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Analysis of an Integrated Pest Management Model with Impulsive Diffusion between Two Regions

Airen Zhou

School of Mathematics and Statistics, Guizhou University of Finance and Economics, Guiyang 550025, China; zhouairen@mail.gufe.edu.cn

Abstract: This paper investigates an integrated pest management model with pulsed diffusion. As we all know, humans have been fighting against pests since they entered the age of farming. When pests are controlled, humans can achieve better harvests. We use the stroboscopic mapping of discrete dynamic system to obtain some important lemmas. Based on the lemmas, firstly, we give the conditions for the global asymptotic stability of the periodic solution of the pest eradication boundary; secondly, the conditions for the permanence of the investigated system are derived; thirdly, numerical simulations are used to verify our obtained theoretical results; finally, increased dispersal was found to have the opposite effect on integrated pest management. We conclude that a combination of impulsive diffusion, spraying pesticides, and releasing natural enemies can play a crucial role in integrated pest management.

Keywords: integrated pest management; releasing natural enemies; impulsive diffusion; pest eradication; permanence

MSC: 34A37; 92B05

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1. Introduction

Integrated pest management (IPM) is a management system for pests which uses appropriate techniques and methods as cooperatively as possible in the light of the population dynamics of pests and their related environmental relationships, so that pest populations can be kept below the level of economic harm. By using an integrated pest management approach, we can effectively manage pests while minimizing the negative impact on the environment. The measures taken by integrated pest management mainly include tools of chemical control, biological control, cultural control, mechanical or physical control, genetic control, etc. In recent years, the models of integrated pest management have been extensively and deeply studied by many scholars [1–18].

In 1972, the United States Council on Environmental Quality proposed the concept of "integrated pest management" (IPM). In terms of the definition of the Food and Agriculture Organization of the United Nations: Integrated management is a pest management system that keeps pest populations below economic hazard levels in accordance with the population dynamics of pests and the environmental relationships associated with them, using appropriate techniques and control methods in as coordinated a manner as possible [19].

According to the definition of IPM, the aim of integrated control is to control pest populations within a certain number, so that the damage caused by pests is below the economically permissible level, rather than to eliminate pests completely. This means that measures to control the growth of pests are implemented only when pest populations reach a certain level (i.e., a critical level) and cause damage to crops that humans cannot tolerate. This so-called critical level is the "economic threshold" (ET) in pest management [20]. Biological control serves as an ecological management strategy aimed at regulating the population of host pests, rather than eradicating them entirely. The objective is to maintain the number of host pests below the economically significant threshold, which ensures the survival of natural enemies and restores a favorable ecological balance in farmland. Ultimately, this approach maximizes benefits and promotes a harmonious agricultural ecosystem [20]. In crops, the presence of a small number of pests can provide food and intermediate hosts for natural enemies, thereby increasing the natural control ability of natural enemies. For example, if the number of leaf-eating pests in rice fields is controlled (less than ET), it is the intermediate host of the parasitic bee of rice bracts, red-eyed bees, and their presence can maintain the number of red-eyed bees. If they all are killed, rice bracts will be flooded in the later stages of the rice field [20].

In this paper, we investigate a pest management model with impulsive diffusion, spraying pesticides, and releasing natural enemies. We aim to uncover the dynamical properties of the system under investigation. Additionally, we anticipate that employing impulsive diffusion, spraying pesticides, and releasing natural enemies will establish a solid foundation for effective pest management.

The structure of this paper is outlined as follows. In Section 2, we introduce the model and provide some background concepts. Section 3 presents several important lemmas. In Section 4, we examine the globally asymptotically stable conditions for the periodic solution of system (1) at the pest eradication boundary, along with the permanent conditions of system (1). In Section 5, we present simulation analyses and offer a brief discussion. Finally, we obtain a concluding statement regarding integrated pest management to summarize our findings.

2. The Model

Based on the basic principles of pest control [1,7,11], in this paper, we hypothesize that the system consists of two patches connected by diffusion, which are divided by rivers, highways, or railways. Predator populations can transcend rivers, highways, or railways, while pest populations cannot. we build a class of pest management model with pesticide spraying, natural enemies release, and dispersal at different pulse moments as follows:

$$
\begin{cases}
\left.\begin{aligned}
\frac{dx_1(t)}{dt} &= x_1(t)(a_1 - b_1 x_1(t)) - \beta_1 x_1(t) y_1(t),\\
\frac{dy_1(t)}{dt} &= k_1 \beta_1 x_1(t) y_1(t) - d_1 y_1(t),\\
\frac{dx_2(t)}{dt} &= x_2(t)(a_2 - b_2 x_2(t)) - \beta_2 x_2(t) y_2(t),\\
\frac{dy_2(t)}{dt} &= k_2 \beta_2 x_2(t) y_2(t) - d_2 y_2(t),
\end{aligned}\right\} t \neq (n+p)u, t \neq (n+p+q)u, t \neq (n+1)u,\\
\left.\begin{aligned}
\Delta x_1(t) &= 0,\\
\Delta y_1(t) &= D(y_2(t) - y_1(t)),\\
\Delta x_2(t) &= 0,\\
\Delta y_2(t) &= D(y_1(t) - y_2(t)),
\end{aligned}\right\} t = (n+p)u, n \in Z^+,\\
\left.\begin{aligned}
\Delta x_1(t) &= -\mu_{11} x_1(t),\\
\Delta y_1(t) &= -\mu_{12} y_1(t),\\
\Delta x_2(t) &= -\mu_{21} x_2(t),\\
\Delta y_2(t) &= -\mu_{22} y_2(t),
\end{aligned}\right\} t = (n+p+q)u, n \in Z^+,\\
\left.\begin{aligned}
\Delta x_1(t) &= 0,\\
\Delta y_1(t) &= \mu_1,\\
\Delta x_2(t) &= 0,\\
\Delta y_2(t) &= \mu_2,
\end{aligned}\right\} t = (n+1)u, n \in Z^+,
\end{cases}
\tag{1}
$$

where $x_i(t)$ and $y_i(t)$ denote the densities of the pest and predator populations in patch $i\,(i=1,2)$ at time t, D represents the dispersal rate of the predator between two patches; it is assumed that the net exchange of the predator from the jth patch to the ith patch is proportional to the difference of y_j, y_i of the predator densities at time $t=(n+p)u$, $0<p<1, n\in Z^+$. $\mu_{11}, \mu_{12}, \mu_{21}, \mu_{22}$ are, respectively, the killing rates of $x_i(t), y_i(t), i=1,2$ due to pesticide spraying at time $t=(n+p+q)u, 0<p+q<1, n\in Z^+$. $\mu_i(i=1,2)$ is the amount of natural enemies released $y_i(i=1,2)$ at time $t=(n+1)u$. $a_i>0$ represents the intrinsic growth rate of the prey population in patch $i\,(i=1,2)$, and $b_i>0$ represents the coefficient of the intraspecific competition of the prey population in patch $i\,(i=1,2)$. $d_i>0$ stands for the natural death rate $i\,(i=1,2)$, $\beta_i(i=1,2)$ is the capture rate of predators in the ith patch, and $k_i(i=1,2)$ is the rate of conversion of nutrients into the reproduction rate of predators in the ith patch [18].

3. The Lemmas

Firstly, similar to the literature [18], we can demonstrate that all solutions of (1) are uniformly ultimately bounded.

