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Finite-Time Ruin Probabilities of Bidimensional Risk Models with Correlated Brownian Motions

Dan Zhu [1], Ming Zhou [2] and Chuancun Yin [1,*]

[1] School of Statistics and Data Science, Qufu Normal University, Qufu 273165, China; zhudanspring@qfnu.edu.cn
[2] Center for Applied Statistics, School of Statistics, Renmin University of China, Beijing 100872, China; mingzhou@ruc.edu.cn
* Correspondence: ccyin@qfnu.edu.cn

Abstract: The present work concerns the finite-time ruin probabilities for several bidimensional risk models with constant interest force and correlated Brownian motions. Under the condition that the two Brownian motions $\{B_1(t), t \geq 0\}$ and $\{B_2(t), t \geq 0\}$ are correlated, we establish new results for the finite-time ruin probabilities. Our research enriches the development of the ruin theory with heavy tails in unidimensional risk models and the dependence theory of stochastic processes.

Keywords: bidimensional perturbed risk model; correlated brownian motions; finite-time ruin probability; heavy-tailed risk model; interest force

MSC: 60H05

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1. Introduction

In traditional studies, many researchers have investigated ruin probability problems of insurers under unidimensional models. For example, ref. [1] studied ruin probability problems with constant interest force. Other studies about these problems can be found in [2–5]. An assumption behind these models is that the insured businesses homogeneous and can be described by a unidimensional model; however, this assumption is too strong. Thus, bidimensional or multidimensional insurance risk models have received growing interest in recent years, such as [6–8]. Various assumptions have been considered regarding the claim arrival process and the distribution of claim amounts; see, e.g., [9–12]. Ref. [13] considered finite-time ruin probabilities for nonstandard bidimensional renewal risk models with constant interest forces and diffusion generated by Brownian motions; they assumed that the two Brownian motions $\{B_1(t), t \geq 0\}$ and $\{B_2(t), t \geq 0\}$ are mutually independent. Similar results were obtained by [14], although they considered dependent subexponential claims. More papers can be found in [15,16], and the references therein. In this paper, we consider uniform asymptotics for the finite-time ruin probabilities for several bidimensional risks models with constant interest force and correlated Brownian motions, meaning that the businesses of the insurer have a relationship with each other. We introduce risk models and different types of ruin times with corresponding ruin probabilities as follows.

The bidimensional risk model $\vec{U}(t) = (U_1(t), U_2(t))^\tau$ is the surplus vector of an insurance company at time $t \geq 0$; in this paper, we state this formally as

$$U_i(t) = u_i e^{rt} + \int_0^t e^{r(t-s)} dC_i(s) - \int_0^t e^{r(t-s)} dS_i(s) + \sigma_i \int_0^t e^{r(t-s)} dB_i(s),\ t \geq 0, \tag{1}$$

where $\vec{u} = (u_1, u_2)^\tau$ stands for the initial surplus vector and $\vec{C}(t) = (C_1(t), C_2(t))^\tau$ for the total premiums received up to time t; here, $\{C_1(t), t \geq 0\}$, $\{C_2(t), t \geq 0\}$ are mutually independent. Moreover, $r \geq 0$ stands for the interest rate and $(S_1(t), S_2(t)) =$

$(\sum_{i=1}^{N_1(t)} X_{1i}, \sum_{i=1}^{N_2(t)} X_{2i})$ for the total amount of claims vector up to time t. Here, $\vec{X}_i = (X_{1i}, X_{2i})^\tau, i = 1, 2, \cdots$ denote pairs of claims with arrival times that constitute a counting process vector $\{\vec{N}(t), t \geq 0\}$, where $\vec{N}(t) = (N_1(t), N_2(t))$, while $\{N_1(t), t \geq 0\}$, $\{N_2(t), t \geq 0\}$ are mutually independent. The process $\{N_i(t), t \geq 0\}$ is a Poisson process with intensity $\lambda_i > 0$, and $\{\vec{X}_i, i = 1, 2, \cdots\}$ is a sequence of independent copies of the random pair $\vec{X} = (X_1, X_2)^\tau$ with the joint distribution function $F(x_1, x_2)$ and the marginal distribution functions $F_1(x_1)$ and $F_2(x_2)$. For all vectors, the $\vec{X}_i$s and $\vec{C}$ consist of only non-negative components $\vec{C}(0) = (0, 0)^\tau$. Moreover, each $C_i(t)$ is a non-decreasing and right-continuous stochastic process. The vector $\vec{B}(t) = (B_1(t), B_2(t))^\tau$ denotes a standard bidimensional Brownian motion with a constant correlation coefficient $\rho \in [-1, 1]$, while $\sigma_1 \geq 0$ and $\sigma_2 \geq 0$ are constants. For simplicity, we assume that $\{\vec{X}_i, i = 1, 2, \cdots\}$, $\{\vec{N}(t), t \geq 0\}$ and $\{\vec{C}(t), t \geq 0\}$ are independent and that both of them are independent of $\{\vec{B}(t), t \geq 0\}$. To avoid the certainty of ruin in each class, we assume that the following safety loading conditions hold when $r = 0$:

$$EC_i(t) - \lambda_i EX_{i1} > 0, \ i = 1, 2.$$

In this paper, we consider the following four types of ruin probabilities. For a finite horizon $T > 0$, we define

$$\psi_{\max}(\vec{u}, T) = P(T_{\max} \leq T | \vec{U}(0) = \vec{u}), \tag{2}$$

where

$$T_{\max} = \inf\{t > 0 | \max\{U_1(t), U_2(t)\} < 0\};$$

$$\psi_{\min}(\vec{u}, T) = P(T_{\min} \leq T | \vec{U}(0) = \vec{u}), \tag{3}$$

where

$$T_{\min} = \inf\{t > 0 | \min\{U_1(t), U_2(t)\} < 0\};$$

and

$$\psi_{\text{sum}}(\vec{u}, T) = P(T_{\text{sum}} \leq T | \vec{U}(0) = \vec{u}), \tag{4}$$

where

$$T_{\text{sum}} = \inf\{t > 0 | U_1(t) + U_2(t) < 0\};$$

$$\psi_{\text{and}}(\vec{u}, T) = P(T_{\text{and}} \leq T | \vec{U}(0) = \vec{u}), \tag{5}$$

where $T_{\text{and}} = \max\{T_1, T_2\}$ and

$$T_i = \inf\{t > 0 | U_i(t) < 0 \text{ for some } 0 \leq t \leq T), \ i = 1, 2,$$

with $\inf \emptyset = \infty$ by convention.

We remark that the probability in (2) denotes the probability of ruin occurring when both $U_1(t)$ and $U_2(t)$ are below zero at the same time within finite time $T > 0$, the probability in (3) denotes the probability of ruin occurring when at least one of $\{U_i(t), i = 1, 2\}$ is below zero within finite time $T > 0$, the probability in (4) denotes the probability of ruin occurring when the total of $U_1(t)$ and $U_2(t)$ is negativ within finite time $T > 0$, and the probability in (5) denotes the probability of ruin occurring when both $U_1(t)$ and $U_2(t)$ are below zero, not necessarily simultaneously, within a finite time $T > 0$. T_{and} represents a more critical time than $T_{\max}$, and the ruin probability defined by T_{sum} is reduced to that in the unidimensional model. The following relation between the four ruin probabilities defined above holds:

$$\psi_{\max}(\vec{u}, T) \leq \psi_{\text{and}}(\vec{u}, T) \leq \psi_{\min}(\vec{u}, T), \ \psi_{\text{sum}}(\vec{u}, T) \leq \psi_{\min}(\vec{u}, T),$$

and

$$\psi_{\min}(\vec{u},T) + \psi_{\text{and}}(\vec{u},T) = P(T_1 \le T | U_1(0) = u_1) + P(T_2 \le T | U_2(0) = u_2). \quad (6)$$

The rest of this paper is organized as follows. In Section 2 we review the related results after briefly introducing preliminaries about heavy-tailed distributions, in Section 3 we provide several important definitions and lemmas, and the main results and the proof procedure are presented in Section 4.

2. Review of Related Results

Unless otherwise stated herein, all limit relations are for $(u_1,u_2) \to (\infty,\infty)$. We denote $a \lesssim b$ and $a \gtrsim b$ if $\limsup a/b \le 1$ and $\limsup a/b \ge 1$, respectively, and $a \sim b$ if both, where, $a(\cdot,\cdot)$ and $b(\cdot,\cdot)$ are two positive functions. Let $F_1 * \cdots * F_n$ be the convolution of the distributions $F_1, \cdots, F_n$ and let F^{*n} denote the n-fold convolution of a distribution F.

In this section, we review definitions and properties that are relevant to the results of this paper, considering only the case of the distribution of heavy-tail claims. An r.v. X or its d.f. $F(x) = 1 - \overline{F}(x)$ satisfying $\overline{F}(x) > 0$ for all $x \in (-\infty,\infty)$ is called heavy-tailed to the right, or simply heavy-tailed, if $E[e^{\gamma X}] = \infty$ for all $\gamma > 0$. In the following, we recall several important classes of heavy-tailed distributions.

