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The (α, p)-Golden Metric Manifolds and Their Submanifolds

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Abstract: The notion of a golden structure was introduced 15 years ago by the present authors and has been a constant interest of several geometers. Now, we propose a new generalization apart from that called the metallic structure, which is also considered by the authors. By adding a compatible Riemannian metric, we focus on the study of the structure induced on submanifolds in this setting and its properties. Also, to illustrate our results, some suitable examples of this type of manifold are presented.

Keywords: almost product structure; almost complex structure; $\Phi_{\alpha,p}$ structure; Riemannian manifold; submanifold

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1. Introduction

The *real metallic number*, denoted by $\sigma_{p,q} := \frac{p+\sqrt{p^2+4q}}{2}$, is the positive solution of the equation $x^2 - px - q = 0$, where p and q are positive integers and $p^2 + 4q > 0$ [1]. These $\sigma_{p,q}$ numbers are members of the *metallic means family*, defined by V.W. de Spinadel in [2,3], which appear as a natural generalization of the golden number $\phi = \frac{1+\sqrt{5}}{2}$. Moreover, A.P. Stakhov gave some new generalizations of the golden section and Fibonacci numbers and developed a scientific principle called the Generalized Principle of the Golden Section in [4,5].

The golden and metallic structures are particular cases of polynomial structures on a manifold which were generally defined by S. I. Goldberg, K. Yano and N. C. Petridis in [6,7].

If $\overline{M}$ is a smooth manifold, then an endomorphism J of the tangent bundle $T\overline{M}$ is called a *metallic structure* on $\overline{M}$ if it satisfies $J^2 = pJ + qId$, where Id stands for the identity (or Kronecker) endomorphism and p and q are positive integers [1]. Moreover, the pair $(\overline{M}, J)$ is called an *almost metallic manifold*. In particular, for $p = q = 1$, the metallic structure J becomes a *golden structure* as defined in [8].

The complex version of the above numbers (namely the *complex metallic numbers*), $\sigma^c_{p,q} = \frac{p+\sqrt{p^2-6q}}{2}$, appears as a solution to the equation $x^2 - px + \frac{3}{2}q = 0$, where p and q are now real numbers satisfying the conditions $q \geq 0$ and $p^2 < 6q$. Moreover, an almost complex metallic structure is defined as an endomorphism J_M which satisfies the relation $J_M^2 - pJ_M + \frac{3}{2}qId = 0$ [9]. For $p = q = 1$, the almost complex metallic structure becomes a *complex golden structure*.

F. Etayo et al. defined in [10] the *α-metallic numbers* of the form $\frac{p+\sqrt{\alpha(p^2+4q)}}{2}$, where p and q are positive integers which satisfy $p^2 + 4q > 0$ and $\alpha \in \{1, -1\}$. Moreover, they introduced the α-metallic metric manifolds using the α-metallic structure, defined by the identity

$$\varphi^2 = p\varphi + \frac{p^2(\alpha-1)+4q\alpha}{4}Id. \tag{1}$$

Some similar manifolds, such as holomorphic golden Norden–Hessian manifolds [11], almost golden Riemannian manifolds [12,13] and α-golden metric manifolds [14], have been studied.

The geometry of submanifolds in Riemannian manifolds was widely studied by many geometers. The properties of the submanifolds in golden Riemannian manifolds were studied in [15]. By generalizing the geometry of the golden Riemannian manifods, we presented in [1,16] the properties of the submanifolds in metallic Riemannian manifolds. The properties of the submanifolds in almost complex metallic manifolds were studied in [17].

The aim of the present paper is to propose a new generalization of the golden structure called the *almost (α, p)-golden structure* and to investigate the geometry of a Riemannian manifold endowed by this structure. This manifold is a natural generalization of the golden Riemannian manifold, presented in [8] and of almost Hermitian golden manifold, studied in [18].

In Section 2, we consider several frameworks in which almost product and almost complex structures are treated in our language of the $(\alpha$, p)-golden structure. These two structures can be unified under the notion of the α-structure, denoted by J_α, which was defined and studied in [10,19].

In Section 3, we study the properties of a Riemannian manifold endowed by a $\Phi_{\alpha,p}$ structure and a compatible Riemannian metric g, called an *almost (α, p)-golden Riemannian manifold*.

In Section 4, we obtain a characterization of the structure induced on a submanifold by the almost (α, p)-golden structure. Finally, we find the necessary and sufficient conditions of a submanifold in an almost (α, p)-golden Riemannian manifold to be an invariant submanifold.

2. The Almost (α, p)-Golden Structure

In order to state the main results of this paper, we need some definitions and notations.

Let us consider the (α, p)-golden means family, which contains the *(α, p)-golden numbers* obtained as the solutions of the equation

$$x^2 - px - \frac{5\alpha-1}{4}p^2 = 0, \tag{2}$$

where $\alpha \in \{-1, 1\}$ and p is a real nonzero number. The *(α, p)-golden numbers* have the form

$$\varphi_{\alpha,p} = p\frac{1+\sqrt{5\alpha}}{2}, \quad \overline{\varphi}_{\alpha,p} = p\frac{1-\sqrt{5\alpha}}{2}. \tag{3}$$

Using these numbers, we define a new structure on a smooth manifold $\overline{M}$ (of even dimensions) which generalizes both the almost golden structure and the almost complex golden structure.

An endomorphism J_1 of the tangent bundle $T\overline{M}$, such as $J_1^2 = Id$, is called an *almost product structure*, where Id is the identity or Kronecker endomorphism. Moreover, the pair $(\overline{M}, J_1)$ is called an *almost product manifold*.

An endomorphism J_{-1} of the tangent bundle $T\overline{M}$ is called an *almost complex structure* on $\overline{M}$ if it satisfies $J_{-1}^2 = -Id$, and $(\overline{M}, J_{-1})$ is called an *almost complex manifold*. In this case, the dimension of $\overline{M}$ is even (e.g., $2m$).

Definition 1. *An endomorphism J_α of the tangent bundle $T\overline{M}$ is called an α-structure on $\overline{M}$ if it satisfies the equality*

$$J_\alpha^2 = \alpha \cdot Id, \tag{4}$$

on the even dimensional manifold $\overline{M}$, where $\alpha \in \{-1, 1\}$ [19].