Lemma 1. *For every solution $(x_1(t), y_1(t), x_2(t), y_2(t))$ of (1), there exists a positive constant M such that $M\geq x_i(t)$ and $M\geq y_i(t)$ (where $i=1,2$) for sufficiently large t.*

If $x_i(t)=0$ for $i=1,2$, we have a subsystem of Equation (1).

$$\begin{cases} \left.\begin{aligned} \frac{dy_1(t)}{dt} &= -d_1y_1(t), \\ \frac{dy_2(t)}{dt} &= -d_2y_2(t), \end{aligned}\right\} t\neq(n+p)u, t\neq(n+p+q)u, t\neq(n+1)u, \\ \left.\begin{aligned} \Delta y_1(t) &= D(y_2(t)-y_1(t)), \\ \Delta y_2(t) &= D(y_1(t)-y_2(t)), \end{aligned}\right\} t=(n+p)u, \\ \left.\begin{aligned} \Delta y_1(t) &= -\mu_{12}y_1(t), \\ \Delta y_2(t) &= -\mu_{22}y_2(t), \end{aligned}\right\} t=(n+p+q)u, \\ \left.\begin{aligned} \Delta y_1(t) &= \mu_1, \\ \Delta y_2(t) &= \mu_2, \end{aligned}\right\} t=(n+1)u, n=1,2,\ldots. \end{cases} \tag{2}$$

The analytic solution of Equation (2) among pulses can be obtained easily as follows.

$$\begin{cases} y_1(t) = \begin{cases} y_1(nu^+)e^{-d_1(t-nu)}, t\in(nu,(n+p)u], \\ y_1((n+p)u^+)e^{-d_1(t-(n+p)u)}, t\in((n+p)u,(n+p+q)u], \\ y_1((n+p+q)u^+)e^{-d_1(t-(n+p+q)u)}, t\in((n+p+q)u,(n+1)u], \end{cases} \\ y_2(t) = \begin{cases} y_2(nu^+)e^{-d_2(t-nu)}, t\in(nu,(n+p)u], \\ y_2((n+p)u^+)e^{-d_2(t-(n+p)u)}, t\in((n+p)u,(n+p+q)u], \\ y_2((n+p+q)u^+)e^{-d_2(t-(n+p+q)u)}, t\in((n+p+q)u,(n+1)u]. \end{cases} \end{cases} \tag{3}$$

Contemplating the third and fourth equations of (2), we have

$$\begin{cases} y_1((n+p)u^+) = (1-D)e^{-d_1pu}y_1(nu^+) + De^{-d_2pu}y_2(nu^+), \\ y_2((n+p)u^+) = De^{-d_1lu}y_1(nu^+) + (1-D)e^{-d_2lu}y_2(nu^+). \end{cases} \tag{4}$$

Contemplating the fifth and sixth equations of (2), we also have

$$\begin{cases} y_1((n+p+q)u^+) = (1-\mu_{12})e^{-d_1qu}y_1((n+p)u^+), \\ y_2((n+p+q)u^+) = (1-\mu_{22})e^{-d_2qu}y_2((n+p)u^+). \end{cases} \tag{5}$$

Contemplating the seventh and eighth equations of (2), we also have

$$\begin{cases} y_1((n+1)u^+) = y_1((n+p+q)u^+)e^{-d_1(1-p-q)u} + \mu_1, \\ y_2((n+1)u^+) = y_2((n+p+q)u^+)e^{-d_2(1-p-q)u} + \mu_2. \end{cases} \tag{6}$$

By substituting (4) and (5) into (6), we have the stroboscopic map of (2)

$$\begin{cases} y_1((n+1)u^+) = (1-\mu_{12})(1-D)e^{-d_1u}y_1(nu^+) \\ \qquad\qquad +(1-\mu_{12})De^{-[d_1(1-p)+d_2p]u}y_2(nu^+) + \mu_1, \\ \\ y_2((n+1)u^+) = (1-\mu_{22})De^{-[d_1p+d_2(1-p)]u}y_1(nu^+) \\ \qquad\qquad +(1-\mu_{22})(1-D)e^{-d_2u}y_2(nu^+) + \mu_2. \end{cases} \tag{7}$$

System (7) has one fixed point

$$\begin{cases} y_1^* = \dfrac{\mu_2F_1 + \mu_1(1-F_2)}{(1-E_1)(1-F_2) - E_2F_1} > 0, \\ y_2^* = \dfrac{\mu_1E_2 + \mu_2(1-E_1)}{(1-E_1)(1-F_2) - E_2F_1} > 0, \end{cases} \tag{8}$$

where

$$E_1 = (1-\mu_{12})(1-D)e^{-d_1u} < 1,$$
$$F_1 = (1-\mu_{12})De^{-[d_1(1-p)+d_2p]u} < 1,$$
$$E_2 = (1-\mu_{22})De^{-[d_1p+d_2(1-p)]u} < 1,$$
$$F_2 = (1-\mu_{22})(1-D)e^{-d_2u} < 1.$$

Lemma 2. *(y_1^*, y_2^*) of (7) has global asymptotic stability.*

Proof. For the sake of convenience, we denote $(y_1^n, y_2^n) = (y_1(nu^+), y_2(nu^+))$. Equation (7) can be expressed in linear form as follows:

$$\begin{pmatrix} y_1^{n+1} \\ y_2^{n+1} \end{pmatrix} = M \begin{pmatrix} y_1^n \\ y_2^n \end{pmatrix}. \tag{9}$$

By *Jury* criteria [21], we obtain

$$1 - \mathrm{tr}M + \det M > 0. \tag{10}$$

We are prone to see that (y_1^*, y_2^*) represents the sole fixed point of Equation (7) and

$$M = \begin{pmatrix} E_1 & F_1 \\ E_2 & F_2 \end{pmatrix}. \tag{11}$$

Since

$$
\begin{aligned}
& 1 - \mathrm{tr}M + \det M \\
= \ & 1 - (E_1 + F_2) + (E_1F_2 - E_2F_1) \\
= \ & (1 - E_1)(1 - F_2) - E_2F_1 \\
= \ & [1 - (1 - \mu_{12})(1 - D)e^{-d_1u}] \times [1 - (1 - \mu_{22})(1 - D)e^{-d_2u}] \\
& -(1 - \mu_{22})De^{-[d_1p + d_2(1-p)]u} \cdot (1 - \mu_{12})De^{-[d_1(1-p) + d_2p]u} \\
= \ & [1 - (1 - \mu_{12})e^{-d_1u}] \times [1 - (1 - \mu_{22})e^{-d_2u}] + (1 - \mu_{22})De^{-d_2u}[1 - (1 - \mu_{12})e^{-d_1u}] \\
& +(1 - \mu_{12})De^{-d_1u}[1 - (1 - \mu_{22})e^{-d_2u}] \\
= \ & (1 - e^{-d_1u}) \times (1 - e^{-d_2u}) + \mu_{22}e^{-d_2u}(1 - e^{-d_1u}) + \mu_{12}e^{-d_1u}(1 - e^{-d_2u}) \\
& +\mu_{12}\mu_{22}e^{-(d_1+d_2)u} + (1 - \mu_{22})De^{-d_2u}[1 - (1 - \mu_{12})e^{-d_1u}] \\
& +(1 - \mu_{12})De^{-d_1u}[1 - (1 - \mu_{22})e^{-d_2u}] \\
> \ & 0.
\end{aligned}
$$

The local stability of (y_1^*, y_2^*) implies its global asymptotic stability, thus concluding the proof. □

According to the stroboscopic mapping of discrete dynamical systems, we can obtain the following lemma.

Lemma 3. *The periodic solution $(\widetilde{y_1(t)}, \widetilde{y_2(t)})$ of system (2) is globally asymptotically stable, where*

$$
\begin{cases}
\widetilde{y_1(t)} = \begin{cases} y_1^* e^{-d_1(t-nu)}, t \in (nu, (n+p)u], \\ y_1^{**} e^{-d_1(t-(n+p)u)}, t \in ((n+p)u, (n+p+q)u], \\ y_1^{***} e^{-d_1(t-(n+p+q)u)}, t \in ((n+p+q)u, (n+1)u], \end{cases} \\
\widetilde{y_2(t)} = \begin{cases} y_2^* e^{-d_2(t-nu)}, t \in (nu, (n+p)u], \\ y_2^{**} e^{-d_2(t-(n+p)u)}, t \in ((n+p)u, (n+p+q)u], \\ y_2^{***} e^{-d_2(t-(n+p+q)u)}, t \in ((n+p+q)u, (n+1)u], \end{cases}
\end{cases}
\tag{12}
$$

where y_1^* and y_2^* are determined as in (8), y_1^{**} and y_2^{**} are defined as

$$
\begin{cases}
y_1^{**} = (1 - D)e^{-d_1pu}y_1^* + De^{-d_2pu}y_2^*, \\
y_2^{**} = De^{-d_1pu}y_1^* + (1 - D)e^{-d_2pu}y_2^*.
\end{cases}
\tag{13}
$$

y_1^{***} and y_2^{***} are defined as

$$
\begin{cases}
y_1^{***} = (1 - \mu_{12})e^{-d_1qu}y_1^{**}, \\
y_2^{***} = (1 - \mu_{22})e^{-d_2qu}y_2^{**}.
\end{cases}
\tag{14}
$$

4. The Dynamics

Theorem 1. *If*

$$
D < \frac{1}{2} \tag{15}
$$

and

$$\max_{i=1,2}\left\{a_i u-\frac{\beta_i}{d_i}\left[y_i^*(1-e^{-d_i p u})+y_i^{**}(1-e^{-d_i q u})+y_i^{***}(1-e^{-d_i(1-(p+q))u})\right]\right\}<0\ (i=1,2), \tag{16}$$

then the pest eradication boundary periodic solution $(0,\widetilde{y_1(t)},0,\widetilde{y_2(t)})$ of (1) has global asymptotic stability, where y_i^* $(i=1,2)$, y_i^{**} $(i=1,2)$ and y_i^{***} $(i=1,2)$ are determined by (8), (13), and (14), respectively.