F is a long tailed distribution, written as $F \in \mathcal{L}$, if $\lim_{x\to\infty} \frac{\overline{F}(x-t)}{\overline{F}(x)} = 1$ holds for some $t > 0$. Note that the convergence is uniform over t in compact intervals. If $\lim_{x\to\infty} \frac{\overline{F^{*n}}(x)}{\overline{F}(x)} = n$ holds $(n = 2,3,\cdots)$, then F is a subexponential distribution on $(0,\infty)$, written as $F \in \mathcal{S}$. For some $0 < t < 1$, if $\limsup_{x\to\infty} \frac{\overline{F}(tx)}{\overline{F}(x)} < \infty$ holds, F is said to have a dominatedly varying tailed distribution, written as $F \in \mathcal{D}$. We call F a consistently varying tailed distribution, written as $F \in \mathcal{C}$, if

$$\lim_{t\downarrow 1} \liminf_{x\to\infty} \frac{\overline{F}(tx)}{\overline{F}(x)} = 1, \quad \text{or equivalently if} \quad \lim_{t\uparrow 1} \limsup_{x\to\infty} \frac{\overline{F}(tx)}{\overline{F}(x)} = 1$$

holds. A distribution F is extended regularly-varying tailed, written as $F \in \mathcal{ERV}(-\alpha,-\beta)$ for some $0 \le \alpha \le \beta < \infty$, if $s^{-\beta} \le \liminf_{x\to\infty} \frac{\overline{F}(sx)}{\overline{F}(x)} \le \limsup_{x\to\infty} \frac{\overline{F}(sx)}{\overline{F}(x)} \le s^{-\alpha}$ holds for $s \ge 1$.

It is obvious that the following formula holds:

$$\mathcal{ERV}(-\alpha,-\beta) \subset \mathcal{C} \subset \mathcal{D} \cap \mathcal{L} \subset \mathcal{S} \subset \mathcal{L}.$$

There are many other references to heavy-tailed distributions; readers may refer to [17–22] among others.

The asymptotic behavior of the finite-time ruin probability of bidimensional or multi-dimensional risk models has previously been investigated by [23]. They proved that under the conditions $F_1, F_2 \in \mathcal{S}$, $N_1(t) = N_2(t)$, and $\sigma_1 = \sigma_2 = 0$, it is the case that $r > 0$ and the claim vector $\vec{X}$ consist of independent components

$$\psi_{\max}(\vec{u};T) \sim \frac{\lambda(\lambda + \frac{1}{T})}{r^2} \int_{u_1}^{u_1 e^{rT}} \frac{\overline{F_1}(y)}{y} dy \int_{u_2}^{u_2 e^{rT}} \frac{\overline{F_2}(y)}{y} dy, \quad \text{as} \quad (u_1,u_2) \to (\infty,\infty).$$

Under the conditions $F_1, F_2 \in \mathcal{S}$, $r = 0$, and $N_1(t) = N_2(t)$, it is the case that $C_i(\cdot)$ are deterministic linear functions, and both the claim vector $\vec{X}$ and the bidimensional Brownian motion $\vec{B}$ consist of independent components. Li et al. [12] found that for each fixed time $T > 0$,

$$\psi_{\max}(\vec{u};T) \sim \lambda T(1+\lambda T)\overline{F_1}(u_1)\overline{F_2}(u_2), \quad \text{as} \quad (u_1,u_2) \to (\infty,\infty).$$

Chen et al. [11] investigated the uniform asymptotics of $\psi_{\text{and}}(\vec{u},T)$ and $\psi_{\min}(\vec{u},T)$ for an ordinary renewal risk model with the claim amounts belonging to the consistently varying tailed distributions class for large T. Zhang and Wang [24] considered model (1) with

$r = 0$ and assumed that all sources of randomness, $\{X_{1k}, k = 1, 2, \cdots\}$, $\{X_{2k}, k = 1, 2, \cdots\}$, $\{N_1(t) = N_2(t), t \geq 0\}$, $\{B_1(t), t \geq 0\}$ and $\{B_2(t), t \geq 0\}$ are mutually independent. They obtained that if $F_1, F_2 \in \mathcal{EVR}(-\alpha, -\beta)$ for some $0 < \alpha \leq \beta < \infty$, then, for each fixed time $T \geq 0$,

$$\psi_{\max}(\vec{u}; T) \sim \lambda T(1 + \lambda T)\overline{F_1}(u_1)\overline{F_2}(u_2), \text{ as } (u_1, u_2) \to (\infty, \infty).$$

The analogous result for multidimensional risk models can be found in Asmussen and Albrecher [17].

3. Some Lemmas

Before providing the main results, we first provide several lemmas.

Lemma 1. *If $F \in \mathcal{S}$, then for each $\varepsilon > 0$ there exists some constant $C_\varepsilon > 0$ such that the inequality*

$$\overline{F^{*n}}(x) \leq C_\varepsilon (1 + \varepsilon)^n \overline{F}(x)$$

holds for all $n = 1, 2, \cdots$ and $x \geq 0$.

Proof. See Lemma 1.3.5 of Embrechts et al. [25]. □

Lemma 2. *Let G_1 and G_2 be two distribution functions. If $G_1 \in \mathcal{S}$ and $\overline{G_2}(x) = o(\overline{G_1}(x))$, then we have $\overline{G_1 * G_2}(x) \sim \overline{G_1}(x)$ as $x \to \infty$.*

Proof. See Proposition 1 of Embrechts et al. [25]. □

Lemma 3. *Consider a unidimensional risk model*

$$U_i(t) = u_i + C_i(t) - S_i(t) + \sigma_i B_i(t),\ t \geq 0, i = 1, 2. \tag{7}$$

If $F_i \in \mathcal{S}$, then the ruin probability with finite-horizon T satisfies

$$\psi_i(u_i; T) = P(U_i(t) < 0 \text{ for some } t \leq T | U_i(0) = u_i) \sim \lambda T \overline{F_i}(u_i), u_i \to \infty.$$

Proof. Clearly, on the one hand,

$$\begin{aligned}
\psi_i(u_i; T) &\geq P(S_i(T) \geq u_i + C_i(T) + \sigma_i \sup_{0 \leq t \leq T} B_i(t)) \\
&= \int_0^\infty P(S_i(T) \geq u_i + C_i(T) + \sigma_i z) dP(\sup_{0 \leq t \leq T} B_i(t)) \leq z) \\
&= P(S_i(T) \geq u_i) \int_0^\infty \int_0^\infty \frac{P(S_i(T) \geq u_i + l_i + \sigma_i z)}{P(S_i(T) \geq u_i)} dP(\sup_{0 \leq t \leq T} B_i(t)) \leq z) \\
&\quad \times dP(C_i(T) \leq l_i) \\
&\sim P(S_i(T) \geq u_i),
\end{aligned} \tag{8}$$

where we have used the fact that $P(S_i(T) \geq u_i + l_i + \sigma_i z) \leq P(S_i(T) \geq u_i)$ and the dominated convergence theorem.

On the other hand,

$$\begin{aligned}
\psi_i(u_i; T) &\leq P(S_i(T) + \sigma_i \sup_{0 \leq t \leq T} (-B_i(t)) \geq u_i) \\
&\sim P(S_i(T) \geq u_i),
\end{aligned} \tag{9}$$

where we have used Lemma 2 and the fact that

$$P(\sigma_i \sup_{0 \leq t \leq T} (-B_i(t)) \geq u_i) = o(P(S_i(T) \geq u_i)).$$

Per Lemma 1 and dominated convergence theorem, we have

$$P(S_i(T) \geq u_i) \sim \overline{F_i}(u_i) \sum_{n=1}^{\infty} nP(N(T) = n) = \lambda T \overline{F_i}(u_i),\ as\ u_i \to \infty.$$

The result follows from (8) and (9). $\square$

Lemma 4. *Consider a unidimensional risk model*

$$U_i(t) = u_i e^{rt} + \int_0^t e^{r(t-s)} C_i(ds) - \int_0^t e^{r(t-s)} dS_i(s) + \sigma_i \int_0^t e^{r(t-s)} dB_i(s),\ t \geq 0, i = 1, 2.$$

If $F_i \in \mathcal{S}$, then the ruin probability with finite-horizon T satisfies

$$\psi_i(u_i; T) = P(U_i(t) < 0 \text{ for some } t \leq T | U_i(0) = u_i) \sim \frac{\lambda}{r} \int_{u_i}^{u_i e^{rT}} \frac{\overline{F_i}(y)}{y} dy,\ u_i \to \infty.$$

Proof. By simply modifying the proof of Lemma 3, we have

$$\psi_i(u_i; T) \sim P\left(\sum_{j=1}^{N(T)} X_{ij} e^{-r\tau_j} \geq u_i \right) \sim \lambda \int_0^T P(X_{i1} e^{-rz} > u_i) dz,\ u_i \to \infty,$$

where in the last step we use (28) from [26]. Here, τ_j are the arrival times of the Poisson process $N(t)$. In fact,

$$z = \frac{1}{r} \log \frac{y}{u_i},$$

and we have that

$$dz = d\left(\frac{1}{r} \log \frac{y}{u_i} \right) = \frac{1}{r} \cdot \frac{u_i}{y} \cdot \frac{1}{u_i} dy = \frac{1}{ry} dy.$$

Then,

$$\lambda \int_0^T P(X_{i1} > u_i e^{rz}) dz = \frac{\lambda}{r} \int_{u_i}^{u_i e^{rT}} \frac{\overline{F_i}(y)}{y} dy.$$

Upon a trivial substitution, the required result is implied. $\square$

Definition 1.

