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Subclasses of p-Valent κ-Uniformly Convex and Starlike Functions Defined by the q-Derivative Operator

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Abstract: The potential for widespread applications of the geometric and mapping properties of functions of a complex variable has motivated this article. On the other hand, the basic or quantum (or q-) derivatives and the basic or quantum (or q-) integrals are extensively applied in many different areas of the mathematical, physical and engineering sciences. Here, in this article, we first apply the q-calculus in order to introduce the q-derivative operator $\mathfrak{S}^{n,m}_{\eta,p,q}$. Secondly, by means of this q-derivative operator, we define an interesting subclass $\mathcal{T}\aleph^{n,m}_{\lambda,p}(\eta,\alpha,\kappa)$ of the class of normalized analytic and multivalent (or p-valent) functions in the open unit disk $\mathbb{U}$. This p-valent analytic function class is associated with the class κ-$\mathcal{UCV}$ of κ-uniformly convex functions and the class κ-$\mathcal{UST}$ of κ-uniformly starlike functions in $\mathbb{U}$. For functions belonging to the normalized analytic and multivalent (or p-valent) function class $\mathcal{T}\aleph^{n,m}_{\lambda,p}(\eta,\alpha,\kappa)$, we then investigate such properties as those involving (for example) the coefficient bounds, distortion results, convex linear combinations, and the radii of starlikeness, convexity and close-to-convexity. We also consider a number of corollaries and consequences of the main findings, which we derived herein.

Keywords: a nalytic functions; multivalent (or p-valent) functions; uniformly convex functions; uniformly starlike functions; basic or quantum (or q-) analysis; q-derivative operator; hadamard product (or convolution); generalized q-hypergeometric function

MSC: 30C45; 05A30; 30C80; 33D05

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1. Introduction and Definitions

We first denote by $\mathcal{A}(p)$ the class of functions of the following form:

$$f(z) = z^p + \sum_{k=p+1}^{\infty} a_k z^k \qquad \left(p \in \mathbb{N} := \{1,2,3,\cdots\}\right), \tag{1}$$

which are analytic and multivalent (or p-valent) in the open unit disk $\mathbb{U}$ given by

$$\mathbb{U} = \{z : z \in \mathbb{C} \quad \text{and} \quad |z| < 1\}.$$

Additionally, we let $\mathcal{T}(p)$ denote the subclass of $\mathcal{A}(p)$, which consists of functions f with the following power-series expansion:

$$f(z) = z^p - \sum_{k=p+1}^{\infty} a_k z^k \qquad \big(z \in \mathbb{U};\ a_k \geqq 0 \quad (k \geqq p+1)\big), \tag{2}$$

which, in the univalent case when $p = 1$, was studied by Silverman [1].

For the function $f(z)$ involved in (1) and the function $g(z)$ given as follows:

$$g(z) = z^p + \sum_{k=p+1}^{\infty} b_k z^k \qquad (z \in \mathbb{U}),$$

the convolution (or, equivalently, the Hadamard product) of the functions $f(z)$ and $g(z)$ is defined below:

$$(f * g)(z) := z^p + \sum_{k=p+1}^{\infty} a_k b_k z^k =: (g * f)(z) \qquad (z \in \mathbb{U}).$$

Motivated by some pioneering developments by Goodman (see [2,3]), involving such function classes as the uniformly convex function class and the uniformly starlike function class in $\mathbb{U}$, Ma and Minda [4], Rønning [5,6], and others, Kanas et al. (see [7–10]; see also the related recent work of Srivastava et al. [11]) introduced and studied the various properties and characteristics of the class κ-$\mathcal{UCV}$ of κ-uniformly convex functions and the class κ-$\mathcal{UST}$ of κ-uniformly starlike functions, which are defined as follows:

$$\kappa\text{-}\mathcal{UCV} := \left\{ f : f \in \mathcal{S} \quad \text{and} \quad \Re\left(1 + \frac{zf''(z)}{f'(z)}\right) > \kappa \left| \frac{zf''(z)}{f'(z)} \right| \right.$$
$$\left. \big(z \in \mathbb{U};\ 0 \leqq \kappa < \infty\big) \right\} \tag{3}$$

and

$$\kappa\text{-}\mathcal{UST} := \left\{ f : f \in \mathcal{S} \quad \text{and} \quad \Re\left(\frac{zf'(z)}{f(z)}\right) > \kappa \left| \frac{zf'(z)}{f(z)} - 1 \right| \right.$$
$$\left. \big(z \in \mathbb{U};\ 0 \leqq \kappa < \infty\big) \right\}, \tag{4}$$

where, as usual, $\mathcal{S}$ denotes the subclass of functions in the normalized analytic function class $\mathcal{A} := \mathcal{A}(1)$ which are also univalent in $\mathbb{U}$. For a remarkably systematic presentation of the properties and characteristics of the normalized analytic and univalent function class $\mathcal{S}$, one should refer to the widely-cited monograph by Duren [12].

Remark 1. *For fixed $\kappa \quad (0 \leqq \kappa < \infty)$, the class κ-$\mathcal{UCV}$ is defined purely geometrically as a subclass of the normalized univalent function class $\mathcal{S}$, the members of which map the intersection of the unit disk $\mathbb{U}$ with any disk centered at the point $\zeta \in \mathbb{C} \quad (|\zeta| \leqq \kappa)$ onto a convex domain. In fact, the notion of κ-uniform convexity happens to be a natural extension of the notion of the classical convexity, for which $\kappa = 0$. It stemmed naturally from the attempt to extend the family of circular arcs, which are contained in $\mathbb{U}$, with center also in $\mathbb{U}$, to a family of circular arcs which are contained in $\mathbb{U}$, but with the center at any point in the complex z-plane. We note also that the class κ-$\mathcal{UCV}$ is invariant under the rotation $e^{i\theta} f\big(e^{-i\theta} z\big)$.*

Remark 2. *We choose here to denote by $\mathbb{U}^*(\zeta, r)$ the open disk with center at $\zeta \in \mathbb{C}$ and radius r so that $\mathbb{U}^*(0,1) =: \mathbb{U}$. If γ is an arc with center at $\zeta \in \mathbb{C}$, and if the function f is analytic on* *the arc γ, then the image arc $\mathbf{\Gamma} = f(\gamma)$ is starlike with respect to a point w_0 if and only if (see, for example,* [13,14])

$$\Re\left(\frac{(z-\zeta)f^*(z)}{f(z)-w_0}\right) \geqq 0 \qquad (z \in \gamma).$$

It is also known that every function $f \in \kappa$-$\mathcal{UST}$ possesses a continuous extension to

$$\widetilde{\mathbb{U}} := \mathbb{U} \cup \{z : z \in \mathbb{C} \quad \text{and} \quad |z| = 1\},$$

$f(\mathbb{U})$ is bounded, and $f(\partial\mathbb{U})$ is a rectifiable curve (see, for details, [9]).

In recent years, the concepts and notions associated with the above-defined function classes κ-$\mathcal{UCV}$ and κ-$\mathcal{UST}$ have been extended not only to the subclasses of the analytic and univalent function classes of order α $(0 \leqq \alpha < 1)$, but also to the subclasses of the analytic and multivalent (or p-valent) function classes of order α $(0 \leqq \alpha < p)$ (see, for example, [15,16]). Here, in this paper, we begin by recalling each of the following definitions in their *duly corrected* forms.