Proof. To establish the local stability of the periodic solution $(0,\widetilde{y_1(t)},0,\widetilde{y_2(t)})$ for Equation (1), we introduce new variables and define $x_1(t)=x_1(t), y_{11}(t)=y_1(t)-\widetilde{y_1(t)}$, $x_2(t)=x_2(t)$, $y_{12}(t)=y_2(t)-\widetilde{y_2(t)}$. This leads to a linearly similar system for Equation (1) with a single periodic solution $(0,\widetilde{y_1(t)},0,\widetilde{y_2(t)})$:

$$\begin{pmatrix}\frac{dx_1(t)}{dt}\\ \frac{dy_{11}(t)}{dt}\\ \frac{dx_2(t)}{dt}\\ \frac{dy_{12}(t)}{dt}\end{pmatrix}=\begin{pmatrix}a_1-\beta_1\widetilde{y_1(t)} & 0 & 0 & 0\\ k_1\beta_1\widetilde{y_1(t)} & -d_1 & 0 & 0\\ 0 & 0 & a_2-\beta_2\widetilde{y_2(t)} & 0\\ 0 & 0 & k_2\beta_2\widetilde{y_2(t)} & -d_2\end{pmatrix}\begin{pmatrix}x_1(t)\\ y_{11}(t)\\ x_2(t)\\ y_{12}(t)\end{pmatrix}.$$

Acquiring the fundamental matrix is a simple endeavor:

$$\Phi(t)=\begin{pmatrix}\exp\left[\int_0^t(a_1-\beta_1\widetilde{y_1(s)})ds\right] & 0 & 0 & 0\\ \exp\left[\int_0^t k_1\beta_1\widetilde{y_1(s)}ds\right] & \exp(-d_1t) & 0 & 0\\ 0 & 0 & \exp\left[\int_0^t(a_2-\beta_2\widetilde{y_2(s)})ds\right] & 0\\ 0 & 0 & \exp\left[\int_0^t k_2\beta_2\widetilde{y_2(s)}ds\right] & \exp(-d_2t)\end{pmatrix}.$$

The linearization of Equation (1) for the fifth, sixth, seventh, and eighth terms results in the following:

$$\begin{pmatrix}x_1((n+p)u^+)\\ y_{11}((n+p)u^+)\\ x_2((n+p)u^+)\\ y_{12}((n+p)u^+)\end{pmatrix}=\begin{pmatrix}1 & 0 & 0 & 0\\ 0 & 1-D & 0 & D\\ 0 & 0 & 1 & 0\\ 0 & D & 0 & 1-D\end{pmatrix}\begin{pmatrix}x_1((n+p)u)\\ y_{11}((n+p)u)\\ x_2((n+p)u)\\ y_{12}((n+p)u)\end{pmatrix}.$$

The linearization of Equation (1) for the ninth, tenth, eleventh, and twelfth terms yields the following:

$$\begin{pmatrix}x_1((n+p+q)u^+)\\ y_{11}((n+p+q)u^+)\\ x_2((n+p+q)u^+)\\ y_{12}((n+p+q)u^+)\end{pmatrix}=\begin{pmatrix}1-\mu_{11} & 0 & 0 & 0\\ 0 & 1-\mu_{12} & 0 & 0\\ 0 & 0 & 1-\mu_{21} & 0\\ 0 & 0 & 0 & 1-\mu_{22}\end{pmatrix}\begin{pmatrix}x_1((n+p+q)u)\\ y_{11}((n+p+q)u)\\ x_2((n+p+q)u)\\ y_{12}((n+p+q)u)\end{pmatrix}.$$

The linearization of Equation (1) for equations involving the thirteenth, fourteenth, fifteenth, and sixteenth terms is as follows:

$$\begin{pmatrix} x_1((n+1)u^+) \\ y_{11}((n+1)u^+) \\ x_2((n+1)u^+) \\ y_{12}((n+1)u^+) \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_1((n+1)u) \\ y_{11}((n+1)u) \\ x_2((n+1)u) \\ y_{12}((n+1)u) \end{pmatrix}.$$

The stability of the periodic solution $(0, \widetilde{y_1(t)}, 0, \widetilde{y_2(t)})$ is determined by the eigenvalues of the system, i.e.,

$$M = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1-D & 0 & D \\ D & 0 & 1 & 0 \\ 0 & D & 0 & 1-D \end{pmatrix} \begin{pmatrix} 1-\mu_{11} & 0 & 0 & 0 \\ 0 & 1-\mu_{12} & 0 & 0 \\ 0 & 0 & 1-\mu_{21} & 0 \\ 0 & 0 & 0 & 1-\mu_{22} \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \Phi(u),$$

which are

$$\begin{aligned} \lambda_1 &= (1-\mu_{11}) \exp\left[\int_0^u \left(a_1 - \beta_1 \widetilde{y_1(s)}\right) ds\right], \\ \lambda_2 &= (1-D)(1-\mu_{12}) \exp(-d_1 u), \\ \lambda_3 &= (1-\mu_{21}) \exp\left[\int_0^u (a_2 - \beta_2 \widetilde{y_2(s)}) ds\right], \\ \lambda_4 &= (1-D)(1-\mu_{22}) \exp(-d_2 u), \end{aligned}$$

where $\lambda_2 < 1$, $\lambda_4 < 1$, and condition (15) holds. In accordance with conditions (15), (16), and the Floquet theory [22], if

$$\exp\left[\int_0^u \left(a_i - \beta_i \widetilde{y_i(s)}\right) ds\right] < 1 \quad (i = 1, 2),$$

then

$$\lambda_1 < 1$$

and

$$\lambda_3 < 1,$$

Consequently, the local stability of the pest eradication boundary periodic solution $(0, \widetilde{y_1(t)}, 0, \widetilde{y_2(t)})$ of (1) is ensured.

In the subsequent analysis, we will demonstrate the global attraction property. By utilizing condition (16), we are able to select $\varepsilon > 0$ such that

$$\rho_i = \exp\left[\int_0^u (a_i - \beta_i (\widetilde{y_i(s)} - \varepsilon) ds\right] < 1 \quad (i = 1, 2).$$

Upon examining the second and fourth equations of (1), it becomes evident that $\frac{dy_i(t)}{dt} \geq -d_i y_i(t)$ $(i = 1, 2)$. Based on this observation, we consider the subsequent impulsive comparative differential equations:

$$\begin{cases} \left.\begin{aligned} \frac{dy_{21}(t)}{dt} &= -d_1 y_{21}(t), \\ \frac{dy_{22}(t)}{dt} &= -d_2 y_{22}(t), \end{aligned}\right\} t \neq (n+p)u, t \neq (n+p+q)u, t \neq (n+1)u, \\ \left.\begin{aligned} \Delta y_{21}(t) &= D(y_{22}(t) - y_{21}(t)), \\ \Delta y_{22}(t) &= D(y_{21}(t) - y_{22}(t)), \end{aligned}\right\} t = (n+p)u, \\ \left.\begin{aligned} \Delta y_{21}(t) &= -\mu_{12} y_{21}(t), \\ \Delta y_{22}(t) &= -\mu_{22} y_{22}(t), \end{aligned}\right\} t = (n+p+q)u. \\ \left.\begin{aligned} \Delta y_{21}(t) &= \mu_1, \\ \Delta y_{22}(t) &= \mu_2, \end{aligned}\right\} t = (n+1)u. \end{cases} \tag{17}$$

Based on Lemma 3 and the comparison theorem of impulsive equations (refer to Theorem 3.1.1 in [23]), the following inequalities hold: $y_1(t) \geq y_{21}(t)$,$y_2(t) \geq y_{22}(t)$, and as t approaches infinity, $y_{21}(t)$ converges to $\widetilde{y_1(t)}$ and $y_{22}(t)$ converges to $\widetilde{y_2(t)}$. Consequently, we can conclude that

$$\begin{cases} y_1(t) \geq y_{21}(t) \geq \widetilde{y_1(t)} - \varepsilon, \\ y_2(t) \geq y_{22}(t) \geq \widetilde{y_2(t)} - \varepsilon, \end{cases} \tag{18}$$

for sufficiently large values of t. To simplify the analysis, we can consider that Equation (18) holds for all t greater than or equal to zero. By combining Equations (1) and (18), we obtain

$$\frac{dx_i(t)}{dt} \leq \left[a_i - \beta_i(\widetilde{y_i(t)} - \varepsilon)\right] x_i(t) \quad (i = 1, 2). \tag{19}$$

Therefore, for each $i = 1, 2$, we have $x_i((n+1)u) \leq x_i(nu^+) \exp[\int_{nu}^{(n+1)u} (a_i - \beta_i(\widetilde{y_i(s)} - \varepsilon)) ds]$. Consequently, it follows that $x_i(nu) \leq x_i(0^+) \rho_i^n$ $(i = 1, 2)$, and as n approaches infinity, $x_i(nu)$ tends to 0 $(i = 1, 2)$. As a result, $x_i(t)$ approaches 0 for each $i = 1, 2$ as t tends to infinity.