(i) *Two processes $\{X_1(t); t \geq 0\}$ and $\{X_2(t); t \geq 0\}$ are said to be positively associated if*

$$\mathrm{Cov}(f(X_1(t_1), X_2(t_2)), g(X_1(t_1), X_2(t_2)) | X_1(0) = x_1, X_2(0) = x_2) \geq 0$$

for all non-decreasing real valued functions f and g such that covariance exists, all $t_1, t_2 \geq 0$, and all $x_1, x_2 \in \mathbb{R}$.

(ii) *Two processes $\{X_1(t); t \geq 0\}$ and $\{X_2(t); t \geq 0\}$ are said to be negatively associated if*

$$\mathrm{Cov}(f(X_1(t_1)), g(X_2(t_2)) | X_1(0) = x_1, X_2(0) = x_2) \leq 0,$$

for all non-decreasing real valued functions f and g such that covariance exists, all $t_1, t_2 \geq 0$, and all $x_1, x_2 \in \mathbb{R}$.

Definition 2. *Two processes $\{X_1(t); t \geq 0\}$ and $\{X_2(t); t \geq 0\}$ are said to be positively (negatively) quadrant-dependent if*

$$\begin{aligned} &P(X_1(t_1) > y_1, X_2(t_2) > y_2 | X_1(0) = x_1, X_2(0) = x_2) \\ &\geq (\leq) P(X_1(t_1) > y_1 | X_1(0) = x_1) P(X_2(t_2) > y_2 | X_2(0) = x_2) \end{aligned} \tag{10}$$

for all $t_1, t_2 \geq 0$ and for all $y_1, y_2, x_1, x_2 \in \mathbb{R}$.

It is well known (cf. Ebrahimi [27]) that $(X_1(t), X_2(t))$ being positively (negatively) associated implies that $X_1(t)$ and $X_2(t)$ are positively (negatively) quadrant-dependent.

Let $\vec{B}(t) = (B_1(t), B_2(t))^\tau$ be a standard bidimensional Brownian motion with constant correlation coefficient $\rho \in (-1, 1)$. For notional convenience, for $t \geq 0$ we write $\underline{B}_i(t) = \inf_{0\leq s\leq t} B_i(s)$, $\overline{B}_i(t) = \sup_{0\leq s\leq t} B_i(s)$, $i = 1, 2$. It is well known that $P(\underline{B}_i(t) < -x) = P(\overline{B}_i(t) > x) = 2P(B_i(t) > x)$ for $x > 0$. The following lemma is essential to proving our main results. Moreover, it is of independent interest.

Lemma 5. *For any $x_1 > 0, x_2 > 0$, if $\rho \in [0, 1)$, then*

$$P(\overline{B}_1(t) > x_1, \overline{B}_2(t) > x_2) \geq P(\overline{B}_1(t) > x_1)P(\overline{B}_2(t) > x_2), \tag{11}$$

and

$$P(\underline{B}_1(t) < -x_1, \underline{B}_2(t) < -x_2) \geq P(\underline{B}_1(t) < -x_1)P(\underline{B}_2(t) < -x_2); \tag{12}$$

If $\rho \in (-1, 0]$, then

$$P(\overline{B}_1(t) > x_1, \overline{B}_2(t) > x_2) \leq P(\overline{B}_1(t) > x_1)P(\overline{B}_2(t) > x_2), \tag{13}$$

and

$$P(\underline{B}_1(t) < -x_1, \underline{B}_2(t) < -x_2) \leq P(\underline{B}_1(t) < -x_1)P(\underline{B}_2(t) < -x_2). \tag{14}$$

Proof. For any $t_1, t_2 \geq 0$, we have $\mathrm{Cov}(B_1(t_1), B_2(t_2)) = \rho \min\{t_1, t_2\}$. It follows from the Theorem in Pitt [28] that $\rho \geq 0$ is necessary and sufficient for $(B_1(t), B_2(t))^\tau$ to be positively associated, as $(B_1(t_1), B_2(t_2))^\tau$ is bivariate normal, which implies that $(B_1(t), B_2(t))^\tau$ is positively quadrant-dependent. Thus, (11) holds. To prove (12), we use (11) and the facts that $-\sup_{0\leq s\leq t} B_i(s) = \inf_{0\leq s\leq t}(-B_i(s))$ and $(-B_1(t), -B_2(t))^\tau$ is a standard bidimensional Brownian motion with correlation coefficient ρ. Inequalities (13) and (14) can be proved similarly. This completes the proof.

For $r \geq 0$, consider a bidimensional Gaussian process $(\int_0^t e^{-rs} dB_1(s), \int_0^t e^{-rs} dB_2(s))^\tau$, where $\vec{B}(t) = (B_1(t), B_2(t))^\tau$ is a standard bidimensional Brownian motion with constant correlation coefficient $\rho \in (-1, 1)$. For $t \geq 0$, we can write

$$\underline{\Delta}_i(t) = \inf_{0\leq s\leq t} \int_0^s e^{-rl} dB_1(l),\ \overline{\Delta}_i(t) = \sup_{0\leq s\leq t} \int_0^s e^{-rl} dB_2(l),\ i = 1, 2.$$

The following lemma is an extension of Lemma 5. □

Lemma 6. *For any $x_1 > 0, x_2 > 0$, if $\rho \in [0, 1)$, then*

$$P\big(\overline{\Delta}_1(t) > x_1, \overline{\Delta}_2(t) > x_2\big) \geq P\big(\overline{\Delta}_1(t) > x_1\big)P\big(\overline{\Delta}_2(t) > x_2\big),$$

and

$$P(\underline{\Delta}_1(t) < -x_1, \underline{\Delta}_2(t) < -x_2) \geq P(\underline{\Delta}_1(t) < -x_1)P(\underline{\Delta}_2(t) < -x_2);$$

If $\rho \in (-1, 0]$, then

$$P\big(\overline{\Delta}_1(t) > x_1, \overline{\Delta}_2(t) > x_2\big) \leq P\big(\overline{\Delta}_1(t) > x_1\big)P\big(\overline{\Delta}_2(t) > x_2\big),$$

and

$$P(\underline{\Delta}_1(t) < -x_1, \underline{\Delta}_2(t) < -x_2) \leq P(\underline{\Delta}_1(t) < -x_1)P(\underline{\Delta}_2(t) < -x_2).$$

Remark 1. *Several distributions of interest are available in closed form (see, e.g., He, Keirstead, and Rebholz [29]). These include the joint distributions of $(\underline{X}_1(t), \underline{X}_2(t))$, $(\overline{X}_1(t), \overline{X}_2(t))$, $(\underline{X}_1(t), \overline{X}_1(t))$, and so on. However, those closed-form results cannot apply our proofs to the main results. The results of Lemmas 5 and 6 cannot be obtained from the results of Shao and Wang [30].*

Lemma 7. *Let $\{N(t), t \geq 0\}$ be a Poisson process with arrival times $\tau_k, k = 1, 2, \cdots$. Considering $N(T) = n$ for arbitrarily fixed $T > 0$ and $n = 1, 2, \cdots$, the random vector $(\tau_1, \cdots, \tau_n)$ is equal in distribution to the random vector $(TU_{(1,n)}, \cdots, TU_{(n,n)})$, where $U_{(1,n)}, \cdots, U_{(n,n)}$ denote the order statistics of n i.i.d. (0, 1) uniformly distributed random variables $U_1, \cdots, U_n$.*

Proof. See Theorem 2.3.1 of Ross [26]. □

Lemma 8. *Let X and Y be two independent and non-negative random variables. If X is subexponentially distributed while Y is bounded and non-degenerate at 0, then the product XY is subexponentially distributed.*

Proof. See Corollary 2.3 of Cline and Samorodnitsky [19]. □

The following result is due to Tang [1].

Lemma 9. *Let X and Y be two independent random variables with distributions F_X and F_Y. Moreover, let Y be non-negative and non-degenerate at 0. Then,*

$$F_{X-Y} \in \mathcal{L} \Leftrightarrow F_X \in \mathcal{L} \Leftrightarrow \overline{F}_{X-Y}(x) \sim \overline{F}_X(x).$$

4. Main Results and Proofs

In this paper, we establish new results for the finite-time ruin probabilities. Unlike the above-motioned articles, we assume that the two Brownian motions $\{B_1(t), t \geq 0\}$ and $\{B_2(t), t \geq 0\}$ are correlated with a constant correlation coefficient $\rho \in (-1, 1)$. The following are the main results of this paper.

