}^3A_{020}^4A_{030}^2 + 5305284A_{030}^3 - 3657312A_{010}A_{030}^3 + 177876A_{010}^2A_{030}^3 \\ & + 39646584A_{020}^2A_{030}^3 - 67794084A_{010}A_{020}^2A_{030}^3 + 21073932A_{010}^2A_{020}^2A_{030}^3 \\ & + 94439034A_{020}^4A_{030}^3 - 260839116A_{010}A_{020}^4A_{030}^3 + 144079560A_{010}^2A_{020}^4A_{030}^3 \\ & - 436941A_{030}^4 + 88938A_{010}A_{030}^4 - 6418602A_{020}^2A_{030}^4 + 4150926A_{010}A_{020}^2A_{030}^4 \\ & - 21736593A_{020}^4A_{030}^4 + 24013260A_{010}A_{020}^4A_{030}^4 + 9882A_{030}^5 + 319302A_{020}^2A_{030}^5 \\ & + 1600884A_{020}^4A_{030}^5,\end{aligned}$$

H_4 and H_5 are given in Appendix A. For every μ_k that was calculated, we already imposed that $\mu_1 = \mu_2 = \cdots = \mu_{k-1} = 0, k = 2, 3, \ldots, 5$.

Because μ_3, μ_4 and μ_5 are so complicated, we can simplify them appropriately. According to Theorem 1, the simplification of μ_3, μ_4, and μ_5 is equivalent to the simplification of H_3, H_4 and H_5. To simplify the calculation, we can first consider whether u_3 can be simplified. Obviously, with the program command *Factor*, *PolynomialReduce*, we can easily find the greatest common factor (G_3) of F_1 and H_3. Then $\mu_3 = \frac{1}{124416} i A_{020} H_3$ can be reduced to $\mu_3 = -\frac{1}{331776} i A_{020} F_2$ by factorization. The same method is adopted for the simplification of μ_4 and μ_5, so we can obtain the following theorem.

Theorem 2. *Through factorization and polynomial reduction, the first five order singular point values of system (8) can be reduced to the following form:*

$$\mu_1 = \frac{1}{4} i(-9A_{020} + 9A_{010}A_{020} + 3A_{020}A_{030} + 2B_{030}).$$

If $B_{030} = -\frac{3}{2}(-3A_{020} + 3A_{010}A_{020} + A_{020}A_{030})$, *then* $\mu_2 = -\frac{1}{144} i A_{020} F_1$.

Case 1 : *if* $A_{020} = 0$, *then* $B_{030} = 0$ *and*

$$\mu_2 = \mu_3 = \mu_4 = \mu_5 = 0.$$

Case 2 : *if* $A_{020} \neq 0$, *then*

$$\begin{aligned} \mu_3 &= -\frac{1}{331776} i A_{020} F_2, \\ \mu_4 &= \frac{1}{1911029760} i A_{020} F_3, \\ \mu_5 &= -i A_{020} F_4, \end{aligned}$$

where

$$\begin{aligned} F_2 = &-4830390 - 19659213A_{010} + 23735349A_{010}^2 + 6510942A_{010}^3 - 3726648A_{010}^4 - 629856A_{010}^5 \\ &- 53937738A_{020}^2 + 184334211A_{010}A_{020}^2 - 138172473A_{010}^2A_{020}^2 + 491287680A_{020}^4 \\ &- 944784000A_{010}A_{020}^4 + 453496320A_{010}^2A_{020}^4 - 14484727A_{030} + 37293750A_{010}A_{030} \\ &- 6818742A_{010}^2A_{030} - 6928416A_{010}^3A_{030} - 1049760A_{010}^4A_{030} + 171803025A_{020}^2A_{030} \\ &- 293577534A_{010}A_{020}^2A_{030} + 93177864A_{010}^2A_{020}^2A_{030} - 314928000A_{020}^4A_{030} \\ &+ 302330880A_{010}A_{020}^4A_{030} + 12608709A_{030}^2 - 10066734A_{010}A_{030}^2 - 3845232A_{010}^2A_{030}^2 \\ &- 419904A_{010}^3A_{030}^2 - 82506681A_{020}^2A_{030}^2 + 62118576A_{010}A_{020}^2A_{030}^2 + 50388480A_{020}^4A_{030}^2 \\ &- 2356794A_{030}^3 - 806112A_{010}A_{030}^3 + 46656A_{010}^2A_{030}^3 + 10353096A_{020}^2A_{030}^3 \\ &- 52056A_{030}^4 + 54432A_{010}A_{030}^4 + 7776A_{030}^5, \end{aligned}$$

F_3 and F_4 are given in Appendix A. For every μ_k that was calculated, we already imposed that $\mu_1 = \mu_2 = \cdots = \mu_{k-1} = 0, k = 2, 3, \ldots, 5$.

4. Center Condition

In this section, we obtain the necessary and sufficient condition for the singular point $(1, 0, 0)$ of system (1) to be a center by using the singular point values and the Darboux integral method.

From Theorem 2, we obtain the following result.

Theorem 3. *For system (7), the first 5 singular point values at the origin vanish if and only if $B_{030} = -\frac{3}{2}(-3A_{020} + 3A_{010}A_{020} + A_{020}A_{030})$ and $A_{020} = 0$.*

Proof. First, we prove its sufficiency, and substitute $B_{030} = -\frac{3}{2}(-3A_{020} + 3A_{010}A_{020} + A_{020}A_{030}), A_{020} = 0$ into the expressions of the first five singular point values in Theorem 2. It is clear that $\mu_1 = \mu_2 = \mu_3 = \mu_4 = \mu_5 = 0$ is established. The sufficiency of Theorem 3 is proved. Next, we prove that the condition in Theorem 3 is also necessary. Taking the first singular point value $\mu_1 = 0$, then we obtain $B_{030} = -\frac{3}{2}(-3A_{020} + 3A_{010}A_{020} + A_{020}A_{030})$. Next, taking the second singular point value $\mu_2 = 0$, then we obtain $A_{020} = 0$ or $A_{020} \neq 0, F_1 = 0$, so we need to have a case-by-case discussion. Case A : If $A_{020} = 0$, that is $B_{030} = 0$, then under this condition, Case 1 of theorem 2 holds, so this situation discussion is completed. Case B : If $A_{020} \neq 0$, then Case 2 of Theorem 2 holds. Next, we need to determine whether $\mu_2 = \mu_3 = \mu_4 = \mu_5 = 0$ is true, that is, to discuss whether or not the polynomials F_1, F_2, F_3 and F_4 have common zeros. For this purpose, we apply the command "GroebnerBasis" in Mathematica to calculate the Gröbner basis of the $< F_1, F_2, F_3, F_4 >$ about the independent variables A_{010}, A_{020} and A_{030}. We obtain

$$\text{GroebnerBasis}[\{F_1, F_2, F_3, F_4\}, \{A_{010}, A_{020}, A_{030}\}] = \{1\}.$$

This indicates that the polynomials F_1, F_2, F_3 and F_4 have no common zeros. Therefore, in Case 2, the first five singular values of system (8) (i.e., (7)) cannot be zero at the same time. In conclusion, the proof of Theorem 3 is complete. □

It is necessary that the system's first finite focal values are equal to zero for a singular point to become a center. In order to obtain the necessary and sufficient center condition, its sufficiency needs to be proved. The common methods of proof include using the symmetry principle, finding the first integral means, determining the integral factor method, etc. It is not difficult to see that Theorem 3 gives the necessary condition for the singular point $(0,0,0)$ to be a center of system (7).

Theorem 4. *For system (7), if and only if $B_{030} = -\frac{3}{2}(-3A_{020} + 3A_{010}A_{020} + A_{020}A_{030})$ and $A_{020} = 0$, the surface $\Gamma : u_1 = 0$ is an invariant algebraic surface, which defines a global center manifold, and on this center manifold, the origin of system (7) is a center.*

Proof. The necessity of Theorem 4 is obvious. Now we use the Darboux integrable theory [30,47] to demonstrate the sufficiency of this condition.

When condition($B_{030} = -\frac{3}{2}(-3A_{020} + 3A_{010}A_{020} + A_{020}A_{030})$ and $A_{020} = 0$) in Theorem 4 holds, system (7) can be rewritten in the form

$$\begin{aligned} \frac{dx_1}{dt} &= -y_1 + u_1^2 + u_1^2 x_1 - \frac{5x_1y_1}{3} - 3A_{010}x_1y_1 - A_{030}x_1y_1 - \frac{2x_1^2y_1}{3} - 2A_{010}x_1^2y_1 \\ &\quad - A_{030}x_1^2y_1 + A_{030}y_1^3, \\ \frac{dy_1}{dt} &= x_1 + x_1^2 + u_1y_1 + 2y_1^2 + 2x_1y_1^2, \\ \frac{du_1}{dt} &= -u_1 - u_1x_1 + u_1y_1^2. \end{aligned} \tag{9}$$

It is known from [47] that for the following system

$$\frac{dx}{dt} = P(x,y,u), \frac{dy}{dt} = Q(x,y,u), \frac{du}{dt} = R(x,y,u), \tag{10}$$

where the degree of $P(x,y,u)$,$Q(x,y,u)$,$R(x,y,u)$ does not exceed n. The polynomial equation $F(x,y,u) = 0$ is an invariant algebraic surface of system (10) if and only if there exists a polynomial $K(x,y,u)$, which satisfies the following condition:

$$\left.\frac{dF}{dt}\right|_{(9)} = \frac{\partial F}{\partial x}P + \frac{\partial F}{\partial y}Q + \frac{\partial F}{\partial u}R = KF, \tag{11}$$

where the polynomial F is called the Dardoux polynomial of system (10). $F(x,y,u)$ is generally defined as

$$F(x,y,u) = \sum_{i=1}^{m} F_i(x,y,u),$$

where m is an integer not less than 1, $F_i(x,y,u)$ is a homogeneous polynomial of degree i, and $F_i(x,y,u) \neq 0$. The polynomial K is the cofactor of $F(x,y,u) = 0$, and its degree does not exceed $n-1$.

