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Article

On the Unique Solvability of Inverse Problems of Magnetometry and Gravimetry

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Abstract: This article deals with the question of the unique solvability of systems of linear algebraic equations, to the solution of which many inverse problems of geophysics are reduced as a result of discretization when applying the methods of integral equations or integral representations. Examples are given of degenerate and nondegenerate systems of different dimensions that arise in the processing of magnetometric and gravimetric data from experimental observations. Conclusions are drawn about the methods for constructing the optimal grid of experimental observation points.

Keywords: degenerate system; integral representations; unique solvability

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1. Introduction

When solving direct and inverse problems of magnetometry and gravimetry, methods based on integral representations of physical fields are used in many cases. Such representations allow one to find the values of some functions by approximately determined values of some other functions and, thereby, calculate the parameters of the geological environment (magnetic susceptibility, rock density, acoustic stiffness coefficients, etc.); build analytical continuations of fields; separate fields; analyze the spectrum signals; etc. [1]. Almost all statements of inverse problems for the described class of problems, as a result of discretization, are reduced to the need to solve systems of linear algebraic equations.

The processing of magnetometric and/or gravimetric data from a mathematical point of view is either the solution of Fredholm integral equations of the first kind with convolution-type kernels, or the solution of boundary value problems for the Laplace and Poisson equations. But in this form it is possible to represent the process of finding the sources of magnetic and/or gravitational fields only in the case when the localization of the sources of the fields is known, and only the density of their distribution within the study area is subject to restoration. In this case, it is assumed that the values of the field components are measured in the outer region of space in relation to the region under study. In this case, mathematical formulations arise related to the determination of externally equivalent mass sources (gravitational and/or magnetic) [2–8]. In recent years, the so-called STAR method [9] has become a popular and effective method for magnetic data inversion.

When solving linear inverse problems, continuous statements are discretized and reduced to systems of linear algebraic equations (SLAEs) with both the right side and the matrix approximately determined. If we are talking about nonlinear inverse problems, then in the general case it is impossible to reduce the formulation of the problem to the solution of a system of linear algebraic equations. However, one can try to investigate the properties of a certain system of linear algebraic equations regarding the density of gravitating masses or components of the magnetization vector in a dia- and paramagnetic medium. In this case, the elements of the matrix of such a system are considered as nonlinear functions of the parameters of the geological environment, i.e., the three-dimensional geometry of the sources is taken into account.

In this paper, we formulate some sufficient conditions for the degeneracy of SLAEs that arise when solving geophysical problems of an interpretative nature and give examples of SLAEs that have a unique solution. Our approach is both analytical and constructive: we aimed to create methods for constructing an optimal (in some sense) network of sensors that perform measurements of experimental data, guaranteeing the nondegeneracy of the SLAE.

Systems of linear algebraic equations, to which geophysical problems are reduced, as a rule, have large and super-large dimensions (tens and hundreds of thousands, or even millions, of unknowns) [10]. But at present, due to the rapid growth of computing power and the development of supercomputer technologies, block methods for solving SLAEs are widely used [11]. Therefore, it is highly important to study the properties of individual blocks, which, like "bricks", form huge matrices that carry information about the sources of the field. The nondegeneracy of the composite matrix block makes it possible to increase the stability of the approximate solution of the inverse problem to input data errors. Our computational experiments have shown that block methods for solving SLAEs based on the regularization of the Cholesky matrix decomposition method are effective precisely in the case of the nondegeneracy of individual (mainly diagonal) blocks [10]. As for the system as a whole, the question of the necessary and sufficient conditions for its nondegeneracy depending on the geometry of the field sources is still open: the elements of the SLAE matrices arising in the framework of the method of integral equations are neither symmetric nor skew-symmetric in the general case. If we consider formulations within the framework of the method of linear integral representations, then the matrices begin to show specific properties—they are necessarily symmetric and positive semidefinite and can be reduced to the form of doubly stochastic matrices (when the sums of all elements of each column and each row are equal to 1). But even in this case, it is very difficult to estimate its rank for an arbitrary matrix dimension. Therefore, in this work we will try to formulate approaches to a partial solution of this problem.

Thus, the novelty of this work is as follows. For the first time, questions of the unique solvability of SLAEs are considered, to which the linear inverse problems of magnetometry and gravimetry are reduced. At the same time, it should be emphasized that in the case of recovering the parameters of magnetic field sources (the first part of the article—Sections 2 and 3), SLAEs arise when discretizing the "continuous" statement of the problem, which, as a rule, cannot be uniquely resolved. In the second part of the article (Sections 4 and 5), we study SLAEs that arise as a result of applying the method of linear integral representations. This method allows, according to the finite information about the studied continuously distributed carriers of magnetic or gravitational masses, i.e., according to the results of measuring the physical fields induced by them, the identification of the equivalents (in terms of external field distributions) of sources given on carriers, the shape and location of which are considered known (but can vary, and then a multi-parameter family of linear inverse problems arises). Previously, such problems have not been considered: as a rule, it was necessary to find only in a certain sense optimal solutions to inverse problems with certain properties (the mass densities must be smooth functions in the support region, take given values at the boundary, etc.).