Theorem 1. *Consider the insurance risk model introduced in Section 1. Assume that $N_1(t) = N_2(t) = N(t)$, $\rho \in (-1, 0]$, $r = 0$ and that $\{X_{1k}, k = 1, 2, \cdots\}$, $\{X_{2k}, k = 1, 2, \cdots\}$, $\{C_1(t), t \geq 0\}$, $\{C_2(t), t \geq 0\}$, $\{N(t), t \geq 0\}$, $\{(B_1(t), B_2(t)), t \geq 0\}$ are mutually independent.*

(a) If $F_1, F_2 \in \mathcal{S}$, then, for each fixed time $T \geq 0$,

$$\psi_{\max}(\vec{u}; T) \sim \lambda T(1 + \lambda T)\overline{F_1}(u_1)\overline{F_2}(u_2), \text{ as } (u_1, u_2) \to (\infty, \infty), \tag{15}$$

$$\psi_{\min}(\vec{u}; T) \sim \lambda T\left(\overline{F_1}(u_1) + \overline{F_2}(u_2)\right), \text{ as } (u_1, u_2) \to (\infty, \infty). \tag{16}$$

(b) If $F_1 * F_2 \in \mathcal{S}$, then, for each fixed time $T \geq 0$,

$$\psi_{\text{sum}}(\vec{u}; T) \sim \lambda T\left(\overline{F_1}(u_1 + u_2) + \overline{F_2}(u_1 + u_2)\right), \text{ as } u_1 + u_2 \to \infty. \tag{17}$$

Proof. First, we establish the asymptotic upper bound for $\psi_{\max}(\vec{u}; T)$. Clearly,

$$\begin{aligned}
\psi_{\max}(\vec{u}; T) &\leq P\left(\sum_{i=1}^{N(T)} \vec{X}_i - \begin{pmatrix} \sigma_1 \underline{B}_1(T) \\ \sigma_2 \underline{B}_2(T) \end{pmatrix} > \vec{u}\right) \\
&= \sum_{n=0}^{\infty} P(N(T) = n) P\left(\sum_{i=1}^{n} \vec{X}_i - \begin{pmatrix} \sigma_1 \underline{B}_1(T) \\ \sigma_2 \underline{B}_2(T) \end{pmatrix} > \vec{u}\right) \\
&= \sum_{n=0}^{\infty} P(N(T) = n) \int_0^{\infty} \int_0^{\infty} P(\sum_{i=1}^{n} \vec{X}_i \in d\vec{z}) \\
&\quad \times P\left(\vec{z} - \begin{pmatrix} \sigma_1 \underline{B}_1(T) \\ \sigma_2 \underline{B}_2(T) \end{pmatrix} > \vec{u}\right).
\end{aligned} \tag{18}$$

Because $\rho \in (-1, 0]$, by using (14) we have

$$P\left(\vec{z} - \begin{pmatrix} \sigma_1 \underline{B}_1(T) \\ \sigma_2 \underline{B}_2(T) \end{pmatrix} > \vec{u}\right) \\ \leq P(z_1 - \sigma_1 \underline{B}_1(T) > u_1) P(z_2 - \sigma_2 \underline{B}_2(T) > u_2). \tag{19}$$

Using the independence of $\{X_{1k}, k = 1, 2, \cdots\}$ and $\{X_{2k}, k = 1, 2, \cdots\}$, we have

$$P\left(\sum_{i=1}^{n} \vec{X}_i \in \vec{z}\right) = P\left(\sum_{i=1}^{n} X_{1i} \in dz_1\right) P\left(\sum_{i=1}^{n} X_{2i} \in dz_2\right). \tag{20}$$

Substituting (19) and (20) into (18) and using the dominated convergence theorem, we obtain

$$\begin{aligned} \psi_{\max}(\vec{u}; T) &\leq \sum_{n=0}^{\infty} P(N(T) = n) P\left(\sum_{i=1}^{n} X_{1i} - \sigma_1 \underline{B}_1(T) > u_1\right) P\left(\sum_{i=1}^{n} X_{2i} - \sigma_2 \underline{B}_2(T) > u_1\right) \\ &\sim \sum_{n=0}^{\infty} P(N(T) = n) n^2 \overline{F_1}(u_1) \overline{F_2}(u_2) \\ &= \lambda T(1 + \lambda T) \overline{F_1}(u_1) \overline{F_2}(u_2), \ \text{ as } \ (u_1, u_2) \to (\infty, \infty), \end{aligned} \tag{21}$$

where in the second step we have used Lemma 2 and the fact that

$$P\left(\sigma_j \sup_{0 \leq t \leq T} (-B_j(t)) \geq u_j\right) = o\left(P(\sum_{i=1}^{n} X_{ji} \geq u_j)\right), \ j = 1, 2.$$

Next, we establish the asymptotic lower bound for $\psi_{\max}(\vec{u}; T)$. Clearly,

$$\begin{aligned} \psi_{\max}(\vec{u}; T) &\geq P\left(\sum_{i=1}^{N(T)} \vec{X}_i - \vec{C}(T) - \begin{pmatrix} \sigma_1 \overline{B}_1(T) \\ \sigma_2 \overline{B}_2(T) \end{pmatrix} > \vec{u}\right) \\ &= \sum_{n=0}^{\infty} P(N(T) = n) P\left(\sum_{i=1}^{n} \vec{X}_i - \begin{pmatrix} \sigma_1 \overline{B}_1(T) \\ \sigma_2 \overline{B}_2(T) \end{pmatrix} - \vec{C}(T) > \vec{u}\right) \\ &\equiv \sum_{n=0}^{\infty} P(N(T) = n) I_1, \end{aligned} \tag{22}$$

where I_1 can be written as

$$I_1 = \int_0^{\infty} \int_0^{\infty} P(\overline{B_1}(T) \in dy_1, \overline{B_2}(T) \in dy_2) J_1 J_2. \tag{23}$$

Here,

$$J_1 = P\left(\sum_{i=1}^{n} X_{1i} - C_1(T) - \sigma_1 y_1 > u_1\right),$$

and

$$J_2 = P\left(\sum_{i=1}^{n} X_{2i} - C_2(T) - \sigma_2 y_2 > u_2\right).$$

For large constants $a > 0$ and $b > 0$, we can further write I_1 as

$$\begin{aligned} I_1 &= \left(\int_0^a \int_0^b + \int_0^a \int_b^{\infty} + \int_a^{\infty} \int_0^b + \int_a^{\infty} \int_b^{\infty}\right) P(\overline{B_1}(T) \in dy_1, \overline{B_2}(T) \in dy_2) J_1 J_2 \\ &\equiv k_1 + k_2 + k_3 + k_4. \end{aligned} \tag{24}$$

First, we consider k_1. Then, per Lemma 9, it holds uniformly for all $y_1 \in [0,a]$ that

$$J_1 \sim n\overline{F}_1(u_1), \text{ as } u_1 \to \infty \tag{25}$$

and it holds uniformly for all $y_2 \in [0,b]$ that

$$J_2 \sim n\overline{F}_2(u_2), \text{ as } u_2 \to \infty. \tag{26}$$

Using Lemma 1 and the dominated convergence theorem, we obtain

$$k_1 \sim n^2\overline{F}_1(u_1)\overline{F}_2(u_2)\int_0^a\int_0^b P(\overline{B_1}(T) \in dy_1, \overline{B_2}(T) \in dy_2), \text{ as } (u_1,u_2) \to (\infty,\infty).$$

Thus,

$$\lim_{(a,b)\to(\infty,\infty)}\lim_{(u_1,u_2)\to(\infty,\infty)}\frac{k_1}{n^2\overline{F}_1(u_1)\overline{F}_2(u_2)} = 1. \tag{27}$$

Now, we consider k_2. Using (25), Lemma 1, and the dominated convergence theorem,

$$\begin{aligned} k_2 &\sim n\overline{F}_1(u_1)\int_0^a\int_b^\infty P(\overline{B_1}(T) \in dy_1, \overline{B_2}(T) \in dy_2)J_2 \\ &\le n\overline{F}_1(u_1)P\left(\sum_{i=1}^n X_{2i} - C_2(T) - \sigma_2 b > u_2\right)\int_0^a\int_b^\infty P(\overline{B_1}(T) \in dy_1, \overline{B_2}(T) \in dy_2) \\ &\sim n^2\overline{F}_1(u_1)\overline{F}_2(u_2)\int_0^a\int_b^\infty P(\overline{B_1}(T) \in dy_1, \overline{B_2}(T) \in dy_2), \text{ as } (u_1,u_2) \to (\infty,\infty). \end{aligned}$$

Thus,

$$\lim_{(a,b)\to(\infty,\infty)}\lim_{(u_1,u_2)\to(\infty,\infty)}\frac{k_2}{n^2\overline{F}_1(u_1)\overline{F}_2(u_2)} = 0. \tag{28}$$

Likewise,

$$\lim_{(a,b)\to(\infty,\infty)}\lim_{(u_1,u_2)\to(\infty,\infty)}\frac{k_3}{n^2\overline{F}_1(u_1)\overline{F}_2(u_2)} = 0. \tag{29}$$