Definition 1 (See, for example, [15,16])**.** *A function $f \in \mathcal{A}(p)$ is said to be in the class κ-$\mathcal{UCV}(p;\alpha)$ of p-valent κ-uniformly convex functions of order α in $\mathbb{U}$ if it satisfies the following condition*:

$$\Re\left(1 + \frac{zf''(z)}{f'(z)} - \alpha\right) > \kappa\left|1 + \frac{zf''(z)}{f'(z)} - p\right| \tag{5}$$
$$(z \in \mathbb{U};\ 0 \leqq \alpha < p;\ 0 \leqq \kappa < \infty).$$

Definition 2 (See, for example, [15,16])**.** *A function $f \in \mathcal{A}(p)$ is said to be in the class κ-$\mathcal{UST}(p;\alpha)$ of p-valent κ-uniformly starlike functions of order α in $\mathbb{U}$ if it satisfies the following condition*:

$$\Re\left(\frac{zf'(z)}{f(z)} - \alpha\right) > \kappa\left|\frac{zf''(z)}{f'(z)} - p\right| \tag{6}$$
$$(z \in \mathbb{U};\ 0 \leqq \alpha < p;\ 0 \leqq \kappa < \infty).$$

Remark 3. *It is easily seen from* (5) *and* (6) *that*

$$f \in \kappa\text{-}\mathcal{UST}(p;\alpha) \iff \frac{zf'(z)}{p} \in \kappa\text{-}\mathcal{UCV}(p;\alpha).$$

Indeed, for different choices of the parameters α, κ and p, we obtain many new or known subclasses, some of which were studied in the earlier literature (see, for example, [7–10,17–19]). *In particular, if we set $p = 1$ and $\alpha = 0$, Definitions 1 and 2 would reduce to the definitions in* (3) *and* (4)*, respectively, which were introduced by Kanas et al. (see* [8–10]).

Remark 4. *The comments and observations, which we have already made in Remarks 1 and 2, would apply, with some modifications, also to the corresponding p-valent function classes κ-$\mathcal{UCV}(p;\alpha)$ and κ-$\mathcal{UST}(p;\alpha)$, respectively. One can thus formulate specific functions, which belong to the p-valent function classes κ-$\mathcal{UCV}(p;\alpha)$ and κ-$\mathcal{UST}(p;\alpha)$ from those in the widely-investigated univalent function classes κ-$\mathcal{UCV} := \kappa$-$\mathcal{UCV}(1;0)$ and κ-$\mathcal{UST} := \kappa$-$\mathcal{UST}(1;0)$ (see, for example,* [6,8–10,13,14]).

A considerable literature exists in which the basic (or quantum or q-) calculus is extensively investigated and applied in the realm of the geometric function theory of complex analysis and in the realm of the special (or higher transcendental) functions in one and more variables. In this connection, the attention of the interested reader should be drawn toward the recent survey-cum-expository review articles [20,21]), in each of which one can find a detailed historical and introductory overview. Indeed, extensive usage of the q-calculus is found also in the modeling and analysis of a wide variety of applied problems (see, for details, [22]; see also [23] (pp. 350–351) and the references cited therein). In particular, several interesting applications of Jackson's q-derivative operator $\mathfrak{D}_q \ \ (0 < q < 1)$ defined by (see [24,25])

$$\mathfrak{D}_q f(z) := \begin{cases} \dfrac{f(z) - f(qz)}{(1-q)z} & \big(z \neq 0;\ 0 < q < 1\big) \\ \\ f'(0) & \big(z = 0\big). \end{cases} \tag{7}$$

can be found in the works by Carmichael [26], Mason [27], Trjitzinsky [28] and other authors.

In the area of geometric function theory of complex analysis, the q-derivative operator $\mathfrak{D}_q$ was used in the year 1990 by Ismail et al. [29] to study a class of q-starlike functions and by Srivastava in his book chapter [30], which was published in 1989, to investigate the univalence, starlikeness and convexity properties of the generalized q-hypergeometric function ${}_r\Phi_s$ involving r numerator parameters and s denominator parameters $\big(r, s \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}\big)$. Some recent studies, in which the q-derivative operator $\mathfrak{D}_q$ was applied to various subclasses of of the function classes $\mathcal{A}(p) \ \ (p \geqq 1)$, can be found in, for example, [31–34].

It is easily seen from the definition (7) that

$$\lim_{q \to 1-} \big\{\mathfrak{D}_q f(z)\big\} = f^*(z).$$

Moreover, if $f \in \mathcal{A}(p)$ has the form (1), then we have

$$\begin{aligned} \mathfrak{D}_q f(z) &= \mathfrak{D}_q \left(z^p + \sum_{k=p+1}^{\infty} a_k z^k \right) \\ &= [p]_q z^{p-1} + \sum_{k=p+1}^{\infty} [k]_q a_k z^{k-1},\ z \in D, \end{aligned}$$

where, in general, the q-number $[\lambda]_q$ is given by

$$[\lambda]_q = \begin{cases} \dfrac{1 - q^\lambda}{1 - q} & \big(\lambda \in \mathbb{C} \setminus \{0\}\big) \\ \\ 0 & \big(\lambda = 0\big), \end{cases}$$

so that

$$\lim_{q \to 1-} \big\{[\lambda]_q\big\} = \lambda \qquad \big(\lambda \in \mathbb{C}\big).$$

Now, in terms of Jackson's q-derivative operator $\mathfrak{D}_q$, we introduce the operator $\mathscr{D}^n_{p,q} : \mathcal{A}(p) \to \mathcal{A}(p) \ \ (n \in \mathbb{N}_0)$, which is defined by

$$\mathscr{D}^0_{p,q} f(z) := f(z) \quad \text{and} \quad \mathscr{D}^n_{p,q} f(z) := z D_q \Big(\mathscr{D}^{n-1}_{p,q} f(z)\Big) \quad \big(n \in \mathbb{N}\big). \tag{8}$$

Thus, if $f \in \mathcal{A}(p)$ has the form (1), it follows that

$$\mathscr{D}^n_{p,q} f(z) = \Big(f * G^n_{p,q}\Big)(z) \qquad \big(z \in \mathbb{U};\ p \in \mathbb{N};\ n \in \mathbb{N}_0\big),$$

where

$$G^n_{p,q}(z) := z^p + \sum_{k=p+1}^{\infty} \Big([k]_q\Big)^n z^k \qquad \big(z \in \mathbb{U};\ p \in \mathbb{N};\ n \in \mathbb{N}_0\big).$$

Moreover, we find that

$$\mathscr{D}^{n}_{p,q}f(z) = z^{p} + \sum_{k=p+1}^{\infty} \Big([k]_q\Big)^{n} a_k z^k \qquad \big(z \in \mathbb{U};\ p \in \mathbb{N};\ n \in \mathbb{N}_0\big).$$

Hence, we have

$$\lim_{q\to 1-} \Big\{\mathscr{D}^{n}_{p,q}f(z)\Big\} = z^{p} + \sum_{k=p+1}^{\infty} k^{n}\, a_k z^k \qquad \big(z \in \mathbb{U};\ p \in \mathbb{N};\ n \in \mathbb{N}_0\big).$$

The q-derivative operator $\mathfrak{S}^{n,m}_{\eta,p,q} : \mathcal{A}(p) \to \mathcal{A}(p)$, which was studied in [31], is defined by

$$\mathfrak{S}^{n,m}_{\eta,p,q}f(z) := \begin{cases} (1-\eta)\mathfrak{S}^{n,m-1}_{\eta,p,q}f(z) + \eta\, \dfrac{z}{p}\Big(\mathfrak{S}^{n,m-1}_{\eta,p,q}f(z)\Big)' & (m \in \mathbb{N};\ \eta \geqq 0) \\ \mathscr{D}^{n}_{p,q}f(z) & (m = 0;\ \eta \geqq 0). \end{cases} \tag{9}$$

which, for a function $f \in \mathcal{A}(p)$ given by (1), yields

$$\mathfrak{S}^{n,m}_{\eta,p,q}f(z) = z^{p} + \sum_{k=p+1}^{\infty} \Big([k]_q\Big)^{n} \left(\frac{p+\eta(k-p)}{p}\right)^{m} a_k z^k$$

$$\big(z \in \mathbb{U};\ m \in \mathbb{N}_0\big).$$

Remark 5. *The parameters η, m, n and p, which are involved in the definition (9) of the q-derivative operator $\mathfrak{S}^{n,m}_{\eta,p,q} : \mathcal{A}(p)$, can be appropriately specialized to deduce several other simpler operators, which were investigated in earlier works (see, for example, [35–40]).*