The structure of this work is as follows. Section 2 formulates the statement of the inverse problem of magnetometry using the method of integral equations. The corresponding purpose of the article is to determine the conditions under which the systems of linear algebraic equations that arise in solving typical geophysical problems of magnetometry are solvable. In Section 3 the corresponding theorems are formulated and proved both for special cases and for a fairly general case. Section 4 formulates the statement of the inverse problem using the method of linear integral representations, which is applicable for solving inverse problems of both magnetometry and gravimetry. It is demonstrated how the corresponding statement of the inverse problem is reduced to the need to solve a system of linear algebraic equations. In Section 5, theorems are formulated on the solvability of such systems of equations for a particular case that is quite common in practice. Conclusions are also drawn about ways to construct the optimal geometry of a grid of observation points.

2. Statement of the Inverse Problem of Magnetometry

The expression for the magnetic field under the assumption that the medium is paramagnetic [12,13] has the form

$$\boldsymbol{B}(\boldsymbol{r}_s) = \frac{\mu_0}{4\pi} \iiint\limits_V \mathbf{K}(\boldsymbol{r}_s, \boldsymbol{r}) \boldsymbol{M}(\boldsymbol{r}) dv. \quad (1)$$

Here, $\boldsymbol{B}(\boldsymbol{r}_s)$ is a vector function that characterizes the magnetic field induction at a point with radius vector $\boldsymbol{r}_s = (x_s, y_s, z_s)$; $\boldsymbol{M}(\boldsymbol{r})$ is a vector function characterizing the density of the magnetic moment of the elementary volume dv in a small neighborhood of the point $\boldsymbol{r} = (x, y, z)$ of a region V; μ_0 is a magnetic constant; and $\mathbf{K}(\boldsymbol{r}_s, \boldsymbol{r})$ is a matrix function defining the kernel of the integral Equation (1) that has the form

$$\mathbf{K}(\boldsymbol{r}_s, \boldsymbol{r}) = \frac{1}{r^5} \begin{bmatrix} 3(x_s - x)^2 - r^2 & 3(x_s - x)(y_s - y) & 3(x_s - x)(z_s - z) \\ 3(y_s - y)(x_s - x) & 3(y_s - y)^2 - r^2 & 3(y_s - y)(z_s - z) \\ 3(z_s - z)(x_s - x) & 3(z_s - z)(y_s - y) & 3(z_s - z)^2 - r^2 \end{bmatrix},$$

where

$$r = |\boldsymbol{r} - \boldsymbol{r}_s| = \sqrt{(x_s - x)^2 + (y_s - y)^2 + (z_s - z)^2}.$$

Thus, the **inverse problem** is to define the function $\boldsymbol{M}(\boldsymbol{r})$, $\boldsymbol{r} \in V$, from Equation (1) according to experimental data $\boldsymbol{B}(\boldsymbol{r}_s)$, $s = \overline{1, N}$.

The representation of the magnetic field in the form (1) is often used in the interpretation of data obtained by various satellite missions in the study of planets in the solar system (see, for example, [5,10,14–27]). However, the same idea is also applicable to the consideration of small areas of polygons on the Earth's surface when conducting geophysical studies. In this case, the surface of the Earth can be considered flat, and the planet itself the lower half-space. Statements of inverse problems under such assumptions are called "local" in what follows. The sphericity of the Earth is usually not taken into account if the linear size of the polygon does not exceed 111 km, which corresponds to approximately one degree in latitude.

At the same time, in practice, researchers often use the formula for expressing the magnetic field created by one magnetic dipole with a magnetic moment $\boldsymbol{m} = (m_x, m_y, m_z)^T$, which is located at a point with the radius vector $\boldsymbol{r}_d = (x_d, y_d, z_d)$. In this case, the expression for the magnetic field has the form

$$\boldsymbol{B}(\boldsymbol{r}_s) = \frac{\mu_0}{4\pi} \left(\frac{3\big(\boldsymbol{m}(\boldsymbol{r}_d), \boldsymbol{r}_s - \boldsymbol{r}_d\big)(\boldsymbol{r}_s - \boldsymbol{r}_d)}{|\boldsymbol{r}_s - \boldsymbol{r}_d|^5} - \frac{\boldsymbol{m}(\boldsymbol{r}_d)}{|\boldsymbol{r}_s - \boldsymbol{r}_d|^3} \right). \quad (2)$$

We assume that there is a set of "sensors" s_j, $j = \overline{1, N}$, each of which is located at a point with coordinates $\boldsymbol{r}_{s_j} = (x_{s_j}, y_{s_j}, z_{s_j})$ and measures at this point the magnetic field induction $\boldsymbol{B}(\boldsymbol{r}_{s_j}) = \left(B_x^{(s_j)}, B_y^{(s_j)}, B_z^{(s_j)}\right)^T$. At each point $\boldsymbol{r}_{s_j}$, the field is induced by a set of magnetic dipoles d_i, $i = \overline{1, N}$, each of which is located at a point with coordinates $\boldsymbol{r}_{d_i} = (x_{d_i}, y_{d_i}, z_{d_i})$ and has a magnetic moment $\boldsymbol{m}(\ \boldsymbol{r}_{d_i}) = \left(m_x^{(d_i)}, m_y^{(d_i)}, m_z^{(d_i)}\right)^T$.