Finally, we deal with k_4:

$$\begin{aligned} k_4 &\le P\left(\sum_{i=1}^n X_{1i} - C_1(T) - \sigma_1 a > u_1\right)P\left(\sum_{i=1}^n X_{2i} - C_2(T) - \sigma_2 b > u_2\right) \\ &\quad \times \int_a^\infty\int_b^\infty P(\overline{B_1}(T) \in dy_1, \overline{B_2}(T) \in dy_2) \\ &\sim n^2\overline{F}_1(u_1)\overline{F}_2(u_2)\int_a^\infty\int_b^\infty P(\overline{B_1}(T) \in dy_1, \overline{B_2}(T) \in dy_2), \text{ as } (u_1,u_2) \to (\infty,\infty), \end{aligned}$$

from which we obtain

$$\lim_{(a,b)\to(\infty,\infty)}\lim_{(u_1,u_2)\to(\infty,\infty)}\frac{k_4}{n^2\overline{F}_1(u_1)\overline{F}_2(u_2)} = 0. \tag{30}$$

From (23) and (27)–(30), we obtain

$$\lim_{(u_1,u_2)\to(\infty,\infty)}\frac{I_1}{n^2\overline{F}_1(u_1)\overline{F}_2(u_2)} = 1. \tag{31}$$

Now, it follows from (22), (31), and the dominated convergence theorem that

$$\lim_{(u_1,u_2)\to(\infty,\infty)}\frac{\psi_{\max}(\vec{u};T)}{\lambda T(1+\lambda T)\overline{F_1}(u_1)\overline{F_2}(u_2)} \ge 1,$$

from which, along with (21), we obtain (15).

Note that

$$\psi_{\text{and}}(\vec{u};T) \le P\left(\sum_{i=1}^{N(T)} X_{1i} - \sigma_1 \underline{B}_1(T) > u_1, \sum_{i=1}^{N(T)} X_{2i} - \sigma_2 \underline{B}_2(T) > u_2\right),$$

from which, along with (18) and (21), we have

$$\lim_{(u_1,u_2)\to(\infty,\infty)} \frac{\psi_{\text{and}}(\vec{u};T)}{\overline{F_1}(u_1)+\overline{F_2}(u_2)} \le \lim_{(u_1,u_2)\to(\infty,\infty)} \frac{\lambda T(1+\lambda T)\overline{F_1}(u_1)\overline{F_2}(u_2)}{\overline{F_1}(u_1)+\overline{F_2}(u_2)} = 0.$$

Thus, it is the case that $\psi_{and}(\vec{u};T) \sim 0$, as $(u_1,u_2) \to (\infty,\infty)$. From (6), we have

$$\psi_{min}(\vec{u};T) \sim P(T_1 \le T \mid U_1(0)=u_1) + P(T_2 \le T \mid U_2(0)=u_1) = \psi_1(u_1;T) + \psi_2(u_2;T).$$

From Lemma 3, we can obtain (16).

Next, we prove relation (17). Using Theorem 7.2 in Ikeda and Watanabe [31] (and see Yin and Wen [32]), for all $t \ge 0$ we have

$$\sqrt{\sigma_1^2+\sigma_2^2+2\rho\sigma_1\sigma_2}W(t) \stackrel{d}{=} \sigma_1 B_1(t) + \sigma_2 B_2(t),$$

where '$\stackrel{d}{=}$' denotes equality in distribution, W is a standard Brownian motion independent of $\{X_{1k}, k=1,2,\cdots\}$, $\{X_{2k}, k=1,2,\cdots\}$, $\{C_1(t), t\ge 0\}$, $\{C_2(t), t\ge 0\}$, and $\{N(t), t\ge 0\}$. Thus, for all $t \ge 0$, $U_1(t)+U_2(t)$ can be written as

$$U_1(t)+U_2(t) \stackrel{d}{=} u_1+u_2+C_1(t)+C_2(t) - \sum_{i=1}^{N(t)}(X_{1i}+X_{2i}) + \sqrt{\sigma_1^2+\sigma_2^2+2\rho\sigma_1\sigma_2}W(t).$$

Applying Lemma 3 to this model, we find that if $F_1 * F_2 \in \mathcal{S}$, then

$$\psi_{\text{sum}}(\vec{u};T) \sim \lambda T\overline{F_1 * F_2}(u_1+u_2) \sim \lambda T(\overline{F_1}(u_1+u_2)+\overline{F_2}(u_1+u_2)),\ u_1+u_2 \to \infty,$$

where, in the last step, we have relied on the statement in [33] (and see Geluk and Tang [34]) that

$$F_1 * F_2 \in \mathcal{S} \text{ if and only if } P(X_1+X_2 > x) \sim \overline{F_1}(x)+\overline{F_2}(x).$$

This ends the proof of Theorem 1. □

Remark 2. *Letting $\{C_i(t) = c_i t, i=1,2$ and $\rho = 0$ in Theorem 1, we obtain Theorem 1 in [12].*

Theorem 2. *Consider the insurance risk model introduced in Section 1. Assume that $N_1(t) = N_2(t) = N(t)$, $\rho \in (-1,0]$, $r > 0$ and that $\{X_{1k}, k=1,2,\cdots\}$, $\{X_{2k}, k=1,2,\cdots\}$, $\{C_1(t), t\ge 0\}$, $\{C_2(t), t\ge 0\}$, $\{N(t), t\ge 0\}$, $\{(B_1(t),B_2(t)), t\ge 0\}$ are mutually independent.*

(a) If $F_1, F_2 \in \mathcal{S}$, then for each fixed time $T \ge 0$,

$$\psi_{\max}(\vec{u};T) \sim \frac{\lambda(\lambda+\frac{1}{T})}{r^2}\int_{u_1}^{u_1 e^{rT}} \frac{\overline{F_1}(y)}{y}dy \int_{u_2}^{u_2 e^{rT}} \frac{\overline{F_2}(y)}{y}dy, \text{ as } (u_1,u_2)\to(\infty,\infty), \quad (32)$$

$$\psi_{\min}(\vec{u};T) \sim \frac{\lambda}{r}\left(\int_{u_1}^{u_1 e^{rT}} \frac{\overline{F_1}(y)}{y}dy + \int_{u_2}^{u_2 e^{rT}} \frac{\overline{F_2}(y)}{y}dy\right), \text{ as } (u_1,u_2)\to(\infty,\infty). \quad (33)$$

(b) If $F_1 * F_2 \in \mathcal{S}$, then for each fixed time $T \ge 0$,

$$\psi_{\text{sum}}(\vec{u};T) \sim \lambda T\int_0^1 \overline{F_1 * F_2}(e^{rTz}(u_1+u_2))dz, \text{ as } u_1+u_2 \to \infty. \quad (34)$$

In particular, if there are two positive constants l_1 and l_2 such that $\overline{F}_i(x) \sim l_i\overline{F}(x)$, $i = 1,2$, then

$$\psi_{\text{sum}}(\vec{u};T) \sim \lambda T\left(\int_0^1 \overline{F_1}(e^{rTz}(u_1+u_2)) + \int_0^1 \overline{F_2}(e^{rTz}(u_1+u_2))\right), \text{ as } u_1+u_2 \to \infty. \quad (35)$$

Proof. We can write $\psi_{\max}(\vec{u};T)$ as

$$\psi_{\max}(\vec{u};T) = P(e^{-rt}U_i(t) < 0, i = 1,2 \text{ for some } 0 < t \leq T|\vec{U}(0) = \vec{u}).$$

For $t \in [0,T]$ and each $i = 1$ or 2, we have

$$\begin{aligned} u_i - \int_0^t e^{-rs}dS_i(s) + \sigma_i\int_0^t e^{-rs}dB_i(s) \quad &\leq \quad e^{-rt}U_i(t) \leq u_i + \int_0^T e^{-rs}dC_i(s) \\ &\quad - \int_0^t e^{-rs}dS_i(s) + \sigma_i\int_0^t e^{-rs}dB_i(s). \end{aligned}$$

It follows that $\psi_{\max}(\vec{u};T)$ satisfies

$$\begin{aligned} \psi_{\max}(\vec{u};T) \quad &\leq \quad P\left(\sum_{i=1}^{N(T)}\vec{X}_ie^{-r\tau_i} - \begin{pmatrix}\sigma_1\underline{\Delta}_1(T)\\ \sigma_2\underline{\Delta}_2(T)\end{pmatrix} > \vec{u}\right)) \\ &\leq \quad \sum_{n=0}^{\infty}P(N(T)=n)P\left(\sum_{i=1}^{n}\vec{X}_ie^{-r\tau_i} - \begin{pmatrix}\sigma_1\underline{\Delta}_1(T)\\ \sigma_2\underline{\Delta}_2(T)\end{pmatrix} > \vec{u}\Big|N(t)=n\right) \\ &\leq \quad \sum_{n=0}^{\infty}P(N(T)=n)\int_0^{\infty}\int_0^{\infty}P\left(\sum_{i=1}^{n}\vec{X}_ie^{-rTU_i} \in d\vec{z}\right) \\ &\quad \times P\left(\vec{z} - \begin{pmatrix}\sigma_1\underline{\Delta}_1(T)\\ \sigma_2\underline{\Delta}_2(T)\end{pmatrix} > \vec{u}\right). \end{aligned} \quad (36)$$

where we have used Lemma 7 in the last steps. Because $\rho \in (-1,0]$, using Lemma 6, we have

$$\begin{aligned} &P\left(\vec{z} - \begin{pmatrix}\sigma_1\underline{\Delta}_1(T)\\ \sigma_2\underline{\Delta}_2(T)\end{pmatrix} > \vec{u}\right) \\ &\leq P(z_1 - \sigma_1\underline{\Delta}_1(T) > u_1)P(z_2 - \sigma_2\underline{\Delta}_2(T) > u_2). \end{aligned} \quad (37)$$

