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Traveling Wave Optical Solutions for the Generalized Fractional Kundu–Mukherjee–Naskar (gFKMN) Model

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Abstract: The work considers traveling wave optical solutions for the nonlinear generalized fractional KMN equation. This equation is considered for describing pulse propagation in optical fibers and communication systems using two quite similar approaches, based on the expansion of these solutions in the exponential or polynomial forms. Both approaches belong to the direct solving class of methods for PDEs and suppose the use of an auxiliary equation. The solutions acquired in this paper are obtained from first- and second-order differential equations that act as auxiliary equations. In addition, we generated 3D, contour, and 2D plots to illustrate the characteristics of the obtained soliton solutions. To create these plots, we carefully selected appropriate values for the relevant parameters.

Keywords: generalized fractional derivative; KMN model; auxiliary equation; exponential expansion; traveling wave optical solution

MSC: 35C07

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1. Introduction

Fractional differential equations (FDEs) have become increasingly important in describing and modeling complex phenomena in various fields of science and engineering. OFDEs, involving fractional derivatives with respect to a single variable, and PFDEs, which involve fractional derivatives with respect to multiple variables, can accurately capture the memory and hereditary properties of the system being modeled. These equations have wide-ranging applications in fields such as physics, chemistry, engineering, biology, finance, and economics, and are particularly useful for describing anomalous diffusion phenomena. Therefore, the study of FDEs is essential for understanding and predicting the behavior of complex systems in many fields of application. However, the non-locality and non-linearity of fractional derivatives present unique challenges in solving and analyzing FDEs. Traditional analytical and numerical methods may not be directly applicable, and new techniques and tools need to be developed to solve and analyze these equations. Despite these challenges, the study of FDEs has led to much important advancement in various fields of science and engineering. The development of new analytical and numerical methods for solving and analyzing FDEs has opened up new avenues for research and has led to a better understanding of complex phenomena. So far, abundant efficient techniques have been proposed for obtaining exact solutions of nonlinear fractional problems such as the generalized projective Riccati equation method [1], the exponential rational function method [2], the sine-Gordon expansion method [3], the auxiliary ordinary differential equation method and the generalized Riccati method [4], the first integral method [5], the Lie symmetry approach [6], the modified Kudryashov method [7], the modified auxiliary equation method [8], the extended $\exp(-\Phi(\xi))$-expansion technique [9], the unified method [10], and so on [11–14].

In the present research, we aim to derive traveling wave solutions to the generalized fractional Kundu–Mukherjee–Naskar (gFKMN) model, which has a dimensionless display as follows [15–19]:

$$iq_t^{(\Theta)} + \alpha q_{xy} + i\beta q(qq_x^* - q^* q_x) = 0, \tag{1}$$

where the quantity $q(x,y,t)$ is a complex solution to the model and q^* is the complex conjugation of $q(x,y,t)$. Moreover, α and β are two parameters to denote the dispersion term and the nonlinearity term, respectively. This equation is considered for describing pulse propagation in optical fibers and communication systems.

The generalized fractional derivative [20,21] is used here. With a function $f:(0,+\infty) \to R$, the generalized fractional operator of order $0 < \Theta \leq 1$ for f is defined as

$$^{t}D_{\Theta}^{GFD} f(t) = \lim_{\varepsilon \to 0} \frac{f(t + \Gamma(\varrho)/\Gamma(\varrho - \Theta + 1)\varepsilon t^{1-\Theta}) - f(t)}{\varepsilon}, \quad \varrho > -1,\ \varrho \in R^{+}. \tag{2}$$

The generalized fractional derivative satisfies the properties given in the following theorem:

Theorem 1. *Let $\alpha \in (0,1]$, $\beta > -1$, $\beta \in R$, and f, g be α-differentiable at a point t; then:*

$$(i)\ ^{t}D_{\Theta}^{GFD}(af + bg) = a\,^{t}D_{\Theta}^{GFD}(f) + b\,^{t}D_{\Theta}^{GFD}(g), \text{for all } a, b \in R.$$

$$(ii)\ ^{t}D_{\Theta}^{GFD}(t^p) = \frac{p\Gamma(\beta)}{\Gamma(\beta - \alpha + 1)} t^{p-\alpha}, \text{for all } p \in R.$$

$$(iii)\ ^{t}D_{\Theta}^{GFD}(fg) = f\,^{t}D_{\Theta}^{GFD}(g) + g\,^{t}D_{\Theta}^{GFD}(f).$$

$$(iv)\ ^{t}D_{\Theta}^{GFD}(\frac{f}{g}) = \frac{g\,^{t}D_{\Theta}^{GFD}(f) - f\,^{t}D_{\Theta}^{GFD}(g)}{g^2}.$$

If, in addition, f is differentiable, then $D_{\Theta}^{GFD}(f)(t) = \left(\frac{\Gamma(\beta)}{\Gamma(\beta-\alpha+1)}\right) t^{1-\alpha} \frac{df}{dt}$.

Equation (1) frequently arises in weather science, tidal waves, river and irrigation flows, tsunami prediction, and other applications. Günerhan et al. [22] obtained new exact solutions for Equation (1) using a new extended direct algebraic method. Rizviet al. [23] obtained the singular soliton, dark soliton, combined dark-singular soliton and other hyperbolic solutions for Equation (1) using the csch method, extended tanh–coth method, and extended rational sinh–cosh method. Talarposhti et al. [24] utilized the exp-function method to derive optical soliton solutions for the KMN model under consideration. Onder et al. [25] introduced optical soliton solutions for the KMN equation using the Sardar sub-equation and the new Kudryashov methods. Using the extended Jacobi's elliptic expansion function and the exp_a function methods, Zafar et al. [26] obtained novel soliton solutions for the KMN equation. Kumar et al. [27] found dark, bright, periodic U-shaped, and singular soliton solutions for Equation (1) using the generalized Kudryashov and the new auxiliary equation methods. In addition, other authors studied this equation using novel exact solution methods such as the extended trial function method [28], sine-Gordon/sinh-Gordon expansion methods [29], semi-inverse method [30], and Hamiltonian-based algorithm [31].