For system (9), in order to prove the existence of the Dardoux polynomial $F(x_1,y_1,u_1)$ by searching from $m = 1$, for $m = 1$, let

$$F(x_1,y_1,u_1) = c_1x_1 + c_2y_1 + c_3u_1. \tag{12}$$

Since system (9) is of degree 3, the corresponding cofactor $K(x_1,y_1,u_1)$ has a degree of at most two, and we assume that

$$K(x_1,y_1,u_1) = h_0 + h_1x_1 + h_2y_1 + h_3u_1 + h_4x_1^2 + h_5y_1^2 + h_6u_1^2 + h_7x_1y_1 + h_8x_1u_1 + h_9y_1u_1. \tag{13}$$

After we substitute (13) and (12) into (11) and obtain an algebraic equation, then comparing the coefficients of the same powers of the equation, we can obtain the following algebraic equations:

$$\begin{aligned}
&c_3h_0 = -c_3, \quad c_1 = c_3h_3, \quad 0 = c_3h_6, \quad -c_1 = c_2h_0, \quad c_3h_2 + c_2h_3 = c_2,\\
&c_2h_2 = 2c_2, \quad c_2h_6 + c_3h_9 = 0, \quad c_3h_5 + c_2h_9 = c_3, \quad c_2h_5 = A_{030}c_1,\\
&c_1h_0 = c_2, \quad c_3h_1 + c_1h_3 = -c_3, \quad c_1h_6 + c_3h_8 = c_1, \quad c_3h_7 + c_2h_8 + c_1h_9 = 0\\
&c_2h_1 + c_1h_2 = -\frac{5c_1}{3} - 3A_{010}c_1 - A_{030}c_1, \quad c_1h_5 + c_2h_7 = 2c_2, \quad c_1h_1 = c_2,\\
&c_3h_4 + c_1h_8 = 0, \quad c_2h_4 + c_1h_7 = -\frac{2c_1}{3} - 2A_{010}c_1 - A_{030}c_1, \quad c_1h_4 = 0.
\end{aligned} \tag{14}$$

Solving the algebraic equations (14), we can obtain $c_1 = 0, c_2 = 0$ and $c_3 = 1$, which results in $F(x_1, y_1, u_1) = u_1$ so that the surface $\Gamma : F(x_1, y_1, u_1) = u_1 = 0$ is truly an invariant algebraic surface of system (9).

Next, we will demonstrate that the surface $\Gamma : F(x_1, y_1, u_1) = u_1 = 0$ is actually a global center manifold of system (9). First, we calculate the normal vector of the surface $F(x_1, y_1, u_1) = u_1$ at the origin, and obtain $\nabla F(0,0,0) = (0,0,1)$. Since the eigenvalues of the linear matrix of system (9) at the origin are two purely imaginary eigenvalues $\pm i$ and a negative eigenvalue $-d$, it is obvious that the tangent space of the center manifold at the origin is spanned by these vectors:

$$\vec{e}_1 = (0, -1, 0), \quad \vec{e}_2 = (1, 0, 0).$$

In addition, one can verify that

$$\nabla F(0,0,0) \cdot \vec{e}_1 = 0, \quad \nabla F(0,0,0) \cdot \vec{e}_2 = 0,$$

which means that the surface $F(x_1, y_1, u_1) = u_1 = 0$ is truly a global center manifold of system (9). Finally, it is necessary to prove that the origin of system (9) is a center on the center manifold $F(x_1, y_1, u_1) = u_1 = 0$. When $u_1 = 0$, system (9) is simplified to the following planar system:

$$\begin{aligned}
\frac{dx_1}{dt} =& -y_1 - \frac{5x_1y_1}{3} - 3A_{010}x_1y_1 - A_{030}x_1y_1 - \frac{2x_1^2y_1}{3} - 2A_{010}x_1^2y_1\\
&- A_{030}x_1^2y_1 + A_{030}y_1^3,\\
\frac{dy_1}{dt} =& \, x_1 + x_1^2 + 2y_1^2 + 2x_1y_1^2.
\end{aligned} \tag{15}$$

It is easy to see that system (15) is a symmetric system about the X-axis, and according to the principle of symmetry, the origin of the system is a center (i.e., the singular point $(0,0,0)$ of system (7) is a center). This completes the proof of Theorem 4. □

Next, we use the system (9) to simulate the case of center at $(0,0,0)$. All trajectories starting from the initial points on the invariant surface will remain on the surface. For example, when the initial points are chosen as $(x_1, y_1, u_1) = (0.05, 0.02, 0)$, the periodic orbits are located on the invariant surface as shown in Figure 1.

However, if the initial points are chosen from outside the invariant surface, then all trajectories first converge to the invariant surface $u_1 = 0$, and once they reach the invariant surface, they become periodic orbits on the invariant surface, as shown in Figure 1, where one initial point is chosen below the invariant surface, and another initial point is chosen above the invariant surface given by $(x_1, y_1, u_1) = (-0.1, -0.1, -0.01), (0.07, 0.03, 0.01)$.

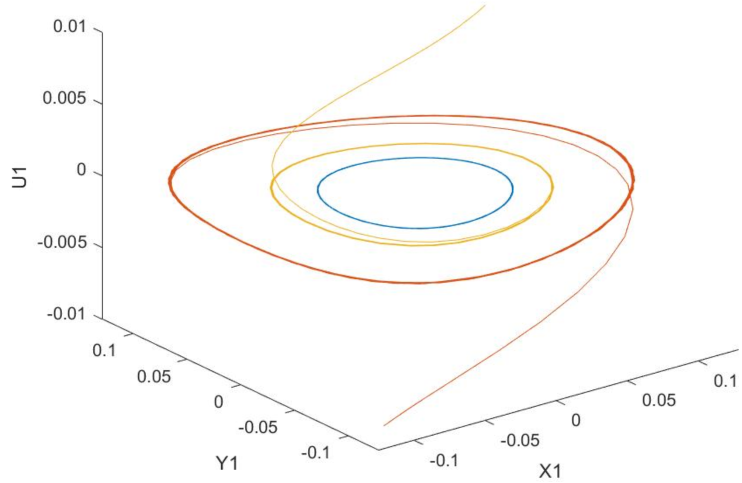

Figure 1. Phase portrait of (9), The blue line is the trajectory of the initial point $(x_1, y_1, u_1) = (0.05, 0.02, 0)$, the yellow line is the trajectory of the initial point $(x_1, y_1, u_1) = (0.07, 0.03, 0.01)$ and the red line is the trajectory of the initial point (x_1, y_1, u_1)=$(-0.1, -0.1, -0.01)$.

According to Theorem 4, we can obtain the following.

Theorem 5. *For system* (1)*, if and only if* $B_{030} = -\frac{3}{2}(-3A_{020} + 3A_{010}A_{020} + A_{020}A_{030})$ *and* $A_{020} = 0$*, the singular point* $(1, 0, 0)$ *is a center on the center manifold.*

5. Bifurcation of Limit Cycles

In this section, we will discuss the maximum number of limit cycles for system (1) branching from the singular point $(1, 0, 0)$.

From Lemma 4, we first need to consider the highest order of the fine focus of the origin of system (7), i.e., find the value of k such that $v_3 = \cdots = v_{2k} = 0$, but $v_{2k+1} \neq 0$ holds. From Theorem 1 and Lemma 3, we obtain the first nine focal values of system (7) at the origin: $v_{2m+1} = i\pi\mu_m$ $(m = 1, 2, \ldots, 5)$.

According to the discussion of Theorem 3 and Theorem 4, we can know that the value of k is 5, so the highest degree of the fine focus at the origin of system (7) is 5. Then, can the order of the fine focus at the origin of system (7) reach 5? That is, we discuss whether system (7) has such a set of parameter conditions that $v_3 = v_5 = v_7 = v_9 = 0$, but $v_{11} \neq 0$. Here is what we will study next.

From the proof of Theorem 3, as long as the condition ($B_{030} = -\frac{3}{2}(-3A_{020} + 3A_{010}A_{020} + A_{020}A_{030})$) is met, we can simply find the values of the coefficients that make $v_3 = 0$ true.

