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Stabilization of *n*-Order Function Differential Equations by Parametric Distributed Control Function with Palindromic Parameters Set

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Abstract: Stabilization by a parametric distributed control function plays a very important role in aeronautics, aerospace and physics. Choosing the right parameters is necessary for handling the distributed control. In the current paper, we introduce stabilization criteria for an n-order functional-differential equation with a parametric distributed control function in n-term integrals and $2n$ parameter sets. In our article, we use properties of unimodal and log-concave polynomials.

Keywords: functional differential equations; exponential stability; feedback control; palindromes; unimodality; log-concave

MSC: 34K20; 34D99; 93-10

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1. Introduction

Stabilization of general n order functional-differential equation by parametric distributed feedback control plays a very important role in various problems in aeronautics, aerospace and physics.

The noise in feedback control is the main reason for investigating mathematical models with distributed inputs. It is impossible to control the value of $x(t_j)$ at a single moment t_j. The distributed control function in integral form is used for the average of the process $x(t)$ in the neighborhood of t_j.

In the paper [1], the absolute stability of neutral system using Lyapunov-Krasovskii functional was studied

Stabilization of non-linear systems with distributed input was presented in [2] and stabilization of linear systems with distributed input was presented in [3].

Asymptotic stability criteria of the zero solution of second-order linear delay differential equations were presented in [4].

Results on boundedness of solutions were obtained in [5].

Stability of second order equations with damping terms was obtained in [6–8].

In [9], linear systems with delayed control action were, transformed into systems without delays. Under a continuity condition, the new system is an ODE control.

The stability of third order differential equations is presented in [10]. The stability of third order neutral delay differential equations is presented in [11].

There are various applications of models described by equations with distributed feedback control in aeronautics, aerospace, ship navigation and network traffic regularization.

Distributed feedback control stabilization of mathematical models, is also significant in medicine (see, for example, [12–15], where the mathematical model of testosterone regulation and model of the hepatitis B virus were researched).

In the current article, we consider the system which can be described in the terms of palindromic polynomials. Palindromic polynomials are an important topic in algebraic combinatorics and have been studied by many mathematicians over the years. Palindromes have important properties that make them useful in various fields of mathematics, including algebra, number theory, and combinatorics. One of the key features of palindromic polynomials is their connection to symmetric functions [16–18]. The combinatorial interpretation of the coefficients has been used to prove many other important results in algebraic [19]. Another area where palindromic polynomials have important applications is in the study of algebraic curves [20]. Palindromic polynomials also have connections to other areas of mathematics, including the theory of special functions, the theory of Lie algebras, and the theory of elliptic curves. They have also been used in various applications, including the design of error-correcting codes, digital signal processing, and mathematical physics.

The palindromes play an important role in numerical theory, cryptography, and time-reverse systems. One important application of palindromic polynomials is in the study of exponential stability of linear time-invariant (LTI) systems. The study of palindromic polynomials and their applications to stability analysis has been an active area of research in control theory. A number of important results have been developed in this area, including connection to the Routh–Hurwitz stability criterion, which is a classical method for determining the stability of a polynomial based on the signs of its coefficients [21]. Other important contributions of analyzing palindromic polynomials and their roots, as well as the development of algorithms for computing the roots of palindromic polynomials and their multiplicities [22,23]. In this article, we discuss the connection of palindromic polynomials to exponential stability of functional-differential system.

Consider the n order differential equation

$$x^{(n)}(t) + \sum_{i=1}^{n-1} p_i(t)x^{(i)}(t) + u(t) = g(t), \tag{1}$$

with distributed feedback control defined by

$$u(t) = \sum_{j=1}^{k} \int_0^t K_j(t,s)x(s)ds, \tag{2}$$

where the kernel function defined in the following form

$$K_j(t,s) = \beta_j e^{-\alpha_j(t-s)}, \;\; \alpha_j, \beta_j \in \mathbb{R}, \; \alpha_j > 0, \;\; 1 \le j \le k \tag{3}$$

and $g(t), p_i(t)$ $1 \le i \le n-1$ are continuous functions.

The results of the exponential stability of Equation (1) for $n = 2, 3$ were presented in [24,25].