2. Mathematical Analysis of the Model

To construct new solutions for Equation (1), the following transformations are utilized:

$$q(x,y,t) = u(\xi)\exp[i\Phi(x,y,t)], \tag{3}$$

where $u(\xi)$ represents the amplitude portion. We will look for a specific class of traveling wave solutions, which impose the reduction of the PDE (1) to an ODE, by introducing the so-called wave variable:

$$\xi = b_1 x + b_2 y - \frac{v\Gamma(\varrho - \Theta + 1)}{\Theta\Gamma(\varrho)} t^{\Theta}. \tag{4}$$

The phase portion of the solution (3) will be considered to be:

$$\Phi(x, y, t) = -\kappa_1 x - \kappa_2 y + \frac{\omega\Gamma(\varrho - \Theta + 1)}{\Theta\Gamma(\varrho)} t^{\Theta} + \theta_0. \tag{5}$$

In this model, κ_1 and κ_2 denote wave numbers in the x-and y-directions respectively. Moreover,ω stands for the frequency of the wave and θ_0 is a constant. Similarly, the parameters b_1 and b_2 in (4) represent inverse width along the x and y directions respectively, while v is used for the velocity. Inserting (3) along with (4) and (5) into (1), we arrive at two real and imaginary parts, respectively

$$\alpha b_1 b_2 u'' - (\omega + \alpha k_1 k_2) u - 2\beta k_1 u^3 = 0, \tag{6}$$

$$v = -\alpha(\kappa_1 b_2 + \kappa_2 b_1). \tag{7}$$

It will be solved below using two different approaches, both of them based on expansions of the solutions in the form of the given auxiliary equations.

3. Expansion Methods

Let us consider the following fractional NLPDE

$$G(u, u_t^{(\Theta)}, u_{xx}, \ldots,) = 0. \tag{8}$$

Then, if we apply the transformation

$$u(x, t) = u(\eta),\ \eta = x - \frac{v\Gamma(\varrho - \Theta + 1)}{\Theta\Gamma(\varrho)} t^{\Theta}, \tag{9}$$

on NLPDE (8), it reduces to a nonlinear problem

$$N(u, u', u'', \ldots,) = 0, \tag{10}$$

where N is a nonlinear and v is an unknown constant to be determined.

In this case, the following general structure will be assumed for the solution of Equation (10):

$$u(\eta) = \sum_{j=-M}^{M} B_j (\exp(-\varphi(\eta)))^j, \tag{11}$$

where the coefficients $B_j(-M \le j \le M)$ are unknown parameters. In addition, the number of M is calculated by using some balance rules.

Let us consider the first-order differential equation:

$$\varphi'(\eta) = \exp(-\varphi(\eta) + \mu \exp(\varphi(\eta) + \lambda, \tag{12}$$

Then, the solutions of Equation (12) are [32–37]

$$\varphi(\eta) = \ln\left(\frac{-\sqrt{\lambda^2 - 4\mu}\tanh\left(\frac{1}{2}\sqrt{\lambda^2 - 4\mu}(\eta + \eta_0)\right) - \lambda}{2\mu}\right),\ \mu \neq 0,\ \lambda^2 - 4\mu > 0, \tag{13}$$

$$\varphi(\eta)=\ln\left(\frac{-\sqrt{\lambda^2-4\mu}\coth\left(\frac{1}{2}\sqrt{\lambda^2-4\mu}(\eta+\eta_0)\right)-\lambda}{2\mu}\right),\ \mu\neq 0,\ \lambda^2-4\mu>0, \tag{14}$$

$$\varphi(\eta)=\ln\left(\frac{\sqrt{-(\lambda^2-4\mu)}\tan\left(\frac{1}{2}\sqrt{-(\lambda^2-4\mu)}(\eta+\eta_0)\right)-\lambda}{2\mu}\right),\ \mu\neq 0,\ \lambda^2-4\mu<0, \tag{15}$$

$$\varphi(\eta)=\ln\left(\frac{\sqrt{-(\lambda^2-4\mu)}\cot\left(\frac{1}{2}\sqrt{-(\lambda^2-4\mu)}(\eta+\eta_0)\right)-\lambda}{2\mu}\right),\ \mu\neq 0,\ \lambda^2-4\mu<0, \tag{16}$$

$$\varphi(\eta)=-\ln\left(\frac{\lambda}{\exp(\lambda(\eta+\eta_0))-1}\right),\ \mu=0,\ \lambda\neq 0,\ \lambda^2-4\mu>0, \tag{17}$$

$$\varphi(\eta)=\ln\left(-\frac{2(\lambda(\eta+\eta_0))+2}{\lambda^2(\eta+\eta_0)}\right),\ \mu\neq 0,\ \lambda\neq 0,\ \lambda^2-4\mu=0, \tag{18}$$

$$\varphi(\eta)=\ln(\eta+\eta_0),\ \lambda=0,\ \mu=0, \tag{19}$$

where η_0 is the integration constant.