Next, we aim to demonstrate the convergence of $y_i(t)$ to $\widetilde{y_i(t)}$ as t approaches infinity, where i takes the values 1 and 2. Let ε_1 be a positive value. It follows that there exists a $t_0 > 0$ such that $0 < x_i(t) < \varepsilon_1$ holds for all $t \geq 0$. Without loss of generality, we can assume that $0 < x_i(t) < \varepsilon_1$ for all $t \geq 0$. Considering system (1), we obtain the following expression:

$$-d_i y_i(t) \leq \frac{dy_i(t)}{dt} \leq -(d_i - k_i \beta_i \varepsilon_1) y_i(t) \quad (i = 1, 2), \tag{20}$$

Consequently, it follows that $y_{21}(t) \leq y_1(t) \leq y_{31}(t)$, $y_{22}(t) \leq y_2(t) \leq y_{32}(t)$. Furthermore, as t approaches infinity, we have the convergence of $y_{21}(t)$ to $\widetilde{y_1(t)}$, $y_{22}(t)$ to $\widetilde{y_2(t)}$, $y_{31}(t)$ to $\widetilde{y_{31}(t)}$, and $y_{32}(t)$ to $\widetilde{y_{32}(t)}$. Here, $(y_{21}(t), y_{22}(t))$ and $(y_{31}(t), y_{32}(t))$ represent the solutions of Equation (17) and

$$\begin{cases} \left.\begin{aligned} \frac{dy_{31}(t)}{dt} &= -(d_1 - k_1\beta_1\varepsilon_1)y_{31}(t), \\ \frac{dy_{32}(t)}{dt} &= -(d_2 - k_2\beta_2\varepsilon_1)y_{32}(t), \end{aligned}\right\} t \neq (n+p)u, t \neq (n+p+q)u, t \neq (n+1)u, \\ \left.\begin{aligned} \Delta y_{31}(t) &= D(y_{32}(t) - y_{31}(t)), \\ \Delta y_{32}(t) &= D(y_{31}(t) - y_{32}(t)), \end{aligned}\right\} t = (n+p)u, \\ \left.\begin{aligned} \Delta y_{31}(t) &= -\mu_{12}y_{31}(t), \\ \Delta y_{32}(t) &= -\mu_{22}y_{32}(t), \end{aligned}\right\} t = (n+p+q)u, \\ \left.\begin{aligned} \Delta y_{31}(t) &= \mu_1, \\ \Delta y_{32}(t) &= \mu_2, \end{aligned}\right\} t = (n+1)u, \end{cases} \tag{21}$$

respectively.

$$\begin{cases} \widetilde{y_{31}(t)} = \begin{cases} y_{31}^* e^{-(d_1-k_1\beta_1\varepsilon_1)(t-nu)}, t \in (nu, (n+p)u], \\ y_{31}^{**} e^{-(d_1-k_1\beta_1\varepsilon_1)(t-(n+p)u)}, t \in ((n+p)u, (n+p+q)u], \\ y_{31}^{***} e^{-(d_1-k_1\beta_1\varepsilon_1)(t-(n+p+q)u)}, t \in ((n+p+q)u, (n+1)u], \end{cases} \\ \widetilde{y_{32}(t)} = \begin{cases} y_{32}^* e^{-(d_2-k_2\beta_2\varepsilon_1)(t-nu)}, t \in (nu, (n+p)u], \\ y_{32}^{**} e^{-(d_2-k_2\beta_2\varepsilon_1)(t-(n+p)u)}, t \in ((n+p)u, (n+p+q)u], \\ y_{32}^{***} e^{-(d_2-k_2\beta_2\varepsilon_1)(t-(n+p+q)u)}, t \in ((n+p+q)u, (n+1)u], \end{cases} \end{cases} \tag{22}$$

where y_{31}^* and y_{32}^* are determined as

$$\begin{cases} y_{31}^* = \dfrac{\mu_2 F_{31} + \mu_1(1 - F_{32})}{(1 - E_{31})(1 - F_{32}) - E_{32}F_{31}} > 0, \\ y_{32}^* = \dfrac{\mu_1 E_{32} + \mu_2(1 - E_{31})}{(1 - E_{31})(1 - F_{32}) - E_{32}F_{31}} > 0, \end{cases} \tag{23}$$

The definitions of y_{31}^{**} and y_{32}^{**} are as follows:

$$\begin{cases} y_{31}^{**} = (1 - D)e^{-(d_1-k_1\beta_1\varepsilon_1)pu} y_{31}^* + De^{-(d_2-k_2\beta_2\varepsilon_1)pu} y_{32}^*, \\ y_{32}^{**} = De^{-(d_1-k_1\beta_1\varepsilon_1)pu} y_{31}^* + (1 - D)e^{-(d_2-k_2\beta_2\varepsilon_1)pu} y_{32}^*, \end{cases} \tag{24}$$

y_{31}^{***} and y_{32}^{***} are defined as

$$\begin{cases} y_{31}^{***} = (1 - \mu_{12})e^{-(d_1-k_1\beta_1\varepsilon_1)qu} y_{31}^{**}, \\ y_{32}^{***} = (1 - \mu_{22})e^{-(d_2-k_2\beta_2\varepsilon_1)qu} y_{32}^{**}, \end{cases} \tag{25}$$

where

$$\begin{aligned} E_{31} &= (1 - \mu_{12})(1 - D)e^{-(d_1-k_1\beta_1\varepsilon_1)u} < 1, \\ F_{31} &= (1 - \mu_{12})De^{-[(d_1-k_1\beta_1\varepsilon_1)(1-p)+(d_2-k_2\beta_2\varepsilon_1)p]u} < 1, \\ E_{32} &= (1 - \mu_{22})De^{-[(d_1-k_1\beta_1\varepsilon_1)p+(d_2-k_2\beta_2\varepsilon_1)(1-p)]u} < 1, \\ F_{32} &= (1 - \mu_{22})(1 - D)e^{-(d_2-k_2\beta_2\varepsilon_1)u} < 1. \end{aligned}$$

Given any $\varepsilon_2 > 0$, there exists a value t_1 such that for all $t > t_1$, we have

$$\widetilde{y_{21}(t)} - \varepsilon_2 < y_1(t) < \widetilde{y_{31}(t)} + \varepsilon_2$$

and

$$\widetilde{y_{22}(t)} - \varepsilon_2 < y_2(t) < \widetilde{y_{32}(t)} + \varepsilon_2.$$

By letting ε_1 approach 0, we obtain the following result

$$\widetilde{y_1(t)} - \varepsilon_2 < y_1(t) < \widetilde{y_1(t)} + \varepsilon_2$$

and

$$\widetilde{y_2(t)} - \varepsilon_2 < y_2(t) < \widetilde{y_2(t)} + \varepsilon_2$$

for sufficiently large values of t, it can be deduced that $y_1(t)$ approaches $\widetilde{y_1(t)}$ and $y_2(t)$ approaches $\widetilde{y_2(t)}$ as t approaches infinity. This conclusion signifies the completion of the proof. □

Our next task is to examine the permanence of system (1).