Using the Equation (1), for $q = p^2$, we obtain the following definition:

Definition 2. *An endomorphism $\Phi_{\alpha,p}$ of the tangent bundle $T\overline{M}$ is called an almost (α, p)-golden structure on $\overline{M}$ if it satisfies the equality*

$$\Phi_{\alpha,p}^2 = p\Phi_{\alpha,p} + \frac{5\alpha - 1}{4} p^2 \cdot Id, \tag{5}$$

where p is a nonzero real number and $\alpha \in \{-1, 1\}$. The pair $(\overline{M}, \Phi_{\alpha,p})$ is called an almost $(\alpha$, $p)$-golden manifold.

In particular, the $\Phi_{\alpha,1}$ structure is named an α-golden structure, and it was studied in [14].

Remark 1. *The eigenvalues of the almost $(\alpha$, $p)$-golden structure $\Phi_{\alpha,p}$ are $\varphi_{\alpha,p}$ and $\overline{\varphi}_{\alpha,p} = p - \varphi_{\alpha,p}$, given in Equation (3).*

In particular, for $\alpha = 1$, we obtain $\varphi_{1,p} = p\frac{1+\sqrt{5}}{2} = p\phi$ as a zero of the polynomial $X^2 - pX - p^2$, and we remark that $\varphi_{1,p}$ is a member of the metallic numbers family, where $q = p^2$ and ϕ is the golden number.

For $\alpha = -1$, we obtain $\varphi_{-1,p} = p\frac{1+i\sqrt{5}}{2} = p\phi_c$ as a zero of the polynomial $X^2 - pX + \frac{3}{2}p^2$, and $\varphi_{-1,p}$ is a member of the complex metallic numbers family, where $q = p^2$ and ϕ_c is the complex golden number.

Moreover, if $(\alpha, p) = (1, 1)$, then one obtains the *golden structure* determined by an endomorphism Φ with $\Phi^2 = \Phi + Id$, as studied in [8]. The same structure was studied in [12], using the name of the *almost golden structure*. In this case, $(\overline{M}, \Phi)$ is called the *almost golden manifold*.

If $(\alpha, p) = (-1, 1)$, then one obtains the *almost complex golden structure* determined by an endomorphism Φ_c, which verifies $\Phi_c^2 = \Phi_c + \frac{3}{2}Id$. In this case, $(\overline{M}, \Phi_c)$ is called the *almost complex golden manifold*, as studied in [11,18].

An important remark is that (α, p)-golden structures appear in pairs. In particular, if $\Phi_{\alpha,p}$ is an (α, p)-golden structure, then $\overline{\Phi}_{\alpha,p} = pId - \Phi_{\alpha,p}$ is also an (α, p)-golden structure. Thus is the case for the almost product structures (J_1 and $-J_1$) and for the almost complex structures (J_{-1} and $-J_{-1}$).

We point out that the almost (α, p)-golden structure $\Phi_{\alpha,p}$ and the α-structure J_α are closely related. Thus, we obtain the correspondence $\Phi_{\alpha,p} \longleftrightarrow J_\alpha$, and we have

$$\overline{\Phi}_{\alpha,p} = pId - \Phi_{\alpha,p} \longleftrightarrow \bar{J}_\alpha = -J_\alpha,$$

where $\Phi_{\alpha,p} =: \Phi^+_{\alpha,p}$, $\overline{\Phi}_{\alpha,p} =: \Phi^-_{\alpha,p}$, $J_\alpha =: J^+_\alpha$ and $\overline{J}_\alpha =: J^-_\alpha$.

Proposition 1. *Every α-structure J_α on $\overline{M}$ defines two almost (α, p)-golden structures, given by the equality*

$$\Phi^\pm_{\alpha,p} = \frac{p}{2}(Id \pm \sqrt{5}J_\alpha); \tag{6}$$

Conversely, two α-structures can be associated to a given almost (α, p)-golden structure as follows:

$$J^\pm_\alpha = \pm\frac{2}{p\sqrt{5}}\left(\Phi_{\alpha,p} - \frac{p}{2}Id\right). \tag{7}$$

Proof. First of all, we seek the real numbers a and b such that $\Phi_{\alpha,p} = aId + bJ_\alpha$. Considering $\Phi^2_{\alpha,p}$, from the identities (4) and (5), we obtain $a = \frac{p}{2}$ and $b = \pm\frac{\sqrt{5}p}{2}$, which implies identity (6). Moreover, $\Phi^{\pm}_{\alpha,p}$ verifies the identity (5).

On the other hand, if $\Phi^{\pm}_{\alpha,p}$ verifies identity (6), then we obtain that $J^{\pm}_\alpha$ verifies identities (4) and (7). Conversely, if $J^{\pm}_\alpha$ verifies identity (7), then $\Phi^{\pm}_{\alpha,p}$ verifies the identity (6). □

Example 1. *(i) An almost product structure J_1 induces two almost $(1, p)$-golden structures:*

$$\Phi^{\pm}_{1,p} = p\frac{Id \pm \sqrt{5}J_1}{2}; \tag{8}$$

(ii) An almost complex structure J_{-1} induces two almost $(-1, p)$-golden structures:

$$\Phi^{\pm}_{-1,p} = p\frac{Id \pm \sqrt{5}J_{-1}}{2}. \tag{9}$$

A straightforward computation using the Equations (5) and (6) gives us the following property:

Proposition 2. *An (α, p)-golden structure $\Phi_{\alpha,p}$ is an isomorphism on the tangent space of the manifold $T_x\overline{M}$ for every $x \in \overline{M}$. It follows that $\Phi_{\alpha,p}$ is invertible, and its inverse is a structure given by the equality*

$$\Phi^{-1}_{\alpha,p} = \frac{4}{p^2(5\alpha - 1)}\Phi_{\alpha,p} - \frac{4}{p(5\alpha - 1)}Id. \tag{10}$$