Definition 3. *For $0 \leqq \alpha < p$, $p \in \mathbb{N}$, $0 \leqq \mu \leqq 1$ and $0 \geqq \kappa < \infty$, let $\aleph^{n,m}_{\mu,p}(\eta,\alpha,\kappa)$ be the subclass of $\mathcal{A}(p)$ consisting of functions f of the form (1) and satisfying the following inequality:*

$$\Re\left(\frac{\left(1-\mu+\frac{\mu}{p}\right)z\left(\mathfrak{S}^{n,m}_{\eta,p,q}f(z)\right)' + \frac{\mu}{p}z^2\left(\mathfrak{S}^{n,m}_{\eta,p,q}f(z)\right)''}{(1-\mu)\mathfrak{S}^{n,m}_{\eta,p,q}f(z) + \frac{\mu}{p}z\left(\mathfrak{S}^{n,m}_{\eta,p,q}f(z)\right)'} - \alpha\right)$$
$$> \kappa\left|\frac{\left(1-\mu+\frac{\mu}{p}\right)z\left(\mathfrak{S}^{n,m}_{\eta,p,q}f(z)\right)' + \frac{\mu}{p}z^2\left(\mathfrak{S}^{n,m}_{\eta,p,q}f(z)\right)''}{(1-\mu)\mathfrak{S}^{n,m}_{\eta,p,q}f(z) + \frac{\mu}{p}z\left(\mathfrak{S}^{n,m}_{\eta,p,q}f(z)\right)'} - p\right| \tag{10}$$

$$\big(z \in \mathbb{U}\big).$$

Furthermore, in terms of the p-valent function class $\mathcal{T}(p)$ of functions $f(z)$ given by (2), we define the p-valent function class $\mathcal{T}\aleph^{n,m}_{\mu,p}(\eta,\alpha,\kappa)$ by

$$\mathcal{T}\aleph^{n,m}_{\mu,p}(\eta,\alpha,\kappa) := \aleph^{n,m}_{\mu,p}(\eta,\alpha,\kappa) \cap \mathcal{T}(p).$$

Remark 6. *Each of the following special cases of the p-valent function class $\mathcal{T}\aleph^{n,m}_{\mu,p}(\eta,\alpha,\kappa)$, given by Definition 3, is worthy of note:*

(i) $\mathcal{T}\aleph^{0,m}_{\mu,p}(\eta,\alpha,\kappa) =: TS_{\mu,p}(\alpha,\kappa)$, *which was studied in* [38,40];

(ii) $\mathcal{T}\aleph^{0,0}_{\mu,p}(\eta,\alpha,\kappa) =: TS_{\mu,p}(f,g,\alpha,\kappa)$, *which was investigated by Aouf et al.* [41] *with the function f replaced by $f*g$;*

(iii) $\mathcal{T}\aleph^{0,0}_{\mu,1}(\eta,\alpha,\kappa) =: TS_{\mu}(f,g,\alpha,\kappa)$, *which was studied by Aouf et al.* [42] *with the function f replaced by $f*g$.*

By means of the following example:

$$f(z) = z^p + \lambda z^{p+1} \qquad (z \in \mathbb{U};\ \lambda \in \mathbb{C}),$$

so that

$$a_{p+1} = \lambda \quad \text{and} \quad a_{p+n} = 0 \qquad (\forall\ n \in \mathbb{N} \setminus \{1\}),$$

it is fairly straightforward to show that

$$\aleph^{n,m}_{\mu,p}(\eta,\alpha,\kappa) \neq \varnothing$$

at least for some admissible values of the parameters involved. Indeed, in this case, we have

$$\mathfrak{S}^{n,m}_{\eta,p,q} f(z) = z^p + \Big([p+1]_q\Big)^n \left(\frac{p+\eta}{p}\right)^m \lambda\, z^{p+1} \qquad (z \in \mathbb{U};\ m \in \mathbb{N}_0),$$

that is,

$$\mathfrak{S}^{n,m}_{\eta,p,q} f(z) = z^p + \mathfrak{a}\, z^{p+1} \qquad (z \in \mathbb{U};\ m \in \mathbb{N}_0),$$

where

$$\mathfrak{a} := \Big([p+1]_q\Big)^n \left(\frac{p+\eta}{p}\right)^m \lambda\, z^{p+1} = \left(\frac{1-q^{p+1}}{1-q}\right)^n \left(\frac{p+\eta}{p}\right)^m \lambda.$$

From the above relation, we find that

$$z\Big(\mathfrak{S}^{n,m}_{\eta,p,q} f(z)\Big)' = pz^p + \mathfrak{a}(p+1)\, z^{p+1}$$

and

$$z^2\Big(\mathfrak{S}^{n,m}_{\eta,p,q} f(z)\Big)'' = p(p-1)z^p + \mathfrak{a}p(p+1)\, z^{p+1}.$$

Therefore, we have

$$\begin{aligned}
&\Re\left(\frac{\Big(1-\mu+\frac{\mu}{p}\Big)z\big(\mathfrak{S}^{n,m}_{\eta,p,q} f(z)\big)' + \frac{\mu}{p}z^2\big(\mathfrak{S}^{n,m}_{\eta,p,q} f(z)\big)''}{(1-\mu)\mathfrak{S}^{n,m}_{\eta,p,q} f(z) + \frac{\mu}{p}z\big(\mathfrak{S}^{n,m}_{\eta,p,q} f(z)\big)'} - \alpha\right) \\
&\quad = \Re\left(\frac{p + \mathfrak{a}\Big(1+\frac{\mu}{p}+\mu\, p\Big)(p+1)\, z}{1+\mathfrak{a}\Big(1+\frac{\mu}{p}\Big)\, z} - \alpha\right)
\end{aligned}$$

and

$$\begin{aligned}
&\kappa\left|\frac{\Big(1-\mu+\frac{\mu}{p}\Big)z\big(\mathfrak{S}^{n,m}_{\eta,p,q} f(z)\big)' + \frac{\mu}{p}z^2\big(\mathfrak{S}^{n,m}_{\eta,p,q} f(z)\big)''}{(1-\mu)\mathfrak{S}^{n,m}_{\eta,p,q} f(z) + \frac{\mu}{p}\, z\big(\mathfrak{S}^{n,m}_{\eta,p,q} f(z)\big)'} - p\right| \\
&\quad = \kappa\left|\frac{p + \mathfrak{a}\Big(1+\frac{\mu}{p}+\mu\, p\Big)(p+1)\, z}{1+\mathfrak{a}\Big(1+\frac{\mu}{p}\Big)\, z} - p\right|.
\end{aligned}$$

Hence, obviously, the inequality (10) becomes

$$\varphi(z) := \Re\left(\frac{p + \mathfrak{a}\left(1 + \frac{\mu}{p} + \mu\, p\right)(p+1)\, z}{1 + \mathfrak{a}\left(1 + \frac{\mu}{p}\right) z} - \alpha\right) - \kappa\left|\frac{p + \mathfrak{a}\left(1 + \frac{\mu}{p} + \mu\, p\right)(p+1)\, z}{1 + \mathfrak{a}\left(1 + \frac{\mu}{p}\right) z} - p\right| > 0 \qquad \big(z \in \mathbb{U}\big).$$

Some further examples of functions in the class $\aleph^{n,m}_{\mu,p}(\eta,\alpha,\kappa)$ are included as the extremal functions of the sharp inequalities, which are asserted by Theorems 4, 5 and 8, and also by Corollaries 1, 3 and 4.