In the current paper we generalize these results: we present a new approach for stabilization of an n order functional-differential equation by distributed feedback control.

In [26], we researched the impossibility of a stabilization system (1) by distributed control function, where the number of terms was insufficient for stabilization. The number of terms was less than the order of the functional-differential equation.

In the current paper, we analyze the case when the number of integral components in a distributed control function is equal to the order of the functional-differential equation. We propose a new stabilization criteria based on choosing the $2n$ set of parameters in a parametric distributed control function.

2. Stabilization Criteria in the Case of $k = n$

Let us introduce the following integro-differential equation, where $n \ge 2$.

$$x^{(n)}(t) + \sum_{i=j}^{n} \beta_j \int_0^t e^{-\alpha_j(t-s)}x(s)ds = 0. \tag{4}$$

Denoting $x_n = x$, $x_{n-1} = x'_n = x'$, ..., $x_1 = x^{(n-1)}$ and $x_{n+j} = \beta_j \int_0^t e^{-\alpha_j(t-s)} x(s)ds$, where $1 \leq j \leq n-1$, differentiating we reduce Equation (4) to a differential system of the first order with $2n$ equations.

$$\begin{cases} x'_1 = -\sum_{j=n+1}^{2n-1} x_j \\ x'_2 = x_1 \\ \ldots \\ x'_n = x_{n-1} \\ x'_{n+1} = \beta_1 x_n - \alpha_1 x_{n+1} \\ \ldots \\ x'_{2n} = \beta_n x_n - \alpha_n x_{2n} \end{cases} . \tag{5}$$

The coefficient matrix $(2n \times 2n)$of this system is

$$M = \begin{pmatrix} 0 & 0 & 0 & \cdots & 0 & 0 & -1 & \cdots & -1 \\ 1 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 & \beta_1 & -\alpha_1 & \cdots & 0 \\ \cdots & \cdots & \cdots & \ddots & \ldots & \cdots & \cdots & \ddots & \cdots \\ 0 & 0 & 0 & \cdots & 0 & \beta_n & 0 & \cdots & -\alpha_n \end{pmatrix}.$$

Let us find the characteristic polynomial of matrix M

$$\det(I\lambda - M) = \begin{pmatrix} \lambda & 0 & \cdots & 0 & 0 & 1 & \cdots & 1 \\ -1 & \lambda & \cdots & 0 & 0 & 0 & \cdots & 0 \\ 0 & -1 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & -1 & \lambda & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 & -\beta_1 & \lambda + \alpha_1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & -\beta_n & 0 & \cdots & \lambda + \alpha_n \end{pmatrix} = \det\begin{pmatrix} A & B \\ C & D \end{pmatrix},$$

where

$$
\begin{aligned}
[A]_{n\times n} &= \begin{pmatrix} \lambda & 0 & \cdots & 0 & 0 \\ -1 & \lambda & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & -1 & \lambda \end{pmatrix}, \ [B]_{n\times n} = \begin{pmatrix} 1 & \cdots & 1 \\ 0 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 0 \end{pmatrix}, \\
[C]_{n\times n} &= \begin{pmatrix} 0 & 0 & \cdots & 0 & -\beta_1 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & -\beta_n \end{pmatrix}, \ [D]_{n\times n} = \begin{pmatrix} \lambda+\alpha_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \lambda+\alpha_n \end{pmatrix}, \\
D^{-1} &= \begin{pmatrix} \frac{1}{\lambda+\alpha_1} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \frac{1}{\lambda+\alpha_n} \end{pmatrix}, \ BD^{-1}C = \begin{pmatrix} 0 & \cdots & -\sum_{i=1}^{n} \frac{\beta_i}{\lambda+\alpha_i} \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 0 \end{pmatrix}, \\
A - BD^{-1}C &= \begin{pmatrix} \lambda & 0 & \cdots & 0 & \sum_{i=1}^{n} \frac{\beta_i}{\lambda+\alpha_i} \\ -1 & \lambda & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & -1 & \lambda \end{pmatrix}.
\end{aligned}
$$

$$
\begin{aligned}
\det\Big(A - BD^{-1}C\Big) &= (-1)^{n+1} \sum_{i=1}^{n} \frac{\beta_i}{\lambda+\alpha_i} \cdot \det \begin{pmatrix} -1 & \lambda & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & -1 \end{pmatrix} + \\
& (-1)^{n+n} \lambda \det \begin{pmatrix} \lambda & 0 & \cdots & 0 & 0 \\ -1 & \lambda & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & -1 & \lambda \end{pmatrix} = \sum_{i=1}^{n} \frac{\beta_i}{\lambda+\alpha_i} + \lambda^n.
\end{aligned}
$$