Lemma 1. *A fixed α-structure J_α yields two complementary projectors P and Q, given by*

$$P = \frac{1}{2}(Id + \sqrt{\alpha}J_\alpha), \quad Q = \frac{1}{2}(Id - \sqrt{\alpha}J_\alpha). \tag{11}$$

Then, we can easily see that

$$P + Q = Id, \quad P^2 = P, \quad Q^2 = Q, \quad PQ = QP = 0, \tag{12}$$

and

$$\sqrt{\alpha}J_\alpha = P - Q. \tag{13}$$

Taking into account the identities (11) and (12), one has the following remark:

Remark 2. *The operators P and Q are orthogonal complementary projection operators and define the complementary distributions $\mathcal{D}_1$ and $\mathcal{D}_2$, where $\mathcal{D}_1$ contains the eigenvectors corresponding to the eigenvalue $\sqrt{\alpha}$ and $\mathcal{D}_2$ contains the eigenvectors corresponding to the eigenvalue $-\sqrt{\alpha}$.*

If the multiplicity of the eigenvalue $\sqrt{\alpha}$ (or $-\sqrt{\alpha}$) is a (or b), where $a + b = dim(\overline{M}) = 2m$, then the dimension of $\mathcal{D}_1$ is a, while the dimension of $\mathcal{D}_2$ is b.

Conversely, if there exist in $\overline{M}$ two complementary distributions $\mathcal{D}_1$ and $\mathcal{D}_2$ of dimensions $a \geq 1$ and $b \geq 1$, respectively, where $a + b = dim(\overline{M}) = 2m$, then we can define an α structure J_α on $\overline{M}$, which verifies identity (13).

A straightforward computation using the identities (7), (11) and (12) gives us the following property:

Proposition 3. *The projection operators $P_{\alpha,p}$ and $Q_{\alpha,p}$ on the almost (α, p)-golden manifold $(\overline{M}, \Phi_{\alpha,p})$ have the form*

$$P_{\alpha,p} = \frac{\sqrt{5\alpha}}{5p} \cdot \Phi_{\alpha,p} + \frac{5 - \sqrt{5\alpha}}{10}Id, \quad Q_{\alpha,p} = -\frac{\sqrt{5\alpha}}{5p} \cdot \Phi_{\alpha,p} + \frac{5 + \sqrt{5\alpha}}{10}Id \tag{14}$$

which verifies

$$P_{\alpha,p} + Q_{\alpha,p} = Id, \quad P_{\alpha,p}^2 = P_{\alpha,p}; \quad Q_{\alpha,p}^2 = Q_{\alpha,p}, \quad P_{\alpha,p} \cdot Q_{\alpha,p} = Q_{\alpha,p} \cdot P_{\alpha,p} = 0 \tag{15}$$

and

$$\Phi_{\alpha,p} = \frac{p\alpha\sqrt{5\alpha}}{2}(P_{\alpha,p} - Q_{\alpha,p}) - \frac{p}{2} Id. \tag{16}$$

Remark 3. *The operators $P_{\alpha,p}$ and $Q_{\alpha,p}$ given in the identities (14) are orthogonal complementary projection operators and define the complementary distributions $\mathcal{D}_1$ and $\mathcal{D}_2$ on $\overline{M}$, which contain the eigenvectors of $\Phi_{\alpha,p}$, corresponding to the eigenvalues $\varphi_{\alpha,p}$ and $\overline{\varphi}_{\alpha,p} = p - \varphi_{\alpha,p}$, respectively.*

3. Almost (α, p)-Golden Riemannian Geometry

Let $\overline{M}$ be an even dimensional manifold endowed with an α-structure J_α. We fix a Riemannian metric $\overline{g}$ such that

$$\overline{g}(J_\alpha X, Y) = \alpha \overline{g}(X, J_\alpha Y), \tag{17}$$

which is equivalent to

$$\overline{g}(J_\alpha X, J_\alpha Y) = \overline{g}(X, Y), \tag{18}$$

for any vector fields $X, Y \in \Gamma(T\overline{M})$, where $\Gamma(T\overline{M})$ is the set of smooth sections of $T\overline{M}$.

Definition 3. *The Riemannian metric $\overline{g}$, defined on an even dimensional manifold $\overline{M}$ and endowed with an α-structure J_α which verifies the equivalent identities (17) and (18), is called a metric (α, J_α)-compatible.*

Thus, by using the identities (7) and (17), we obtain that the Riemannian metric $\overline{g}$ verifies the identity

$$\overline{g}(\Phi_{\alpha,p} X, Y) - \alpha \overline{g}(X, \Phi_{\alpha,p} Y) = \frac{p}{2}(1 - \alpha)\overline{g}(X, Y), \tag{19}$$

for any $X, Y \in \Gamma(T\overline{M})$.

Moreover, from identities (7) and (18), we remark that $\overline{g}$ and $(\Phi_{\alpha,p})$ are related by

$$\overline{g}(\Phi_{\alpha,p} X, \Phi_{\alpha,p} Y) = \frac{p}{2}\left(\overline{g}(\Phi_{\alpha,p} X, Y) + \overline{g}(X, \Phi_{\alpha,p} Y)\right) + p^2 \overline{g}(X, Y), \tag{20}$$

for any $X, Y \in \Gamma(T\overline{M})$.

Definition 4. *An almost (α, p)-golden Riemannian manifold is a triple $(\overline{M}, \Phi_{\alpha,p}, \overline{g})$, where $\overline{M}$ is an even dimensional manifold, $\Phi_{\alpha,p}$ is an almost (α, p)-golden structure and $\overline{g}$ is a Riemannian metric which verifies identities (19) and (20).*

Remark 4. *For $\alpha = 1$ in the identities (19) and (20), we obtain*

$$\overline{g}(\Phi_{1,p} X, Y) = \overline{g}(X, \Phi_{1,p} Y)), \tag{21}$$

which is equivalent to

$$\overline{g}(\Phi_{1,p} X, \Phi_{1,p} Y) = p\overline{g}(\Phi_{1,p} X, Y) + p^2 \overline{g}(X, Y), \tag{22}$$

and the triple $(\overline{M}, \Phi_{1,p}, \overline{g})$ is a particular case of an almost metallic Riemannian manifold, which was studied in [1,16].