Next, for the special case when

$$\begin{aligned} &p = 3, \quad \mu = 0.3, \quad \alpha = 0.2, \quad n = 4, \quad q = 0.1, \\ &m = 5, \quad \eta = 0.01, \quad \lambda = 0.1 \quad \text{and} \quad \kappa = 0.7, \end{aligned} \tag{11}$$

the image of the unit circle $\partial\mathbb{U}$ given by

$$\partial\mathbb{U} := \{z : z \in \mathbb{C} \quad \text{and} \quad |z| = 1\}$$

by the function φ is non-negative. Hence, we have

$$\varphi(z) > 1.2 > 0 \qquad \big(z \in \mathbb{U}\big). \tag{12}$$

Figure 1 below is made with the aid of the *Maple* software.

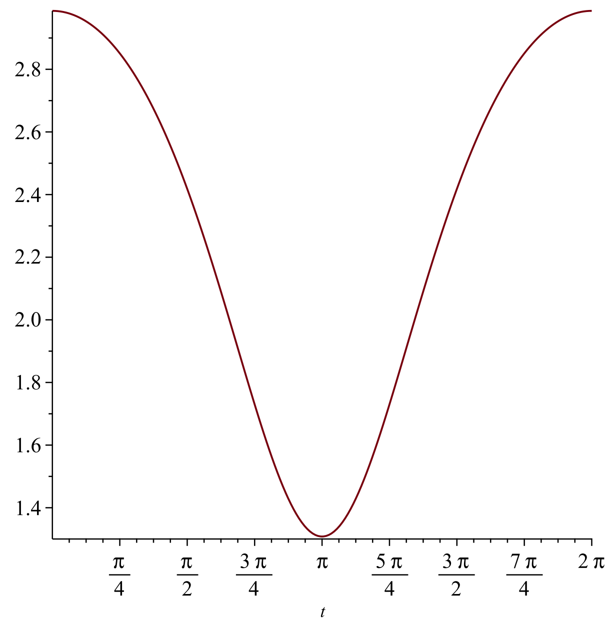

Figure 1. The graph of $v := \varphi(e^{it})$ for $t \in [0, 2\pi]$ and $v = 1.2$.

Thus, for the values given by (11), we have

$$f(z) = z^p + \lambda z^{p+1} \in \aleph^{n,m}_{\mu,p}(\eta,\alpha,\kappa).$$

Hence, there exist values for the parameters such that $\aleph^{n,m}_{\mu,p}(\eta,\alpha,\kappa) \neq \emptyset$.

By using some linear operators and fractional calculus, several new subclasses of the families of p-valent κ-uniformly convex of order α in $\mathbb{U}$ and p-valent κ-uniformly starlike of order α in $\mathbb{U}$ were studied in (for example) [15,16]. Various interesting properties and characteristics of functions in such function classes were obtained in these articles. In light of these earlier developments, it is of interest to consider the behavior of functions in the classes κ-$\mathcal{UCV}(p;\alpha)$ and κ-$\mathcal{UST}(p;\alpha)$, which we introduced in Definition 3 by means of q-derivative operators.

It is our purpose in this article is to obtain coefficient bounds, distortion properties, convex linear combinations and the radii of starlikeness, convexity and close-to-convexity for functions belonging to the following class:

$$\mathcal{T}\aleph_{\mu,p}^{n,m}(\eta,\alpha,\kappa)$$

of p-valent κ-uniformly starlike and κ-uniformly starlike convex functions.

Remark 7. *In the sequel, we shall assume that the coefficients of the functions, which belong to the above-defined function classes*

$$\aleph_{\mu,p}^{n,m}(\eta,\alpha,\kappa) \qquad \text{and} \qquad \mathcal{T}\aleph_{\mu,p}^{n,m}(\eta,\alpha,\kappa),$$

are given by (1) *and* (2)*, respectively.*

2. Coefficient Estimates

Henceforth, unless stated otherwise, it is assumed in this article that $0 \leqq \alpha < p$, $p \in \mathbb{N}, 0 \leqq \mu \leqq 1$ and $0 \leqq \kappa < \infty$.

We shall now prove the results which are stated below.

Theorem 1. *If $f \in \mathcal{A}(p)$ is of the form* (1)*, and if its coefficients satisfy each of the following inequalities*:

$$\sum_{k=p+1}^{\infty} \left([k]_q\right)^n \left(\frac{p+\eta(k-p)}{p}\right)^m \left(\frac{p+\mu(k-p)}{p}\right) |a_k| < 1 \tag{13}$$

and

$$\begin{aligned} \sum_{k=p+1}^{\infty} \left([k]_q\right)^n \left(\frac{p+\eta(k-p)}{p}\right)^m \left(\frac{p+\mu(k-p)}{p}\right) \\ \cdot [k(1+\kappa)-(\alpha+p\kappa)]|a_k| \\ < p-\alpha, \end{aligned} \tag{14}$$

then $f \in \aleph_{\mu,p}^{n,m}(\eta,\alpha,\kappa)$.

Proof. According to Definition 3, we need to prove that, under the assumptions (13) and (14), it follows that inequality (5) holds true. Since inequality (5) is equivalent to

$$\begin{aligned} &\kappa\left|\frac{\left(1-\mu+\frac{\mu}{p}\right)z\left(\mathfrak{S}_{\eta,p,q}^{n,m}f(z)\right)'+\frac{\mu}{p}z^2\left(\mathfrak{S}_{\eta,p,q}^{n,m}f(z)\right)''}{(1-\mu)\mathfrak{S}_{\eta,p,q}^{n,m}f(z)+\frac{\mu}{p}z\left(\mathfrak{S}_{\eta,p,q}^{n,m}f(z)\right)'}-p\right| \\ &\qquad -\Re\left(\frac{\left(1-\mu+\frac{\mu}{p}\right)z\left(\mathfrak{S}_{\eta,p,q}^{n,m}f(z)\right)'+\frac{\mu}{p}z^2\left(\mathfrak{S}_{\eta,p,q}^{n,m}f(z)\right)''}{(1-\mu)\mathfrak{S}_{\eta,p,q}^{n,m}f(z)+\frac{\mu}{p}z\left(\mathfrak{S}_{\eta,p,q}^{n,m}f(z)\right)'}-p\right) \\ &< p-\alpha \qquad (z\in\mathbb{U}), \end{aligned} \tag{15}$$

by using the triangle inequality, together with (13) and (14), we find that

$$\kappa\left|\frac{\left(1-\mu+\frac{\mu}{p}\right)z\left(\mathfrak{S}^{n,m}_{\eta,p,q}f(z)\right)'+\frac{\mu}{p}z^2\left(\mathfrak{S}^{n,m}_{\eta,p,q}f(z)\right)''}{(1-\mu)\mathfrak{S}^{n,m}_{\eta,p,q}f(z)+\frac{\mu}{p}z\left(\mathfrak{S}^{n,m}_{\eta,p,q}f(z)\right)'}-p\right| \tag{16}$$

$$-\Re\left(\frac{\left(1-\mu+\frac{\mu}{p}\right)z\left(\mathfrak{S}^{n,m}_{\eta,p,q}f(z)\right)'+\frac{\mu}{p}z^2\left(\mathfrak{S}^{n,m}_{\eta,p,q}f(z)\right)''}{(1-\mu)\mathfrak{S}^{n,m}_{\eta,p,q}f(z)+\frac{\mu}{p}z\left(\mathfrak{S}^{n,m}_{\eta,p,q}f(z)\right)'}-p\right)$$

$$\leqq (1+\kappa)\left|\frac{\left(1-\mu+\frac{\mu}{p}\right)z\left(\mathfrak{S}^{n,m}_{\eta,p,q}f(z)\right)'+\frac{\mu}{p}z^2\left(\mathfrak{S}^{n,m}_{\eta,p,q}f(z)\right)''}{(1-\mu)\mathfrak{S}^{n,m}_{\eta,p,q}f(z)+\frac{\mu}{p}z\left(\mathfrak{S}^{n,m}_{\eta,p,q}f(z)\right)'}-p\right| \tag{17}$$

$$\leqq \frac{(1+\kappa)\sum\limits_{k=p+1}^{\infty}\left([k]_q\right)^n\left(\frac{p+\eta(k-p)}{p}\right)^m\left(\frac{p+\mu(k-p)}{p}\right)(k-p)|a_k|}{1-\sum\limits_{k=p+1}^{\infty}\left([k]_q\right)^n\left(\frac{p+\eta(k-p)}{p}\right)^m\left(\frac{p+\mu(k-p)}{p}\right)|a_k|} \tag{18}$$

$$(z\in\mathbb{U}).$$

We now observe that, if the assumption (14) holds true, then the last expression in (16) is bounded above by $p-\alpha$. Hence, according to (15), we conclude that $f\in\aleph^{n,m}_{\mu,p}(\eta,\alpha,\kappa)$. The proof of Theorem 1 is thus completed. □