Another first-order equation that can be considered an auxiliary equation is:

$$\varphi'(\eta)=-\sqrt{\mu(\exp(-\varphi(\eta))^2+\lambda}, \tag{20}$$

Then, the solutions of Equation (20) are [38–40]

$$\varphi(\eta)=-\ln\left(-\sqrt{\frac{\lambda}{\mu}}\operatorname{csch}\left(\sqrt{\lambda}(\eta+\eta_0)\right)\right),\ \mu>0,\ \lambda>0, \tag{21}$$

$$\varphi(\eta)=-\ln\left(\sqrt{-\frac{\lambda}{\mu}}\sec\left(\sqrt{-\lambda}(\eta+\eta_0)\right)\right),\ \mu>0,\ \lambda<0, \tag{22}$$

$$\varphi(\eta)=-\ln\left(\sqrt{-\frac{\lambda}{\mu}}\operatorname{sech}\left(\sqrt{\lambda}(\eta+\eta_0)\right)\right),\ \mu<0,\ \lambda>0, \tag{23}$$

$$\varphi(\eta)=-\ln\left(\sqrt{-\frac{\lambda}{\mu}}\csc\left(\sqrt{-\lambda}(\eta+\eta_0)\right)\right),\ \mu>0,\ \lambda<0, \tag{24}$$

$$\varphi(\eta)=-\ln\left(\frac{1}{\pm\sqrt{\mu}(\eta+\eta_0)}\right),\ \mu>0,\ \lambda=0, \tag{25}$$

$$\varphi(\eta)=-\ln\left(\frac{i}{\pm\sqrt{-\mu}(\eta+\eta_0)}\right),\ \mu<0,\ \lambda=0, \tag{26}$$

where η_0 is the constant of integration.

We will consider below that $\varphi(\eta)$ is an alternative solution of the auxiliary Equation (12), or, respectively, (20), where μ and λ are arbitrary constants.

The substitution of (11) into (10) leads to a system of nonlinear equations for $B_j(-M\leq j\leq M),\mu,\lambda$ and v. Via symbolic software such as Maple, the solution of the system in terms of $B_j(-M\leq j\leq M),\mu,\lambda$ and v can be determined.

4. Solving Equation (1)

In our specific case of Equation (6), balancing between u'' and u^3 gives $M = 1$, both for (12) and for (20). Thus, the following symbolic structure can be considered for the solution of the problem:

$$u(\eta) = B_{-1}(\exp(-\varphi(\eta)))^{-1} + B_0 + B_1 \exp(-\varphi(\eta)). \tag{27}$$

4.1. Solutions via First Exponential Expansion

First of all, we insert (27) into (6) and collect all the terms with the same power of $\exp(-\varphi(\eta))$. Assuming all coefficients equal to zero, a set of nonlinear equations is derived:

$$\begin{aligned}
&\exp(-\varphi(\eta))^{-3}: \ -2\beta\kappa_1 {B_{-1}}^3 + 2\alpha b_2 b_1 \mu^2 B_{-1},\\
&\exp(-\varphi(\eta))^{-2}: \ 3\alpha b_2 b_1 \mu B_{-1}\lambda - 6\beta\kappa_1 B_0 {B_{-1}}^2,\\
&\exp(-\varphi(\eta))^{-1}: \ 2\alpha b_2 b_1 \mu B_{-1} + \alpha b_2 b_1 B_{-1}\lambda^2 - \omega B_{-1} - \alpha\kappa_1\kappa_2 B_{-1}\\
&\qquad -6\beta\kappa_1 {B_0}^2 B_{-1} - 6\beta\kappa_1 B_1 {B_{-1}}^2,\\
&\text{const}: \ -2\beta\kappa_1 {B_0}^3 + \alpha b_2 b_1 B_{-1}\lambda - 12\beta\kappa_1 B_0 B_1 B_{-1}\\
&\qquad -\omega B_0 + \alpha b_2 b_1 \mu B_1 \lambda - \alpha\kappa_1\kappa_2 B_0,\\
&\exp(-\varphi(\eta)): \ 2\alpha b_2 b_1 \mu B_1 + \alpha b_2 b_1 B_1 \lambda^2 - \omega B_1 - \alpha\kappa_1\kappa_2 B_1 - 6\beta\kappa_1 {B_0}^2 B_1\\
&\qquad\quad -6\beta\kappa_1 {B_1}^2 B_{-1},\\
&\exp(-\varphi(\eta))^2: \ 3\alpha b_2 b_1 B_1 \lambda - 6\beta\kappa_1 B_0 {B_1}^2,\\
&\exp(-\varphi(\eta))^3: \ -2\beta\kappa_1 {B_1}^3 + 2\alpha b_2 b_1 B_1.
\end{aligned} \tag{28}$$

Taking as zero the coefficients of all powers of $\exp(-\varphi(\eta))$, we obtain a set of polynomial equations in terms of B_{-1}, B_0, B_1 and ω. In what follows, we outline the solutions for the system obtained using Maple.

Case 1:

$$B_{-1} = \pm\mu\sqrt{\frac{\alpha b_2 b_1}{\beta\kappa_1}}, \ B_0 = \pm\frac{\lambda}{2}\sqrt{\frac{\alpha b_2 b_1}{\beta\kappa_1}}, \ B_1 = 0, \ \omega = -\frac{1}{2}\alpha b_2 b_1(\lambda^2 - 4\mu) - \alpha\kappa_1\kappa_2. \tag{29}$$

Case 2:

$$B_{-1} = 0, \ B_0 = \pm\frac{\lambda}{2}\sqrt{\frac{\alpha b_2 b_1}{\beta\kappa_1}}, \ B_1 = \pm\sqrt{\frac{\alpha b_2 b_1}{\beta\kappa_1}}, \ \omega = -\frac{1}{2}\alpha b_2 b_1(\lambda^2 - 4\mu) - \alpha\kappa_1\kappa_2. \tag{30}$$

Subsequently, utilizing the secured values (20), (30), exact solutions to Equation (1) are obtained.