By the Schur formula [27]

$$
\det(I\lambda - M) = \det(D)\det\Big(A - BD^{-1}C\Big) = \prod_{j=1}^{n}\Big(\lambda+\alpha_j\Big)\left(\lambda^n + \sum_{i=1}^{n} \frac{\beta_i}{\lambda+\alpha_i}\right),
$$

and

$$
P(\lambda) = \lambda^n \prod_{j=1}^{n}\Big(\lambda+\alpha_j\Big) + \sum_{i=1}^{n} \beta_i \prod_{\substack{j=1 \\ j\neq i}}^{n}\Big(\lambda+\alpha_j\Big). \tag{6}
$$

Consider the polynomial

$$
\widetilde{P}(\lambda) = \lambda^n + a_{n-1}\lambda^{n-1} + a_{n-2}\lambda^{n-2} + \dots + a_1\lambda + 1.
$$

$\widetilde{P}(\lambda)$ is a palindrome polynomial or palindrome if $a_k = a_{n-k}, \quad k = 0, 1, \dots, n$.

Theorem 1 ([28])**.** *Polynomial $\widetilde{P}(\lambda)$ is a palindrome iff for any root r also $\frac{1}{r}$ is the root of $\widetilde{P}(\lambda)$.*

Theorem 2 ([28])**.** *Let $\widetilde{P}(\lambda)$ be a palindrome of even degree $2n$ then $\widetilde{P}(\lambda) = \lambda^n Q\Big(\lambda + \frac{1}{\lambda}\Big)$, where $Q(t)$ is a polynomial of degree n.*

Theorem 3. *Let $\widetilde{P}(\lambda) = \lambda^2 + A\lambda + 1$, $A > 0$ quadric palindrome then $\widetilde{P}(\lambda)$ has all the roots on the left half plane, i.e. $Re(\lambda_1) < 0, Re(\lambda_2) < 0$.*

Proof. $\lambda^2 + A\lambda + 1 = 0 \Rightarrow \lambda_{1,2} = \frac{1}{2}\left(-A \pm \sqrt{A^2 - 4}\right)$.
If $A > 2$, then $A > \sqrt{A^2 - 4}$ and $\lambda_1 < 0, \lambda_2 < 0$.
If $A = 2$, then $\lambda_1 = \lambda_2 = -\frac{A}{2} < 0$.
If $A < 2$, then. $Re(\lambda_1) = Re(\lambda_2) = -\frac{A}{2} < 0$. □

Theorem 4. *Let $\widetilde{P}(\lambda) = \lambda^4 + A\lambda^3 + B\lambda^2 + A\lambda + 1$, $A > 0$, $B > 2$ quartic palindrome then $\widetilde{P}(\lambda)$ has all the roots on the left half plane.*

Proof. By Routh-Hurvitz criterion $M_1 = A > 0$,

$$M_2 = \begin{vmatrix} A & A \\ 1 & B \end{vmatrix} = AB - A = A(B-1) > 0,$$

$$M_3 = \begin{vmatrix} A & A & 0 \\ 1 & B & 1 \\ 0 & A & A \end{vmatrix} = A^2(B-2) > 0,$$

$$M_4 = \begin{vmatrix} A & A & 0 & 0 \\ 1 & B & 1 & 0 \\ 0 & A & A & 0 \\ 0 & 1 & B & 1 \end{vmatrix} = M_3 > 0.$$

All roots of $P(\lambda)$ lie in the left half plane. □

Theorem 5. *(degree reduction)*
Let $\widetilde{P}(\lambda) = \lambda^{2n} + a_{2n-1}\lambda^{2n-1} + a_{2n-2}\lambda^{2n-2} + ... + a_1\lambda + 1$ palindrome of even degree, then $\widetilde{P}(\lambda) = \lambda^n Q(t)$, where $t = \lambda + \frac{1}{\lambda}$, $\deg(Q(t)) = n$ (By Theorem 2).
If all roots of $Q(t)$ are real and negative, then $\widetilde{P}(\lambda)$ is exponentially stable, i.e. all roots of $\widetilde{P}(\lambda)$ in $\mathbb{C}^- = \{\lambda \in \mathbb{C} : Re(\lambda) < 0\}$.