Remark 8. *If we put $n=m=0$, $p=1$ and $\mu=0$ in Theorem 1, we obtain the results derived earlier by Shams et al. [43]. Moreover, for $n=m=0$, $p=1$ and $\mu=\kappa=0$, Theorem 1 reduces to the corresponding result of Silverman [1].*

Theorem 2. *If $f\in\mathcal{T}\aleph^{n,m}_{\mu,p}(\eta,\alpha,\kappa)$, then each of the following conditions is satisfied:*

$$\sum_{k=p+1}^{\infty}\left([k]_q\right)^n\left(\frac{p+\eta(k-p)}{p}\right)^m\left(\frac{p+\mu(k-p)}{p}\right)a_k<1 \tag{19}$$

and

$$\sum_{k=p+1}^{\infty}\left([k]_q\right)^n\left(\frac{p+\eta(k-p)}{p}\right)^m\left(\frac{p+\mu(k-p)}{p}\right)\cdot[k(1+\kappa)-(\alpha+p\kappa)]a_k\leqq p-\alpha. \tag{20}$$

Proof. If $f\in\mathcal{T}\aleph^{n,m}_{\mu,p}(\eta,\alpha,\kappa)$ and $z\in\mathbb{U}$ is real, then

$$\frac{p-\sum\limits_{k=p+1}^{\infty}\left([k]_q\right)^n\left(\frac{p+\eta(k-p)}{p}\right)^m\left(\frac{p+\mu(k-p)}{p}\right)a_kz^{k-p}}{1-\sum\limits_{k=p+1}^{\infty}\left([k]_q\right)^n\left(\frac{p+\eta(k-p)}{p}\right)^m\left(\frac{p+\mu(k-p)}{p}\right)a_kz^{k-p}}-\alpha$$

$$>\kappa\left|\frac{-\sum\limits_{k=p+1}^{\infty}\left([k]_q\right)^n\left(\frac{p+\eta(k-p)}{p}\right)^m\left(\frac{p+\mu(k-p)}{p}\right)(k-p)a_kz^{k-p}}{1-\sum\limits_{k=p+1}^{\infty}\left([k]_q\right)^n\left(\frac{p+\eta(k-p)}{p}\right)^m\left(\frac{p+\mu(k-p)}{p}\right)a_k\,z^{k-p}}\right|.$$

Upon letting $z \to 1-$ along the real axis, we obtain

$$\frac{p - \sum_{k=p+1}^{\infty}\left([k]_q\right)^n \left(\frac{p+\eta(k-p)}{p}\right)^m \left(\frac{p+\mu(k-p)}{p}\right) a_k}{1 - \sum_{k=p+1}^{\infty}\left([k]_q\right)^n \left(\frac{p+\eta(k-p)}{p}\right)^m \left(\frac{p+\mu(k-p)}{p}\right) a_k} - \alpha$$
$$\geqq \kappa \frac{\sum_{k=p+1}^{\infty}\left([k]_q\right)^n \left(\frac{p+\eta(k-p)}{p}\right)^m \left(\frac{p+\mu(k-p)}{p}\right)(k-p)a_k}{\left|1 - \sum_{k=p+1}^{\infty}\left([k]_q\right)^n \left(\frac{p+\eta(k-p)}{p}\right)^m \left(\frac{p+\mu(k-p)}{p}\right) a_k\right|}. \tag{21}$$

Therefore, if we assume that (19) holds true, then (21) is equivalent to (20). This completes the proof of Theorem 2. □

Since the series in (19) has positive coefficients, in view Theorem 2, we can obtain the coefficient estimates for the functions of the class $\mathcal{T}\aleph_{\mu,p}^{n,m}(\eta,\alpha,\kappa)$.

Corollary 1. *If $f \in \mathcal{T}\aleph_{\mu,p}^{n,m}(\eta,\alpha,\kappa)$ such that the inequality (19) is satisfied, then*

$$a_k \leqq \frac{p-\alpha}{\left([k]_q\right)^n \left(\frac{p+\eta(k-p)}{p}\right)^m \left(\frac{p+\mu(k-p)}{p}\right)[k(1+\kappa)-(\alpha+p\kappa)]} \tag{22}$$
$$(k \geqq p+1;\ p \in \mathbb{N}).$$

The result (22) is sharp for the extremal function f given by

$$f(z) = z^p - \frac{p-\alpha}{\left([k]_q\right)^n \left(\frac{p+\eta(k-p)}{p}\right)^m \left(\frac{p+\mu(k-p)}{p}\right)[k(1+\kappa)-(\alpha+p\kappa)]} z^k \tag{23}$$
$$(k \geqq p+1;\ p \in \mathbb{N}).$$

3. Distortion Theorems

The coefficient bounds, which we obtained in the preceding section (Section 2), enable us to prove the following distortion theorem.

Theorem 3. *Let f defined by (2) be in the class $\mathcal{T}\aleph_{\mu,p}^{n,m}(\eta,\alpha,\kappa)$. Suppose also that inequality (19) is satisfied. Then, for $|z| = r < 1$, the following distortion results hold true:*

$$|f(z)| \geqq r^p - \frac{p-\alpha}{\left([p+1]_q\right)^n \left(\frac{p+\eta}{p}\right)^m \left(\frac{p+\mu}{p}\right)(p+1+\kappa-\alpha)} r^{p+1} \tag{24}$$

and

$$|f(z)| \leqq r^p + \frac{p-\alpha}{\left([p+1]_q\right)^n \left(\frac{p+\eta}{p}\right)^m \left(\frac{p+\mu}{p}\right)(p+1+\kappa-\alpha)} r^{p+1}. \tag{25}$$

The equalities in (24) and (25) are attained for the function $\widetilde{f}$ given by

$$\widetilde{f}(z) = z^p - \frac{p-\alpha}{\left([p+1]_q\right)^n \left(\frac{p+\eta}{p}\right)^m \left(\frac{p+\mu}{p}\right)(p+1+\kappa-\alpha)} z^{p+1}. \tag{26}$$

Proof. For $f \in \mathcal{T}\aleph_{\mu,p}^{n,m}(\eta,\alpha,\kappa)$, by using Theorem 2 (i), we have

$$\begin{aligned}\left([p+1]_q\right)^n \left(\frac{p+\eta}{p}\right)^m \left(\frac{p+\mu}{p}\right)(p+1+\kappa-\alpha)\sum_{k=p+1}^{\infty} a_k \\ \leqq \sum_{k=p+1}^{\infty}\left([k]_q\right)^n \left(\frac{p+\eta(k-p)}{p}\right)^m \left(\frac{p+\mu(k-p)}{p}\right) \\ \cdot [k(1+\kappa)-(\alpha+p\kappa)]a_k \leqq p-\alpha,\end{aligned}$$

which yields

$$\sum_{k=p+1}^{\infty} a_k \leqq \frac{p-\alpha}{\left([p+1]_q\right)^n \left(\frac{p+\eta}{p}\right)^m \left(\frac{p+\mu}{p}\right)(p+1+\kappa-\alpha)}. \tag{27}$$