For the convenience of presenting the formulas below, along with the above indexing, we use the following form (under the condition that it does not lead to contradictions in the notation):

$$\{s_j\}\Big|_{j=\overline{1,N}} \equiv \{s_1, s_2, \dots, s_N\} \quad \leftrightarrow \quad \{s\}\Big|_{s=\overline{1,N}} \equiv \{1, 2, \dots, N\},$$
$$\{d_i\}\Big|_{i=\overline{1,N}} \equiv \{d_1, d_2, \dots, d_N\} \quad \leftrightarrow \quad \{d\}\Big|_{d=\overline{1,N}} \equiv \{1, 2, \dots, N\}.$$

Thus, we assume that at the observation points $\boldsymbol{r}_s = (x_s, y_s, z_s)$, $s = \overline{1,N}$, the magnetic field $\boldsymbol{B}(\boldsymbol{r}_s) \equiv \boldsymbol{B}^{(s)} = \left(B_x^{(s)}, B_y^{(s)}, B_z^{(s)}\right)^T$, $s = \overline{1,N}$ is measured, which is induced by magnetic dipoles $\boldsymbol{m}(\boldsymbol{r}_d) \equiv \boldsymbol{m}^{(d)} = \left(m_x^{(d)}, m_y^{(d)}, m_z^{(d)}\right)^T$, $d = \overline{1,N}$, located at points with coordinates $\boldsymbol{r}_d = (x_d, y_d, z_d)$, $d = \overline{1,N}$. As a result, the question arises of the unique solvability of the following system of linear algebraic equations in order to determine the values of the magnetic moments $\boldsymbol{m}(\boldsymbol{r}_d) \equiv \boldsymbol{m}^{\,(d)} = \left(m_x^{(d)}, m_y^{(d)}, m_z^{(d)}\right)^T$, $d = \overline{1,N}$:

$$\begin{cases} \sum\limits_{d=1}^{N} \left(\dfrac{3\left(m_x^{(d)} x_{sd} + m_y^{(d)} y_{sd} + m_z^{(d)} z_{sd}\right) x_{sd}}{r_{sd}^5} - \dfrac{m_x^{(s)}}{r_{sd}^3} \right) = B_x^{(s)}, \quad s = \overline{1,N}, \\ \sum\limits_{d=1}^{N} \left(\dfrac{3\left(m_x^{(d)} x_{sd} + m_y^{(d)} y_{sd} + m_z^{(d)} z_{sd}\right) y_{sd}}{r_{sd}^5} - \dfrac{m_y^{(s)}}{r_{sd}^3} \right) = B_y^{(s)}, \quad s = \overline{1,N}, \\ \sum\limits_{d=1}^{N} \left(\dfrac{3\left(m_x^{(d)} x_{sd} + m_y^{(d)} y_{sd} + m_z^{(d)} z_{sd}\right) z_{sd}}{r_{sd}^5} - \dfrac{m_z^{(s)}}{r_{sd}^3} \right) = B_z^{(s)}, \quad s = \overline{1,N}. \end{cases} \tag{3}$$

Here, r_{sd} denotes the distance between the s-th sensor and the d-th dipole:

$$r_{sd} = \sqrt{(x_s - x_d)^2 + (y_s - y_d)^2 + (z_s - z_d)^2}.$$

The notations x_{sd}, y_{sd}, and z_{sd} have similar meanings:

$$x_{sd} = x_s - x_d, \quad y_{sd} = y_s - y_d, \quad z_{sd} = z_s - z_d.$$

Remark 1. *Note that system (3) also arises when the problem (1) is discretized. In this case,* $\mathbf{m} = \mathbf{M}\,dv$*, and when discretizing, the elementary volume* dv *of the domain* V *is replaced by a volume of a fixed value, to which the corresponding magnetic dipole is assigned.*

System (3) can be rewritten in block notation:

$$\underbrace{\begin{bmatrix} \mathbf{K}_{11} & \mathbf{K}_{12} & \dots & \mathbf{K}_{1N} \\ \mathbf{K}_{21} & \mathbf{K}_{22} & \dots & \mathbf{K}_{2N} \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{K}_{N1} & \mathbf{K}_{N2} & \dots & \mathbf{K}_{NN} \end{bmatrix}}_{\substack{\equiv \\ \mathbf{K}}} \times \begin{pmatrix} \boldsymbol{m}^{(1)} \\ \boldsymbol{m}^{(2)} \\ \vdots \\ \boldsymbol{m}^{(N)} \end{pmatrix} = \begin{pmatrix} \boldsymbol{B}^{(1)} \\ \boldsymbol{B}^{(2)} \\ \vdots \\ \boldsymbol{B}^{(N)} \end{pmatrix}. \tag{4}$$

Here, the block $\mathbf{K}_{sd}$ has the form

$$\mathbf{K}_{sd} = \frac{1}{r_{sd}^5} \begin{bmatrix} 2x_{sd}^2 - y_{sd}^2 - z_{sd}^2 & 3x_{sd}\,y_{sd} & 3x_{sd}\,z_{sd} \\ 3x_{sd}\,y_{sd} & 2y_{sd}^2 - x_{sd}^2 - z_{sd}^2 & 3y_{sd}\,z_{sd} \\ 3x_{sd}\,z_{sd} & 3y_{sd}\,z_{sd} & 2z_{sd}^2 - x_{sd}^2 - y_{sd}^2 \end{bmatrix}.$$

Thus, **the first purpose of this article** is to formulate conditions under which the system of linear algebraic equations of the form (3) or (4) is solvable.