So, we now only need to look for the values of the coefficients such that $v_5 = v_7 = v_9 = 0, v_{11} \neq 0$. According to the proof of Theorem 3, we know that the polynomials F_1, F_2, F_3 and F_4 have no common root, but F_1, F_2 and F_3 should have. Then, finding the coefficient values such that $v_5 = v_7 = v_9 = 0, v_{11} \neq 0$ holds is equivalent to solving the equations $F_1 = 0, F_2 = 0, F_3 = 0$ for finding the solutions of A_{010}, A_{020}, A_{030}. With the help of Mathematica,

$$NSolve[F_1 == 0, F_2 == 0, F_3 == 0, A_{010}, A_{020}, A_{030}],$$

we can obtain that this equation has four groups of real solutions, and take one of them as follows (only showing the first 50 digits):

$$\begin{aligned} A_{010} &= 4.5629298768000231029699982427738544924493861105275\ldots, \\ A_{020} &= 2.3050977788352747166914549291286457145886259511924\ldots, \\ A_{030} &= -10.163059853074365253530894833667331792823007039006\ldots, \end{aligned} \tag{16}$$

under which

$$\begin{aligned} &F_1 = 0, \quad F_2 = 0, \quad F_3 = 0, \\ &F_4 = 9.9753317578388292503592866856452263744010574823802390 58 \ldots \times 10^{35}. \end{aligned}$$

This result implies that there is a common solution such that $F_1 = F_2 = F_3 = 0$ but $F_4 \neq 0$, namely $v_5 = v_7 = v_9 = 0, v_{11} \neq 0$. This means that we found the values of the coefficients that make $v_3 = v_5 = v_7 = v_9 = 0, v_{11} \neq 0$ to be true. Substitute the value of (16) in the condition $B_{030} = -\frac{3}{2}(-3A_{020} + 3A_{010}A_{020} + A_{020}A_{030})$, obtaining

$$B_{030} = -1.8177878129715659867279118826288392922662277542539 1 \ldots . \tag{17}$$

Hence, there exists a set of parameter conditions of system (7) such that $v_3 = v_5 = v_7 = v_9 = 0$, but $v_{11} \neq 0$, that is, the origin of system (7) is a fine focus of the fifth order, i.e., the singular point $(1, 0, 0)$ of system (1) is a fine focus of fifth order.

To sum up, we have the following theorem.

Theorem 6. *For the flow on the center manifold of system (7), the highest order of the fine focus at the origin is five, and the origin becomes a fine focus of the fifth order if and only if the following conditions are satisfied:*

$$\begin{aligned} B_{030} &= -\frac{3}{2}(-3A_{020} + 3A_{010}A_{020} + A_{020}A_{030}), \\ F_1 &= 0, \quad F_2 = 0, \quad F_3 = 0, \end{aligned} \tag{18}$$

where the polynomials F_1, F_2 and F_3 have the same form as in Theorem 2. So for system (1), if and only if condition (18) is satisfied, the highest order of the fine focus is five, and the singular point $(1, 0, 0)$ is a fine focus of fifth order.

By applying Theorem 6 and Lemma 4 to system (1), we can observe that, under suitable perturbations, the singular point $(1, 0, 0)$ can generate at most five small amplitude limit cycles. Nonetheless, it remains to be determined whether the system indeed has five small amplitude limit cycles. The following theorem provides an affirmative answer to this question.

Theorem 7. *For system (1), when the coefficients satisfy the condition of Theorem 6, the singular point $(1, 0, 0)$ is the fifth-order fine focus. Moreover, system (1) can bifurcate five limit cycles at the singular point $(1, 0, 0)$ by suitable perturbations of the parameters.*

Proof. If conditions (18) in Theorem 6 hold, then the singular point $(1, 0, 0)$ of system (1) is a fine focus of the fifth order. In addition, under conditions (16) and (17), the Jacobian determinant of (v_3, v_5, v_7, v_9) with respect to the variables $(A_{010}, A_{020}, A_{030}, B_{030})$ can be easily obtained by using the software Mathematica 12 to calculate

$$\begin{aligned} J &= \det\left[\frac{\partial(v_3, v_5, v_7, v_9)}{\partial((A_{010}, A_{020}, A_{030}, B_{030})}\right]_{(16),(17)} \\ &= -6.8023196169343150899480506292338682089264681533432605 7 \times 10^{7} \neq 0. \end{aligned} \tag{19}$$

According to Lemma 5, after system (1) is perturbed appropriately, four small-amplitude limit cycle bifurcations occur around the singular point. Meanwhile, its linear system also generates a limit cycle after the perturbation. Thus, through proper disturbance, system (1) bifurcates a total of five small-magnitude limit cycles near the singular point $(1, 0, 0)$. This finishes the proof. □

The singular points $(1,0,0)$, $(-\frac{1}{2},\frac{\sqrt{3}}{2},0)$ and $(-\frac{1}{2},-\frac{\sqrt{3}}{2},0)$ are symmetric about the Z-axis. This means that five small-amplitude limit cycles can also be generated around the other two singularities, respectively. To sum up, we have the following theorem:

Theorem 8. *System (1) can bifurcate a total of 15 limit cycles with $5-5-5$ distribution by suitable perturbations of the parameters.*

6. Conclusions

In this paper, we utilize the formal series method to compute the values of singular points and investigate the bifurcation of limit cycles and centers in a specific class of three-dimensional cubic systems with Z_3-equivariant symmetry. We not only establish a necessary and sufficient condition for the singular point of system (1) to transform into a center but also demonstrate that the system can give rise to 15 small amplitude limit cycles with a $5-5-5$ distribution from the singular points following appropriate perturbation. It is worth noting that the obtained result of fifteen limit cycles represents a novel lower bound on the number of limit cycles in three-dimensional cubic systems with Z_3-equivariant symmetry.

However, for more general three-dimensional symmetric systems, the problem remains open. Further investigations are required to explore additional dynamic properties, such as isochronous centers and periodic orbits. The challenges lie in calculating the quantities of singular points and determining the center conditions. Future research efforts may focus on enhancing computational tools to overcome these difficulties and achieve further improvements.

Author Contributions: Methodology, T.H. and J.G.; software, J.G.; validation, T.H., J.G. and Y.O.; writing—original draft preparation, T.H. and Y.O.; writing—review and editing, W.H. All authors have read and agreed to the published version of the manuscript.

Funding: This research was funded by the National Natural Science Foundation of China (No. 12061016), the Youth Innovation Talent Program of Education Department of Guangdong Province (No. 2019KQ NCX211) and the Basic Ability Enhancement Program for Young and Middle-aged Teachers of Guangxi (No. 2022KY0904).

Data Availability Statement: Not applicable.

Conflicts of Interest: The authors declare no conflict of interest.

Appendix A

The polynomials of F_3,F_4, H_4, and H_5 in Theorems 2 and 1 are listed in this appendix.

$$
\begin{aligned}
F_3 = {} & 153095888826 + 1860305695623A_{010} - 1998230280921A_{010}^2 - 638294984676A_{010}^3 \\
& + 267941328516A_{010}^4 + 1430360874792A_{020}^2 - 31204484283510A_{010}A_{020}^2 \\
& + 33087257828550A_{010}^2A_{020}^2 - 106292949284826A_{020}^4 + 348987923959347A_{010}A_{020}^4 \\
& - 259688926254681A_{010}^2A_{020}^4 + 1017460715581440A_{020}^6 - 1956655222272000A_{010}A_{020}^6 \\
& + 939194506690560A_{010}^2A_{020}^6 + 1278101405341A_{030} - 3023729061786A_{010}A_{030} \\
& + 337744106676A_{010}^2A_{030} + 602736838416A_{010}^3A_{030} + 154609142688A_{010}^4A_{030} \\
& - 23127732639234A_{020}^2A_{030} + 66370988256264A_{010}A_{020}^2A_{030} - 35443200166344A_{010}^2A_{020}^2A_{030} \\
& + 339429484206945A_{020}^4A_{030} - 579570891113790A_{010}A_{020}^4A_{030} + 188232401827080A_{010}^2A_{020}^4A_{030} \\
& - 652218407424000A_{020}^6A_{030} + 626129671127040A_{010}A_{020}^6A_{030} - 1081711527693A_{030}^2 \\
& + 428242288068A_{010}A_{030}^2 + 710340936552A_{010}^2A_{030}^2 + 207054755712A_{010}^3A_{030}^2 \\
& + 23076408763458A_{020}^2A_{030}^2 - 28757005607952A_{010}A_{020}^2A_{030}^2 + 1346567276160A_{010}^2A_{020}^2A_{030}^2 \\
& - 164335971898521A_{020}^4A_{030}^2 + 125488267884720A_{010}A_{020}^4A_{030}^2 + 104354945187840A_{020}^6A_{030}^2 \\
& - 60345069588A_{030}^3 + 510443578224A_{010}A_{030}^3 + 35765478720A_{010}^2A_{030}^3
\end{aligned}
$$

$$
\begin{aligned}
& - 5647535184168A_{020}^{2}A_{030}^{3} + 897711517440A_{010}A_{020}^{2}A_{030}^{3} + 20914711314120A_{020}^{4}A_{030}^{3} \\
& + 110236757652A_{030}^{4} - 22269541248A_{010}A_{030}^{4} + 149618586240A_{020}^{2}A_{030}^{4} - 5637182688A_{030}^{5}.
\end{aligned}
$$