Definition 1. *System (1) is considered to be permanent if the definition of persistence from reference [18] is satisfied.*

Theorem 2. *If*

$$\min_{i=1,2}\left\{a_i u - \frac{\beta_i}{d_i}\left[y_i^*(1-e^{-d_i p u}) + y_i^{**}(1-e^{-d_i q u}) + y_i^{***}(1-e^{-d_i(1-(p+q)u)})\right]\right\} > 0 \ (i=1,2), \tag{26}$$

we define y_i^* $(i=1,2)$, y_i^{**} $(i=1,2)$, and y_i^{***} $(i=1,2)$ according to Equations (8), (13), and (14), respectively, then the system (1) can be considered permanent.

Proof. Firstly, according to Lemma 1, uniform boundedness is ensured. By utilizing Equation (1) and invoking Theorem 1, we can deduce that $y_i(t) > \widetilde{y_i(t)} - \varepsilon_2 > y_i^* e^{-d_i p u} + y_i^{**} e^{-d_i q u} + y_i^{***} e^{-d_i(1-(p+q))u} \triangleq m_i$ $(i=1,2)$, for a ε_2 that is small enough. Therefore, it suffices to find $m_3 > 0$ and ε_3 such that $x_i(t) > m_3$ for sufficiently large t.

Suppose the opposite is true, and let us assume $x_i(t) < m_4$ for all $t \geq 0$, where $m_4 > 0$ is selected to be sufficiently small, satisfying $m_4 < \frac{d_i}{k_i\beta_i - d_i}$ $(d_i < k_i\beta_i)$ holds. By utilizing condition (26) and selecting ε_3 to be small enough, we can establish that this assumption is invalid.

Taking

$$\begin{aligned}
\delta_i &= a_i u \\
&\quad - \frac{\beta_i[y_{4i}^*(1-e^{-(d_i-k_i\beta_i m_4)pu}) + y_{4i}^{**}(1-e^{-(d_i-k_i\beta_i m_4)qu}) + y_{4i}^{***}(1-e^{-(d_i-k_i\beta_i m_4)(1-(p+q))u})]}{d_i - k_i\beta_i m_4} \\
&\quad -\beta_i\varepsilon_3 u \\
&> 0,
\end{aligned}$$

where y_{4i}^* $(i=1,2)$, y_{4i}^{**} $(i=1,2)$, and y_{4i}^{***} $(i=1,2)$ are defined in accordance with Equations (30)–(32) below.

Then,

$$\begin{cases} \left.\begin{aligned} \frac{dy_1(t)}{dt} &< -(d_1 - k_1\beta_1 m_4)y_1(t), \\ \frac{dy_2(t)}{dt} &< -(d_2 - k_2\beta_2 m_4)y_2(t), \end{aligned}\right\} t \neq (n+p)u, t \neq (n+p+q)u, t \neq (n+1)u, \\ \left.\begin{aligned} \Delta y_1(t) &= D(y_2(t) - y_1(t)), \\ \Delta y_2(t) &= D(y_1(t) - y_2(t)), \end{aligned}\right\} t = (n+p)u, \\ \left.\begin{aligned} \Delta y_1(t) &= -\mu_{12} y_1(t), \\ \Delta y_2(t) &= -\mu_{22} y_2(t), \end{aligned}\right\} t = (n+p+q)u, \\ \left.\begin{aligned} \Delta y_1(t) &= \mu_1, \\ \Delta y_2(t) &= \mu_2, \end{aligned}\right\} t = (n+1)u. \end{cases} \tag{27}$$

According to Lemma 3, it follows that $y_1(t) \leq y_{41}(t)$, $y_2(t) \leq y_{42}(t)$, with $y_{41}(t)$ converging to $\overline{y_{41}(t)}$ and $y_{42}(t)$ converging to $\overline{y_{42}(t)}$ as t approaches infinity. Here, $(y_{41}(t), y_{42}(t))$ represents the solution of

$$\begin{cases} \left.\begin{aligned} \frac{dy_{41}(t)}{dt} &= -(d_1 - k_1\beta_1 m_4)y_{41}(t), \\ \frac{dy_{42}(t)}{dt} &= -(d_2 - k_2\beta_2 m_4)y_{42}(t), \end{aligned}\right\} t \neq (n+p)u, t \neq (n+p+q)u, t \neq (n+1)u, \\ \left.\begin{aligned} \Delta y_{41}(t) &= D(y_{42}(t) - y_{41}(t)), \\ \Delta y_{42}(t) &= D(y_{41}(t) - y_{42}(t)), \end{aligned}\right\} t = (n+p)u, \\ \left.\begin{aligned} \Delta y_{41}(t) &= -\mu_{12} y_{41}(t), \\ \Delta y_{42}(t) &= -\mu_{22} y_{42}(t), \end{aligned}\right\} t = (n+p+q)u, \\ \left.\begin{aligned} \Delta y_{41}(t) &= \mu_1, \\ \Delta y_{42}(t) &= \mu_2, \end{aligned}\right\} t = (n+1)u, \end{cases} \tag{28}$$

with

$$\begin{cases} \overline{y_{41}(t)} = \begin{cases} y_{41}^{*} e^{-(d_1 - k_1\beta_1 m_4)(t - nu)}, t \in (nu, (n+p)u], \\ y_{41}^{**} e^{-(d_1 - k_1\beta_1 m_4)(t - (n+p)u)}, t \in ((n+p)u, (n+p+q)u], \\ y_{41}^{***} e^{-(d_1 - k_1\beta_1 m_4)(t - (n+p+q)u)}, t \in ((n+p+q)u, (n+1)u], \end{cases} \\ \overline{y_{42}(t)} = \begin{cases} y_{42}^{*} e^{-(d_2 - k_2\beta_2 m_4)(t - nu)}, t \in (nu, (n+p)u], \\ y_{42}^{**} e^{-(d_2 - k_2\beta_2 m_4)(t - (n+p)u)}, t \in ((n+p)u, (n+p+q)u], \\ y_{42}^{***} e^{-(d_2 - k_2\beta_2 m_4)(t - (n+p+q)u)}, t \in ((n+p+q)u, (n+1)u], \end{cases} \end{cases} \tag{29}$$

where y_{41}^{*} and y_{42}^{*} are determined as

$$\begin{cases} y_{41}^{*} = \dfrac{\mu_2 F_{41} + \mu_1(1 - F_{42})}{(1 - E_{41})(1 - F_{42}) - E_{42}F_{41}} > 0, \\ y_{42}^{*} = \dfrac{\mu_1 E_{42} + \mu_2(1 - E_{41})}{(1 - E_{41})(1 - F_{42}) - E_{42}F_{41}} > 0, \end{cases} \tag{30}$$

and y_{41}^{**}, y_{42}^{**} are defined as

$$\begin{cases} y_{41}^{**} = (1 - D)e^{-(d_1 - k_1\beta_1 m_4)pu} y_{41}^{*} + D e^{-(d_2 - k_2\beta_2 m_4)pu} y_{42}^{*}, \\ y_{42}^{**} = D e^{-(d_1 - k_1\beta_1 m_4)pu} y_{41}^{*} + (1 - D)e^{-(d_2 - k_2\beta_2 m_4)pu} y_{42}^{*}, \end{cases} \tag{31}$$

and $y_{41}^{***}, y_{42}^{***}$ are defined as

$$\begin{cases} y_{41}^{***} = (1-\mu_{12})e^{-(d_1-k_1\beta_1 m_4)qu}y_{41}^{**}, \\ y_{42}^{***} = (1-\mu_{22})e^{-(d_2-k_2\beta_2 m_4)qu}y_{42}^{**}, \end{cases} \tag{32}$$

where

$$\begin{aligned} E_{41} &= (1-\mu_{12})(1-D)e^{-(d_1-k_1\beta_1 m_4)u} < 1, \\ F_{41} &= (1-\mu_{12})De^{-[(d_1-k_1\beta_1 m_4)(1-p)+(d_2-k_2\beta_2 m_4)p]u} < 1, \\ E_{42} &= (1-\mu_{22})De^{-[(d_1-k_1\beta_1 m_4)p+(d_2-k_2\beta_2 m_4)(1-p)]u} < 1, \\ F_{42} &= (1-\mu_{22})(1-D)e^{-(d_2-k_2\beta_2 m_4)u} < 1. \end{aligned}$$