Remark 5. *For $\alpha = -1$ in the identities (19) and (20), we have*

$$\overline{g}(\Phi_{-1,p}X, Y) + \overline{g}(X, \Phi_{-1,p}Y) = p\overline{g}(X, Y), \tag{23}$$

which is equivalent to

$$\overline{g}(\Phi_{-1,p}X, \Phi_{-1,p}Y) = \frac{3}{2}p^2\overline{g}(X, Y), \tag{24}$$

and the triple $(\overline{M}, \Phi_{-1,p}, \overline{g})$ is a particular case of an almost complex metallic Riemannian manifold, which was studied in [9].

Proposition 4. *If $(\overline{M}, \Phi_{\alpha,p}, \overline{g})$ is an almost (α, p)-golden Riemannian manifold of dimmension $2m$, then the trace of the $\Phi_{\alpha,p}$ structure satisfies*

$$trace(\Phi^2_{\alpha,p}) = p \cdot trace(\Phi_{\alpha,p}) + \frac{5\alpha - 1}{2}mp^2. \tag{25}$$

Proof. If we denote a local orthonormal basis of $T\overline{M}$ by $\{E_1, E_2, ..., E_{2m}\}$, then from the identity (5), we obtain

$$\overline{g}(\Phi^2_{\alpha,p}E_i, E_i) = p\overline{g}(\Phi_{\alpha,p}E_i, E_i) + \frac{5\alpha - 1}{4}p^2\overline{g}(E_i, E_i),$$

and by summing this equality for $i \in \{1, \ldots 2m\}$, we obtain the claimed relation. □

Example 2. *Using $\varphi_{\alpha,p}$ and $\overline{\varphi}_{\alpha,p}$, defined in Equation (3), let us consider the endomorphism $\Phi_{\alpha,p} : \mathbb{R}^{2m} \to \mathbb{R}^{2m}$, given by*

$$\Phi_{\alpha,p}(X^i, Y^i) := (\varphi_{\alpha,p}X^1, \ldots, \varphi_{\alpha,p}X^m, \overline{\varphi}_{\alpha,p}Y^1, \ldots, \overline{\varphi}_{\alpha,p}Y^m), \tag{26}$$

where $(X^i, Y^i) := (X^1, \ldots, X^m, Y^1, \ldots, Y^m)$ and $i \in \{1, \ldots, m\}$.

Using identities (2) and (26), a straightforward computation yields

$$\Phi^2_{\alpha,p}(X^i, Y^i) := (\varphi^2_{\alpha,p}X^i, \overline{\varphi}^2_{\alpha,p}Y^i) = (p\varphi_{\alpha,p}X^i + \frac{5\alpha - 1}{4}p^2X^i, p\overline{\varphi}_{\alpha,p}Y^i + \frac{5\alpha - 1}{4}p^2Y^i).$$

Thus, we obtain

$$\Phi^2_{\alpha,p}(X^i, Y^i) = p(\varphi_{\alpha,p}X^i, \overline{\varphi}_{\alpha,p}Y^i) + \frac{5\alpha - 1}{4}p^2(X^i, Y^i) = p\Phi_{\alpha,p}(X^i, Y^i) + \frac{5\alpha - 1}{4}p^2(X^i, Y^i)$$

and hence $\Phi_{\alpha,p}$ verifies Equation (5).

Let us consider the structure J_α associated with $\Phi_{\alpha,p}$ by identities (6) and (7):

$$J_\alpha(X^i, Y^i) := (X^1, \ldots, X^m, \alpha Y^1, \ldots, \alpha Y^m).$$

Using the identity (17), we remark that the Euclidean metric $\overline{g} := \langle \, , \, \rangle$ on $\mathbb{R}^{2m}$ verifies

$$\overline{g}(J_\alpha Z, Z') = \alpha\Sigma^m_{i=1}(X^iX'^i + Y^iY'^i) = \alpha\overline{g}(Z, J_\alpha Z'),$$

for any $Z := (X^1, \ldots, X^m, Y^1, \ldots, Y^m), Z' = (X'^1, \ldots, X'^m, Y'^1, \ldots, Y'^m) \in \Gamma(\mathbb{R}^{2m})$. Thus, it is (α, J_α)-compatible. Using the identity (7), we obtain

$$\overline{g}(\Phi_{\alpha,p}Z, \Phi_{\alpha,p}Z') = \frac{p}{2}(\overline{g}(\Phi_{\alpha,p}Z, Z') + \overline{g}(Z, \Phi_{\alpha,p}Z')) + p^2\overline{g}(Z, Z').$$

Therefore, $\overline{g}$ verifies the identity (20), which implies that $(\mathbb{R}^{2m}, \Phi_{\alpha,p}, \overline{g})$ is an almost (α, p)-golden Riemannian manifold.

Definition 5. *If ∇ is the Levi-Civita connection on $(\overline{M}, \overline{g})$, then the covariant derivative ∇J_α is a tensor field of the type $(1,2)$, defined by*

$$(\nabla_X J_\alpha)Y := \nabla_X J_\alpha Y - J_\alpha \nabla_X Y, \tag{27}$$

for any $X, Y \in \Gamma(T\overline{M})$.

Hence, from the identity (6), we obtain

$$(\nabla_X \Phi_{\alpha,p})Y = \frac{p\sqrt{5}}{2}(\nabla_X J_\alpha)Y. \tag{28}$$

Let us consider now the Nijenhuis tensor field of J_α. Using a similar approach to that in [19] (Definition 2.8 and Proposition 2.9), we obtain

$$N_{J_\alpha}(X,Y) = J_\alpha^2[X,Y] + [J_\alpha X, J_\alpha Y] - J_\alpha[J_\alpha X, Y] - J_\alpha[X, J_\alpha Y], \tag{29}$$

for any $X, Y \in \Gamma(T\overline{M})$, which is equivalent to

$$N_{J_\alpha}(X,Y) = (\nabla_{J_\alpha X} J_\alpha)Y - (\nabla_{J_\alpha Y} J_\alpha)X + (\nabla_X J_\alpha)J_\alpha Y - (\nabla_Y J_\alpha)J_\alpha X. \tag{30}$$