3. Uniqueness Theorems in the Case of Solving the Inverse Problem of Magnetometry

This section deals with uniqueness theorems for special cases (see Sections 3.1–3.4) and for a fairly general case (see Section 3.5). Consider first the simplest case of two sensors and two magnetic dipoles.

3.1. Two Dipoles and Two Sensors on One Line

Let the sensors s_1 and s_2 and the dipoles d_1 and d_2 be located on the same straight line (we take this as the axis Ox of some Cartesian coordinate system). In this case, system (4) takes the form

$$\begin{bmatrix} \mathbf{K}_{11} & \mathbf{K}_{12} \\ \mathbf{K}_{21} & \mathbf{K}_{22} \end{bmatrix} \times \begin{pmatrix} \boldsymbol{m}^{(1)} \\ \boldsymbol{m}^{(2)} \end{pmatrix} = \begin{pmatrix} \boldsymbol{B}^{(1)} \\ \boldsymbol{B}^{(2)} \end{pmatrix}. \tag{5}$$

The expressions for blocks of system matrix (5) resemble the following:

$$\mathbf{K}_{11} = \begin{bmatrix} \frac{2}{r_{11}^3} & 0 & 0 \\ 0 & -\frac{1}{r_{11}^3} & 0 \\ 0 & 0 & -\frac{1}{r_{11}^3} \end{bmatrix}, \quad \mathbf{K}_{12} = \begin{bmatrix} \frac{2}{r_{12}^3} & 0 & 0 \\ 0 & -\frac{1}{r_{12}^3} & 0 \\ 0 & 0 & -\frac{1}{r_{12}^3} \end{bmatrix},$$
$$\mathbf{K}_{21} = \begin{bmatrix} \frac{2}{r_{21}^3} & 0 & 0 \\ 0 & -\frac{1}{r_{21}^3} & 0 \\ 0 & 0 & -\frac{1}{r_{21}^3} \end{bmatrix}, \quad \mathbf{K}_{22} = \begin{bmatrix} \frac{2}{r_{22}^3} & 0 & 0 \\ 0 & -\frac{1}{r_{22}^3} & 0 \\ 0 & 0 & -\frac{1}{r_{21}^3} \end{bmatrix}.$$

For convenience, let us rename the elements in the matrix $\mathbf{K}$ of system (5):

$$\mathbf{K} \equiv \begin{bmatrix} \mathbf{K}_{11} & \mathbf{K}_{12} \\ \mathbf{K}_{21} & \mathbf{K}_{22} \end{bmatrix} \equiv \begin{bmatrix} a_{11} & 0 & 0 & a_{12} & 0 & 0 \\ 0 & b_{11} & 0 & 0 & b_{12} & 0 \\ 0 & 0 & b_{11} & 0 & 0 & b_{12} \\ a_{21} & 0 & 0 & a_{22} & 0 & 0 \\ 0 & b_{21} & 0 & 0 & b_{22} & 0 \\ 0 & 0 & b_{21} & 0 & 0 & b_{22} \end{bmatrix}. \tag{6}$$

In order for the determinant of the matrix (6) to be equal to zero, it is necessary and sufficient that the coordinates of the sensors coincide (for different coordinates of the dipoles):

$$\begin{aligned} &\det K = 0 \Rightarrow a_{11}a_{22} - a_{12}a_{21} = (b_{11}b_{22} - b_{12}b_{21})^2 = 0 \Rightarrow \\ &\det K = 0 \Leftrightarrow b_{11}b_{22} - b_{12}b_{21} = 0 \Leftrightarrow x_{11}x_{22} = x_{12}x_{21} \Leftrightarrow \\ &(x_{s_1} - x_{d_1})(x_{s_2} - x_{d_2}) = (x_{s_1} - x_{d_2})(x_{s_2} - x_{d_1}) \Rightarrow \\ &x_{s_1}(x_{d_1} - x_{d_2}) = x_{s_2}(x_{d_1} - x_{d_2}) \Rightarrow (x_{s_1} - x_{s_2})(x_{d_1} - x_{d_2}) = 0. \end{aligned}$$

Thus, we prove Theorem 1.