First, we represent the families of hyperbolic function solutions corresponding to $\mu \neq 0, \lambda^2 - 4\mu > 0$,

For Case 1:

$$\begin{aligned}
q_{1,2}(x,y,t) &= \pm\tfrac{1}{2}\sqrt{\tfrac{\alpha b_2 b_1}{\beta\kappa_1}(\lambda^2 - 4\mu)}\tanh\left(\tfrac{1}{2}\sqrt{\lambda^2 - 4\mu}\,\eta\right)\\
&\times \exp\left[i\left(-\kappa_1 x - \kappa_2 y - \left(\tfrac{1}{2}\alpha b_2 b_1(\lambda^2 - 4\mu) + \alpha\kappa_1\kappa_2\right)\tfrac{\Gamma(\varrho - \Theta + 1)}{\Theta\Gamma(\varrho)} t^{\Theta} + \theta_0\right)\right],
\end{aligned} \tag{31}$$

$$\begin{aligned}
q_{3,4}(x,y,t) &= \pm\tfrac{1}{2}\sqrt{\tfrac{\alpha b_2 b_1}{\beta\kappa_1}(\lambda^2 - 4\mu)}\coth\left(\tfrac{1}{2}\sqrt{\lambda^2 - 4\mu}\,\eta\right)\\
&\times \exp\left[i\left(-\kappa_1 x - \kappa_2 y - \left(\tfrac{1}{2}\alpha b_2 b_1(\lambda^2 - 4\mu) + \alpha\kappa_1\kappa_2\right)\tfrac{\Gamma(\varrho - \Theta + 1)}{\Theta\Gamma(\varrho)} t^{\Theta} + \theta_0\right)\right].
\end{aligned} \tag{32}$$

For Case 2:

$$q_{5,6}(x,y,t)=\pm\frac{1}{2}\sqrt{\frac{\alpha b_2 b_1}{\beta\kappa_1}}\frac{\left(\lambda\sqrt{\lambda^2-4\mu}\tanh\left(\frac{1}{2}\sqrt{\lambda^2-4\mu}\eta\right)+\lambda^2-4\mu\right)}{\left(\sqrt{\lambda^2-4\mu}\tanh\left(\frac{1}{2}\sqrt{\lambda^2-4\mu}\eta\right)+\lambda\right)}$$
$$\times\exp\left[i\left(-\kappa_1 x-\kappa_2 y-\left(\frac{1}{2}\alpha b_2 b_1(\lambda^2-4\mu)+\alpha\kappa_1\kappa_2\right)\frac{\Gamma(\varrho-\Theta+1)}{\Theta\Gamma(\varrho)}t^{\Theta}+\theta_0\right)\right], \tag{33}$$

$$q_{7,8}(x,y,t)=\pm\frac{1}{2}\sqrt{\frac{\alpha b_2 b_1}{\beta\kappa_1}}\frac{\left(\lambda\sqrt{\lambda^2-4\mu}\coth\frac{1}{2}\left(1/2\sqrt{\lambda^2-4\mu}\eta\right)+\lambda^2-4\mu\right)}{\left(\sqrt{\lambda^2-4\mu}\coth\left(\frac{1}{2}\sqrt{\lambda^2-4\mu}\eta\right)+\lambda\right)}$$
$$\times\exp\left[i\left(-\kappa_1 x-\kappa_2 y-\left(\frac{1}{2}\alpha b_2 b_1(\lambda^2-4\mu)+\alpha\kappa_1\kappa_2\right)\frac{\Gamma(\varrho-\Theta+1)}{\Theta\Gamma(\varrho)}t^{\Theta}+\theta_0\right)\right], \tag{34}$$

When $\mu=0$, $\lambda\neq 0$,

$$q_{9,10}(x,y,t)=\pm\frac{1}{2}\sqrt{\frac{\alpha b_2 b_1}{\beta\kappa_1}}\frac{\lambda\left(e^{\lambda\ \eta}+1\right)}{\left(e^{\lambda\ \eta}-1\right)}$$
$$\times\exp\left[i\left(-\kappa_1 x-\kappa_2 y-\left(\frac{1}{2}\alpha b_2 b_1\lambda^2+\alpha\kappa_1\kappa_2\right)\frac{\Gamma(\varrho-\Theta+1)}{\Theta\Gamma(\varrho)}t^{\Theta}+\theta_0\right)\right], \tag{35}$$

where $\eta=b_1x+b_2y+(\alpha(\kappa_1 b_2+\kappa_2 b_1))\frac{\Gamma(\varrho-\Theta+1)}{\Theta\Gamma(\varrho)}t^{\Theta}$.

The families of periodic function solutions corresponding to $\mu\neq 0,\lambda^2-4\mu<0$, are:
For Case 1:

$$q_{11,12}(x,y,t)=\pm\frac{1}{2}\sqrt{-\frac{\alpha b_2 b_1}{\beta\kappa_1}(\lambda^2-4\mu)}\tan\left(\frac{1}{2}\sqrt{-(\lambda^2-4\mu)}\eta\right)$$
$$\times\exp\left[i\left(-\kappa_1 x-\kappa_2 y-\left(\frac{1}{2}\alpha b_2 b_1(\lambda^2-4\mu)+\alpha\kappa_1\kappa_2\right)\frac{\Gamma(\varrho-\Theta+1)}{\Theta\Gamma(\varrho)}t^{\Theta}+\theta_0\right)\right], \tag{36}$$

$$q_{13,14}(x,y,t)=\pm\frac{1}{2}\sqrt{-\frac{\alpha b_2 b_1}{\beta\kappa_1}(\lambda^2-4\mu)}\cot\left(\frac{1}{2}\sqrt{-(\lambda^2-4\mu)}\eta\right)$$
$$\times\exp\left[i\left(-\kappa_1 x-\kappa_2 y-\left(\frac{1}{2}\alpha b_2 b_1(\lambda^2-4\mu)+\alpha\kappa_1\kappa_2\right)\frac{\Gamma(\varrho-\Theta+1)}{\Theta\Gamma(\varrho)}t^{\Theta}+\theta_0\right)\right]. \tag{37}$$