The Nijenhuis tensor field corresponding to the (α, p)-golden structure $\Phi := \Phi_{\alpha,p}$ is given by the equality

$$N_\Phi(X,Y) := \Phi^2[X,Y] + [\Phi X, \Phi Y] - \Phi[\Phi X, Y] - \Phi[X, \Phi Y]. \tag{31}$$

Thus, from the identity (31), we obtain

$$N_\Phi(X,Y) = (\nabla_{\Phi X}\Phi)Y - (\nabla_{\Phi Y}\Phi)X - \Phi(\nabla_X \Phi)Y + \Phi(\nabla_Y \Phi)X, \tag{32}$$

for any $X, Y \in \Gamma(T\overline{M})$. Moreover, from identities (28), (30) and (32), we obtain

$$N_\Phi(X,Y) = \frac{5p^2}{4} N_{J_\alpha}(X,Y). \tag{33}$$

Recall that a structure J on a differentiable manifold is *integrable* if the Nijenhuis tensor field N_J corresponding to the structure J vanishes identically (i.e., $N_J = 0$). We point out that necessary and sufficient conditions for the integrability of a polynomial structure whose characteristic polynomial has only simple roots were given in [20].

For an *integrable almost (α, p)-golden structure* (i.e., $N_{\Phi_{\alpha,p}} = 0$), we drop the adjective *"almost"* and then simply call it an *(α, p)-golden structure*. From Equation (6), it is found that $\Phi_{\alpha,p}$ is integrable if and only if the associated almost α structure J_α is integrable. The distribution $\mathcal{D}_1$ is integrable if $Q_{\alpha,p}\left[P_{\alpha,p}X, P_{\alpha,p}Y\right] = 0$ and also analogous, the distribution $\mathcal{D}_2$ is integrable if $P_{\alpha,p}\left[Q_{\alpha,p}X, Q_{\alpha,p}Y\right] = 0$, for any $X, Y \in \Gamma(T\overline{M})$.

Let us consider now the second fundamental form Ω, which is a 2-form on $(\overline{M}, J_\alpha, \overline{g})$, where J_α is an α structure defined in Equation (4) and the metric $\overline{g}$ is (α, J_α)-compatible. The 2-form Ω is defined as follows:

$$\Omega(X,Y) := \overline{g}(J_\alpha X, Y), \tag{34}$$

for any $X, Y \in \Gamma(T\overline{M})$. From Equaitons (17) and (34), we obtain the following property:

Proposition 5. *If $\overline{M}$ is a Riemannian manifold endowed by an α structure J_α and the metric $\overline{g}$, which is (α, J_α)-compatible, then for any $X, Y \in \Gamma(T\overline{M})$, we have*

$$\Omega(X,Y) = \alpha\Omega(Y,X). \tag{35}$$

By using the correspondence between $\Phi_{\alpha,p}$ and J_α given in the identities (6) and (7), we obtain the following Lemma:

Lemma 2. *If $(\overline{M}, \Phi_{\alpha,p}, \overline{g})$ is an almost (α, p)-golden Riemannian manifold, then*

$$\Omega(X,Y) = \pm \frac{2}{p\sqrt{5}}[\overline{g}(\Phi_{\alpha,p}X, Y) - \frac{p}{2}\overline{g}(X,Y)], \tag{36}$$

$$\Omega(\Phi_{\alpha,p}X, Y) = \frac{p}{2}\Omega(X,Y) + \frac{p\alpha\sqrt{5}}{2}\overline{g}(X,Y), \tag{37}$$

for any $X, Y \in \Gamma(T\overline{M})$.

Hence, by inverting $X \leftrightarrow Y$ in Equation (37), we obtain

$$\Omega(\Phi_{\alpha,p}Y, X) = \frac{p}{2}\Omega(Y,X) + \frac{p\alpha\sqrt{5}}{2}\overline{g}(X,Y). \tag{38}$$

Using the identity (35) in the equality (38) and multiplying by $\alpha = \pm 1$, we obtain

$$\Omega(X, \Phi_{\alpha,p}Y) = \frac{p}{2}\Omega(X,Y) + \frac{p\sqrt{5}}{2}\overline{g}(X,Y). \tag{39}$$

Proposition 6. *Let $(\overline{M}, \Phi_{\alpha,p}, \overline{g})$ be an almost (α, p)-golden Riemannian manifold. Then, we have*

$$\Omega(\Phi_{\alpha,p}X, Y) - \Omega(X, \Phi_{\alpha,p}Y) = \frac{p(\alpha-1)\sqrt{5}}{2}\overline{g}(X,Y), \tag{40}$$

$$\Omega(\Phi_{\alpha,p}X, Y) + \Omega(X, \Phi_{\alpha,p}Y) = p\Omega(X,Y) + \frac{p(\alpha+1)\sqrt{5}}{2}\overline{g}(X,Y), \tag{41}$$

for any $X, Y \in \Gamma(T\overline{M})$.

Remark 6. *Let $(\overline{M}, \Phi_{\alpha,p}, \overline{g})$ be an almost (α, p)-golden Riemannian manifold. In particular, for any $X, Y \in \Gamma(T\overline{M})$, we have the following:*
(1) For $\alpha = 1$, we have

$$\Omega(\Phi_{1,p}X, Y) = \Omega(X, \Phi_{1,p}Y) = \frac{p}{2}\Omega(X,Y) + \frac{p\sqrt{5}}{2}\overline{g}(X,Y) \tag{42}$$

(2) For $\alpha = -1$, we have

$$\Omega(\Phi_{-1,p}X, Y) + \Omega(X, \Phi_{-1,p}Y) = p\Omega(Y,X). \tag{43}$$

Lemma 3. *Let $\overline{M}$ be a Riemannian manifold endowed with an α structure J_α and the metric $\overline{g}$, which is (α, J_α)-compatible. Then, for any $X, Y, Z \in \Gamma(T\overline{M})$, we obtain*

$$\overline{g}((\nabla_X J_\alpha)Y, Z) = \alpha\overline{g}(Y, (\nabla_X J_\alpha)Z). \tag{44}$$

Also, from Equations (28) and (44), we obtain the following:

Proposition 7. *If $(\overline{M}, \Phi_{\alpha,p}, \overline{g})$ is an almost (α, p)-golden Riemannian manifold, then for any $X, Y, Z \in \Gamma(T\overline{M})$, the structure $\Phi_{\alpha,p}$ satisfies*

$$\overline{g}((\nabla_X \Phi_{\alpha,p})Y, Z) = \alpha\overline{g}(Y, (\nabla_X \Phi_{\alpha,p})Z). \tag{45}$$

4. Submanifolds in the Almost (α, p)-Golden Riemannian Manifold

In this section, we assume that M is a $2n$-dimensional submanifold isometrically immersed in a $2m$-dimensional almost (α, p)-golden Riemannian manifold $(\overline{M}, \Phi_{\alpha,p}, \overline{g})$. We study some properties of the submanifold M in the almost (α, p)-golden Riemannian geometry regarding the structure induced by the given $\Phi_{\alpha,p}$ structure.

We shall denote with $\Gamma(TM)$ the set of smooth sections of TM. Let us denote with T_xM (and with $T_x^{\perp}M$) the tangent space (and the normal space) of M in a given point $x \in M$. For any $x \in \overline{M}$, we have the direct sum decomposition:

$$T_x\overline{M} = T_xM \oplus T_x^{\perp}M.$$

If g is the induced Riemannian metric on M, then it is given by $g(X,Y) = \overline{g}(i_*X, i_*Y)$ for any $X, Y \in \Gamma(TM)$, where i_* is the differential of the immersion $i : M \to \overline{M}$. We shall assume that all of the immersions are injective. In the rest of the paper, we shall denote with X the vector field i_*X for any $X \in \Gamma(TM)$ in order to simplify the notations.

From Equations (17) and (18), we remark that the induced metric on the submanifold M verifies the following equalities:

$$(i)\ g(J_\alpha X, Y) = \alpha g(X, J_\alpha Y), \quad (ii)\ g(J_\alpha X, J_\alpha Y) = g(X,Y), \tag{46}$$

for any $X, Y \in \Gamma(TM)$.

The decomposition into the tangential and normal parts of $\Phi_{\alpha,p}X$ and $\Phi_{\alpha,p}V$ for any $X \in \Gamma(TM)$ and $U \in \Gamma(T^{\perp}M)$ is given by

$$(i)\ \Phi_{\alpha,p}X = \mathcal{T}X + \mathcal{N}X, \quad (ii)\ \Phi_{\alpha,p}U = \mathfrak{t}U + \mathfrak{n}U, \tag{47}$$

where $\mathcal{T} : \Gamma(TM) \to \Gamma(TM)$, $\mathcal{N} : \Gamma(TM) \to \Gamma(T^{\perp}M)$, $\mathfrak{t} : \Gamma(T^{\perp}M) \to \Gamma(TM)$ and $\mathfrak{n} : \Gamma(T^{\perp}M) \to \Gamma(T^{\perp}M)$.

In the next considerations, we denote with $\overline{\nabla}$ and ∇ the Levi-Civita connections on $(\overline{M}, \overline{g})$ and on the submanifold (M, g), respectively.

The Gauss and Weingarten formulas are given by the respective equalities

$$(i)\ \overline{\nabla}_X Y = \nabla_X Y + h(X,Y), \quad (ii)\ \overline{\nabla}_X U = -A_U X + \nabla_X^{\perp} U, \tag{48}$$

for any tangent vector fields $X, Y \in \Gamma(TM)$ and any normal vector field $U \in \Gamma(T^{\perp}M)$, where h is the second fundamental form and A_U is the shape operator of M with respect to U, while $\nabla^{\perp}$ is the normal connection to the normal bundle $\Gamma(T^{\perp}M)$. Furthermore, the second fundamental form h and the shape operator A_U are related as follows:

$$\overline{g}(h(X,Y), U) = \overline{g}(A_U X, Y), \tag{49}$$

for any $X, Y \in \Gamma(TM)$ and $U \in \Gamma(T^{\perp}M)$.

For the α structure J_α, the decompositions into tangential and normal parts of $J_\alpha X$ and $J_\alpha U$ for any $X \in \Gamma(TM)$ and $U \in \Gamma(T^{\perp}M)$ are given by the respective formulas

$$(i)\ J_\alpha X = fX + \omega X, \quad (ii)\ J_\alpha U = BU + CU, \tag{50}$$

where $f : \Gamma(TM) \to \Gamma(TM), fX := (J_\alpha X)^T$, $\omega : \Gamma(TM) \to \Gamma(T^{\perp}M)$, $\omega X := (J_\alpha X)^{\perp}$, $B : \Gamma(T^{\perp}M) \to \Gamma(TM)$, $BU := (J_\alpha U)^T$ and $C : \Gamma(T^{\perp}M) \to \Gamma(T^{\perp}M)$, $CU := (V)^{J_\alpha \perp}$.

Direct calculus shows that the maps f, ω, B and C satisfy the following identity:

$$(i)\ \overline{g}(fX, Y) = \alpha \overline{g}(X, fY), \quad (ii)\ \overline{g}(CU, V) = \alpha \overline{g}(U, CV) \tag{51}$$

$$\overline{g}(\omega X, V) = \alpha \overline{g}(X, BV), \tag{52}$$

for any $X, Y \in \Gamma(TM)$ and $U, V \in \Gamma(T^{\perp}M)$. Using Equation (47), we obtain the following lemma:

Lemma 4. *Let $(\overline{M}, \overline{g})$ be a Riemannian manifold endowed with an α structure J_α, and let $\Phi_{\alpha,p}$ be the almost (α, p)-golden structure related by J_α through the relationships in Equation (6). Thus, we obtain*

$$(i)\ \mathcal{T}X = \frac{p}{2}X \pm \frac{\sqrt{5\alpha}}{2}fX, \quad (ii)\ \mathcal{N}X = \pm\frac{\sqrt{5\alpha}}{2}\omega X \tag{53}$$

$$(i)\ \mathfrak{t}V = \pm\frac{\sqrt{5\alpha}}{2}BV, \quad (ii)\ \mathfrak{n}V = \frac{p}{2}V \pm \frac{\sqrt{5\alpha}}{2}CV, \tag{54}$$

for any $X \in \Gamma(TM)$ and $V \in \Gamma(T^{\perp}M)$.