$$
\begin{aligned}
F_4 = {} & 68877886387242147094678966461250623783 0 + 1231005045690561766900465775385216914877A_{010} \\
& - 240259091662558278753262598554065251583 3A_{010}^{2} \\
& + 348037208923537857825394686867492211230A_{010}^{3} \\
& + 348166336951752929540419215325732540416A_{010}^{4} \\
& + 908042027816600212878333569489315229538 2A_{020}^{2} \\
& + 7069063685868319763572096614659283150927A_{010}A_{020}^{2} \\
& - 1923216913875969810962147016977408704667 7A_{010}^{2}A_{020}^{2} \\
& + 50593800254744136931455917589351640132860A_{020}^{4} \\
& - 304289436008251175989339699172119267442154A_{010}A_{020}^{4} \\
& + 277975712305470658446181680043793793725070A_{010}^{2}A_{020}^{4} \\
& - 1396652502664536885414402784150908339495864A_{020}^{6} \\
& + 2780299096539092533142828162666048893865508A_{010}A_{020}^{6} \\
& - 1394768485468708965294504930932436908223084A_{010}^{2}A_{020}^{6} \\
& + 665375591788459551015076454145021137387520A_{020}^{8} \\
& - 1279568445747037598105916257971194494976000A_{010}A_{020}^{8} \\
& + 614192853958578047090839803826173357588480A_{010}^{2}A_{020}^{8} \\
& + 126789338108326223609389785524160477698 3A_{030} \\
& - 2874088873754780590366895380736580738102A_{010}A_{030} \\
& + 1093424481530540189328166987697315090586A_{010}^{2}A_{030} \\
& - 458429104516587521302200013179151302 72A_{010}^{3}A_{030} \\
& + 14976116540257939703499335035815634790 61A_{020}^{2}A_{030} \\
& - 44236857255180413391803477488573324467486A_{010}A_{020}^{2}A_{030} \\
& + 37525422044195232805000372851851751379368A_{010}^{2}A_{020}^{2}A_{030} \\
& - 426768857109444157383044020295650797850734A_{020}^{4}A_{030} \\
& + 792832802383380658921719204789850668157620A_{010}A_{020}^{4}A_{030} \\
& - 290779004200677867944326749981256448310864A_{010}^{2}A_{020}^{4}A_{030} \\
& + 1057168539024433592028935151769659920670156A_{020}^{6}A_{030} \\
& - 1165772124358428273118379839961668938168936A_{010}A_{020}^{6}A_{030} \\
& + 108726317249183364933392987986742494861920A_{010}^{2}A_{020}^{6}A_{030} \\
& - 426522815249012532701972085990398164992000A_{020}^{8}A_{030} \\
& + 409461902639052031393893202550782238392320A_{010}A_{020}^{8}A_{030} \\
& - 53282672810931587624340627615101163286 5A_{030}^{2} \\
& + 26771632269456096919024958257569016734 6A_{010}A_{030}^{2} \\
& - 324676778452895444580848309165864124 48A_{010}^{2}A_{030}^{2} \\
& - 164004206707428677940535753842243083 52A_{010}^{3}A_{030}^{2} \\
& - 2507318576113657024299901758180225216815 7A_{020}^{2}A_{030}^{2} \\
& + 42635500339076249806850608144231530802992A_{010}A_{020}^{2}A_{030}^{2} \\
& - 632209486145688181125587802768697816646 4A_{010}^{2}A_{020}^{2}A_{030}^{2} \\
& + 232722431734207427849582473168576163605662A_{020}^{4}A_{030}^{2} \\
& - 190445728571715520105322086367166194713440A_{010}A_{020}^{4}A_{030}^{2}
\end{aligned}
$$

$$
\begin{aligned}
&- 2269624107255252579443534266772302178304 A_{010}^{2} A_{020}^{4} A_{030}^{2} \\
&- 2336923680806918800675809350022596917640 76 A_{020}^{6} A_{030}^{2} \\
&+ 7263024250792802673926921978341963751328 0 A_{010} A_{020}^{6} A_{030}^{2} \\
&- 7009488406677605616346934028382765056 0 A_{010}^{2} A_{020}^{6} A_{030}^{2} \\
&+ 6824365043984200523231553375846370639872 0 A_{020}^{8} A_{030}^{2} \\
&- 699633528118432954646643023100467370 6 A_{030}^{3} \\
&+ 2276264453307633755522596671361029072 0 A_{010} A_{030}^{3} \\
&- 7304668247287611683449889498657735040 A_{010}^{2} A_{030}^{3} \\
&- 2867585019691216717166740174150041 60 A_{010}^{3} A_{030}^{3} \\
&+ 9756450036904207698300731137028979055272 A_{020}^{2} A_{030}^{3} \\
&- 3787787182129968824314891879255361354880 A_{010} A_{020}^{2} A_{030}^{3} \\
&- 1667519791049702530772671542155224611 84 A_{010}^{2} A_{020}^{2} A_{030}^{3} \\
&- 3118978191019303416994126004888676542088 0 A_{020}^{4} A_{030}^{3} \\
&- 1482748541403348711150521726684928307200 A_{010} A_{020}^{4} A_{030}^{3} \\
&- 140483449271569237884919416807279820 80 A_{010}^{2} A_{020}^{4} A_{030}^{3} \\
&+ 1212937891940007947604607459594626863088 0 A_{020}^{6} A_{030}^{3} \\
&- 4672992271118403744231289352255176704 0 A_{010} A_{020}^{6} A_{030}^{3} \\
&+ 5274829256045156347685312239999313184 A_{030}^{4} \\
&+ 5134268245993103486156724629150987 52 A_{010} A_{030}^{4} \\
&- 4274101938713584183617330324563558 40 A_{010}^{2} A_{030}^{4} \\
&+ 1134201129767466249845387720392704 0 A_{010}^{3} A_{030}^{4} \\
&- 5604862268907634979036746177204808787 84 A_{020}^{2} A_{030}^{4} \\
&- 1107852528158714146750237208270502144 00 A_{010} A_{020}^{2} A_{030}^{4} \\
&- 1004983274903755585522945377278361 60 A_{010}^{2} A_{020}^{2} A_{030}^{4} \\
&- 2420690574394215060008923236980536381 44 A_{020}^{4} A_{030}^{4} \\
&- 9365563284771282525661294453818654720 A_{010} A_{020}^{4} A_{030}^{4} \\
&- 7788320451864006240385482253758627840 A_{020}^{6} A_{030}^{4} \\
&+ 3859413645259942957727377284371961 60 A_{030}^{5} \\
&- 2041387716465489377623259464417935 36 A_{010} A_{030}^{5} \\
&+ 1643321305654407220054438365364224 0 A_{010}^{2} A_{030}^{5} \\
&- 1840041992696044343864488980729202022 4 A_{020}^{2} A_{030}^{5} \\
&- 6699888499358370570152969181855744 0 A_{010} A_{020}^{2} A_{030}^{5} \\
&- 1560927214128547087610215742303109120 A_{020}^{4} A_{030}^{5} \\
&- 3117692093236988357379290511192883 2 A_{030}^{6} \\
&+ 7174804938471160634211630034452480 A_{010} A_{030}^{6} \\
&- 1116648083226395095025494863642624 0 A_{020}^{2} A_{030}^{6} \\
&+ 9857635768252922816564961686323 20 A_{030}^{7},
\end{aligned}
$$