For Case 2:

$$q_{15,16}(x,y,t)=\pm\frac{1}{2}\sqrt{\frac{\alpha b_2 b_1}{\beta\kappa_1}}\frac{\left(\lambda\sqrt{-(\lambda^2-4\mu)}\tan\left(\frac{1}{2}\sqrt{-(\lambda^2-4\mu)}\eta\right)+\lambda^2-4\mu\right)}{\left(\sqrt{-(\lambda^2-4\mu)}\tan\left(\frac{1}{2}\sqrt{-(\lambda^2-4\mu)}\eta\right)+\lambda\right)}$$
$$\times\exp\left[i\left(-\kappa_1 x-\kappa_2 y-\left(\frac{1}{2}\alpha b_2 b_1(\lambda^2-4\mu)+\alpha\kappa_1\kappa_2\right)\frac{\Gamma(\varrho-\Theta+1)}{\Theta\Gamma(\varrho)}t^{\Theta}+\theta_0\right)\right], \tag{38}$$

$$q_{17,18}(x,y,t)=\pm\frac{1}{2}\sqrt{\frac{\alpha b_2 b_1}{\beta\kappa_1}}\frac{\left(\lambda\sqrt{-(\lambda^2-4\mu)}\cot\left(\frac{1}{2}\sqrt{-(\lambda^2-4\mu)}\eta\right)+\lambda^2-4\mu\right)}{\left(\sqrt{-(\lambda^2-4\mu)}\cot\left(\frac{1}{2}\sqrt{-(\lambda^2-4\mu)}\eta\right)+\lambda\right)}$$
$$\times\exp\left[i\left(-\kappa_1 x-\kappa_2 y-\left(\frac{1}{2}\alpha b_2 b_1(\lambda^2-4\mu)+\alpha\kappa_1\kappa_2\right)\frac{\Gamma(\varrho-\Theta+1)}{\Theta\Gamma(\varrho)}t^{\Theta}+\theta_0\right)\right], \tag{39}$$

where $\eta=b_1x+b_2y+(\alpha(\kappa_1 b_2+\kappa_2 b_1))\frac{\Gamma(\varrho-\Theta+1)}{\Theta\Gamma(\varrho)}t^{\Theta}$.

The families of rational function solutions of type 1, corresponding to $\lambda^2-4\mu=0$, are:
For Case 1:

$$q_{19,20}(x,y,t)=\pm\sqrt{\frac{\alpha b_2 b_1}{\beta\kappa_1}}\frac{1}{b_1x+b_2y+(\alpha(\kappa_1 b_2+\kappa_2 b_1))t}$$
$$\times\exp\left[i\left(-\kappa_1 x-\kappa_2 y-\alpha\kappa_1\kappa_2\frac{\Gamma(\varrho-\Theta+1)}{\Theta\Gamma(\varrho)}t^{\Theta}+\theta_0\right)\right], \tag{40}$$

For Case 2:

$$q_{21,22}(x,y,t)=\pm\sqrt{\frac{\mu\alpha b_2 b_1}{\beta\kappa_1}}\frac{1}{\left(\sqrt{\mu}(b_1x+b_2y+(\alpha(\kappa_1 b_2+\kappa_2 b_1))t)-1\right)}$$
$$\times\exp\left[i\left(-\kappa_1 x-\kappa_2 y-\alpha\kappa_1\kappa_2\frac{\Gamma(\varrho-\Theta+1)}{\Theta\Gamma(\varrho)}t^{\Theta}+\theta_0\right)\right]. \tag{41}$$

while that of type 2, corresponding to $\lambda = 0$, $\mu = 0$, is:

For Case 2:

$$\begin{aligned} q_{23,24}(x,y,t) &= \pm\sqrt{\tfrac{\alpha b_2 b_1}{\beta\kappa_1}}(b_1x + b_2y + (\alpha(\kappa_1 b_2 + \kappa_2 b_1))t) \\ &\times \exp\Big[i\Big(-\kappa_1 x - \kappa_2 y - \alpha\kappa_1\kappa_2\tfrac{\Gamma(\varrho-\Theta+1)}{\Theta\Gamma(\varrho)}t^{\Theta} + \theta_0\Big)\Big]. \end{aligned} \quad (42)$$

4.2. Solutions via Second Exponential Expansion

Again, by substituting (27) into (6) and using (20) recurrently, we obtain a system of algebraic equations and equate the coefficients of $(\exp(-\varphi(\eta)))^j, i = -3, -2, \ldots, 2, 3$ to zero as follows:

$$\begin{aligned} &\exp(-\varphi(\eta))^{-3}: \ -2\beta\kappa_1 {B_{-1}}^3, \\ &\exp(-\varphi(\eta))^{-2}: \ -6\beta\kappa_1 B_0 {B_{-1}}^2, \\ &\exp(-\varphi(\eta))^{-1}: \ -\alpha\kappa_1\kappa_2 B_{-1} - 6\beta\kappa_1 {B_{-1}}^2 B_1 + \alpha b_2 b_1 B_{-1}\lambda - \omega B_{-1} - 6\beta\kappa_1 {B_0}^2 B_{-1}, \\ &\text{const}: \ -\omega B_0 - 12\beta\kappa_1 B_0 B_1 B_{-1} - 2\beta\kappa_1 {B_0}^3 - \alpha\kappa_1\kappa_2 B_0, \\ &\exp(-\varphi(\eta))^{1}: \ -6\beta\kappa_1 B_{-1}{B_1}^2 - 6\beta\kappa_1 {B_0}^2 B_1 - \alpha\kappa_1\kappa_2 B_1 + \alpha b_2 b_1 B_1\lambda - \omega B_1, \\ &\exp(-\varphi(\eta))^{2}: \ -6\beta\kappa_1 B_0 {B_1}^2, \\ &\exp(-\varphi(\eta))^{3}: \ 2\alpha b_2 b_1 B_1\mu - 2\beta\kappa_1 {B_1}^3. \end{aligned} \quad (43)$$