Using the independence of $\{X_{1k}, k = 1,2,\cdots\}$ and $\{X_{2k}, k = 1,2,\cdots\}$, we have

$$\begin{aligned} P\left(\sum_{i=1}^{n}\vec{X}_ie^{-rTU_i} \in d\vec{z}\right) \quad &= \quad \int_0^1\cdots\int_0^1 P\left(\sum_{i=1}^{n}X_{1i}e^{-rTv_i} \in dz_1\right)P\left(\sum_{i=1}^{n}X_{2i}e^{-rTv_i} \in dz_2\right) \\ &\quad \times\prod_{j=1}^{n}P(U_j \in dv_j). \end{aligned} \quad (38)$$

Substituting (37) and (38) into (36) and using

$$P\left(\sum_{i=1}^{n}X_{1i}e^{-rTv_i} - \sigma_1\underline{\Delta}_1(T) > u_1\right) \sim P\left(\sum_{i=1}^{n}X_{1i}e^{-rTv_i} > u_1\right), u_1 \to \infty,$$

and

$$P\left(\sum_{i=1}^{n}X_{2i}e^{-rTv_i} - \sigma_2\underline{\Delta}_2(T) > u_2\right) \sim P\left(\sum_{i=1}^{n}X_{2i}e^{-rTv_i} > u_2\right), u_2 \to \infty,$$

uniformly for $(v_1, \cdots, v_n) \in [0,1]^n$, we obtain

$$\begin{aligned} \psi_{\max}(\vec{u};T) &\lesssim \sum_{n=0}^{\infty} P(N(T)=n) P\left(\sum_{i=1}^{n} X_{1i} e^{-rTU_i} > u_1, \sum_{i=1}^{n} X_{2i} e^{-rTU_i} > u_2\right) \\ &\equiv \sum_{n=0}^{\infty} P(N(T)=n) k_5. \end{aligned} \tag{39}$$

We apply Proposition 5.1 of Tang and Tsitsiashvili [22], which says that for i.i.d. subexponential random variables $\{X_k\}$ and for arbitrarily a and b where $0 < a \le b < \infty$, the relation

$$P\left(\sum_{i=1}^{n} c_i X_i > x\right) \sim \sum_{i=1}^{n} P(c_i X_i > x)$$

holds uniformly for $(c_1, \cdots, c_n) \in [a,b] \times \cdots \times [a,b]$. Hence, by conditioning on $(U_1, \cdots, U_n)$, we find that where

$$k_5 \sim n^2 P\left(X_{11} e^{-rTU_1} > u_1\right) P\left(X_{21} e^{-rTU_1} > u_2\right), \tag{40}$$

by substituting (40) into (39) and using the dominated convergence theorem, we obtain

$$\limsup_{(u_i,u_2)\to(\infty,\infty)} \frac{\psi_{\max}(\vec{u};T)}{\frac{\lambda(\lambda+\frac{1}{T})}{r^2} \int_{u_1}^{u_1 e^{rT}} \frac{\overline{F_1}(y)}{y} dy \int_{u_2}^{u_2 e^{rT}} \frac{\overline{F_2}(y)}{y} dy} \le 1. \tag{41}$$

Next, we establish the asymptotic lower bound for $\psi_{\max}(\vec{u};T)$. Clearly,

$$\begin{aligned} \psi_{\max}(\vec{u};T) &\ge P\left(\sum_{i=1}^{N(T)} \vec{X}_i e^{-r\tau_i} - \int_0^T e^{-rs} d\vec{C}(s) - \begin{pmatrix} \sigma_1 \overline{\Delta}_1(T) \\ \sigma_2 \overline{\Delta}_2(T) \end{pmatrix} > \vec{u}\right) \\ &= \sum_{n=0}^{\infty} P(N(T)=n) P\left(\sum_{i=1}^{n} \vec{X}_i e^{-rTU_i} - \begin{pmatrix} \sigma_1 \overline{\Delta}_1(T) \\ \sigma_2 \overline{\Delta}_2(T) \end{pmatrix} - \int_0^T e^{-rs} d\vec{C}(s) > \vec{u}\right) \\ &\equiv \sum_{n=0}^{\infty} P(N(T)=n) I_2, \end{aligned} \tag{42}$$

where, for some positive constants c and d,

$$I_2 = \left(\int_0^c \int_0^d + \int_0^c \int_d^\infty + \int_c^\infty \int_0^d + \int_c^\infty \int_d^\infty\right) P(\overline{\Delta_1}(T) \in dy_1, \overline{\Delta_2}(T) \in dy_2) J_3 J_4.$$

Here,

$$J_3 = P\left(\sum_{i=1}^{n} X_{1i} e^{-rTU_i} - \int_0^T e^{-rs} dC_1(s) - \sigma_1 y_1 > u_1\right),$$

and

$$J_4 = P\left(\sum_{i=1}^{n} X_{2i} e^{-rTU_i} - \int_0^T e^{-rs} dC_2(s) - \sigma_2 y_2 > u_2\right).$$

Per Lemma 8, we know that $\sum_{i=1}^{n} X_{ji} e^{-rTU_i} \in \mathcal{S}, j = 1,2$, as all $X_{ji} \in \mathcal{S}$. Then, invoking Lemma 9, we obtain

$$J_3 \sim nP(X_{11} e^{-rTU_1} > u_1), \text{ as } u_1 \to \infty, \ J_4 \sim nP(X_{21} e^{-rTU_1} > u_2) \text{ as } u_2 \to \infty$$

uniformly for all $y_1 \in [0,c]$ and $y_2 \in [0,d]$, respectively. Now, using the same argument by which we reached (31), we have

$$\lim_{(u_1,u_2)\to(\infty,\infty)} \frac{I_2}{n^2 P(X_{11} e^{-rTU_1} > u_1) P(X_{21} e^{-rTU_1} > u_2)} = 1. \tag{43}$$

Now, it follows from (42), (43), Lemma 1, and the dominated convergence theorem that

$$\lim_{(u_i,u_2)\to(\infty,\infty)} \frac{\psi_{\max}(\vec{u};T)}{\lambda T(1+\lambda T)P(X_{11}e^{-rTU_1} > u_1)P(X_{21}e^{-rTU_1} > u_2)} \geq 1,$$

or, equivalently,

$$\lim_{(u_i,u_2)\to(\infty,\infty)} \frac{\psi_{\max}(\vec{u};T)}{\frac{\lambda(\lambda+\frac{1}{T})}{r^2}\int_{u_1}^{u_1e^{rT}}\frac{\overline{F_1}(y)}{y}dy\int_{u_2}^{u_2e^{rT}}\frac{\overline{F_2}(y)}{y}dy} \geq 1,$$

from which, along with (41), we obtain (32).

The relation (33) follows from (6) and Lemma 4 because, as above,

$$\begin{aligned}
&\lim_{(u_1,u_2)\to(\infty,\infty)} \frac{\psi_{\text{and}}(\vec{u};T)}{\int_{u_1}^{u_1e^{rT}}\frac{\overline{F_1}(y)}{y}dy+\int_{u_2}^{u_2e^{rT}}\frac{\overline{F_2}(y)}{y}dy} \\
&\qquad\leq \lim_{(u_1,u_2)\to(\infty,\infty)} \frac{\frac{\lambda(\lambda+\frac{1}{T})}{r^2}\int_{u_1}^{u_1e^{rT}}\frac{\overline{F_1}(y)}{y}dy\int_{u_2}^{u_2e^{rT}}\frac{\overline{F_2}(y)}{y}dy}{\int_{u_1}^{u_1e^{rT}}\frac{\overline{F_1}(y)}{y}dy+\int_{u_2}^{u_2e^{rT}}\frac{\overline{F_2}(y)}{y}dy} = 0.
\end{aligned}$$

From (6), we have

$$\psi_{min}(\vec{u};T) \sim \psi_1(u_1;T)+\psi_2(u_2;T), \text{ as } (u_1,u_2)\to(\infty,\infty).$$

From Lemma 4, we have

$$\psi_i(u_i;T) \sim \frac{\lambda}{r}\int_{u_i}^{u_ie^{rT}}\frac{\overline{F_i}(y)}{y}dy,\ u_i\to\infty,\ i=1,2.$$

Then,

$$\psi_{min}(\vec{u};T) \sim \frac{\lambda}{r}\left(\int_{u_1}^{u_1e^{rT}}\frac{\overline{F_1}(y)}{y}dy+\int_{u_2}^{u_2e^{rT}}\frac{\overline{F_2}(y)}{y}dy\right), \text{ as } (u_1,u_2)\to(\infty,\infty).$$

Thus, we have completed the proof of (33).