Theorem 1. *The solution of system (5) is unique if two sensors and two dipoles are located at different points on the same straight line and the condition* $(x_{s_1} - x_{d_1})(x_{s_2} - x_{d_2}) = (x_{s_1} - x_{d_2})(x_{s_2} - x_{d_1})$ *is fulfilled, which corresponds to the case of "unseparated" dipoles (see Figure 1a); if* $(x_{s_1} - x_{d_1})(x_{s_2} - x_{d_2}) = (x_{s_1} - x_{d_2})(x_{d_1} - x_{s_2})$*, then the matrix system of equations can be degenerate, and the components of the two dipoles cannot be uniquely determined (dipoles and sensors "separate" each other, see Figure 1b).*

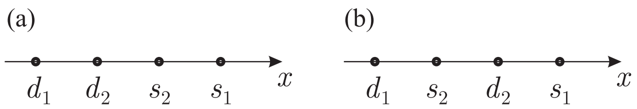

Figure 1. (**a**) The case in which dipoles and sensors "do not separate" each other; (**b**) the case in which dipoles and sensors "separate" each other.

3.2. Two Dipoles and Two Sensors Located in the Same Plane

Let us draw a straight line through two dipoles and place two sensors, s_1 and s_2, on some straight line in a plane perpendicular to the straight line containing the dipoles d_1 and d_2 and passing through the middle of the segment connecting the dipoles (see Figure 2). In this case, system (4) takes the form

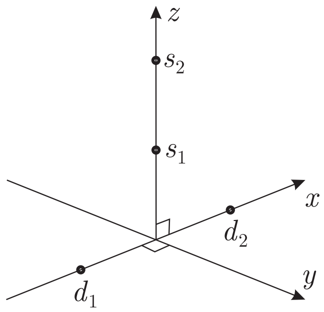

Figure 2. The second case of the mutual arrangement of sensors and dipoles in three-dimensional space.

$$\begin{bmatrix} \mathbf{K}_{11} & \mathbf{K}_{12} \\ \mathbf{K}_{21} & \mathbf{K}_{22} \end{bmatrix} \times \begin{pmatrix} \boldsymbol{m}^{(1)} \\ \boldsymbol{m}^{(2)} \end{pmatrix} = \begin{pmatrix} \boldsymbol{B}^{(1)} \\ \boldsymbol{B}^{(2)} \end{pmatrix}. \tag{7}$$

The expressions for the matrix blocks (7) resemble the following:

$$\begin{aligned}
\mathbf{K}_{11} &= \frac{1}{r_{11}^5} \begin{bmatrix} 2x_{11}^2 - z_{11}^2 & & 3x_{11}\,z_{11} \\ 0 & -r_{11}^2 & 0 \\ 3x_{11}\,z_{11} & 0 & 2z_{11}^2 - x_{11}^2 \end{bmatrix}, \\
\mathbf{K}_{12} &= \frac{1}{r_{sd}^5} \begin{bmatrix} 2x_{12}^2 - z_{12}^2 & 0 & 3x_{12}\,z_{12} \\ 0 & -r_{11}^2 & 0 \\ 3x_{12}\,z_{12} & 0 & 2z_{12}^2 - x_{12}^2 \end{bmatrix}, \\
\mathbf{K}_{21} &= \frac{1}{r_{21}^5} \begin{bmatrix} 2x_{21}^2 - z_{21}^2 & 0 & 3x_{21}\,z_{21} \\ 0 & -r_{11}^2 & 0 \\ 3x_{21}\,z_{21} & 0 & 2z_{21}^2 - x_{21}^2 \end{bmatrix}, \\
\mathbf{K}_{22} &= \frac{1}{r_{22}^5} \begin{bmatrix} 2x_{22}^2 - z_{22}^2 & 0 & 3x_{22}\,z_{22} \\ 0 & -r_{11}^2 & 0 \\ 3x_{22}\,z_{22} & 0 & 2z_{22}^2 - x_{22}^2 \end{bmatrix}.
\end{aligned} \tag{8}$$

The distances from the sensor located in this way to each of the two dipoles are the same.

Taking into account that $y_{1d} = y_{2d} = 0$, $d = \overline{1,2}$, and $r_{s1} = r_{s2}$, $s = \overline{1,2}$, we determine that in system matrix (7), two rows are proportional:

$$\begin{bmatrix} \mathbf{K}_{11} & \mathbf{K}_{12} \\ \mathbf{K}_{21} & \mathbf{K}_{22} \end{bmatrix} \equiv \begin{bmatrix} \square & \square & \square & \square & \square & \square \\ 0 & -\frac{1}{r_{11}^3} & 0 & 0 & -\frac{1}{r_{12}^3} & 0 \\ \square & \square & \square & \square & \square & \square \\ \square & \square & \square & \square & \square & \square \\ 0 & -\frac{1}{r_{21}^3} & 0 & 0 & -\frac{1}{r_{22}^3} & 0 \\ \square & \square & \square & \square & \square & \square \end{bmatrix}.$$

Thus, system (7) does not have unique solvability.

We obtain a similar result if we place two sensors at the same distance from the straight line connecting the two dipoles (it is not required that the distances from the sensor to each of the two dipoles be the same). Sensors s_1 and s_2 and dipoles d_1 and d_2 in Figure 2 change places. The quasi-solution in this case is also determined ambiguously.