Therefore, there exist $T_1 > 0$ and $\varepsilon_3 > 0$ meeting

$$\overline{y_{41}(t)} + \varepsilon_3 \geq y_{41}(t) \geq y_1(t)$$

and

$$\overline{y_{42}(t)} + \varepsilon_3 \geq y_{42}(t) \geq y_2(t).$$

Then,

$$\left[a_i - \beta_i(\overline{y_{4i}(t)} + \varepsilon_3)\right]x_i(t) \leq \frac{dx_i(t)}{dt} \quad (i = 1,2), \tag{33}$$

for $T_1 \leq t$, let $N_1 \in Z^+$ and $T_1 < N_1 u$. By taking the integral of Equation (33) over the interval $(nu, (n+1)u)$ for $N_1 \leq n$, we have

$$\begin{aligned} x_i((n+1)u) &\geq x_i(nu^+)\exp\left(\int_{nu}^{(n+1)u}\left[a_i - \beta_i(\overline{y_{4i}(t)} + \varepsilon_3)\right]dt\right) \\ &= x_i(nu)e^{\delta_i} \quad (i = 1,2). \end{aligned}$$

Consequently, we have $x_i((N_1+k)u) \geq x_i(N_1 u^+)e^{k\delta_i}$ goes to infinity, which contradicts the boundedness of $x_1(t)$ and $x_2(t)$. Thus, there exists a positive constant t_1 satisfying $m_3 \leq x_i(t)$ $(i = 1,2)$. This concludes the proof. □

5. Simulation Analysis and Discussion

This paper presents a pest management model that incorporates pesticide spraying, natural enemies release, and dispersal at different pulse moments. This integrated pest management model includes the diffusion of predator populations between two regions, providing a comprehensive representation of pest management dynamics. Our analysis establishes that all solutions of the system under investigation are uniformly ultimately bounded. Additionally, by Theorem 1, if $D < \frac{1}{2}$ and

$$\max_{i=1,2}\left\{a_i u - \frac{\beta_i}{d_i}\left[y_i^*(1-e^{-d_i pu}) + y_i^{**}(1-e^{-d_i qu}) + y_i^{***}(1-e^{-d_i(1-(p+q))u})\right]\right\} < 0 \ (i = 1,2),$$

then solution $(0, \widetilde{y_1(t)}, 0, \widetilde{y_2(t)})$ of system (1) possesses global asymptotic stability. By Theorem 2, if

$$\min_{i=1,2}\left\{a_i u - \frac{\beta_i}{d_i}\left[y_i^*(1-e^{-d_i pu}) + y_i^{**}(1-e^{-d_i qu}) + y_i^{***}(1-e^{-d_i(1-(p+q)u)})\right]\right\} > 0 \ (i = 1,2),$$

then system (1) possesses permanence.

5.1. The Dynamical Behaviors Influenced by Parameter D

Considering the following parameter values: $x_1(0) = 1.0, y_1(0) = 1.0, x_2(0) = 1.0, y_2(0) = 1.0, a_1 = 0.7, b_1 = 0.2, a_2 = 0.7, b_2 = 0.2$, and

β_1	β_2	k_1	k_2	μ_1	μ_2	d_1	d_2	u	p	q	D	μ_{11}	μ_{12}	μ_{21}	μ_{22}
0.5	0.5	0.5	0.5	0.5	0.5	0.3	0.3	1.0	0.25	0.3	0.05	0.1	0.1	0.1	0.1

It is evident that conditions (15) and (16) are satisfied. Consequently, the periodic solution representing pest eradication in system (1) is globally asymptotically stable (see Figure 1). Let us consider the initial values $x_1(0) = 1.0, y_1(0) = 1.0, x_2(0) = 1.0, y_2(0) = 1.0$, along with the parameter values $a_1 = 0.9, b_1 = 0.2, a_2 = 09, b_2 = 0.2, \beta_1 = 0.5, \beta_2 = 0.5$, and

β_1	β_2	k_1	k_2	μ_1	μ_2	d_1	d_2	u	p	q	D	μ_{11}	μ_{12}	μ_{21}	μ_{22}
0.5	0.5	0.5	0.5	0.5	0.5	0.3	0.3	1.0	0.25	0.3	0.95	0.1	0.1	0.1	0.1

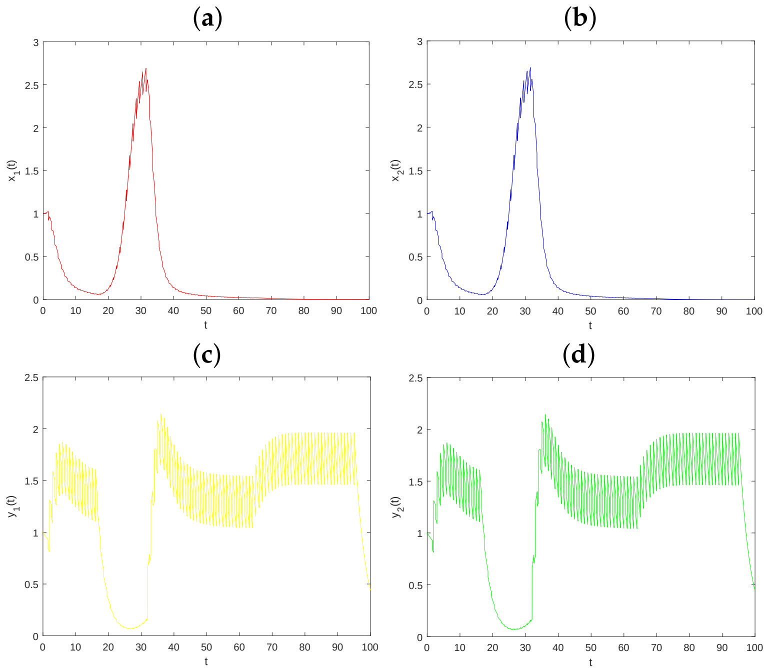

Figure 1. The global asymptotic stability from verification of the impulsive diffusion parameter D. System (1) with initial conditions $x_1(0) = 1.0,\ y_1(0) = 1.0,\ x_2(0) = 1.0,\ y_2(0) = 1.0$, as well as parameter values $a_1 = 0.7,\ b_1 = 0.2,\ a_2 = 0.7,\ b_2 = 0.2,\ \beta_1 = 0.5,\ \beta_2 = 0.5,\ k_1 = 0.5,\ k_2 = 0.5,\ \mu_1 = 0.5,\ \mu_2 = 0.5,\ d_1 = 0.3,\ d_2 = 0.3,\ u = 1,\ p = 0.25,\ q = 0.3,\ D = 0.05,\ \mu_{11} = 0.1,\ \mu_{12} = 0.1,\ \mu_{21} = 0.1$, and $\mu_{22} = 0.1$, exhibits a globally asymptotically stable pest eradication periodic solution. The time-series of $x_1(t),\ x_2(t),\ y_1(t),\ y_2(t)$ are shown in (**a**–**d**), respectively.

It is evident that condition (26) is satisfied, indicating that system (1) is permanent (see Figure 2). By evaluating Equations (16) and (26), we can determine the existence of a threshold, denoted as D^*, which satisfies the following condition:

$$\max_{i=1,2}\left\{a_i u - \frac{\beta_i}{d_i}\left[y_i^*(1-e^{-d_i p u}) + y_i^{**}(1-e^{-d_i q u}) + y_i^{***}(1-e^{-d_i(1-(p+q))u})\right]\right\} < 0\ (i = 1,2)$$

or

$$\min_{i=1,2}\left\{a_i u - \frac{\beta_i}{d_i}\left[y_i^*(1-e^{-d_i p u}) + y_i^{**}(1-e^{-d_i q u}) + y_i^{***}(1-e^{-d_i(1-(p+q)u)})\right]\right\} > 0\ (i = 1,2).$$

When the value of D is less than D^*, the pest populations will tend towards extinction. Conversely, if D is greater than D^*, the system will exhibit its permanence.