Proof. Let s root of $Q(t)$ and $s = -p < 0$. Then $\lambda + \frac{1}{\lambda} = s \Rightarrow \lambda + \frac{1}{\lambda} = -p \Rightarrow \lambda^2 + p\lambda + 1 = 0$ quadric palindrome, so by Theorem 3, $\lambda \in \mathbb{C}^-$. □

Theorem 6 ([28])**.** *(palindrome factorization)*
Let $f(\lambda), g(\lambda)$ be two palindromes of degree n and m respectively. Then
1. $f(\lambda) \cdot g(\lambda)$ is a palindrome of degree $m + n$.
2. If $g(\lambda) | f(\lambda)$ and $f(\lambda) = g(\lambda) \cdot h(\lambda)$, then $h(\lambda)$ is also a palindrome. So we obtain that an irreducible palindrome must be linear, quadric or quartic, i.e. a palindrome factorized to
$\lambda + A$, $\lambda^2 + A\lambda + 1$ or $\lambda^4 + A\lambda^3 + B\lambda^2 + A\lambda + 1$.

Corollary 1. *The palindrome of even degree can be factorized by irreducible palindromes $\lambda^2 + A\lambda + 1$ or $\lambda^4 + A\lambda^3 + B\lambda^2 + A\lambda + 1$.*

Consider system (5), where $n \geq 2$, with characteristic polynomial (6).

Theorem 7. *If characteristic polynomial (6) is a palindrome, then parameters β_i $1 \leq i \leq n$ can be uniquely represented via $\alpha_1, ..., \alpha_n$.*

Proof. Recall

$$P(\lambda) = \lambda^{2n} + \lambda^{2n-1}\sum_{j=1}^{n}\alpha_j + \lambda^{2n-2}\sum_{\substack{j,k=1 \\ j\neq k}}^{n}\alpha_j\alpha_k + ... + \lambda^n\prod_{j=1}^{n}\alpha_j + ... + \lambda\sum_{j=1}^{n}\alpha_j + 1$$

is a palindrome.

Let us denote:

$$\sum_{j=1}^{n} \alpha_j = A_{2n-1}, \sum_{\substack{j,k=1 \\ j\neq k}}^{n} \alpha_j\alpha_k = A_{2n-2}, ..., \prod_{j=1}^{n} \alpha_j = A_n.$$

$\beta_i \;\; 1 \leq i \leq n$ is solution of the following system

$$\begin{cases} \sum_{i=1}^{n} \beta_i = A_{n-1} \\ \sum_{i=1}^{n} \beta_i \sum_{\substack{j=1 \\ i\neq j}}^{n} \alpha_j = A_{n-2} \\ \vdots \\ \sum_{i=1}^{n} \beta_i \prod_{\substack{j=1 \\ j\neq i}}^{n} \alpha_j = 1 \end{cases} . \tag{7}$$

$$A \cdot \begin{pmatrix} \beta_1 \\ \vdots \\ \beta_n \end{pmatrix} = \begin{pmatrix} A_{n-1} \\ \vdots \\ 1 \end{pmatrix},$$

where

$$A = \begin{pmatrix} 1 & 1 & ... & 1 \\ \sum_{j=2}^{n} \alpha_j & \sum_{\substack{j=1 \\ j\neq 2}}^{n} \alpha_j & ... & \sum_{j=1}^{n-1} \alpha_j \\ ... & ... & ... & ... \\ \prod_{j=2}^{n} \alpha_j & \prod_{\substack{j=1 \\ j\neq 2}}^{n} \alpha_j & ... & \prod_{j=1}^{n-1} \alpha_j \end{pmatrix}.$$

Let us note that $\det(A) \neq 0$, because $\alpha_1 \neq \alpha_2 \neq ... \neq \alpha_n$ and positive. So system (7) has a unique solution. □

Consider system (5) with characteristic polynomial (6) in the case when $n = 2$.