Thus true:

Theorem 2. *The pseudo-solution of system (7) is determined ambiguously if the following two cases of the mutual arrangement of dipoles and sensors take place: (a) two dipoles and N sensors are located in the same plane, while the sensors lie on a straight line passing through the middle of the segment, connecting the dipoles and perpendicular to them; (b) two sensors and N dipoles lie in the* *same plane, and the dipoles are located on a straight line perpendicular to the segment connecting the two sensors and passing through the center of this segment.*

3.3. Two Dipoles and N Sensors Located on a Straight Line Perpendicular to the Segment Connecting the Dipoles and Passing through Its Middle

Let us now assume that the measurements of the magnetic induction components are performed at the points of a straight line passing through the middle of the segment connecting the two dipoles and perpendicular to this segment (this case is similar to that described in Section 3.2). Then, system (3) takes the form

$$\begin{cases} \sum\limits_{d=1}^{2} \left(\dfrac{3\left(m_x^{(d)} x_{sd} + m_y^{(d)} y_{sd} + m_z^{(d)} z_{sd}\right) x_{sd}}{r_{sd}^5} - \dfrac{m_x^{(s)}}{r_{sd}^3} \right) = B_x^{(s)}, \quad s = \overline{1,N}, \\ \sum\limits_{d=1}^{2} \left(\dfrac{3\left(m_x^{(d)} x_{sd} + m_y^{(d)} y_{sd} + m_z^{(d)} z_{sd}\right) y_{sd}}{r_{sd}^5} - \dfrac{m_y^{(s)}}{r_{sd}^3} \right) = B_y^{(s)}, \quad s = \overline{1,N}, \\ \sum\limits_{d=1}^{2} \left(\dfrac{3\left(m_x^{(d)} x_{sd} + m_y^{(d)} y_{sd} + m_z^{(d)} z_{sd}\right) z_{sd}}{r_{sd}^5} - \dfrac{m_z^{(s)}}{r_{sd}^3} \right) = B_z^{(s)}, \quad s = \overline{1,N}. \end{cases} \tag{9}$$

Obviously, system (9) is an overdetermined system of linear algebraic equations, which, in general, is inconsistent. As is known [13], in the finite-dimensional case, there always exists a pseudo-solution of the system, which can be obtained by solving the normal system of equations $\mathbf{K}^T\mathbf{K}\boldsymbol{m} = \mathbf{K}^T\boldsymbol{B}$ (here, $\mathbf{K}$ is the matrix of the original system, and $\boldsymbol{B}$ is the right side of system). Since the rank of $\mathbf{K}^T\mathbf{K}$ is equal to the rank of the original matrix $\mathbf{K}$, with the arrangement of dipoles and sensors described above, we obtain a system of linear algebraic equations with rows proportional to each other in each block (taking into account that $r_{s1} = r_{s2}$ for all $s = \overline{1,N}$). Such lines resemble the following:

$$\begin{bmatrix} 0 & -\frac{1}{r_{s1}^3} & 0 & 0 & -\frac{1}{r_{s2}^3} & 0 \end{bmatrix}.$$

Thus, we prove Theorem 3.

Theorem 3. *If all sensors are located on a straight line lying in the plane of symmetry of two unknown dipoles d_1 and d_2, then the quasi-solution of the system (9) is determined ambiguously. However, the normal pseudo-solution is unique.*

3.4. Two Dipoles and N Sensors Located on the Same Straight Line

For this case, the analog of Theorem 1 is true.

Theorem 4. *If two dipoles and N sensors are located on the same straight line, the coordinates of any two sensors do not coincide, and these two sensors "do not separate" the selected two dipoles; then, the quasi-solution of system (3) is unique, regardless of the values that the coordinates of the remaining $N-2$ sensors take.*

The system of equations from which, in the described case, the components of two dipoles lying on the same line as the sensors are determined has the form described in (9), whose first two pairs of blocks are expressed using Formula (6). The rank of such a system cannot be greater than 6.

Consider two dipoles and two sensors that are both located either to the right or to the left of both dipoles. According to Theorem 1, the components of the dipoles are uniquely determined, and, therefore, the remaining sensors do not add information about the magnetic field.

3.5. N Dipoles and N Sensors in 3D Space

Let us now consider the case in which it is required to determine the components of N magnetic dipoles, d_i and $i = \overline{1,N}$, from the three components of the magnetic induction vector measured at N arbitrary points in three-dimensional space. A process similar to that carried out in Section 3.2 allows us to conclude that the solution of system (3) is non-unique if, for any two indices s_1 and s_2, the following relations are valid:

$$\begin{aligned} y_{s_1 d} &= y_{s_2 d} = 0, \quad d = \overline{1,N}, \\ r_{s_1 d} &= r_{s_2 d}, \qquad\quad d = \overline{1,N}. \end{aligned} \tag{10}$$

Then, the two rows in (4) match: in the s_1-th block

$$\begin{bmatrix} 0 & -\frac{1}{r^3_{s_1 1}} & 0 & 0 & -\frac{1}{r^3_{s_1 2}} & 0 & \dots & 0 & -\frac{1}{r^3_{s_1 N}} & 0 \end{bmatrix}$$

and in the s_2-th block

$$\begin{bmatrix} 0 & -\frac{1}{r^3_{s_2 1}} & 0 & 0 & -\frac{1}{r^3_{s_2 2}} & 0 & \dots & 0 & -\frac{1}{r^3_{s_2 N}} & 0 \end{bmatrix}.$$

Thus, we prove Theorem 5.