Equating to zero the coefficients of all powers of $\exp(-\varphi(\eta))$, yields a set of algebraic equations for B_{-1}, B_0, B_1 and ω. Solving the system of algebraic equations with the aid of Maple, we obtain

$$B_{-1} = 0, \ B_0 = 0, \ B_1 = \pm\sqrt{\frac{\alpha b_2 b_1\mu}{\beta\kappa_1}}, \ \omega = \alpha(b_2 b_1 - \kappa_1\kappa_2). \quad (44)$$

First, we represent the families of hyperbolic function solutions corresponding to $\lambda > 0$

$$\begin{aligned} q_{25,26}(x,y,t) &= \pm\sqrt{\tfrac{\alpha\lambda b_2 b_1}{\beta\kappa_1}}\operatorname{csch}\Big(\sqrt{\lambda}(b_1x + b_2y + \alpha(\kappa_1 b_2 + \kappa_2 b_1)\tfrac{\Gamma(\varrho-\Theta+1)}{\Theta\Gamma(\varrho)}t^{\Theta})\Big) \\ &\times \exp\Big[i\Big(-\kappa_1 x - \kappa_2 y + \alpha(b_2 b_1 - \kappa_1\kappa_2)\tfrac{\Gamma(\varrho-\Theta+1)}{\Theta\Gamma(\varrho)}t^{\Theta} + \theta_0\Big)\Big], \end{aligned} \quad (45)$$

$$\begin{aligned} q_{27,28}(x,y,t) &= \pm\sqrt{-\tfrac{\alpha\lambda b_2 b_1}{\beta\kappa_1}}\operatorname{sech}\Big(\sqrt{\lambda}(b_1x + b_2y + \alpha(\kappa_1 b_2 + \kappa_2 b_1)\tfrac{\Gamma(\varrho-\Theta+1)}{\Theta\Gamma(\varrho)}t^{\Theta})\Big) \\ &\times \exp\Big[i\Big(-\kappa_1 x - \kappa_2 y + \alpha(b_2 b_1 - \kappa_1\kappa_2)\tfrac{\Gamma(\varrho-\Theta+1)}{\Theta\Gamma(\varrho)}t^{\Theta} + \theta_0\Big)\Big]. \end{aligned} \quad (46)$$

Second, the families of periodic function solutions corresponding to $\lambda > 0$ are:

$$\begin{aligned} q_{29,30}(x,y,t) &= \pm\sqrt{-\tfrac{\alpha\lambda b_2 b_1}{\beta\kappa_1}}\sec\Big(\sqrt{-\lambda}(b_1x + b_2y + \alpha(\kappa_1 b_2 + \kappa_2 b_1)\tfrac{\Gamma(\varrho-\Theta+1)}{\Theta\Gamma(\varrho)}t^{\Theta})\Big) \\ &\times \exp\Big[i\Big(-\kappa_1 x - \kappa_2 y + \alpha(b_2 b_1 - \kappa_1\kappa_2)\tfrac{\Gamma(\varrho-\Theta+1)}{\Theta\Gamma(\varrho)}t^{\Theta} + \theta_0\Big)\Big]. \end{aligned} \quad (47)$$

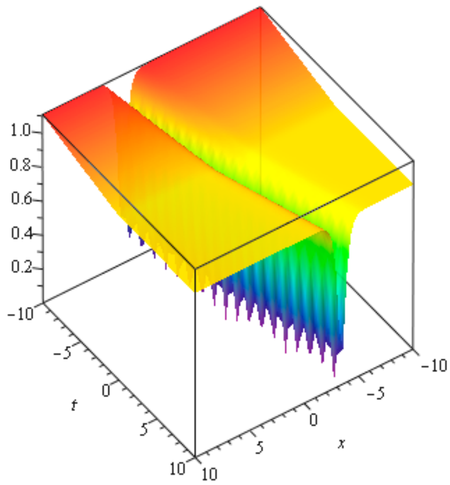
$$\begin{aligned} q_{31,32}(x,y,t) &= \pm\sqrt{-\tfrac{\alpha\lambda b_2 b_1}{\beta\kappa_1}}\csc\Big(\sqrt{-\lambda}(b_1x + b_2y + \alpha(\kappa_1 b_2 + \kappa_2 b_1)\tfrac{\Gamma(\varrho-\Theta+1)}{\Theta\Gamma(\varrho)}t^{\Theta})\Big) \\ &\times \exp\Big[i\Big(-\kappa_1 x - \kappa_2 y + \alpha(b_2 b_1 - \kappa_1\kappa_2)\tfrac{\Gamma(\varrho-\Theta+1)}{\Theta\Gamma(\varrho)}t^{\Theta} + \theta_0\Big)\Big]. \end{aligned} \quad (48)$$

Third, the families of rational function solutions are:

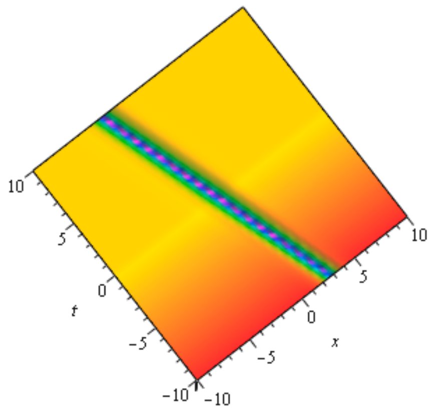
$$\begin{aligned} q_{33,34}(x,y,t) &= \pm\sqrt{\tfrac{\alpha b_2 b_1}{\beta\kappa_1}}\tfrac{1}{(b_1x + b_2y + \alpha(\kappa_1 b_2 + \kappa_2 b_1)t)} \\ &\times \exp\Big[i\Big(-\kappa_1 x - \kappa_2 y + \alpha(b_2 b_1 - \kappa_1\kappa_2)\tfrac{\Gamma(\varrho-\Theta+1)}{\Theta\Gamma(\varrho)}t^{\Theta} + \theta_0\Big)\Big]. \end{aligned} \quad (49)$$