Next, we prove relation (34). Similarly, for all $t \geq 0$, we have

$$\begin{aligned}
U_1(t)+U_2(t) \stackrel{d}{=} &\ (u_1+u_2)e^{rt}+\int_0^t e^{r(t-s)}d(C_1(s)+C_2(s)) \\
&-\int_0^t e^{r(t-s)}d\sum_{i=1}^{N(s)}(X_{1i}+X_{2i}) \\
&+\sqrt{\sigma_1^2+\sigma_2^2+2\rho\sigma_1\sigma_2}\int_0^t e^{r(t-s)}dW(s),
\end{aligned} \tag{44}$$

where $\{W(t), t\geq 0\}$ is a standard Brownian motion independent of $\{X_{1k}, k=1,2,\cdots\}$, $\{X_{2k}, k=1,2,\cdots\}$, $\{C_1(t), t\geq 0\}$, $\{C_2(t), t\geq 0\}$, and $\{N(t), t\geq 0\}$.

From Lemma 4, we have

$$\psi_{sum}(\vec{u};T) \sim \frac{\lambda}{r}\int_{u_1+u_2}^{(u_1+u_2)e^{rT}}\frac{\overline{F_1*F_2}(y)}{y}dy,\ u_1+u_2\to\infty.$$

Let $y = (u_1 + u_2)e^{rTz}$; then, $dy = rT(u_1 + u_2)e^{rTz}dz$. Therefore,

$$\begin{aligned}
\psi_{sum}(\vec{u};T) &\sim \frac{\lambda}{r}\int_0^1 \frac{\overline{F_1 * F_2}((u_1+u_2)e^{rTz})}{(u_1+u_2)e^{rTz}} rT(u_1+u_2)e^{rTz}dz \\
&= T\lambda \int_0^1 \overline{F_1 * F_2}((u_1+u_2)e^{rTz})dz, \text{as } \ (u_1,u_2) \to (\infty,\infty).
\end{aligned}$$

This completes the proof of (34). The result (35) follows from (34) and Lemma 3.1 in [5]. This ends the proof of Theorem 2. □

Remark 3. *When letting $\{C_i(t) = c_i t, i = 1,2,\ \rho = 0,\ \sigma_1 = 0, \sigma_2 = 0$ in Theorem 2, we obtain the result in Liu et al. [23].*

Theorem 3. *Consider the insurance risk model introduced in Section 1. Assume that $\rho \in (-1,0]$, $r = 0$ and $\{X_{1k}, k = 1,2,\cdots\}$, $\{X_{2k}, k = 1,2,\cdots\}$, $\{C_1(t), t \geq 0\}$, $\{C_2(t), t \geq 0\}$, $\{N_i(t), t \geq 0\}$, and $i = 1,2$, $\{(B_1(t), B_2(t)), t \geq 0\}$ are mutually independent.*
(a) If $F_1, F_2 \in \mathcal{S}$, then for each fixed time $T \geq 0$,

$$\psi_{\max}(\vec{u};T) \sim \lambda_1\lambda_2 T^2 \overline{F_1}(u_1)\overline{F_2}(u_2), \ \text{ as } \ (u_1,u_2) \to (\infty,\infty), \tag{45}$$

$$\psi_{\min}(\vec{u};T) \sim T\big(\lambda_1\overline{F_1}(u_1) + \lambda_2\overline{F_2}(u_2)\big), \ \text{ as } \ (u_1,u_2) \to (\infty,\infty). \tag{46}$$

(b) If $F_{\xi X_{11}+(1-\xi)X_{21}} \in \mathcal{S}$, where ξ is a random variable independent of $\{X_{1k}, k = 1,2,\cdots\}$ and $\{X_{2k}, k = 1,2,\cdots\}$ and $P(\xi = 1) = 1 - P(\xi = 0) = \frac{\lambda_1}{\lambda_1+\lambda_2}$; then, for each fixed time $T \geq 0$,

$$\psi_{\text{sum}}(\vec{u};T) \sim T\big(\lambda_1\overline{F_1}(u_1+u_2) + \lambda_2\overline{F_2}(u_1+u_2)\big), \ \text{ as } \ u_1+u_2 \to \infty. \tag{47}$$

Proof. As the proof is similar to that of Theorem 1, we only provide the main steps. First, we establish the asymptotic upper bound for $\psi_{\max}(\vec{u};T)$. Clearly,

$$\begin{aligned}
\psi_{\max}(\vec{u};T) &\leq P\left(\begin{pmatrix}\sum_{i=1}^{N_1(T)} X_{1i} \\ \sum_{i=1}^{N_2(T)} X_{2i}\end{pmatrix} - \begin{pmatrix}\sigma_1\underline{B}_1(T) \\ \sigma_2\underline{B}_2(T)\end{pmatrix} > \begin{pmatrix}u_1 \\ u_2\end{pmatrix}\right) \\
&= \int_0^\infty\int_0^\infty P\left(\sum_{i=1}^{N_1(T)} X_{1i} \in dz_1\right)P\left(\sum_{i=1}^{N_2(T)} X_{2i} \in dz_2\right) \\
&\quad \times P\left(\begin{pmatrix}z_1 \\ z_2\end{pmatrix} - \begin{pmatrix}\sigma_1\underline{B}_1(T) \\ \sigma_2\underline{B}_2(T)\end{pmatrix} > \begin{pmatrix}u_1 \\ u_2\end{pmatrix}\right).
\end{aligned} \tag{48}$$

Because $\rho \in (-1,0]$, using (14), we have

$$\begin{aligned}
&P\left(\begin{pmatrix}z_1 \\ z_2\end{pmatrix} - \begin{pmatrix}\sigma_1\underline{B}_1(T) \\ \sigma_2\underline{B}_2(T)\end{pmatrix} > \begin{pmatrix}u_1 \\ u_2\end{pmatrix}\right) \\
&\leq P(z_1 - \sigma_1\underline{B}_1(T) > u_1)P(z_2 - \sigma_2\underline{B}_2(T) > u_2).
\end{aligned} \tag{49}$$

Substituting (49) into (48), we obtain

$$\begin{aligned}
\psi_{\max}(\vec{u};T) &\leq P\left(\sum_{i=1}^{N_1(T)} X_{1i} - \sigma_1\underline{B}_1(T) > u_1\right)P\left(\sum_{i=1}^{N_2(T)} X_{2i} - \sigma_2\underline{B}_2(T) > u_1\right) \\
&\sim \lambda_1\lambda_2T^2\overline{F_1}(u_1)\overline{F_2}(u_2), \ \text{ as } \ (u_1,u_2) \to (\infty,\infty),
\end{aligned} \tag{50}$$

where in the last step we have used Lemma 3.

Next, we establish the asymptotic lower bound for $\psi_{\max}(\vec{u};T)$. Clearly,

$$\begin{aligned}\psi_{\max}(\vec{u};T) &\geq P\left(\begin{pmatrix}\sum_{i=1}^{N_1(T)} X_{1i} \\ \sum_{i=1}^{N_2(T)} X_{2i}\end{pmatrix} - \begin{pmatrix}C_1(T) \\ C_2(T)\end{pmatrix} - \begin{pmatrix}\sigma_1\overline{B}_1(T) \\ \sigma_2\overline{B}_2(T)\end{pmatrix} > \begin{pmatrix}u_1 \\ u_2\end{pmatrix}\right) \\ &= \sum_{n=0}^{\infty} P(N_1(T)=n)\sum_{m=0}^{\infty} P(N_1(T)=m) I_3, \end{aligned} \tag{51}$$

where

$$I_3 = P\left(\begin{pmatrix}\sum_{i=1}^{n} X_{1i} \\ \sum_{i=1}^{m} X_{2i}\end{pmatrix} - \begin{pmatrix}C_1(T) \\ C_2(T)\end{pmatrix} - \begin{pmatrix}\sigma_1\overline{B}_1(T) \\ \sigma_2\overline{B}_2(T)\end{pmatrix} > \begin{pmatrix}u_1 \\ u_2\end{pmatrix}\right).$$

Using the same arguments as those used to prove (31), we obtain

$$\lim_{(u_1,u_2)\to(\infty,\infty)} \frac{I_3}{nm\overline{F}_1(u_1)\overline{F}_2(u_2)} = 1,$$

from which, together with (51), we have

$$\lim_{(u_1,u_2)\to(\infty,\infty)} \frac{\psi_{\max}(\vec{u};T)}{\lambda_1\lambda_2 T^2\overline{F_1}(u_1)\overline{F_2}(u_2)} \geq 1.$$