$$
\begin{aligned}
H_4 = {} & 221218352160 + 731888205624A010 - 3274050925314A_{010}^2 + 3705930423855A_{010}^3 \\
& - 1504278394767A_{010}^4 + 2964082653A_{010}^5 + 125620783551A_{010}^6 - 14983447554A_{010}^7 \\
& + 6193484383608A_{020}^2 - 8340759404082A_{010}A_{020}^2 - 22638714256422A_{010}^2A_{020}^2 \\
& + 57508838332749A_{010}^3A_{020}^2 - 47758958765523A_{010}^4A_{020}^2 + 17472871138743A_{010}^5A_{020}^2 \\
& - 2173513442409A_{010}^6A_{020}^2 - 179418733128A_{010}^7A_{020}^2 + 35274035939904A_{020}^4 \\
& - 97259857756002A_{010}A_{020}^4 + 636147513486A_{010}^2A_{020}^4 + 261086437048113A_{010}^3A_{020}^4 \\
& - 359696560295817A_{010}^4A_{020}^4 + 215772452693217A_{010}^5A_{020}^4 - 63839679086979A_{010}^6A_{020}^4 \\
& + 8051372602398A_{010}^7A_{020}^4 + 54041325463008A020^6 - 165699813295512A_{010}A_{020}^6 \\
& + 33204719595258A_{010}^2A_{020}^6 + 460180940964507A_{010}^3A_{020}^6 - 757292594763861A_{010}^4A_{020}^6 \\
& + 527204513374623A_{010}^5A_{020}^6 - 173484957871755A_{010}^6A_{020}^6 + 21845866533732A_{010}^7A_{020}^6 \\
& + 714639390760A_{030} - 3954719109708A_{010}A_{030} + 6330620684763A_{010}^2A_{030} \\
& - 3928882015764A_{010}^3A_{030} + 737702588403A_{010}^4A_{030} + 112140546390A_{010}^5A_{030} \\
& - 24972412590A_{010}^6A_{030} - 480704260662A_{020}^2A_{030} - 26779482155124A_{010}A_{020}^2A_{030} \\
& + 80008978158693A_{010}^2A_{020}^2A_{030} - 85292733754620A_{010}^3A_{020}^2A_{030} + 40291787129697A_{010}^4A_{020}^2A_{030} \\
& - 7390087811514A_{010}^5A_{020}^2A_{030} - 60915893184A_{010}^6A_{020}^2A_{030} - 31796207028342A_{020}^4A_{030} \\
& - 10161191049804A_{010}A_{020}^4A_{030} + 294127655276781A_{010}^2A_{020}^4A_{030} - 523370096940852A_{010}^3A_{020}^4A_{030} \\
& + 388246832456751A_{010}^4A_{020}^4A_{030} - 136688635109862A_{010}^5A_{020}^4A_{030} + 19858055051394A_{010}^6A_{020}^4A_{030} \\
& - 55233271098504A_{020}^6A_{030} + 22136479730172A_{010}A_{020}^6A_{030} + 460180940964507A_{010}^2A_{020}^6A_{030} \\
& - 1009723459685148A_{010}^3A_{020}^6A_{030} + 878674188957705A_{010}^4A_{020}^6A_{030} - 346969915743510A_{010}^5A_{020}^6A_{030} \\
& + 50973688578708A_{010}^6A_{020}^6A_{030} - 998437719666A_{030}^2 + 3096671171301A_{010}A_{030}^2 \\
& - 3028096214442A_{010}^2A_{030}^2 + 1019258483994A_{010}^3A_{030}^2 - 27890122095A_{010}^4A_{030}^2 \\
& - 14983447554A_{010}^5A_{030}^2 - 6621988031622A_{020}^2A_{030}^2 + 34774719861879A_{010}A_{020}^2A_{030}^2 \\
& - 54117192208698A_{010}^2A_{020}^2A_{030}^2 + 34636122388926A_{010}^3A_{020}^2A030^2 - 8755138483911A_{010}^4A_{020}^2A_{030}^2 \\
& + 296811924264A_{010}^5A_{020}^2A_{030}^2 - 3457746740322A_{020}^4A_{030}^2 + 109056291168483A_{010}A_{020}^4A_{030}^2 \\
& - 283572390076974A_{010}^2A_{020}^4A_{030}^2 + 277915273616538A_{010}^3A_{020}^4A_{030}^2 - 121414926704805A_{010}^4A_{020}^4A_{030}^2 \\
& + 20929574030526A_{010}^5A_{020}^4A_{030}^2 + 3689413288362A_{020}^6A_{030}^2 + 153393646988169A_{010}A_{020}^6A_{030}^2 \\
& - 504861729842574A_{010}^2A_{020}^6A_{030}^2 + 585782792638470A_{010}^3A_{020}^6A_{030}^2 - 289141596452925A_{010}^4A_{020}^6A_{030}^2 \\
& + 50973688578708A_{010}^5A_{020}^6A_{030}^2 + 466978687857A_{030}^3 - 933190268148A_{010}A_{030}^3 \\
& + 528955995006A_{010}^2A_{030}^3 - 68735004300A_{010}^3A_{030}^3 \\
& - 2774712510A_{010}^4A_{030}^3 + 4831643800503A_{020}^2A_{030}^3 \\
& - 14722618186188A_{010}A_{020}^2A_{030}^3 + 14246364724218A_{010}^2A_{020}^2A_{030}^3 - 5072319256428A_{010}^3A_{020}^2A_{030}^3 \\
& + 363633189840A_{010}^4A_{020}^2A_{030}^3 + 13341114508671A_{020}^4A_{030}^3 - 67879940374116A_{010}A_{020}^4A_{030}^3 \\
& + 98999775198414A_{010}^2A_{020}^4A_{030}^3 - 57298958882100A_{010}^3A_{020}^4A030^3 + 12222829449810A_{010}^4A_{020}^4A_{030}^3 \\
& + 17043738554241A_{020}^6A_{030}^3 - 112191495520572A_{010}A_{020}^6A_{030}^3 + 195260930879490A_{010}^2A_{020}^6A_{030}^3 \\
& - 128507376201300A_{010}^3A_{020}^6A_{030}^3 + 28318715877060A_{010}^4A_{020}^6A_{030}^3 - 101551098303A_{030}^4 \\
& + 121990806297A_{010}A_{030}^4 - 31871161935A_{010}^2A_{030}^4 + 924904170A_{010}^3A_{030}^4 \\
& - 1463891403051A_{020}^2A_{030}^4 + 2842783709499A_{010}A_{020}^2A_{030}^4 - 1570127337639A_{010}^2A_{020}^2A_{030}^4 \\
& + 187456955400A_{010}^3A_{020}^2A_{030}^4 - 6061982838705A_{020}^4A_{030}^4 + 17560187642997A_{010}A_{020}^4A_{030}^4 \\
& - 15158932029405A_{010}^2A_{020}^4A_{030}^4 + 4272705923850A_{010}^3A_{020}^4A_{030}^4 - 9349291293381A_{020}^6A_{030}^4 \\
& + 32543488479915A_{010}A_{020}^6A_{030}^4 - 32126844050325A_{010}^2A_{020}^6A_{030}^4 + 9439571959020A_{010}^3A_{020}^6A_{030}^4 \\
& + 10545786495A_{030}^5 - 6288151338A_{010}A_{030}^5 + 554942502A_{010}^2A_{030}^5 + 221959852533A_{020}^2A_{030}^5 \\
& - 250524764730A_{010}A_{020}^2A_{030}^5 + 50740569504A_{010}^2A_{020}^2A_{030}^5 + 1241360850195A_{020}^4A_{030}^5
\end{aligned}
$$

$$
\begin{aligned}
&- 2132416578438A_{010}A_{020}^4A_{030}^5 + 894227072886A_{010}^2A_{020}^4A_{030}^5 + 2169565898661A_{020}^6A_{030}^5\\
&- 4283579206710A_{010}A_{020}^6A_{030}^5 + 1887914391804A_{010}^2A_{020}^6A_{030}^5 - 467064225A_{030}^6\\
&+ 102767130A_{010}A_{030}^6 - 16255592649A_{020}^2A_{030}^6 + 7109971992A_{010}A_{020}^2A_{030}^6\\
&- 124646789979A_{020}^4A_{030}^6 + 103768106778A_{010}A_{020}^4A_{030}^6 - 237976622595A_{020}^6A_{030}^6\\
&+ 209768265756A_{010}A_{020}^6A_{030}^6 + 6851142A_{030}^7 + 408671568A_{020}^2A_{030}^7\\
&+ 5151316662A_{020}^4A_{030}^7 + 9988965036A_{020}^6A_{030}^7,
\end{aligned}
$$