Theorem 5. *When the conditions (10) are met, the solution of system (3) is determined ambiguously.*

4. Statement of the Inverse Problem of Magnetometry or Gravimetry in the Case of Using the Method of Integral Representations

The potential U of a magnetic or gravitational field can be represented as the sum of the single- and double-layer potentials created by a set of horizontal planes located below a given surface. If the coordinate system is chosen in such a way that the surface (for example, the Earth's surface in the local version of the S-approximation method (see [10,26,27])) is given by the equation $z = 0$, then the potential U can be represented as follows [1]:

$$\begin{aligned} U(\boldsymbol{r}_s) = &\sum_{l=1}^{L} \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} \frac{\rho_1^{(l)}(x,y)dxdy}{\sqrt{(x_s-x)^2+(y_s-y)^2+(z_s-H_l)^2}} + \\ &+ \sum_{l=1}^{L} \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} \frac{\rho_2^{(l)}(x,y)(z_s-H_l)dxdy}{\sqrt{(x_s-x)^2+(y_s-y)^2+(z_s-H_l)^2}}. \end{aligned} \tag{11}$$

Note that in Section 2, the role of the sources of the field $\boldsymbol{B}$ is performed by the vector function $\boldsymbol{M}(\boldsymbol{r})$, which characterizes the density of the magnetic moment. Now, we consider sources that are equivalent in potential U (to the field $\boldsymbol{B}$ in the case of the magnetometry problem)—i.e., scalar functions $\rho_1^{(l)}(x,y)$ and $\rho_2^{(l)}(x,y)$, $l = \overline{1,L}$, which determine the surface density of these equivalent field sources on the set ($l = \overline{1,L}$) of parallel planes, each of which is located at a depth of $z = H_l$. In the case of the gravitational field potential, the functions $\rho_1^{(l)}(x,y)$ and $\rho_2^{(l)}(x,y)$ have a similar meaning.

The derivative with respect to the variable z_s of the potential U, taken with the opposite sign, has the form:

$$\begin{aligned} -\frac{\partial U}{\partial z_s}(\boldsymbol{r}_s) = &\sum_{l=1}^{L} \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} \frac{\rho_1^{(l)}(x,y)(z_s-H_l)dxdy}{\left((x_s-x)^2+(y_s-y)^2+(z_s-H_l)^2\right)^{3/2}} + \\ &+ \sum_{l=1}^{L} \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} \frac{\rho_2^{(l)}(x,y)\left(2(z_s-H_l)^2-(x_s-x)^2-(y_s-y)^2\right)dxdy}{\left((x_s-x)^2+(y_s-y)^2+(z_s-H_l)\right)^{3/2}}. \end{aligned} \tag{12}$$

Let us introduce notation for the integrands in (12):

$$Q_1^{(sl)}(x,y) \equiv \frac{z_s - H_l}{\left((x_s-x)^2+(y_s-y)^2+(z_s-H_l)^2\right)^{3/2}},$$
$$Q_2^{(sl)}(x,y) \equiv \frac{2(z_s-H_l)^2-(x_s-x)^2-(y_s-y)^2}{\left((x_s-x)^2+(y_s-y)^2+(z_s-H_l)\right)^{3/2}}.$$

Then, we obtain:

$$f_s \equiv -\frac{\partial U}{\partial z_s}(\boldsymbol{r}_s) = \sum_{l=1}^{L}\int\limits_{-\infty}^{+\infty}\int\limits_{-\infty}^{+\infty}(\rho_1^{(l)}(x,y)Q_1^{(sl)}(x,y)+ \\ +\rho_2^{(l)}(x,y)Q_2^{(sl)}(x,y))dxdy, \quad s=\overline{1,N}. \tag{13}$$

Thus, the **inverse problem** consists in defining the functions $\rho_1^{(l)}(x,y)$ and $\rho_2^{(l)}(x,y)$, $l=\overline{1,L}$ from the system of Equation (13) according to experimental observations f_s, $s=\overline{1,N}$.

In practice, measurements are carried out with errors, so the input information comprises values of $f_{s\delta}$ such that $\frac{1}{N}\sum\limits_{s=1}^{N}(f_s-f_{s\delta})^2 \le \delta^2$. By solving the variational problem

$$(\rho_1^{(l)},\rho_2^{(l)}) = \underset{\rho_1^{(l)},\rho_2^{(l)}\in L_2(\mathbb{R}^1\times\mathbb{R}^1)}{\operatorname{argmin}} \sum_{l=1}^{L}\int\limits_{-\infty}^{+\infty}\int\limits_{-\infty}^{+\infty}\left(\left(\rho_1^{(l)}(x,y)\right)^2+\left(\rho_2^{(l)}(x,y)\right)^2\right)dxdy$$

with conditions

$$f_{s\delta}-\sum_{l=1}^{L}\int\limits_{-\infty}^{+\infty}\int\limits_{-\infty}^{+\infty}(\rho_1^{(l)}(x,y)Q_1^{(sl)}(x,y)+\rho_2^{(l)}(x,y)Q_2^{(sl)}(x,y))dxdy=0, \quad s=\overline{1,N},$$

we determine that the required functions should resemble the following [1]:

$$\rho_1^{(l)}(x,y)=\sum_{s=1}^{N}\lambda_s Q_1^{(sl)}(x,y), \quad \rho_2^{(l)}(x,y)=\sum_{s=1}^{N}\lambda_s Q_2^{(sl)}(x,y), \quad l=\overline{1,L}. \tag{14}$$