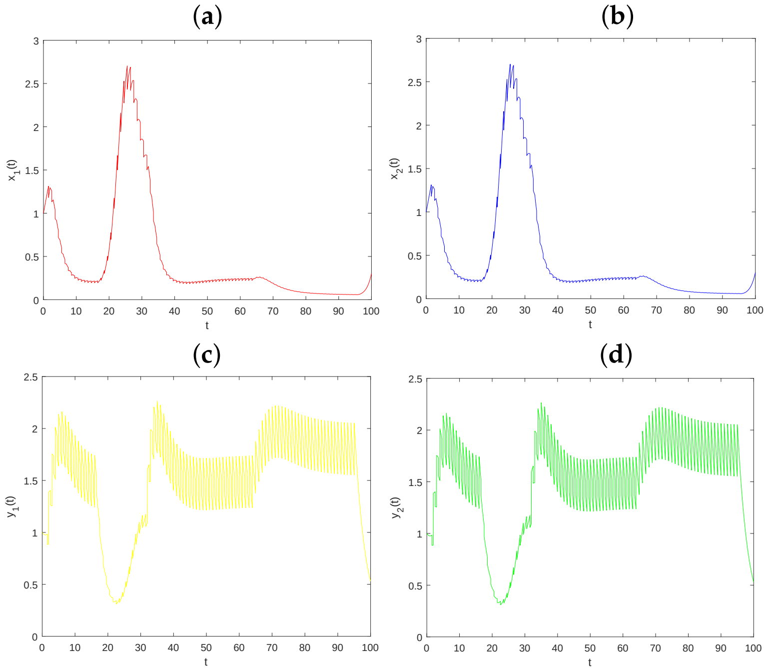

Figure 2. The permanence from verification of the impulsive diffusion parameter D. System (1) with initial conditions $x_1(0) = 1.0$, $y_1(0) = 1.0$, $x_2(0) = 1.0$, $y_2(0) = 1.0$, along with parameter values $a_1 = 0.9$, $b_1 = 0.2$, $a_2 = 09$, $b_2 = 0.2$, $\beta_1 = 0.5$, $\beta_2 = 0.5$, $k_1 = 0.5$, $k_2 = 0.5$, $\mu_1 = 0.5$, $\mu_2 = 0.5$, $d_1 = 0.3$, $d_2 = 0.3$, $u = 1$, $p = 0.25$, $q = 0.3$, $D = 0.95$, $\mu_{11} = 0.1$, $\mu_{12} = 0.1$, $\mu_{21} = 0.1$, and $\mu_{22} = 0.1$, satisfies the condition of permanence. The time-series of $x_1(t)$, $x_2(t)$, $y_1(t)$, and $y_2(t)$ are displayed in (**a**–**d**), respectively.

5.2. *The Dynamical Behaviors Influenced by Parameters μ_1 and μ_2*

In this subsection, we assume that $\mu = \mu_1 = \mu_2$. The initial values are set as $x_1(0) = 1.0$, $y_1(0) = 1.0$, $x_2(0) = 1.0$, and $y_2(0) = 1.0$. We consider various parameter values: $a_1 = 0.8$, $b_1 = 0.2$, $a_2 = 0.8$, $b_2 = 0.2$, and

β_1	β_2	k_1	k_2	μ_1	μ_2	d_1	d_2	u	p	q	D	μ_{11}	μ_{12}	μ_{21}	μ_{22}
0.5	0.5	0.5	0.5	0.5	0.5	0.3	0.3	1.0	0.25	0.3	0.4	0.1	0.1	0.1	0.1

It is obvious that conditions (15) and (16) are satisfied. Consequently, the periodic solution representing pest eradication in system (1) is globally asymptotically stable (see Figure 3). Similarly, assuming initial values $x_1(0) = 1.0, y_1(0) = 1.0, x_2(0) = 1.0$, and $y_2(0) = 1.0$, we set the following parameter values: $a_1 = 0.9, b_1 = 0.2, a_2 = 0.9, b_2 = 0.2$, and

β_1	β_2	k_1	k_2	μ_1	μ_2	d_1	d_2	u	p	q	D	μ_{11}	μ_{12}	μ_{21}	μ_{22}
0.5	0.5	0.5	0.5	0.1	0.1	0.3	0.3	1.0	0.25	0.3	0.2	0.1	0.1	0.1	0.1

It is apparent that condition (26) is satisfied, indicating that system (1) is permanent (see Figure 4). By analysis, we can establish the existence of a threshold value μ^*. If μ is greater than μ^*, the pest population will inevitably go extinct. Conversely, if μ is less than μ^*, the system will exhibit its permanence.

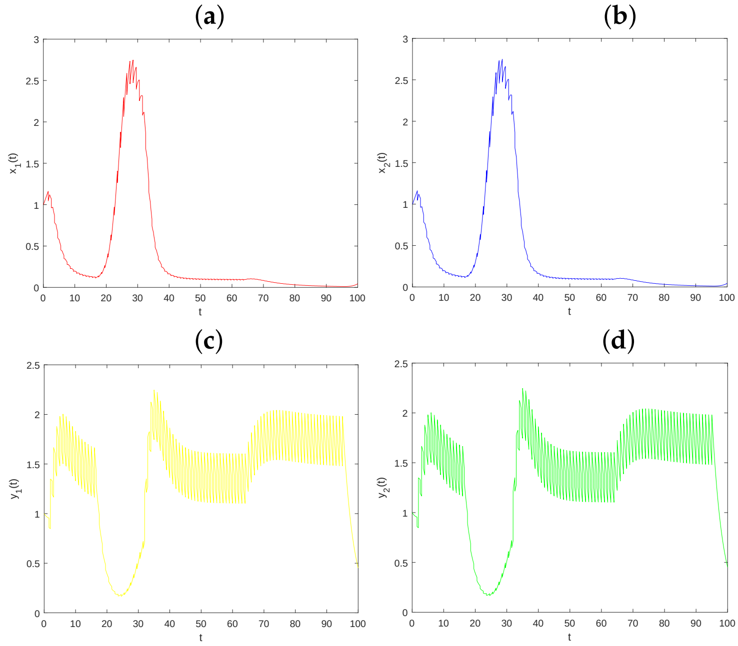

Figure 3. The global asymptotic stability from verification of the natural enemies release parameters μ_1 and μ_2. System (1) with initial conditions $x_1(0) = 1.0$, $y_1(0) = 1.0$, $x_2(0) = 1.0$, $y_2(0) = 1.0$, along with parameter values $a_1 = 0.8$, $b_1 = 0.2$, $a_2 = 0.8$, $b_2 = 0.2$, $\beta_1 = 0.5$, $\beta_2 = 0.5$, $k_1 = 0.5$, $k_2 = 0.5$, $\mu_1 = 0.5$, $\mu_2 = 0.5$, $d_1 = 0.3$, $d_2 = 0.3$, $u = 1$, $p = 0.25$, $q = 0.3$, $D = 0.4$, $\mu_{11} = 0.1$, $\mu_{12} = 0.1$, $\mu_{21} = 0.1$, and $\mu_{22} = 0.1$, exhibits a globally asymptotically stable pest eradication periodic solution. The time-series of $x_1(t)$, $x_2(t)$, $y_1(t)$, and $y_2(t)$ are displayed in (**a**–**d**), respectively.

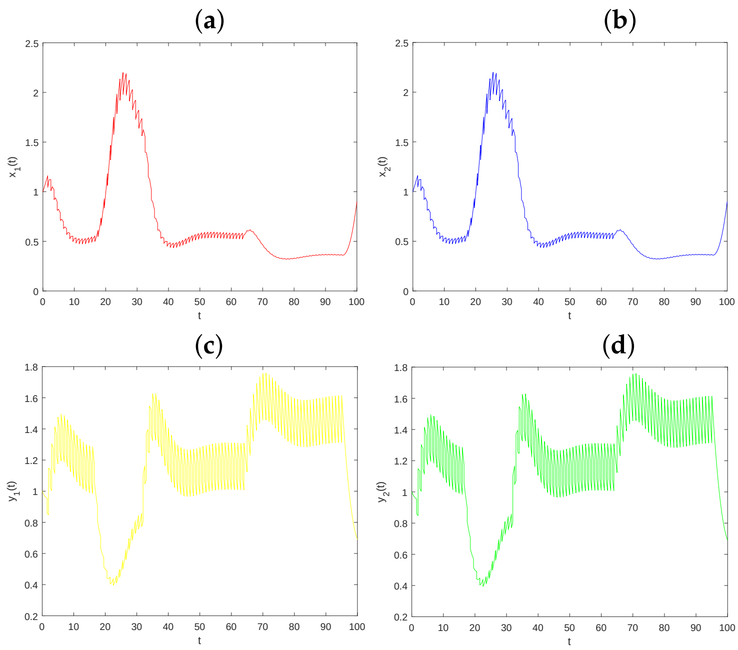

Figure 4. The permanence from verification of the natural enemies release parameters μ_1 and μ_2. System (1) exhibits permanence with initial conditions $x_1(0) = 1.0$, $y_1(0) = 1.0$, $x_2(0) = 1.0$, $y_2(0) = 1.0$, and parameters $a_1 = 0.9$, $b_1 = 0.2$, $a_2 = 0.9$, $b_2 = 0.2$, $\beta_1 = 0.5$, $\beta_2 = 0.5$, $k_1 = 0.5$, $k_2 = 0.5$, $\mu_1 = 0.1$, $\mu_2 = 0.1$, $d_1 = 0.3$, $d_2 = 0.3$, $u = 1$, $p = 0.25$, $q = 0.3$, $D = 0.2$, $\mu_{11} = 0.1$, $\mu_{12} = 0.1$, $\mu_{21} = 0.1$, and $\mu_{22} = 0.1$. The time-series of $x_1(t)$, $x_2(t)$, $y_1(t)$, and $y_2(t)$ are shown in (**a**–**d**), respectively.