$$
\begin{aligned}
H_5 = {}& 26047403632588800 + 62283999673710144A_{010} - 457203483658618416A_{010}^2\\
&+ 802068495449351130A_{010}^3 - 651126352309712091A_{010}^4 + 249098613258091716A_{010}^5\\
&- 19318764392183268A_{010}^6 - 16129408037673384A_{010}^7 + 4412436251425983A_{010}^8\\
&- 311465669010246A_{010}^9 + 826744476573935712A_{020}^2 - 1515596159606865552A_{010}A_{020}^2\\
&- 3656053627391992410A_{010}^2A_{020}^2 + 13957013723046354162A_{010}^3A_{020}^2\\
&- 17793702513143509428A_{010}^4A_{020}^2 + 11556821802995711586A_{010}^5A_{020}^2\\
&- 3942230326032786678A_{010}^6A_{020}^2 + 531397507902744186A_{010}^7A_{020}^2\\
&+ 47650118230995876A_{010}^8A_{020}^2 - 14320653571649790A_{010}^9A_{020}^2 + 6838374750025360968A_{020}^4\\
&- 24571305393662550168A_{010}A_{020}^4 + 13446722048105843172A_{010}^2A_{020}^4\\
&+ 64096289141188739832A_{010}^3A_{020}^4 - 142982026721362172106A_{010}^4A_{020}^4\\
&+ 137982371823677645442A_{010}^5A_{020}^4 - 74348866747077492960A_{010}^6A_{020}^4\\
&+ 23074743966668144028A_{010}^7A_{020}^4 - 3754387579296012642A_{010}^8A_{020}^4 + 218150645127858594A_{010}^9A_{020}^4\\
&+ 20080850316820245504A_{020}^6 - 89973550913773306176A_{010}A_{020}^6 + 107994130891684926102A_{010}^2A_{020}^6\\
&+ 114396829990826733378A_{010}^3A_{020}^6 - 464388625027240942068A_{010}^4A_{020}^6\\
&+ 578354138708740960302A_{010}^5A_{020}^6 - 388834440573414221262A_{010}^6A_{020}^6\\
&+ 152667927380636526762A_{010}^7A_{020}^6 - 33573242649902998356A_{010}^8A_{020}^6\\
&+ 3276205405828876134A_{010}^9A_{020}^6 + 17151446953326830976A_{020}^8 - 80048998096790498784A_{010}A_{020}^8\\
&+ 97703751139075013544A_{010}^2A_{020}^8 + 132343912278512832690A_{010}^3A_{020}^8\\
&- 535337458608086504091A_{010}^4A_{020}^8 + 708217483790272026690A_{010}^5A_{020}^8\\
&- 504976296508333723440A_{010}^6A_{020}^8 + 205925672850387910992A_{010}^7A_{020}^8\\
&- 45066566307116336589A_{010}^8A_{020}^8 + 4087052508752448012A_{010}^9A_{020}^8 + 7561882556498 6048A_{030}\\
&- 4656698209411648A_{010}A_{030} + 131850673562021668 2A_{010}^2A_{030} - 1427248597195074240A_{010}^3A_{030}\\
&+ 774848118804073056A_{010}^4A_{030} - 179111153159893368A_{010}^5A_{030} - 5225619601213200A_{010}^6A_{030}\\
&+ 7711919648564580A_{010}^7A_{030} - 726753227690574A_{010}^8A_{030} - 41875284719667600A_{020}^2A_{030}\\
&- 5263932835992481164A_{010}A_{020}^2A_{030} + 21018793002371036082A_{010}^2A_{020}^2A_{030}\\
&- 33208996980791086740A_{010}^3A_{020}^2A_{030} + 26826017487736141602A_{010}^4A_{020}^2A_{030}\\
&- 11613444242562198900A_{010}^5A_{020}^2A_{030} + 2369071039539461490A_{010}^6A_{020}^2A_{030}\\
&- 67458762534157236A_{010}^7A_{020}^2A_{030} - 28167892478038206A_{010}^8A_{020}^2A_{030}\\
&- 7428103935467459112A_{020}^4A_{030} + 21190562817083090 64A_{010}A_{020}^4A_{030}\\
&+ 86672105911672999848A_{010}^2A_{020}^4A_{030} - 228903371022856518828A_{010}^3A_{020}^4A_{030}\\
&+ 267536187132742189674A_{010}^4A_{020}^4A_{030} - 171031064014255130928A_{010}^5A_{020}^4A_{030}\\
&+ 61854844330798091748A_{010}^6A_{020}^4A_{030} - 11635680985426711116A_{010}^7A_{020}^4A_{030}\\
&+ 799885698802148178A_{010}^8A_{020}^4A_{030} - 30510114216864644160A_{020}^6A_{030}\\
&+ 70806748390788195972A_{010}A_{020}^6A_{030} + 128438842322716610562A_{010}^2A_{020}^6A_{030}\\
&- 654935734180473985116A_{010}^3A_{020}^6A_{030} + 1008529381769682596814A_{010}^4A_{020}^6A_{030}\\
&- 808881789016010805876A_{010}^5A_{020}^6A_{030} + 368606518904713287762A_{010}^6A_{020}^6A_{030}
\end{aligned}
$$

$$- 9209834718330616812 A_{010}^{7} A_{020}^{6} A_{030} + 10041118334646986214 A_{010}^{8} A_{020}^{6}\ A030$$
$$- 2668299936559683292 8A_{020}^{8} A_{030} + 6513583409271667569 6A_{010} A_{020}^{8} A_{030}$$
$$+ 132343912278512832690 A_{010}^{2} A_{020}^{8} A_{030} - 713783278144115338788 A_{010}^{3} A_{020}^{8} A_{030}$$
$$+ 1180362472983786711150 A_{010}^{4} A_{020}^{8} A_{030} - 1009952593016667446880 A_{010}^{5} A_{020}^{8} A_{030}$$
$$+ 480493236650905125648 A_{010}^{6} A_{020}^{8} A_{030} - 120177510152310230904 A_{010}^{7} A_{020}^{8} A_{030}$$
$$+ 12261157526257344036 A_{010}^{8} A_{020}^{8} A_{030} - 143575491157645968 A_{030}^{2} + 633583354199149566 A_{010} A_{030}^{2}$$
$$- 1024879470264570870 A_{010}^{2} A_{030}^{2} + 779980753296234576 A_{010}^{3} A_{030}^{2} - 275846599444332420 A_{010}^{4} A_{030}^{2}$$
$$+ 29139466716573264 A_{010}^{5} A_{030}^{2} + 4111298959155120 A_{010}^{6} A_{030}^{2} - 692145931133880 A_{010}^{7} A_{030}^{2}$$
$$- 140549617470029970 6A_{020}^{2} A_{030}^{2} + 9607762837519840422 A_{010} A_{020}^{2} A_{030}^{2}$$
$$- 21774003308430402924 A_{010}^{2} A_{020}^{2} A_{030}^{2} + 23306886013432080852 A_{010}^{3} A_{020}^{2} A_{030}^{2}$$
$$- 12970395634568003010 A_{010}^{4} A_{020}^{2} A_{030}^{2} + 3545849583724359042 A_{010}^{5} A_{020}^{2} A030^{2}$$
$$- 310820889911636052 A_{010}^{6} A_{020}^{2} A_{030}^{2} - 17831765654836056 A_{010}^{7} A_{020}^{2} A_{030}^{2}$$
$$- 87106051890375850 8A_{020}^{4} A_{030}^{2} + 36847401297164213256 A_{010} A_{020}^{4} A_{030}^{2}$$
$$- 134482393356114213144 A_{010}^{2} A_{020}^{4} A_{030}^{2} + 73454589500810364 A_{010}^{3} A_{020}^{4} A_{030}^{2}$$
$$- 161710234641117618408 A_{010}^{4} A_{020}^{4} A_{030}^{2} + 70043507245526365188 A_{010}^{5} A_{020}^{4} A_{030}^{2}$$
$$- 15491174491697465988 A_{010}^{6} A_{020}^{4} A_{030}^{2} + 1260425949627627432 A_{010}^{7} A_{020}^{4} A_{030}^{2}$$
$$+ 11602901586742184646 A_{020}^{6} A_{030}^{2} + 47493618218202162582 A_{010} A_{020}^{6} A_{030}^{2}$$
$$- 345343317495646690404 A_{010}^{2} A_{020}^{6} A_{030}^{2} + 702090132683197950972 A_{010}^{3} A_{020}^{6} A_{030}^{2}$$
$$- 700078914070994307690 A_{010}^{4} A_{020}^{6} A_{030}^{2} + 380987873921274679746 A_{010}^{5} A_{020}^{6} A_{030}^{2}$$
$$- 110446055183571713676 A_{010}^{6} A_{020}^{6} A_{030}^{2} + 13671493935743125368 A_{010}^{7} A_{020}^{6} A_{030}^{2}$$
$$+ 1085597234878611261 6A_{020}^{8} A_{030}^{2} + 44114637426170944230 A_{010} A_{020}^{8} A_{030}^{2}$$
$$- 356891639072057669394 A_{010}^{2} A_{020}^{8} A_{030}^{2} + 786908315322524474100 A_{010}^{3} A_{020}^{8} A_{030}^{2}$$
$$- 841627160847222872400 A_{010}^{4} A_{020}^{8} A_{030}^{2} + 480493236650905125648 A_{010}^{5} A_{020}^{8} A_{030}^{2}$$
$$- 140207095177695269388 A_{010}^{6} A_{020}^{8} A_{030}^{2} + 16348210035009792048 A_{010}^{7} A_{020}^{8} A_{030}^{2}$$
$$+ 94616234261801406 A_{030}^{3} - 304504138258287912 A_{010} A_{030}^{3} + 356164972648830216 A_{010}^{2} A_{030}^{3}$$
$$- 183286917965521584 A_{010}^{3} A_{030}^{3} + 36395171348162616 A_{010}^{4} A_{030}^{3} - 620161629086556 A_{010}^{5} A_{030}^{3}$$
$$- 323001434529144 A_{010}^{6} A_{030}^{3} + 1385939476816652214 A_{020}^{2} A_{030}^{3} - 6087491097901497756 A_{010} A_{020}^{2} A_{030}^{3}$$
$$+ 9708389340720128676 A_{010}^{2} A_{020}^{2} A_{030}^{3} - 7312605525185682168 A_{010}^{3} A_{020}^{2} A_{030}^{3}$$
$$+ 2650562235241850970 A_{010}^{4} A_{020}^{2} A_{030}^{3} - 365408125727432292 A_{010}^{5} A_{020}^{2} A_{030}^{3}$$
$$+ 1472845625257608 A_{010}^{6} A_{020}^{2} A_{030}^{3} + 5026169743727691096 A_{020}^{4} A_{030}^{3}$$
$$- 34536327707029476132 A_{010} A_{020}^{4} A_{030}^{3} + 77120578657923663708 A_{010}^{2} A_{020}^{4} A_{030}^{3}$$
$$- 80652365244227044224 A_{010}^{3} A_{020}^{4} A_{030}^{3} + 43559476666524346140 A_{010}^{4} A_{020}^{4} A_{030}^{3}$$
$$- 11619304626309521148 A_{010}^{5} A_{020}^{4} A_{030}^{3} + 1131151493255563080 A_{010}^{6} A_{020}^{4} A_{030}^{3}$$
$$+ 5797143221722087590 A_{020}^{6} A_{030}^{3} - 80715281755790308500 A_{010} A_{020}^{6} A_{030}^{3}$$
$$+ 243942448062202501356 A_{010}^{2} A_{020}^{6} A_{030}^{3} - 322706520279398345400 A_{010}^{3} A_{020}^{6} A_{030}^{3}$$
$$+ 218538460521020039850 A_{010}^{4} A_{020}^{6} A_{030}^{3} - 75629359102190826492 A_{010}^{5} A_{020}^{6} A_{030}^{3}$$
$$+ 10853756738188727832 A_{010}^{6} A_{020}^{6} A_{030}^{3} + 4901626380685660470 A_{020}^{8} A_{030}^{3}$$
$$- 79309253127123926532 A_{010} A_{020}^{8} A_{030}^{3} + 262302771774174824700 A_{010}^{2} A_{020}^{8} A_{030}^{3}$$
$$- 374056515932099054400 A_{010}^{3} A_{020}^{8} A_{030}^{3} + 266940687028280625360 A_{010}^{4} A_{020}^{8} A_{030}^{3}$$
$$- 93471396785130179592 A_{010}^{5} A_{020}^{8} A_{030}^{3} + 12715274471674282704 A_{010}^{6} A_{020}^{8} A_{030}^{3}$$
$$- 32450534093177415 A_{030}^{4} + 76868572425088428 A_{010} A_{030}^{4} - 62150051322476748 A_{010}^{2} A_{030}^{4}$$
$$+ 19248377856243768 A_{010}^{3} A_{030}^{4} - 1702051282878570 A_{010}^{4} A_{030}^{4} - 53833572421524 A_{010}^{5} A_{030}^{4}$$
$$- 620117509835760048 A_{020}^{2} A_{030}^{4} + 1964138307087356874 A_{010} A_{020}^{2} A_{030}^{4}$$