5. Physical Explanation

In this section, we interpret some of the fFKMN model and wave solutions from the perspective of their physical meaning. Through utilization of exponential expansion methods, novel types of traveling wave optical solution were discovered, encompassing hyperbolic, trigonometric, and rational functions. We successfully derived soliton solutions for this nonlinear model, including bright, dark, periodic, singular, and other types of solitons. To gain a comprehensive understanding of their physical behavior, we depicted some of the obtained solutions graphically. The following results were obtained and are presented in the accompanying figures to enhance our understanding of the physical phenomenon at hand. Figures 1–3 depict the 3D, contour and 2D plots of the absolute of $q_i(x, y, t)$, $i = 1, 11, 27$. Figure 1 represents the gFKMN model wave solution given in Equation (31). Figure 1a–c demonstrates that the absolute values of $q_1(x, y, t)$ form a dark solitary (peakon soliton) wave solution with the duration $-10 \leq t, x \leq 10$ when $b_1 = 1.5, b_2 = 0.5, \alpha = 0.2, \beta = 1, \kappa_1 = 0.2, \kappa_2 = 1.5, \mu = -1, \lambda = 1$, $\theta_0 = 1$, $\varrho = -0.5, y = 1, \Theta = 0.99$, while $t = 0.2$ (red), $t = 0.5$ (green), $t = 1$ (blue). Figure 2 depicts the complex wave solution given in Equation (36). We observe from Figure 2a–c that the absolute value of $q_{11}(x, y, t)$ is a singular periodic wave solution with the duration $-10 \leq t, x \leq 10$ when $b_1 = 0.5, b_2 = 1, \alpha = 1, \beta = 0.5$, $\kappa_1 = 0.2, \kappa_2 = 0.5$, $\mu = 1, \lambda = 1.5, \theta_0 = 1$, $\varrho = 2, y = 1, \Theta = 0.9$, while $t = 0.2$ (red), $t = 0.5$ (green), $t = 1$ (blue). Figure 3 illustrates the complex solitary wave solution given in Equation (45). We observe from Figure 3a–c that the absolute value of $q_{27}(x, y, t)$ is a bright solitary (cuspon soliton) wave solution with the duration $-10 \leq t, x \leq 10$ when $b_1 = 2, b_2 = 0.5, \alpha = 2$, $\beta = 1.5, \kappa_1 = -1.2, \kappa_2 = 1, \lambda = 0.5, \theta_0 = 1, \varrho = -0.5, y = 1, \Theta = 1$, while $t = 0.2$ (red), $t = 0.5$ (green), $t = 1$ (blue).