Now, by using Equations (53) and (54) in the Equations (51) and (52), respectively, we obtain the following property:

Proposition 8. *Let $(\overline{M}, \overline{g})$ be a Riemannian manifold endowed with an almost (α, p)-golden structure. Thus, for any $X, Y \in \Gamma(TM)$, the maps $\mathcal{T}$ and $\mathfrak{n}$ satisfy*

$$\overline{g}(\mathcal{T}X, Y) = \alpha\overline{g}(X, \mathcal{T}Y) + \frac{p(1-\alpha)}{2}g(X, Y), \tag{55}$$

$$\overline{g}(\mathfrak{n}U, V) = \alpha\overline{g}(U, \mathfrak{n}V) + \frac{p(1-\alpha)}{2}g(U, V). \tag{56}$$

Moreover, for any $U, V \in \Gamma(T^{\perp}M)$, $\mathcal{N}$ and $\mathfrak{t}$ satisfy

$$\overline{g}(\mathcal{N}X, U) = \alpha\overline{g}(X, \mathfrak{t}U). \tag{57}$$

Definition 6. *The covariant derivatives of the tangential and normal parts of $\Phi_{\alpha,p}X$ (and $\Phi_{\alpha,p}V$) are given by*

$$(i)\ (\nabla_X\mathcal{T})Y = \nabla_X\mathcal{T}Y - \mathcal{T}(\nabla_XY), \quad (ii)\ (\overline{\nabla}_X\mathcal{N})Y = \nabla^{\perp}_X\mathcal{N}Y - \mathcal{N}(\nabla_XY), \tag{58}$$

$$(i)\ (\nabla_X\mathfrak{t})U = \nabla_X\mathfrak{t}U - \mathfrak{t}(\nabla^{\perp}_XU), \quad (ii)\ (\overline{\nabla}_X\mathfrak{n})U = \nabla^{\perp}_X\mathfrak{n}U - \mathfrak{n}(\nabla^{\perp}_XU), \tag{59}$$

for any X, $Y \in \Gamma(TM)$ and $U \in \Gamma(T^{\perp}M)$.

Remark 7. *Let M be an isometrically immersed submanifold of a Riemannian manifold $(\overline{M}, \overline{g})$ endowed by a J_α structure and a $\Phi_{(\alpha,p)}$-golden structure. Then, for any $X, Y, Z \in \Gamma(TM)$, we obtain*

$$(i)\ \overline{g}((\nabla_Xf)Y, Z) = \alpha\overline{g}(Y, (\nabla_Xf)Z), \quad (ii)\ \overline{g}((\nabla_X\mathcal{T})Y, Z) = \alpha\overline{g}(Y, (\nabla_X\mathcal{T})Z). \tag{60}$$

The identities (60) result from Equations (51)(i) and (53)(i).

Let M a submanifold of co-dimension $2r$ in $\overline{M}$. We fix a local orthonormal basis $\{N_1, ..., N_{2r}\}$ of the normal space $T^{\perp}_xM$ for any $x \in \overline{M}$. Hereafter, we assume that the indices i, j and k run over the range $\{1, ..., 2r\}$.

Let $\Phi_{\alpha,p} := \Phi$ be the almost (α, p)-golden structure. Then, we obtain the decomposition

$$(i)\ \Phi X = \mathcal{T}X + \sum_{i=1}^{2r} u_i(X)N_i, \ (ii)\ \Phi N_i = \xi_i + \sum_{j=1}^{2r} \mathcal{A}_{ij}N_j, \tag{61}$$

for any $X \in T_xM$, where ξ_i represents the vector fields on M, u_i represents the 1-forms on M and $\mathcal{A} := (\mathcal{A}_{ij})_{2r}$ is a $2r \times 2r$ matrix of smooth real functions on M.

Moreover, from Equations (47) and (61), we remark that

$$\mathcal{N}X = \sum_{i=1}^{2r} u_i(X)N_i, \tag{62}$$

for any $X \in T_xM$ and

$$(i)\ \mathfrak{t}N_i = \xi_i, \quad (ii)\ \mathfrak{n}N_i = \sum_{k=1}^{2r} \mathcal{A}_{ik}N_k. \tag{63}$$

Therefore, we find the structure $\Sigma = (\mathcal{T}, g, u_i, \xi_i, \mathcal{A})$ on the submanifold M through $\Phi_{\alpha,p}$, and we shall obtain a characterization of the structure induced on a submanifold M by the almost (α, p)-golden structure in a similar manner to that in Theorem 3.1. from [15].

Theorem 1. *The structure $\Sigma = (\mathcal{T}, g, u_i, \xi_i, \mathcal{A})$ induced on the submanifold M by the almost (α, p)-golden structure $\Phi_{\alpha,p}$ on $\overline{M}$ satisfies the following equalities:*

$$\mathcal{T}^2X = p\mathcal{T}X + \frac{5\alpha - 1}{4}p^2X - \sum_{i=1}^{2r} u_i(X)\xi_i, \tag{64}$$

$$\mathcal{A}_{ij} = \alpha\mathcal{A}_{ji} + \frac{p(1-\alpha)}{2}\delta_{ij}, \tag{65}$$

$$u_i(X) = \alpha g(X, \xi_i), \tag{66}$$

$$\mathcal{T}\xi_i = p\xi_i - \sum_{j=1}^{2r} \mathcal{A}_{ij}\xi_j, \tag{67}$$

$$u_j(\xi_i) = \frac{5\alpha - 1}{4}p^2\delta_{ij} + p\mathcal{A}_{ij} - \sum_{k=1}^{2r} \mathcal{A}_{ik}\mathcal{A}_{kj}, \tag{68}$$

for any $X \in \Gamma(TM)$, where $\mathcal{T}$ is a (1,1)-tensor field on M, ξ_i represents the tangent vector fields on M, u_i represents the 1-form M and the matrix $\mathcal{A}$ is determined by its entries $\mathcal{A}_{ij}$, which are real functions on M (for any $i, j \in \{1, \ldots, 2r\}$).