$$
\begin{aligned}
&-2236908094564719258A_{010}^2A_{020}^2A_{030}^4+1119501339095052270A_{010}^3A_{020}^2A_{030}^4\\
&-219604384906033500A_{010}^4A_{020}^2A_{030}^4+8407421157693852A_{010}^5A_{020}^2A_{030}^4\\
&-3282313509036732366A_{020}^4A_{030}^4+14406794568961407522A_{010}A_{020}^4A_{030}^4\\
&-22422072017080626192A_{010}^2A_{020}^4A_{030}^4+16101017887676012100A_{010}^3A_{020}^4A_{030}^4\\
&-5385641248045817640A_{010}^4A_{020}^4A_{030}^4+640985846178152412A_{010}^5A_{020}^4A_{030}^4\\
&-7057300339008631224A_{020}^6A_{030}^4+42309141982778725398A_{010}A_{020}^6A_{030}^4\\
&-83566714131810916290A_{010}^2A_{020}^6A_{030}^4+75138997028592122910A_{010}^3A_{020}^6A_{030}^4\\
&-32345006145277990500A_{010}^4A_{020}^6A_{030}^4+5537064652066401300A_{010}^5A_{020}^6A_{030}^4\\
&-6609104427260327211A_{020}^8A_{030}^4+43717128629029137450A_{010}A_{020}^8A_{030}^4\\
&-93514128983024763600A_{010}^2A_{020}^8A_{030}^4+88980229009426875120A_{010}^3A_{020}^8A_{030}^4\\
&-38946415327137574830A_{010}^4A_0 20^8A_{030}^4+6357637235837141352A_{010}^5A_{020}^8A_{030}^4\\
&+6396218688994776A_{030}^5-10624653613105560A_{010}A_{030}^5+5351133244526496A_{010}^2A_{030}^5\\
&-850379819632884A_{010}^3A_{030}^5+17944524140508A_{010}^4A_{030}^5+155594379157647930A_{020}^2A_{030}^5\\
&-355736728154963412A_{010}A_{020}^2A_{030}^5+272943854785152342A_{010}^2A_{020}^2A_{030}^5\\
&-76904568972100188A_{010}^3A_{020}^2A_{030}^5+5359473167586300A_{010}^4A_{020}^2A_{030}^5\\
&+1067611679230014234A_{020}^4A_{030}^5-3299299462277955216A_{010}A_{020}^4A_{030}^5\\
&+3542919371357270076A_{010}^2A_{020}^4A_{030}^5-1582905863125885284A_{010}^3A_{020}^4A_{030}^5\\
&+238798648576174428A_{010}^4A_{020}^4A_{030}^5+2930747285901580038A_{020}^6A_{030}^5\\
&-11527573092502966164A_{010}A_{020}^6A_{030}^5+15486368110035513174A_{010}^2A_{020}^6A_{030}^5\\
&-8847407821682614212A_{010}^3A_{020}^6A_{030}^5+1882416978346146228A_{010}^4A_{020}^6A_{030}^5\\
&+2914475241935275830A_{020}^8A_{030}^5-12468550531069968480A_{010}A_{020}^8A_{030}^5\\
&+17796045801885375024A_{010}^2A_{020}^8A_{030}^5-10385710753903353288A_{010}^3A_{020}^8A_{030}^5\\
&+2119212411945713784A_{010}^4A_{020}^8A_{030}^5-729463314158604A_{030}^6+770291007640992A_{010}A_{030}^6\\
&-209734213378680A_{010}^2A_{030}^6+11963016093672A_{010}^3A_{030}^6-23126694310657134A_{020}^2A_{030}^6\\
&+35984738436764310A_{010}A_{020}^2A_{030}^6-15938516975012892A_{010}^2A_{020}^2A_{030}^6\\
&+1759216136875848A_{010}^3A_{020}^2A_{030}^6-200971208600510184A_{020}^4A_{030}^6\\
&+430234721396425116A_{010}A_{020}^4A_{030}^6-288539739718121052A_{010}^2A_{020}^4A_{030}^6\\
&+58652299650288456A_{010}^3A_{020}^4A_{030}^6-661828757449878342A_{020}^6A_{030}^6\\
&+1771659646039177974A_{010}A_{020}^6A_{030}^6-1511580111881849604A_{010}^2A020^6A030^6\\
&+426476830963738968A_{010}^3A_{020}^6A_{030}^6-692697251726109360A_{020}^8A_{030}^6\\
&+1977338422431708336A_{010}A_{020}^8A_{030}^6-1730951792317225548A_{010}^2A_{020}^8A_{030}^6\\
&+470936091543491952A_{010}^3A_{020}^8A_{030}^6+45481663155384A_{030}^7-26634233714580A_{010}A_{030}^7\\
&+2848337165160A_{010}^2A_{030}^7+1993039033187934A_{020}^2A_{030}^7-1821282030828684A_{010}A_{020}^2A_{030}^7\\
&+332491176644328A_{010}^2A_{020}^2A_{030}^7+22263401805115716A_{020}^4A_{030}^7-29859821032181076A_{010}A_{020}^4A_{030}^7\\
&+9176890421473704A_{010}^2A_{020}^4A_{030}^7+86791034490315822A_{020}^6A_{030}^7\\
&-147484976522215572A_{010}A_{020}^6A_{030}^7+62091253978650936A_{010}^2A_{020}^6A_{030}^7\\
&+94158972496748016A_{020}^8A_{030}^7-164852551649259576A_{010}A_{020}^8A_{030}^7\\
&+67276584506213136A_{010}^2A_{020}^8A_{030}^7-1394863856397A_{030}^8+332306002602A_{010}A_{030}^8\\
&-88778814317784A_{020}^2A_{030}^8+34472146858866A_{010}A_{020}^2A_{030}^8-1344376139736726A_{020}^4A_{030}^8\\
&+831240074408850A_{010}A_{020}^4A_{030}^8-6292080932875704A_{020}^6A_{030}^8+5271437199254454A_{010}A_{020}^6A_{030}^8\\
&-6868856318719149A_{020}^8A_{030}^8+5606382042184428A_{010}A_{020}^8A_{030}^8+15824095362A_{030}^9\\
&+1527285024594A_{020}^2A_{030}^9+33249602976354A_{020}^4A_{030}^9+198837156800790A_{020}^6A_{030}^9\\
&+207643779340164A_{020}^8A_{030}^9.
\end{aligned}
$$