From (2) and (27), we find for $|z| = r < 1$ that

$$\begin{aligned}|f(z)| &\geqq r^p - r^{p+1}\sum_{k=p+1}^{\infty} a_k \\ &\geqq r^p - \frac{p-\alpha}{\left([p+1]_q\right)^n \left(\frac{p+\eta}{p}\right)^m \left(\frac{p+\mu}{p}\right)(p+1+\kappa-\alpha)} r^{p+1}\end{aligned}$$

and

$$\begin{aligned}|f(z)| &\leqq r^p + r^{p+1}\sum_{k=p+1}^{\infty} a_k \\ &\leqq r^p + \frac{p-\alpha}{\left([p+1]_q\right)^n \left(\frac{p+\eta}{p}\right)^m \left(\frac{p+\mu}{p}\right)(p+1+\kappa-\alpha)} r^{p+1}.\end{aligned}$$

Finally, since the function $\widetilde{f}$ given by (26) satisfies the equalities in assertions (24) and (25), the proof of Theorem 3 is complete. □

Theorem 4. *Let $f \in \mathcal{T}\aleph_{\mu,p}^{n,m}(\eta,\alpha,\kappa)$. Then, for $|z| = r < 1$, each of the following assertions holds true*:

$$|f'(z)| \geqq pr^{p-1} - \frac{(p+1)(p-\alpha)}{\left([p+1]_q\right)^n \left(\frac{p+\eta}{p}\right)^m \left(\frac{p+\mu}{p}\right)(p+1+\kappa-\alpha)} r^p \tag{28}$$

and

$$|f'(z)| \leqq pr^{p-1} + \frac{(p+1)(p-\alpha)}{\left([p+1]_q\right)^n \left(\frac{p+\eta}{p}\right)^m \left(\frac{p+\mu}{p}\right)(p+1+\kappa-\alpha)} r^p. \tag{29}$$

The results (28) and (29) are sharp for the extremal the function $\widetilde{f}$ given by (26).

Proof. From Theorem 2, we have

$$\sum_{k=p+1}^{\infty} ka_k \leqq \frac{(p+1)(p-\alpha)}{\left([p+1]_q\right)^n \left(\frac{p+\eta}{p}\right)^m \left(\frac{p+\mu}{p}\right)(p+1+\kappa-\alpha)}.$$

The remainder of the proof of Theorem 4 is essentially analogous to that of Theorem 3, so we choose to omit the details involved. □

Next, upon differentiating both sides of (2) m times, we have

$$f^{(m)}(z) = \frac{p!}{(p-m)!} z^{p-m} - \sum_{k=p+1}^{\infty} \frac{k!}{(k-m)!} a_k z^{k-m} \tag{30}$$

$$\left(k \geqq p+1;\ p \in \mathbb{N};\ m < p;\ m \in \mathbb{N}_0\right).$$

Theorem 5. *Let the function $f(z)$, defined by (2), be in the class $\mathcal{T}\aleph^{n,m}_{\mu,p}(\eta,\alpha,\kappa)$. Then, for $|z| = r < 1$, it is asserted that*

$$\left|f^{(m)}(z)\right| \geqq \left(\frac{p!}{(p-m)!} - \frac{(p+1)!\,(p-\alpha)}{\left([p+1]_q\right)^n \left(\frac{p+\eta}{p}\right)^m \left(\frac{p+\mu}{p}\right)(p+1-m)!\,[p+1+\kappa-\alpha]}\, r\right) r^{p-m} \tag{31}$$

and

$$\left|f^{(m)}(z)\right| \leqq \left(\frac{p!}{(p-m)!} + \frac{(p+1)!\,(p-\alpha)}{\left([p+1]_q\right)^n \left(\frac{p+\eta}{p}\right)^m \left(\frac{p+\mu}{p}\right)(p+1-m)!\,[p+1+\kappa-\alpha]}\, r\right) r^{p-m}. \tag{32}$$

The results (31) and (32) are sharp for the extremal function $\widetilde{f}$ given by (26).

Proof. By using (14), we have

$$\sum_{k=p+1}^{\infty} \frac{k!}{(k-m)!} a_k \leqq \frac{(p+1)!(p-\alpha)}{\left([p+1]_q\right)^n \left(\frac{p+\eta}{p}\right)^m \left(\frac{p+\mu}{p}\right)(p+1-m)![p+1+\kappa-\alpha]}. \tag{33}$$

Additionally, from (3) and (33), we find that

$$\begin{aligned}\left|f^{(m)}(z)\right| &\geqq \frac{p!}{(p-m)!} r^{p-m} - r^{p+1-m} \sum_{k=p+1}^{\infty} \frac{k!}{(k-m)!} a_k \\ &\geqq \left(\frac{p!}{(p-m)!} - \frac{(p+1)!(p-\alpha)}{\left([p+1]_q\right)^n \left(\frac{p+\eta}{p}\right)^m \left(\frac{p+\mu}{p}\right)(p+1-m)!(p+1+\kappa-\alpha)}\, r\right) r^{p-m}\end{aligned}$$

and

$$\begin{aligned}\left|f^{(m)}(z)\right| &\leqq \frac{p!}{(p-m)!} r^{p-m} + r^{p+1-m} \sum_{k=p+1}^{\infty} \frac{k!}{(k-m)!} a_k \\ &\leqq r^{p-m}\left(\frac{p!}{(p-m)!} + \frac{(p+1)!(p-\alpha)}{\left([p+1]_q\right)^n \left(\frac{p+\eta}{p}\right)^m \left(\frac{p+\mu}{p}\right)(p+1-m)!(p+1+\kappa-\alpha)}\, r\right),\end{aligned}$$

which evidently completes the proof of Theorem 5. □

4. Convex Linear Combinations

Theorem 6. *Let $\zeta_\nu \geqq 0$ for $\nu = 1, 2, 3, \cdots, n$. Suppose also that*

$$\sum_{\nu=1}^{n} \zeta_\nu \leqq 1 \qquad (\zeta_\nu \geqq 0). \qquad (\zeta_\nu \geqq 0). \tag{34}$$

If the functions $f_\nu(z)$, defined by

$$f_\nu(z) = z^p - \sum_{k=p+1}^{\infty} a_{k,\nu} z^k \qquad (a_{k,\nu} \geqq 0;\ \nu = 1, 2, 3, \cdots, n), \tag{35}$$

are in the class $\mathcal{T}\aleph_{\mu,p}^{n,m}(\eta, \alpha, \kappa)$ for every $\nu = 1, 2, 3, \cdots, n$, then the function $\mathfrak{f}(z)$ defined by the following linear combination:

$$\mathfrak{f}(z) = z^p - \sum_{k=p+1}^{\infty} \left(\sum_{\nu=1}^{\infty} \zeta_\nu \, a_{k,\nu} \right) z^k \tag{36}$$

is also in the class $\mathcal{T}\aleph_{\mu,p}^{n,m}(\eta, \alpha, \kappa)$.

Proof. Since each of the functions $f_\nu(z) \in \mathcal{T}\aleph_{\mu,p}^{n,m}(\eta, \alpha, \kappa)$ $(\nu = 1, 2, 3, \cdots, n)$ satisfies the assertion (20) of Theorem 2, we obtain

$$\begin{aligned} &\sum_{k=p+1}^{\infty} \left([k]_q\right)^n \left(\frac{p + \eta(k-p)}{p}\right)^m \left(\frac{p + \mu(k-p)}{p}\right) \\ &\qquad \cdot [k(1+\kappa) - (\alpha + p\kappa)] a_{k,\nu} \leqq p - \alpha \end{aligned} \tag{37}$$

for every $\nu = 1, 2, 3, \cdots, n$.