Theorem 8. *Let $\beta_1 + \beta_2 = \alpha_1 + \alpha_2$ and $\beta_1\alpha_2 + \beta_2\alpha_1$ = 1, so*

$$P(\lambda) = \lambda^4 + (\alpha_1 + \alpha_2)\lambda^3 + \alpha_1\alpha_2\lambda^2 + (\alpha_1 + \alpha_2)\lambda + 1$$

is quartic palindrome. If $\alpha_1 > 1, \alpha_2 > 1$ and $\alpha_1\alpha_2 > \alpha_1 + \alpha_2$ then system (5) is exponentially stable for $n = 2$.

Proof. Using Routh–Hurwitz criterion $M_1 = \alpha_1 + \alpha_2 > 0$,
$$M_2 = \begin{vmatrix} \alpha_1 + \alpha_2 & \alpha_1 + \alpha_2 \\ 1 & \alpha_1\alpha_2 \end{vmatrix} = (\alpha_1 + \alpha_2)(\alpha_1\alpha_2 - 1) > 0,$$
because $\alpha_1 > 1, \alpha_2 > 1$, so $\alpha_1\alpha_2 > 1$,
$$M_3 = \begin{vmatrix} \alpha_1 + \alpha_2 & \alpha_1 + \alpha_2 & 0 \\ 1 & \alpha_1\alpha_2 & 1 \\ 0 & \alpha_1 + \alpha_2 & \alpha_1 + \alpha_2 \end{vmatrix} = (\alpha_1 + \alpha_2)^2(\alpha_1\alpha_2 - 2) > 0,$$
because $\alpha_1\alpha_2 > \alpha_1 + \alpha_2 > 2$,
$$M_4 = \begin{vmatrix} \alpha_1 + \alpha_2 & \alpha_1 + \alpha_2 & 0 & 0 \\ 1 & \alpha_1\alpha_2 & 1 & 0 \\ 0 & \alpha_1 + \alpha_2 & \alpha_1 + \alpha_2 & 0 \\ 0 & 1 & \alpha_1\alpha_2 & 1 \end{vmatrix} = M_3 > 0.$$
We obtain that (5) is exponentially stable for $n = 2$. □

Definition 1. *Let $a_0, a_1, ..., a_n$ be real numbers. It is palindromic (or symmetric) with center of symmetry at $n/2$ if $a_i = a_n - i$ for $i = 0, 1, ..., n$. It is unimodal if $a_0 \leq ... \leq a_{m-1} \leq a_m \geq a_{m+1} \geq ... \geq a_n$ for some m.*

Definition 2. *The real numbers $\{d_0, ..., d_m\}$ is considered to be logarithmic concave (log-concave) if $d_{j+1} \cdot d_{j-1} \leq d_j^2 \ 1 \leq j \leq m-1$. If the real numbers is concave than it is an unimodal palindrome.*

Theorem 9 ([29]). *Let $p(x) = \sum_{k=0}^{m} c_k x^k$ be an unimodal polynomial. If $c_j - c_{j-1} \geq c_{j+1} - c_j$ then polynomial $p(x)$ is log concave.*

Definition 3. *$\{a_n\}_{n=1}^{m}$ is $r-$ factor strong log-concave if $a_j^2 > r a_{j-1} a_{j+1}$ where $r > 1$.*

Theorem 10 ([30]). *Let $p(x)$ be a polynomial with positive coefficients and $\deg(p(x)) > 5$. If $p(x)$ r_0 strongly log-concave, where $r_0 \approx 1.466$ is the unique real root of $r^3 - r^2 - 1 = 0$, then all roots of $p(x)$ have a negative real part.*

Theorem 11. *Let us consider a system (5) with a characteristic polynomial (6).*
Consider $\alpha_j > 1$, $1 \leq j \leq n$ and $\alpha_1 < \alpha_2 < ... < \alpha_n$ such that

$$\sum_{j=1}^{n} \alpha_j = A_{2n-1}, \sum_{\substack{j,k=1 \\ j \neq k}}^{n} \alpha_j \alpha_k = A_{2n-2}, ..., \prod_{j=1}^{n} \alpha_j = A_n.$$