5.3. *The Dynamical Behaviors Influenced by Parameters* $\mu_{ij}(i = 1, 2)$

In this subsection, we consider the scenario where all values of μ', μ_{11}, μ_{12}, μ_{21}, and μ_{22} are equal. Initially, we set the following conditions for the pest management system: $x_1(0) = 1.0, y_1(0) = 1.0, x_2(0) = 1.0$, $y_2(0) = 1.0$. Furthermore, we assign parameter values as follows: $a_1 = 0.6, b_1 = 0.2, a_2 = 0.6, b_2 = 0.2$, and

β_1	β_2	k_1	k_2	μ_1	μ_2	d_1	d_2	u	p	q	D	μ_{11}	μ_{12}	μ_{21}	μ_{22}
0.5	0.5	0.5	0.5	0.35	0.35	0.3	0.3	1.0	0.25	0.5	0.2	0.3	0.3	0.3	0.3

It is confirmed that conditions (15) and (16) are satisfied, and the pest eradication periodic solution of system (1) is demonstrated to be globally asymptotically stable, as depicted in Figure 5. Subsequently, we maintain the same initial conditions as before but make adjustments to certain parameter values: $a_1 = 0.6, b_1 = 0.2, a_2 = 0.6, b_2 = 0.2$, and

β_1	β_2	k_1	k_2	μ_1	μ_2	d_1	d_2	u	p	q	D	μ_{11}	μ_{12}	μ_{21}	μ_{22}
0.5	0.5	0.5	0.5	0.35	0.35	0.3	0.3	1.0	0.25	0.5	0.2	0.05	0.05	0.05	0.05

We verify that condition (26) is satisfied and observe that system (1) exhibits its permanence, as depicted in Figure 6. Through calculations, we determine the existence of a threshold value, denoted as μ^{**}, if μ' exceeds μ^{**}, the pest population will become extinct. Conversely, if μ' is lower than μ^{**}, the system will exhibit its permanence.

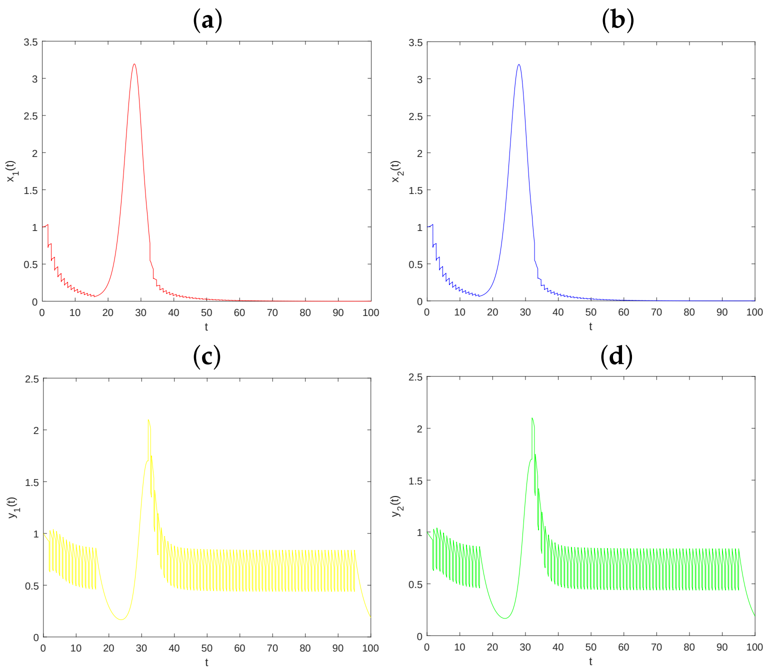

Figure 5. The global asymptotical stability from verification of the pesticide spraying parameters $\mu_{ij}, i = 1, 2$. The periodic solution of system (1) with initial conditions $x_1(0) = 1.0$, $y_1(0) = 1.0$, $x_2(0) = 1.0$, $y_2(0) = 1.0$ is globally asymptotically stable for pest eradication with parameters $a_1 = 0.6$, $b_1 = 0.2$, $a_2 = 0.6$, $b_2 = 0.2$, $\beta_1 = 0.5$, $\beta_2 = 0.5$, $k_1 = 0.5$, $k_2 = 0.5$, $\mu_1 = 0.35$, $\mu_2 = 0.35$, $d_1 = 0.3$, $d_2 = 0.3$, $u = 1$, $p = 0.25$, $q = 0.5$, $D = 0.2$, $\mu_{11} = 0.3$, $\mu_{12} = 0.3$, $\mu_{21} = 0.3$, and $\mu_{22} = 0.3$. The time-series of $x_1(t)$, $x_2(t)$, $y_1(t)$, and $y_2(t)$ are shown in parts (**a**–**d**), respectively.

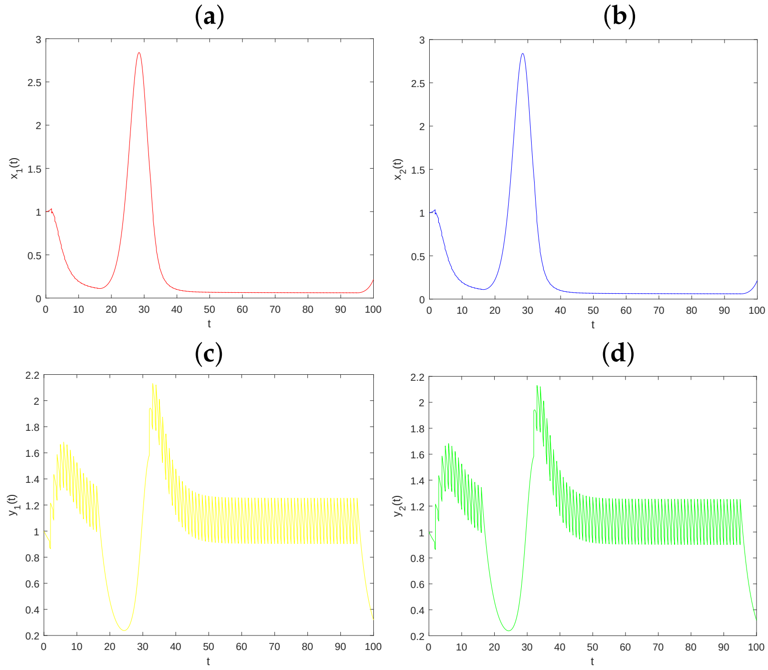

Figure 6. The permanence from verification of the pesticide spraying parameters $\mu_{ij}, i = 1, 2$. System (1) exhibits permanence with initial conditions $x_1(0) = 1.0$, $y_1(0) = 1.0$, $x_2(0) = 1.0$, $y_2(0) = 1.0$, along with the following parameter values: $a_1 = 0.6$, $b_1 = 0.2$, $a_2 = 0.6$, $b_2 = 0.2$, $\beta_1 = 0.5$, $\beta_2 = 0.5$, $k_1 = 0.5$, $k_2 = 0.5$, $\mu_1 = 0.35$, $\mu_2 = 0.35$, $d_1 = 0.3$, $d_2 = 0.3$, $u = 1$, $p = 0.25$, $q = 0.5$, $D = 0.2$, $\mu_{11} = 0.05$, $\mu_{12} = 0.05$, $\mu_{21} = 0.05$, and $\mu_{22} = 0.05$. The time-series plots of $x_1(t)$, $x_2(t)$, $y_1(t)$, and $y_2(t)$ confirm the system's permanence (**a**–**d**).

6. Conclusions

Based on our numerical simulations, we have found that increasing the diffusion has a negative impact on integrated pest management. However, we can achieve optimal pest control at a lower cost by implementing a combination of control strategies such as population dispersal, pesticide spraying, and natural enemies release. This study concludes that impulsive diffusion, pesticide spraying, and natural enemies release provide a solid foundation for effective pest management tactics.

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