Now, upon multiplying both sides of (37) by ζ_ν, we sum each side of the resulting equation from $\nu = 1$ to $\nu = n$. Thus, if we make use of hypotheses (34) and (35), our demonstration of Theorem 6 is completed by appealing to Theorem 2 once again. □

Corollary 2. *The class $\mathcal{T}\aleph_{\mu,p}^{n,m}(\eta, \alpha, \kappa)$ is closed under convex linear combinations.*

Theorem 7. *Let $f_p(z) = z^p$ and*

$$f_k(z) = z^p - \frac{p - \alpha}{\left([k]_q\right)^n \left(\frac{p+\eta(k-p)}{p}\right)^m \left(\frac{p+\mu(k-p)}{p}\right) [k(1+\kappa) - (\alpha + p\kappa)]} z^k \tag{38}$$

$$(k \geqq p+1).$$

Then $f(z) \in \mathcal{T}\aleph_{\mu,p}^{n,m}(\eta, \alpha, \kappa)$ if and only if

$$f(z) = \sum_{k=p+1}^{\infty} \gamma_k f_k(z), \tag{39}$$

where

$$\gamma_k \geqq 0 \qquad \text{and} \qquad \sum_{k=p}^{\infty} \gamma_k = 1.$$

Proof. Let us assume that

$$\begin{aligned} f(z) &= \sum_{k=p}^{\infty} \gamma_k\, f_k(z) \\ &= z^p - \sum_{k=p+1}^{\infty} \frac{p-\alpha}{\left([k]_q\right)^n \left(\frac{p+\eta(k-p)}{p}\right)^m \left(\frac{p+\mu(k-p)}{p}\right)[k(1+\kappa)-(\alpha+p\kappa)]} \gamma_k z^k. \end{aligned}$$

Then it follows that

$$\begin{aligned} &\sum_{k=p+1}^{\infty} \frac{\left([k]_q\right)^n \left(\frac{p+\eta(k-p)}{p}\right)^m \left(\frac{p+\mu(k-p)}{p}\right)[k(1+\kappa)-(\alpha+p\kappa)]}{p-\alpha} \\ &\qquad \cdot \frac{p-\alpha}{\left([k]_q\right)^n \left(\frac{p+\eta(k-p)}{p}\right)^m \left(\frac{p+\mu(k-p)}{p}\right)[k(1+\kappa)-(\alpha+p\kappa)]} \gamma_k \\ &= \sum_{k=p+1}^{\infty} \gamma_k = 1-\gamma_p \leqq 1, \end{aligned}$$

which proves that $f(z) \in \mathcal{T}\aleph_{\mu,p}^{n,m}(\eta,\alpha,\kappa)$.

Conversely, if we let $f(z) \in \mathcal{T}\aleph_{\mu,p}^{n,m}(\eta,\alpha,\kappa)$, then

$$a_k \leqq \frac{p-\alpha}{\left([k]_q\right)^n \left(\frac{p+\eta(k-p)}{p}\right)^m \left(\frac{p+\mu(k-p)}{p}\right)[k(1+\kappa)-(\alpha+p\kappa)]} \quad (k \geqq p+1).$$

Thus, upon setting

$$\gamma_k = \frac{\left([k]_q\right)^n \left(\frac{p+\eta(k-p)}{p}\right)^m \left(\frac{p+\mu(k-p)}{p}\right)[k(1+\kappa)-(\alpha+p\kappa)]}{p-\alpha} \quad (k \geqq p+1)$$

and

$$\gamma_p = 1 - \sum_{k=p+1}^{\infty} \gamma_k,$$

we see that $f(z)$ can be expressed in the form (39). The proof of Theorem 7 is now completed. □

In view of Theorem 7, we can obtain the following corollary.

Corollary 3. *The extreme points of the class $\mathcal{T}\aleph_{\mu,p}^{n,m}(\eta,\alpha,\kappa)$ are the functions $f_p(z) = z^p$ and*

$$f_k(z) = z^p - \frac{p-\alpha}{\left([k]_q\right)^n \left(\frac{p+\eta(k-p)}{p}\right)^m \left(\frac{p+\mu(k-p)}{p}\right)[k(1+\kappa)-(\alpha+p\kappa)]} z^k$$

$$(k \geqq p+1).$$

5. Radii of Close-to-Convexity, Starlikeness and Convexity

In this section, we first introduce the notions of close-to-convexity, starlikeness and convexity.

Let the class $\mathcal{S} \subset \mathcal{A}$ consist of normalized univalent functions in $\mathbb{U}$. The commonly-known subclasses of the normalized analytic and univalent function class $\mathcal{S} \subset \mathcal{A}$ in $\mathbb{U}$ are the classes of convex, starlike and close-to-convex functions of order α $(0 \leqq \alpha < 1)$, which (in the simpler case when $p = 1$) are defined and denoted, respectively, as follows:

$$\mathcal{C}(\alpha) := \left\{ f : f \in \mathcal{S} \quad \text{and} \quad \Re\left(\frac{(zf'(z))'}{f'(z)} \right) > \alpha \quad (z \in \mathbb{U};\ 0 \leqq \alpha < 1) \right\},$$

$$\mathcal{S}^*(\alpha) := \left\{ f : f \in \mathcal{S} \quad \text{and} \quad \Re\left(\frac{zf'(z)}{f(z)} \right) > \alpha \quad (z \in \mathbb{U};\ 0 \leqq \alpha < 1) \right\}$$

and

$$\mathcal{K}(\alpha) := \left\{ f : f \in \mathcal{A},\ g \in \mathcal{C} \ \text{ and } \ \Re\left(\frac{f'(z)}{g'(z)} \right) > \alpha \quad (z \in \mathbb{U};\ 0 \leqq \alpha < 1) \right\}$$

or, equivalently,

$$\mathcal{K}(\alpha) := \left\{ f : f \in \mathcal{A},\ g \in \mathcal{S}^* \ \text{ and } \ \Re\left(\frac{zf'(z)}{g(z)} \right) > \alpha \quad (z \in \mathbb{U};\ 0 \leqq \alpha < 1) \right\}.$$

With a view to defining the radius of close-to-convexity, let the function $f \in \mathcal{C}(\alpha)$ be close-to-convex relative to the normalized convex function $g(z)$ for $|z| < R_{\mathcal{K}(\alpha)}(f, g)$ and in no larger disk. Then the radius of close-to-convexity of order α relative to g for the class $\mathcal{C}(\alpha)$ is denoted by $R_{\mathcal{K}(\alpha)}(g)$ and defined as follows:

$$R_{\mathcal{K}(\alpha)}(g) := \liminf_{f \in \mathcal{K}(\alpha)} \left\{ R_{\mathcal{K}(\alpha)}(f, g) \right\}.$$

We next let the function $f \in \mathcal{S}^*(\alpha)$ be starlike of order α in $|z| < R_{\mathcal{S}^*}(f)$ and in no larger disk. Then the radius of starlikeness of order α for the class $\mathcal{S}^*(\alpha)$ is denoted by $R_{\mathcal{S}^*(\alpha)}$ and defined by

$$R_{\mathcal{S}^*(\alpha)} := \liminf_{f \in \mathcal{S}^*(\alpha)} \left\{ R_{\mathcal{S}^*(\alpha)}(f) \right\}.$$

Similarly, we can define the radius of convexity of order α for the class $\mathcal{C}(\alpha)$, which may be denoted analogously by $R_{\mathcal{C}(\alpha)}$.

Each of the above definitions can readily be extended to the cases of the corresponding multivalent (or p-valent) functions. The details are skipped here.