Proof. Using $\Phi_{\alpha,p} := \Phi$ in the identity (47)(i) and (5), we obtain $p\Phi X + \frac{5\alpha-1}{4}p^2 \cdot X = \Phi\mathcal{T}X + \Phi\mathcal{N}X$. Moreover, using identities (47)(i) and (61)(i), we obtain

$$p\mathcal{T}X + p\sum_{i=1}^{2r} u_i(X)N_i + \frac{5\alpha - 1}{4}p^2 \cdot X = \mathcal{T}^2X + \mathcal{N}\mathcal{T}X + \sum_{i=1}^{2r} u_i(X)\Phi N_i \tag{69}$$

By using the identity (62) and equalizing the tangential part of the identity (69), we obtain equality (64).

Now, using the identity (56), we obtain

$$\overline{g}(\mathfrak{n}N_i, N_j) = \alpha\overline{g}(N_i, \mathfrak{n}N_j) + \frac{p(1-\alpha)}{2}g(N_i, N_j)$$

and from the equality (63)(ii), we obtain the identity (65).

From the identity (57), we obtain $\overline{g}(\mathcal{N}X, N_j) = \alpha\overline{g}(X, \mathfrak{t}N_j)$ and by using identities (62) and (63)(i), we obtain the equality (66).

From the Equation (5), we obtain $\Phi^2N_i = p\Phi N_i + \frac{5\alpha-1}{4}p^2 \cdot N_i$ and from the identity (61)(ii), we obtain

$$\Phi(\xi_i + \sum_{j=1}^{2r} \mathcal{A}_{ij}N_j) = p(\xi_i + \sum_{j=1}^{2r} \mathcal{A}_{ij}N_j) + \frac{5\alpha - 1}{4}p^2 \cdot N_i,$$

Moreover, using identities (61)(i) and (61)(ii), we obtain

$$\mathcal{T}\xi_i + \sum_{j=1}^{2r} u_j(\xi_i)N_j + \sum_{j=1}^{2r} \mathcal{A}_{ij}(\xi_j + \sum_{k=1}^{2r} \mathcal{A}_{jk}N_k) = p\xi_i + p\sum_{j=1}^{2r} \mathcal{A}_{ij}N_j + \frac{5\alpha-1}{4}p^2 \cdot N_i.$$

When comparing the tangential and normal parts of both sides of this last equality, respectively, we infer the identities (67) and (68). □

By using identities (61)(i) and (61)(ii), we obtain the following remark:

Remark 8. *If $(\overline{M}, \Phi, \overline{g})$ is an almost (α, p)-golden Riemannian manifold and $X, Y \in \Gamma(TM)$, then for any $i, j \in \{1, \ldots, 2r\}$, we obtain*

$$\overline{g}(\Phi X, \Phi Y) = g(\mathcal{T}X, \mathcal{T}Y) + \sum_{i=1}^{2r} u_i(X)u_i(Y), \tag{70}$$

$$\overline{g}(\Phi N_i, \Phi N_j) = g(\xi_i, \xi_j) + \sum_{k=1}^{2r} \mathcal{A}_{ik}\mathcal{A}_{kj}, \tag{71}$$

If M is an invariant submanifold of $\overline{M}$ (i.e., $\Phi(T_xM) \subset T_xM$ and $\Phi(T_x^{\perp}M) \subset T_x^{\perp}M$ for all $x \in M$), then from identities (61), we obtain $\Phi X = \mathcal{T}X$, which implies $u_i(X) = 0$ and $\xi_i = 0$ for any $i \in \{1, 2, \ldots, 2r\}$. Therefore, using the identities (64) and (68), we obtain the following property:

Proposition 9. *Let M be an invariant submanifold of co-dimension $2r$ of the almost (α, p)-golden Riemannian manifold $(\overline{M}, \Phi, \overline{g})$, and let $\Sigma = (\mathcal{T}, g, u_i = 0, \xi_i = 0, \mathcal{A})$ be the structure induced on the submanifold M. Then, $\mathcal{T}$ is an (α, p)-golden structure on M; in other words, we have*

$$\mathcal{T}^2X = p\mathcal{T}X + \frac{5\alpha-1}{4}p^2X, \tag{72}$$

for any $X \in \Gamma(TM)$, where p is a real nonzero number and $\alpha \in \{-1, 1\}$. Moreover, the quadratic matrix $\mathcal{A}$ satisfies the equality

$$\mathcal{A}^2 = p\mathcal{A} + \frac{5\alpha-1}{4}p^2I_{2r}, \tag{73}$$

where its entries $\mathcal{A}_{ij}$ are real functions on M ($i, j \in \{1, \ldots, 2r\}$) and I_{2r} is an identical matrix of the order $2r$.

Theorem 2. *A necessary and sufficient condition for the invariance of a submanifold M of co-dimension $2r$ in a $2m$-dimensional Riemannian manifold $(\overline{M}, \overline{g})$ endowed with an almost (α, p)-golden structure Φ is that the structure $\mathcal{T}$ on (M, g) is also an almost (α, p)-golden structure.*

Proof. If $\mathcal{T}$ is an almost (α, p)-golden structure, then from Equation (64), we obtain

$$\sum_{i=1}^{2r} u_i(X)\xi_i = 0, \tag{74}$$

for any $X \in \Gamma(TM)$. By taking the g product with X in Equation (74), we infer that

$$\sum_{i=1}^{2r} u_i(X)g(X, \xi_i) = \sum_i (u_i(X))^2 = 0,$$

which is equivalent to $u_i(X) = 0$ for every $i \in \{1, \ldots, 2r\}$, and this fact implies that M is invariant.

Conversely, if M is an invariant submanifold, then from Equation (72), we obtain that the structure $\mathcal{T}$ on (M, g) is also an almost (α, p)-golden structure. $\square$

5. Conclusions

The world of quadratic endomorphisms of a given manifold is enriched now with a new class. If a Riemmanian metric is added through a compatibility condition, then a new geometry is developed. Its submanifolds also carry remarkable structures, and new studies are expected to enrich this domain of differential geometry.

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