The proof of (46) is straightforward, and is omitted here. Next, we prove (47). Using the properties of two independent compound Poisson processes and two independent Brownian motions, for all $t \geq 0$ we have

$$\begin{aligned} U_1(t)+U_2(t) &\overset{d}{=} u_1+u_2+C_1(t)+C_2(t)-\sum_{i=1}^{N_0(t)}(\xi X_{1i}+(1-\xi)X_{2i}) \\ &\quad +\sqrt{\sigma_1^2+\sigma_2^2+2\rho\sigma_1\sigma_2}W(t), \end{aligned}$$

where $\{W(t), t \geq 0\}$ is a standard Brownian motion, $\{N_0(t), t \geq 0\}$ is a Poisson process with intensity $\lambda_1+\lambda_2$, and ξ is a Bernoulli random variable with $P(\xi=1) = 1-P(\xi=0) = \frac{\lambda_1}{\lambda_1+\lambda_2}$. Moreover, ξ, $\{W(t), t \geq 0\}$, $\{N_0(t), t \geq 0\}$, $\{X_{1k}, k=1,2,\cdots\}$, $\{X_{2k}, k=1,2,\cdots\}$, $\{C_1(t), t \geq 0\}$, $\{C_2(t), t \geq 0\}$, and $\{N(t), t \geq 0\}$ are independent. Applying Lemma 3 to this model, we obtain

$$\psi_{\text{sum}}(\vec{u};T) \sim (\lambda_1+\lambda_2)T\overline{F}_{\xi X_{11}+(1-\xi)X_{21}}(u_1+u_2),\ u_1+u_2\to\infty,$$

and result (47) follows (c.f. Kaas et al. [35].)

$$P(\xi X_{11}+(1-\xi)X_{21} > u_1+u_2) = \frac{\lambda_1}{\lambda_1+\lambda_2}\overline{F}_1(u_1+u_2) + \frac{\lambda_2}{\lambda_1+\lambda_2}\overline{F}_2(u_1+u_2).$$

This ends the proof of Theorem 3. □

Theorem 4. *Consider the insurance risk model introduced in Section 1. Assume that $\rho \in (-1,0]$, $r > 0$, and that $\{X_{1k}, k=1,2,\cdots\}$, $\{X_{2k}, k=1,2,\cdots\}$, $\{C_1(t), t \geq 0\}$, $\{C_2(t), t \geq 0\}$, $\{N_i(t), t \geq 0\}$, $i=1,2$, and $\{(B_1(t), B_2(t)), t \geq 0\}$ are mutually independent.*
(a) If $F_1, F_2 \in \mathcal{S}$, then for each fixed time $T \geq 0$,

$$\psi_{\max}(\vec{u};T) \sim \frac{\lambda_1\lambda_2}{r^2}\int_{u_1}^{u_1 e^{rT}} \frac{\overline{F_1}(y)}{y}dy \int_{u_2}^{u_2 e^{rT}} \frac{\overline{F_2}(y)}{y}dy,\ \text{ as }\ (u_1,u_2)\to(\infty,\infty), \tag{52}$$

$$\psi_{\min}(\vec{u};T) \sim \frac{1}{r}\left(\lambda_1 \int_{u_1}^{u_1e^{rT}} \frac{\overline{F_1}(y)}{y}dy + \lambda_2 \int_{u_2}^{u_2e^{rT}} \frac{\overline{F_2}(y)}{y}dy\right), \text{ as } (u_1,u_2) \to (\infty,\infty). \tag{53}$$

(b) If $F_{\xi X_{11}+(1-\xi)X_{21}} \in \mathcal{S}$, where ξ is defined as in Theorem 3, then for each fixed time $T \geq 0$,

$$\psi_{\text{sum}}(\vec{u};T) \sim \frac{1}{r}\left(\lambda_1 \int_{u_1+u_2}^{(u_1+u_2)e^{rT}} \frac{\overline{F_1}(y)}{y}dy + \lambda_2 \int_{u_1+u_2}^{(u_1+u_2)e^{rT}} \frac{\overline{F_2}(y)}{y}dy\right), \text{ as } u_1+u_2 \to \infty. \tag{54}$$

Proof. As in the proof of Theorem 2, we have

$$\begin{aligned}
\psi_{\max}(\vec{u};T) &\leq P\left(\begin{pmatrix}\sum_{i=1}^{N_1(T)} X_{1i}e^{-r\tau_i} \\ \sum_{i=1}^{N_2(T)} X_{2i}e^{-r\tau_i}\end{pmatrix} - \begin{pmatrix}\sigma_1\underline{\Delta}_1(T) \\ \sigma_2\underline{\Delta}_2(T)\end{pmatrix} > \vec{u}\right) \\
&\leq \sum_{n=0}^{\infty} P(N_1(T)=n) \sum_{m=0}^{\infty} P(N_2(T)=m) \\
&\quad \times P\left(\begin{pmatrix}\sum_{i=1}^{n} X_{1i}e^{-r\tau_i} \\ \sum_{i=1}^{m} X_{2i}e^{-r\tau_i}\end{pmatrix} - \begin{pmatrix}\sigma_1\underline{\Delta}_1(T) \\ \sigma_2\underline{\Delta}_2(T)\end{pmatrix} > \vec{u}\right) \\
&\lesssim \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} nmP(N(T)=n)P(N_2(T)=m)P\left(X_{11}e^{-rTU_1} > u_1\right)P\left(X_{21}e^{-rTU_1} > u_2\right) \\
&= \lambda_1\lambda_2T^2P\left(X_{11}e^{-rTU_1} > u_1\right)P\left(X_{21}e^{-rTU_1} > u_2\right).
\end{aligned}$$

It follows that

$$\limsup_{(u_i,u_2)\to(\infty,\infty)} \frac{\psi_{\max}(\vec{u};T)}{\frac{\lambda_1\lambda_2}{r^2}\int_{u_1}^{u_1e^{rT}} \frac{\overline{F_1}(y)}{y}dy \int_{u_2}^{u_2e^{rT}} \frac{\overline{F_2}(y)}{y}dy} \leq 1.$$

The asymptotic lower bound for $\psi_{\max}(\vec{u};T)$ can be established similarly.
The relation (53) follows from (6), Lemma 4, and the fact that

$$\begin{aligned}
&\lim_{(u_1,u_2)\to(\infty,\infty)} \frac{\psi_{\text{and}}(\vec{u};T)}{\lambda_1\int_{u_1}^{u_1e^{rT}} \frac{\overline{F_1}(y)}{y}dy + \lambda_2\int_{u_2}^{u_2e^{rT}} \frac{\overline{F_2}(y)}{y}dy} \\
&\leq \lim_{(u_1,u_2)\to(\infty,\infty)} \frac{\frac{\lambda_1\lambda_2}{r^2}\int_{u_1}^{u_1e^{rT}} \frac{\overline{F_1}(y)}{y}dy \int_{u_2}^{u_2e^{rT}} \frac{\overline{F_2}(y)}{y}dy}{\lambda_1\int_{u_1}^{u_1e^{rT}} \frac{\overline{F_1}(y)}{y}dy + \lambda_2\int_{u_2}^{u_2e^{rT}} \frac{\overline{F_2}(y)}{y}dy} = 0.
\end{aligned}$$

Finally, we prove (54). Using the same arguments as above, we have

$$\begin{aligned}
U_1(t)+U_2(t) &\overset{d}{=} (u_1+u_2)e^{rt} + \int_0^t e^{r(t-s)}d(C_1(s)+C_2(s)) \\
&\quad - \int_0^t e^{r(t-s)}d\sum_{i=1}^{N_0(t)}(\xi X_{1i}+(1-\xi)X_{2i}) \\
&\quad + \sqrt{\sigma_1^2+\sigma_2^2+2\rho\sigma_1\sigma_2}\int_0^t e^{r(t-s)}dW(s),\ t \geq 0,
\end{aligned} \tag{55}$$

where ξ, $\{W(t), t \geq 0\}$, $\{N_0(t), t \geq 0\}$ are the same as in the proof of Theorem 3. It follows from Lemma 4 that

$$\psi_{\text{sum}}(\vec{u};T) \sim \frac{\lambda_1+\lambda_2}{r}\int_{u_1+u_2}^{(u_1+u_2)e^{rT}} \frac{\overline{F}_{\xi X_{11}+(1-\xi)X_{21}}(y)}{y}dy,\ u_1+u_2 \to \infty,$$

and the result (54) follows, as

$$\overline{F}_{\xi X_{11}+(1-\xi)X_{21}}(y) = \frac{\lambda_1}{\lambda_1+\lambda_2}\overline{F}_1(y) + \frac{\lambda_2}{\lambda_1+\lambda_2}\overline{F}_2(y).$$

This completes the proof of Theorem 4. □

5. Conclusions

In this paper, we have investigated a bidimensional risk model that describes the surplus process of an insurer. We provide new results for the different types of finite-time ruin probabilities under the circumstance of that the Brownian motions are correlated with a constant correlation coefficient. We remark that the extension to multidimensional models is more complicated. However, multidimensional models can better describe different insurance businesses. In addition, we might consider the relationship between different businesses in the future research, which could be an even more interesting problem.

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