References

1. Poincaré, H. Mémoire sur les courbes définies par les équation différentielle. *J. Mathématiques Pures Appliquée* **1881**, *7*, 375–422.
2. Hilbert, D. Mathematical problems. *Bull. Am. Math. Soc.* **1902**, *8*, 437–479. [CrossRef]
3. Shi, S. A concrete example of the existence of four limit cycles for plane quadratic systems. *Sci. Sin.* **1980**, *23*, 153–158.
4. Chen, L.; Wang, M. The relative position, and the number, of limit cycles of a quadratic differential system. *Acta Math. Sin.* **1979**, *22*, 751–758
5. Li, J.; Liu, Y. New results on the study of Z_q-equivariant planar polynomial vector fields. *Qual. Theory Dyn. Syst.* **2010**, *9*, 167–219. [CrossRef]
6. Li, C.; Liu, C.; Yang, J. A cubic system with 13 limit cycles. *J. Differ. Equ.* **2009**, *246*, 3609–3619. [CrossRef]
7. Zhang, T.H.; Han, M.; Zang, H.; Meng, X.Z. Bifurcations of limit cycles for a cubic Hamiltonian system under quartic perturbations. *Chaos Solitons Fractals* **2004**, *22*, 1127–1138. [CrossRef]
8. Wang, S.; Yu, P. Bifurcation of limit cycles in a quintic Hamiltonian system under sixth-order perturbation. *Chaos Solitons Fractals* **2005**, *26*, 1317–1335. [CrossRef]
9. Wang, S.; Yu, P. Existence of 121 limit cycles in a perturbed planar polynomial Hamiltonian vector field of degree 11. *Chaos Solitons Fractals* **2006**, *30*, 606–621. [CrossRef]
10. Du, C.; Liu, Y.; Huang, W. Limit cycles bifurcations for a class of Kolmogorov model in symmetrical vector field. *Int. J. Bifurc. Chaos* **2014**, *24*, 8. [CrossRef]
11. Du, C.; Huang, W.; Zhang, Q. Center problem and the bifurcation of limit cycles for a cubic polynomial system. *Appl. Math. Model.* **2015**, *39*, 5200–5215. [CrossRef]
12. Yu, P.; Han, M. Small limit cycles bifurcating from fine focus points in cubic order Z_2-equivariant vector fields. *Chaos Solitons Fractals* **2005**, *24*, 329–348. [CrossRef]
13. Wang, Q.; Liu, Y.; Du, C. Small limit cycles bifurcating from fine focus points in quartic order Z_3-equivariant vector fields. *J. Math. Anal. Appl.* **2008**, *337*, 524–536. [CrossRef]
14. Yao, W.H.; Yu, P. Bifurcation of small limit cycles in Z_5-equivariant planar vector fields of order 5. *J. Math. Anal. Appl.* **2007**, *328*, 400–413. [CrossRef]
15. Li, J.; Chan, H.S.Y.; Chung, K.W. Bifurcations of limit cycles in a Z_6-equivariant planar vector field of degree 5. *Sci. China Math. Phys. Astron.* **2002**, *45*, 817–826.
16. Li, J.; Zhang, M.Q. Bifurcations of limit cycles in a Z_8-equivariant planar vector field of degree 7. *J. Dyn. Differ. Equ.* **2004**, *16*, 1123–1139. [CrossRef]
17. Wang, S.; Yu, P.; Li, J. Bifurcation of limit cycles in a Z_{10}-equivariant vector fields of degree 9. *Int. J. Bifurc. Chaos Appl. Sci. Eng.* **2006**, *16*, 2309–2324. [CrossRef]
18. Huang, C.; Liu, H.; Shi, X.; Chen, X.; Xiao, M.; Wang, Z.; Cao, J. Bifurcations in a fractional-order neural network with multiple leakage delays. *Neural Netw.* **2020**, *131*, 115–126. [CrossRef] [PubMed]
19. Huang, C.; Wang, J.; Chen, X.; Cao, J. Bifurcations in a fractional-order BAM neural network with four different delays. *Neural Netw.* **2021**, *141*, 344–354. [CrossRef] [PubMed]
20. Xu, C.; Liao, M.; Li, P.; Guo, Y. Bifurcation properties for fractional order delayed BAM neural networks. *Cogn. Comput.* **2021**, *13*, 322–356. [CrossRef]
21. Xu, C.; Mu, D.; Liu, Z.; Pang, Y. Comparative exploration on bifurcation behavior for integer-order and fractional-order delayed BAM neural networks. *Nonlinear Anal. Model. Control.* **2022**, *27*, 1–24. [CrossRef]
22. Xu, C.; Mu, D.; Liu, Z.; Pang, Y.; Liac, M.; Aouiti, C. New insight into bifurcation of fractional-order 4D neural networks incorporating two different time delays. *Commun. Nonlinear Sci. Numer. Simul.* **2023**, *118*, 107043. [CrossRef]
23. Dulac, H. Sur les cycles limites. *Bull. Société Mathématique Fr.* **1923**, *51*, 45–188. [CrossRef]
24. Romanovski, V.G.; Shafer, D.S. Centers and limit cycles in polynomial systems of ordinary differential equations. *Sch. Real Complex Singul. São Carlos* **2016**, *68*, 267–373.
25. Huang, W.; Gu, J.; Wang, Q. Limit cycles and isochronous centers of three dimensional differential systems. *J. Guangxi Norm. Univ. Nat. Sci. Ed.* **2022**, *40*, 104–126.
26. Sanchez-Sanchez, I.; Torregrosa, J. Hopf bifurcation in 3-dimensional polynomial vector fields. *Commun. Nonlinear Sci. Numer. Simul.* **2022**, *105*, 106068. [CrossRef]
27. Gu, J.; Zegeling, A.; Huang, W. Bifurcation of Limit Cycles and Isochronous Centers on Center Manifolds for a Class of Cubic Kolmogorov Systems in R 3. *Qual. Theory Dyn. Syst.* **2023**, *22*, 42. [CrossRef]
28. Lu, J.; Wang, C.; Huang, W.; Wang, Q. Local Bifurcation and Center Problem for a More Generalized Lorenz System. *Qual. Theory Dyn. Syst.* **2022**, *21*, 96. [CrossRef]
29. Guo, L.; Yu, P.; Chen, Y. Twelve limit cycles in 3D quadratic vector fields with Z_3 symmetry. *Int. J. Bifurc. Chaos* **2018**, *28*, 1850139:1–1850139:14. [CrossRef]
30. Guo, L.; Yu, P.; Chen, Y. Bifurcation analysis on a class of three-dimensional quadratic systems with twelve limit cycles. *Appl. Math. Comput.* **2019**, *363*, 12. [CrossRef]
31. Hassard, B.D.; Kazarinoff, N.D.; Wan, Y.H. *Theory and Applications of Hopf Bifurcation*; CUP Archive: Cambridge, UK , 1981.
32. Liu, L.; Aybar, O.O.; Romanovski, V.G.; Zhang, W. Identifying weak foci and centers in the Maxwell–Bloch system. *J. Math. Anal. Appl.* **2015**, *430*, 549–571. [CrossRef]

33. Llibre, J.; Makhlouf, A.; Badi, S. 3-dimensional Hopf bifurcation via averaging theory of second order. *Discret. Contin. Dyn. Syst.-Ser. A (DCDS-A)* **2009**, *25*, 1287. [CrossRef]
34. Barreira, L.; Llibre, J.; Valls, C. Bifurcation of limit cycles from a 4-dimensional center in Rm in resonance 1: N. *J. Math. Anal. Appl.* **2012**, *389*, 754–768. [CrossRef]
35. Buică, A.; García, I.A.; Maza, S. Existence of inverse Jacobi multipliers around Hopf points in R3: Emphasis on the center problem. *J. Differ. Equ.* **2012**, *252*, 6324–6336. [CrossRef]
36. Buică, A.; García, I.A.; Maza, S. Multiple Hopf bifurcation in R3 and inverse Jacobi multipliers. *J. Differ. Equ.* **2014**, *256*, 310–325. [CrossRef]
37. Tian, Y.; Yu, P. An explicit recursive formula for computing the normal form and center manifold of general n-dimensional differential systems associated with Hopf bifurcation. *Int. J. Bifurc. Chaos* **2013**, *23*, 1350104. [CrossRef]
38. Edneral, V.F.; Mahdi, A.; Romanovski, V.G.; Shafer, D.S. The center problem on a center manifold in R3. *Nonlinear Anal. Theory, Methods Appl.* **2012**, *75*, 2614–2622. [CrossRef]
39. García, I.A.; Maza, S.; Shafer, D.S. Cyclicity of polynomial nondegenerate centers on center manifolds. *J. Differ. Equ.* **2018**, *265*, 5767–5808. [CrossRef]
40. Wang, Q.; Liu, Y.; Chen, H. Hopf bifurcation for a class of three-dimensional nonlinear dynamic systems. *Bull. Des Sci. Mathématiques* **2010**, *134*, 786–798. [CrossRef]
41. Llibre, J.; Zhang, X. Invariant algebraic surfaces of the Lorenz system. *J. Math. Phys.* **2002**, *43*, 1622–1645. [CrossRef]
42. Li, J. Hilbert's 16th problem and bifurcations of planar polynomial vector fields. *Int. J. Bifurc. Chaos Appl. Sci. Eng.* **2003**, *13*, 47–106. [CrossRef]
43. Du, C.; Wu, X.; Peng, Y. The limit cycles bifurcation of cubic system with equivariant symmetric structure. *Nat. Sci. J. Xiangtan Univ.* **2009**, *31*, 12.
44. Wang, Q.; Huang, W. The equivalence between singular point quantities and Liapunov constants on center manifold. *Adv. Differ. Equ.* **2012**, *78*, 12. [CrossRef]
45. Du, C.; Liu, Y.; Huang, W. A class of three-dimensional quadratic systems with ten limit cycles. *Int. J. Bifurc. Chaos* **2016**, *26*, 1650149:1–1650149:11. [CrossRef]
46. Huang, W.; Wang, Q.; Chen, A. Hopf bifurcation and the centers on center manifold fora class of three-dimensional circuit system. *Math. Methods Appl. Sci.* **2020**, *43*, 1988–2000. [CrossRef]
47. Llibre, J.; Zhang, X. Darboux theory of integrability in C^n taking into account the multiplicity. *J. Differ. Equ.* **2009**, *246*, 541–551. [CrossRef]