Theorem 8. *Let $f(z) \in \mathcal{T}\aleph_{\mu,p}^{n,m}(\eta, \alpha, \kappa)$. Then each of the following assertions holds true:*

(i) *The function $f(z)$ is p-valent close-to-convex of order δ $(0 \leqq \delta < p)$ in $|z| < r_1$, where*

$$r_1 = \inf_{k \geqq p+1} \left\{ \left([k]_q\right)^n \left(\frac{p + \eta(k-p)}{p} \right)^m \left(\frac{p + \mu(k-p)}{p} \right) (p - \delta) \cdot \left(\frac{k(1+\kappa) - (\alpha + p\kappa)}{k(p-\alpha)} \right) \right\}^{\frac{1}{k-p}}. \tag{40}$$

(ii) *The function $f(z)$ is p-valent starlike of order δ $(0 \leqq \delta < p)$ in $|z| < r_2$, where*

$$r_2 = \inf_{k \geqq p+1} \left\{ \left([k]_q\right)^n \left(\frac{p + \eta(k-p)}{p} \right)^m \left(\frac{p + \mu(k-p)}{p} \right) (p - \delta) \cdot \left(\frac{k(1+\kappa) - (\alpha + p\kappa)}{(k-\delta)(p-\alpha)} \right) \right\}^{\frac{1}{k-p}}. \tag{41}$$

Each of these results is sharp with the extremal function $f(z)$ given by (23).

Proof. We prove each part of Theorem 8 separately.

(i) We need to show that

$$\left|\frac{f'(z)}{z^{p-1}} - p\right| \leqq p - \delta \qquad (|z| < r_1),$$

where r_1 is given by (40). Indeed we find from (2) that

$$\left|\frac{f'(z)}{z^{p-1}} - p\right| \leqq \sum_{k=p+1}^{\infty} k a_k |z|^{k-p}.$$

Thus, clearly, we have

$$\left|\frac{f'(z)}{z^{p-1}} - p\right| \leqq p - \delta,$$

if

$$\sum_{k=p+1}^{\infty} \left(\frac{k}{p-\delta}\right) a_k |z|^{k-p} \leqq 1. \tag{42}$$

However, by Theorem 2, (42) will be true if

$$\left(\frac{k}{p-\delta}\right)|z|^{k-p} \leqq \left([k]_q\right)^n \left(\frac{p+\eta(k-p)}{p}\right)^m \left(\frac{p+\mu(k-p)}{p}\right) \cdot \left(\frac{k(1+\kappa)-(\alpha+p\kappa)}{p-\alpha}\right),$$

that is, if

$$|z| \leqq \left\{\left([k]_q\right)^n \left(\frac{p+\eta(k-p)}{p}\right)^m \left(\frac{p+\mu(k-p)}{p}\right)(p-\delta) \cdot \left(\frac{k(1+\kappa)-(\alpha+p\kappa)}{k(p-\alpha)}\right)\right\}^{\frac{1}{k-p}} \qquad (k \geqq p+1).$$

The proof of Part (i) is completed.

(ii) It is sufficient to show that

$$\left|\frac{zf'(z)}{f(z)} - p\right| \leqq p - \delta \qquad (|z| < r_2),$$

where r_2 is given by (41). In fact, we find from (2) that

$$\left|\frac{zf'(z)}{f(z)} - p\right| \leqq \frac{\sum\limits_{k=p+1}^{\infty} (k-p) a_k |z|^{k-p}}{1 - \sum\limits_{k=p+1}^{\infty} a_k |z|^{k-p}}.$$

Thus, clearly, we have

$$\left|\frac{zf'(z)}{f(z)} - p\right| \leqq p - \delta,$$

if

$$\sum_{k=p+1}^{\infty} \left(\frac{k-\delta}{p-\delta}\right) a_k |z|^{k-p} \leqq 1. \tag{43}$$

Moreover, by Theorem 2, this last inequality (43) will hold true if

$$\left(\frac{k-\delta}{p-\delta}\right)|z|^{k-p} \leqq \left([k]_q\right)^n \left(\frac{p+\eta(k-p)}{p}\right)^m \left(\frac{p+\mu(k-p)}{p}\right) \cdot \left(\frac{k(1+\kappa)-(\alpha+p\kappa)}{p-\alpha}\right),$$

that is, if

$$|z| \leqq \left\{\left([k]_q\right)^n \left(\frac{p+\eta(k-p)}{p}\right)^m \left(\frac{p+\mu(k-p)}{p}\right)(p-\delta) \cdot \left(\frac{k(1+\kappa)-(\alpha+p\kappa)}{(k-\delta)(p-\alpha)}\right)\right\}^{\frac{1}{k-p}} \qquad (k \geqq p+1).$$

This evidently completes the proof of Part (ii) of Theorem 8. □

By appealing appropriately to Theorem 8, we can deduce the following corollary.

Corollary 4. *Let $f(z) \in \mathcal{T}\aleph_{\mu,p}^{n,m}(\eta,\alpha,\kappa)$. Then $f(z)$ is p-valent convex of order δ $(0 \leqq \delta < p)$ in $|z| < r_3$, where*

$$r_3 = \inf_{k \geqq p+1} \left\{\left([k]_q\right)^n \left(\frac{p+\eta(k-p)}{p}\right)^m \left(\frac{p+\mu(k-p)}{p}\right)(p-\delta) \qquad (44)$$

$$\cdot \left(\frac{k(1+\kappa)-(\alpha+p\kappa)}{k(k-\delta)(p-\alpha)}\right)\right\}^{\frac{1}{k-p}}. \qquad (45)$$

The result is sharp for the extremal function $f(z)$ given by (23).

6. Concluding Remarks and Observations

The present investigation stems from the widely- and extensively-studied q-series and q-polynomials, especially the q-functions and the q-polynomials of hypergeometric type, which are known to have potential for applications in many areas of combinatorial analysis and number-theoretic analysis such as (for example) the partition theory. In addition, the q-analysis and the q-calculus are useful also in a considerably-wide variety of scientific disciplines such as, for example, Lie theory and Lie groups, finite vector spaces, particle physics, mechanical engineering, non-linear electric circuit theory, quantum mechanics, theory of heat conduction, cosmology, and statistics (see, for details, [23] (pp. 350–351) and [22] (Section 5)). Here, in this article, we successfully applied the q-calculus to an equally-important and fast-growing area of geometric function theory of complex analysis. With this object in view, we introduced and systematically studied the properties and characteristics of a general subclass $\mathcal{T}\aleph_{\lambda,p}^{n,m}(\eta,\alpha,\kappa)$ of multivalent (or p-valent) κ-uniformly convex and multivalent (or p-valent) κ-uniformly starlike functions, which are associated with the Jackson q-derivative operator, in the open unit disk $\mathbb{U}$. The results, which we presented in this article, answer the questions concerning the coefficient bounds, the distortion properties, convex linear combinations and the radii of starlikeness, convexity and close-to-convexity for functions belonging to the class $\mathcal{T}\aleph_{\mu,p}^{n,m}(\eta,\alpha,\kappa)$ of multivalent (or p-valent) κ-uniformly starlike and convex functions in $\mathbb{U}$. Our results are shown to generalize and extend some of the earlier works of several authors.

Regrettably, a large number of more-or-less amateurishly-produced research works are still being published in the current literature, in which one can find obvious and inconsequential variations and straightforward translations of the corresponding known q-results in terms of the so-called $(\mathfrak{p},q)$-calculus, where the obviously redundant (or superfluous) parameter $\mathfrak{p}$ is simply forced-in rather unnecessarily. Such false claims to "generalization" should be discouraged rather strongly. In this connection, we suggest that the readers should see the recent survey-cum-expository review articles [20] (p. 340) and [21] (Section 5, pp. 1511–1512); see also [22]), in which such trivialities and inconsequential developments are exposed. The tendencies to produce and flood the literature with trivialities involving the so-called $(\mathfrak{p}, q)$-calculus should be discouraged by all means.

For the interest and motivation of the readers, we choose to refer to the classic monograph of Levin [44] (see also [45]) with the aim of exploring the connection between the results presented herein and the estimates for entire (or integral) functions which satisfy some specific conditions of the growth regularity.

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