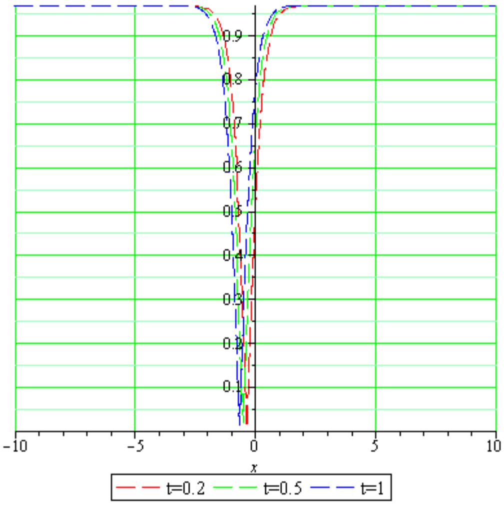


(**a**) 3D



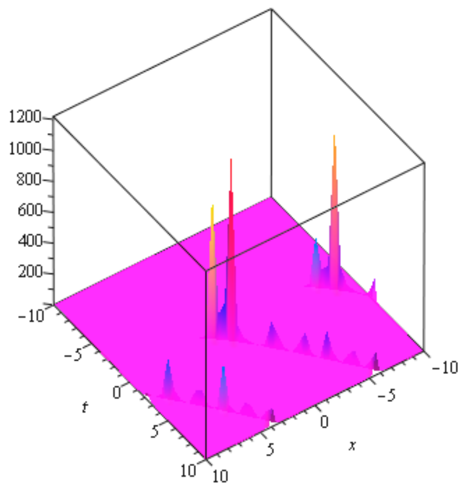
(**b**) Contour

Figure 1. *Cont.*

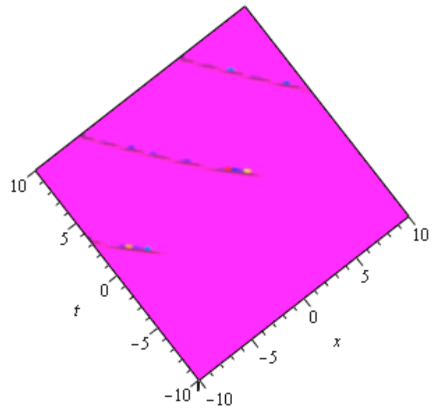


(**c**) 2D

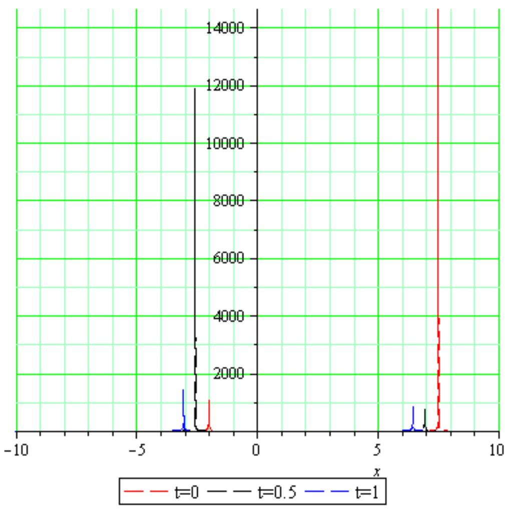
Figure 1. 3D, contour and 2D plots of $|q_1|$ with $b_1 = 1.5,\ b_2 = 0.5,\ \alpha = 0.2,\ \beta = 1,\ \kappa_1 = 0.2,\ \kappa_2 = 1.5,\ \mu = -1, \lambda = 1,\ \theta_0 = 1$, and $\varrho = -0.5$, with $y = 1$ and $\Theta = 0.99$.



(**a**) 3D

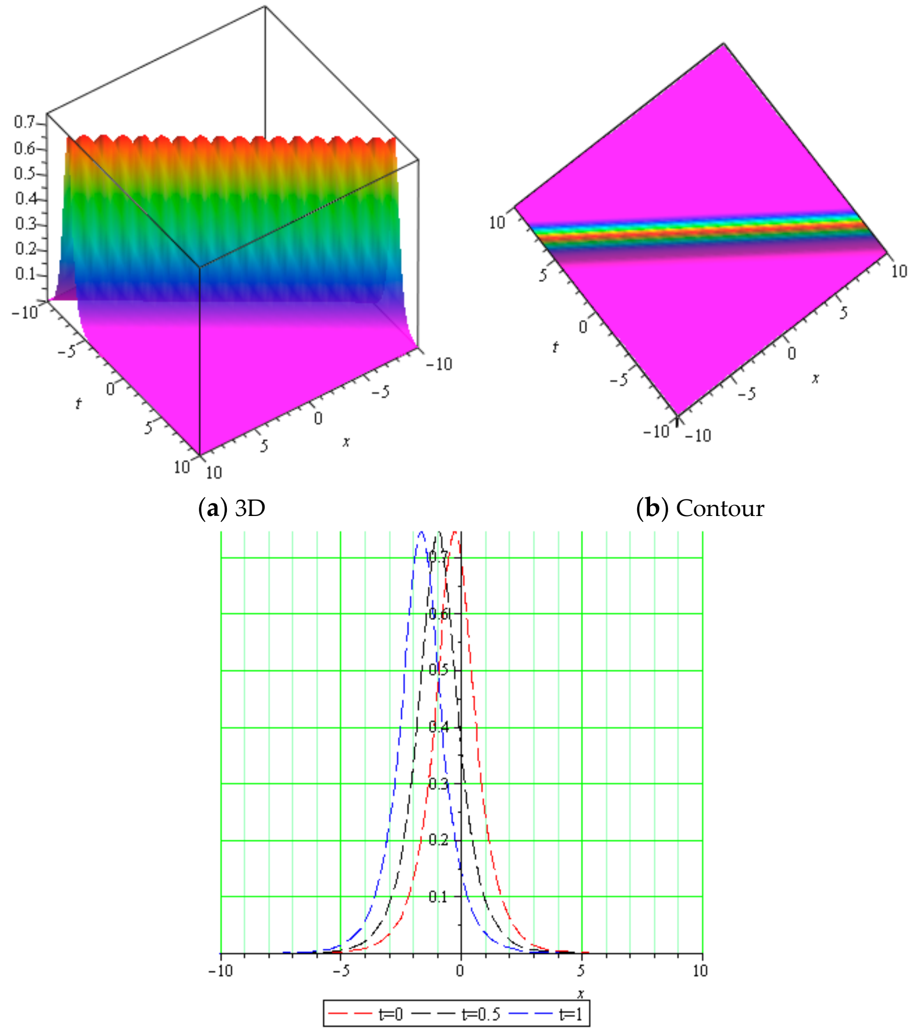


(**b**) Contour



(**c**) 2D

Figure 2. 3D, contour and 2D plots of $|q_{11}|$ with $b_1 = 0.5,\ b_2 = 1,\ \alpha = 1,\ \beta = 0.5,\ \kappa_1 = 0.2,\ \kappa_2 = 0.5,\ \mu = 1, \lambda = 1.5,\ \theta_0 = 1$, and $\varrho = 2$, with $y = 1$ and $\Theta = 0.9$.



(**a**) 3D (**b**) Contour

(**c**) 2D

Figure 3. 3D, contour and 2D plots of $|q_{27}|$ with $b_1 = 2$, $b_2 = 0.5$, $\alpha = 2$, $\beta = 1.5$, $\kappa_1 = -1.2$, $\kappa_2 = 1$, $\lambda = 0.5$, $\theta_0 = 1$, and $\varrho = 1$, with $y = 1$ and $\Theta = 1$.

6. Conclusions

In this work, two exponential expansion methods are well applied to the fractional nonlinear KMN model. A generalized fractional derivative is used. As mentioned earlier in the literature section, several researchers have reported diverse methods for obtaining bright, dark, and singular soliton solutions to integer-order KMN models [15–31]. As a result, the researchers focused solely on obtaining bright, dark, and singular soliton solutions for an integer-order KMN model. Compared to the soliton solutions attained in previous studies [15–31], the bright, dark, periodic, and singular soliton wave solutions generated in this study are novel in terms of their use of the generalized fractional derivative. This approach has not been reported in previously published articles, to the best of the authors' knowledge. The results demonstrate that the employed methods are efficient mathematical techniques to find traveling wave solutions for a fractional NLPDE. Moreover, the studied method can be easily adopted to investigate other fractional NLPDEs arising in mathematics, physics and other applied disciplines.

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