If $P(\lambda)$ is an unimodal palindrome

$$P(\lambda) = \lambda^{2n} + \lambda^{2n-1} \sum_{j=1}^{n} \alpha_j + \lambda^{2n-2} \sum_{\substack{j,k=1 \\ j \neq k}}^{n} \alpha_j \alpha_k + ... + \lambda^n \prod_{j=1}^{n} \alpha_j + ... + \lambda \sum_{j=1}^{n} \alpha_j + 1. \tag{8}$$

If $\left\{A_j\right\}_{j=0}^{n-1}$ r_0 strongly log-concave, then polynomial (8) r_0 strong log-concave (and system (5) is exponentially stable).

Proof. For all $j \leq n-1$, $A_j^2 > r_0 A_{j-1} A_{j+1}$.
However, because $P(\lambda)$ is a palindrome, $A_{2n-j} = A_j$. For all $n \leq j < 2n$, $\ A_j = A_{n+k}$, where $k = 0, ..., n$.
We have

$$A_j^2 = A_{n+k}^2 = A_{n-k}^2 > r_0 \left(A_{n-k-1} \cdot A_{n-k+1}\right) = r_0 \left(A_{n+k-1} \cdot A_{n+k+1}\right) = r_0 A_{j-1} \cdot A_{j+1}.$$

We obtain that $\left\{A_j\right\}_{j=0}^{2n}$ is r_0 strongly log-concave, so $P(\lambda)$ is r_0 strongly log-concave. That is, all roots have a negative real part. □

3. Examples

Example 1. *In the case of a differential equation of the second order and a distributed control function with two integral terms*

$$x''(t) + \beta_1 \int_0^t e^{-\alpha_1 (t-s)} x(s) ds + \beta_2 \int_0^t e^{-\alpha_2 (t-s)} x(s) ds = 0,$$

the characteristic polynomial is

$$P(\lambda) = \lambda^4 + (\alpha_1 + \alpha_2)\lambda^3 + \alpha_1 \alpha_2 \lambda^2 + (\beta_1 + \beta_2)\lambda + (\beta_1 \alpha_2 + \beta_2 \alpha_1).$$

We suppose that $P(\lambda)$ is palindromic polynomial, so the following equations are fulfilled

$$\begin{cases} \beta_1 + \beta_2 = \alpha_1 + \alpha_2 \\ \beta_1\alpha_2 + \beta_2\alpha_1 = 1 \end{cases} \tag{9}$$

Let us choose $\alpha_1 = 20$ and $\alpha_2 = 40$ such that conditions of Theorem 11 are fulfilled, i.e α_1 and α_2 is strongly log-concave. Coefficients of characteristic palindromic polynomial is strongly log-concave, because

$$A_4 = 1,\ A_3 = \alpha_1 + \alpha_2 = 60,\ A_2 = \alpha_1\alpha_2 = 800,\ A_1 = 60,\ A_0 = 1,\ r_0 \approx 1.466$$

and

$$\begin{cases} A_2^2 > r_0 A_1 A_3 \\ A_1^2 > r_0 A_0 A_2 \\ A_3^2 > r_0 A_2 A_4 \end{cases}.$$

Now, we can calculate parameters, solving system (9) (Solution of the following system exists, by Theorem 7) $\beta_1 = -9.9999, \beta_2 = 19.999$, and the roots of the characteristic polynomial are following

$$\lambda_1 = -40.0125, \lambda_2 = -19.975, \lambda_3 = -1 * 10^{-8}, \lambda_4 = -0.0125.$$

In the next examples, we will use the same technique, as in Example 1.

Example 2. *In the case of a differential equation of the third order and a distributed control function with three integral terms*

$$x'''(t) + \beta_1 \int_0^t e^{-\alpha_1(t-s)} x(s)ds + \beta_2 \int_0^t e^{-\alpha_2(t-s)} x(s)ds + \beta_3 \int_0^t e^{-\alpha_3(t-s)} x(s)ds = 0,$$

the characteristic polynomial is

$$\begin{aligned} P(\lambda) \quad &= \quad \lambda^3(\lambda+\alpha_1)(\lambda+\alpha_2)(\lambda+\alpha_3) + \beta_1(\lambda+\alpha_2)(\lambda+\alpha_3) \\ &\quad + \beta_2(\lambda+\alpha_1)(\lambda+\alpha_3) + \beta_3(\lambda+\alpha_1)(\lambda+\alpha_2). \end{aligned}$$

$$\begin{aligned} \alpha_1 &= 2, \alpha_2 = 3, \alpha_3 = 6, \beta_1 = 30.75, \beta_2 = -97.333, \beta_3 = 102.583, \\ \lambda_1 &= -5.0506 - i0.418, \lambda_2 = -5.0506 + i0.418, \lambda_3 = -0.25275 - i0.96753, \lambda_4 = -0.25275 + i0.96753, \\ \lambda_5 &= -0.19665 - i0.01627, \lambda_6 = -0.19665 + i0.01627. \end{aligned}$$

Example 3. *In the case of a differential equation of the fourth order and a distributed control function with four integral terms*

$$x''''(t) + \beta_1 \int_0^t e^{-\alpha_1(t-s)} x(s)ds + \beta_2 \int_0^t e^{-\alpha_2(t-s)} x(s)ds$$

$$+\beta_3 \int_0^t e^{-\alpha_3(t-s)} x(s)ds + \beta_4 \int_0^t e^{-\alpha_4(t-s)} x(s)ds = 0,$$

the characteristic polynomial is

$$\begin{aligned} P(\lambda) \quad &= \quad \lambda^4(\lambda+\alpha_1)(\lambda+\alpha_2)(\lambda+\alpha_3)(\lambda+\alpha_4) + \beta_1(\lambda+\alpha_2)(\lambda+\alpha_3)(\lambda+\alpha_4) \\ &\quad + \beta_2(\lambda+\alpha_1)(\lambda+\alpha_3)(\lambda+\alpha_4) + \beta_3(\lambda+\alpha_1)(\lambda+\alpha_2)(\lambda+\alpha_4) \\ &\quad + \beta_4(\lambda+\alpha_1)(\lambda+\alpha_2)(\lambda+\alpha_3). \end{aligned}$$

$$\begin{aligned}
\alpha_1 &= 3, \alpha_2 = 4, \alpha_3 = 5, \alpha_4 = 6, \beta_1 = -1369.333, \beta_2 = 10027.5, \beta_3 = -19932, \beta_4 = 11615.833, \\
\lambda_1 &= -7.3276, \lambda_2 = -0.13647, \lambda_3 = -0.26795 - i0.96343, \lambda_4 = -0.26795 + i0.96343, \\
\lambda_5 &= -4.8378 - i2.5345, \lambda_6 = -4.8378 + i2.5345, \lambda_7 = -0.16219 - i0.08497, \\
\lambda_8 &= -0.16219 + i0.08497.
\end{aligned}$$

4. Conclusions

In the current article, we propose the stabilization criteria by feedback control (2) of n-order functional-differential equations. We considered control functions in integral form

$$u(t) = \sum_{j=1}^{k} u_j(t)$$

where $u_j(t) = \int_0^t K_j(t,s)x(s)ds$

The vectors $(u_1, ..., u_k)$ define the feedback control of Equation (2). Here vector dimensions define the degree of freedom of the feedback control function.

In the previous paper [26], the stabilizing of n-order functional-differential Equation (1) by a distributed control function (2) was proven, where the number of integral terms is less than n is impossible. The numerical examples for various orders were given. In the current paper, we show that we can stabilize n-order functional-differential Equation (1) by a distributed control function (2), where the number of integral terms is equal to n. We investigated a novel approach for finding the stabilization set of $2n$ parameters of integral terms based on log-concave and palindromic polynomials.

Open Questions:

(1) The existence of more general condition than strong log-concavity which can be applied to parameters set in Theorem 11.

(2) Applying our approach to other types of functional-differential systems, for example, to obtain another criteria for absolute stability of neutral systems studied in paper [1].

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