Substituting (14) into (13), we obtain the following system of linear algebraic equations with respect to the vector $\lambda=(\lambda_1,\ldots,\lambda_N)^T$:

$$\mathbf{A}\lambda = \boldsymbol{f}_\delta. \tag{15}$$

Here, $\boldsymbol{f}_\delta=(f_{1\delta},\ldots,f_{N\delta})^T$ and the elements a_{ij} of the matrix $\mathbf{A}$ have the following form:

$$a_{ij}=\sum_{l=1}^{L}\int\limits_{-\infty}^{+\infty}\int\limits_{-\infty}^{+\infty}(Q_1^{(il)}(x,y)Q_1^{(jl)}(x,y)+Q_2^{(il)}(x,y)Q_2^{(jl)}(x,y))dxdy, \qquad i,j=\overline{1,N}.$$

Moreover, these elements can be calculated explicitly using the Poisson integral:

$$a_{ij}=2\pi\sum_{l=1}^{L}\left(\frac{z_i+z_j-2H_l}{\left((x_i-x_j)^2+(y_i-y_j)^2+(z_i+z_j-2H_l)^2\right)^{3/2}}- \\ -\frac{9(z_i+z_j-2H_l)\left((x_i-x_j)^2+(y_i-y_j)^2\right)-6(z_i+z_j-2H_l)^2}{\left((x_i-x_j)^2+(y_i-y_j)^2+(z_i+z_j-2H_l)^2\right)^{3/2}}\right).$$

Remark 2. *Note again that the coordinate indexing refers only to the position of the sensors* $\mathbf{r}_s=(x_s,y_s,z_s)$, $s=\overline{1,N}$.

Using the values of λ_s, $s = \overline{1, N}$ found from the solution to system (15), the functions $\rho_1^{(l)}(x, y)$ and $\rho_2^{(l)}(x, y)$, $l = \overline{1, L}$ can be found using Formula (14). Then, these functions can be used to determine some functionals [1,10,26,27], which can represent higher derivatives of the potential, the analytical continuation of the magnetic or gravitational field, etc.

Thus, **the second purpose of this article** is to formulate the conditions under which a system of linear algebraic equations of the form (15) is solvable.

5. Local Version of Uniqueness Theorems in the Case of Using the Method of Integral Representations

In this section, we consider uniqueness theorems for a particular case that is quite common in practice, when the sources of the potential of a magnetic or gravitational field are concentrated on only one plane (see Sections 5.1 and 5.2). We call the corresponding problem setting "local". Examples of solvable problems are given (see Section 5.3). Then (see Section 5.4), we draw conclusions about the methods of constructing the optimal geometry of the grid of observation points.

Thus, we assume that the sources of the field potential are located in only one plane, which we denote as D. Thus, $L = 1$, and for convenience, we use the following notation:

$$\rho_1(x, y) \equiv \rho_1^{(1)}(x, y), \quad \rho_2(x, y) \equiv \rho_2^{(1)}(x, y), \quad H \equiv -H_1.$$

The examination of the system matrix (15) for unique solvability, even in such a seemingly simple case, is a very laborious process. Therefore, in this paper, we restrict ourselves to the consideration of two even narrower cases: (1) the representation of physical fields (gravitational or magnetic) in the form of a single-layer potential, and (2) representations of physical fields in the form of a double-layer potential. In the expressions (11) and (13), these are the first and second terms, respectively.

5.1. Taking into Account the Single-Layer Potential Only

If any higher derivative of the magnetic or gravitational potential is approximated by a single-layer potential, then the system matrix (15) takes the form

$$\mathbf{A} \equiv \begin{bmatrix} \dfrac{1}{4(z_1+H)^2} & \dfrac{z_1+z_2+2H}{((z_1+z_2+2H)^2+r_{12}^2)^{3/2}} & \cdots & \dfrac{z_1+z_N+2H}{((z_1+z_N+2H)^2+r_{1N}^2)^{3/2}} \\ \dfrac{z_2+z_1+2H}{((z_2+z_1+2H)^2+r_{21}^2)^{3/2}} & \dfrac{1}{4(z_2+H)^2} & \cdots & \dfrac{z_2+z_N+2H}{((z_2+z_N+2H)^2+r_{2N}^2)^{3/2}} \\ \vdots & \vdots & \ddots & \vdots \\ \dfrac{z_N+z_1+2H}{((z_N+z_1+2H)^2+r_{N1}^2)^{3/2}} & \dfrac{z_N+z_2+2H}{((z_N+z_2+2H)^2+r_{N2}^2)^{3/2}} & \cdots & \dfrac{1}{4(z_N+H)^2} \end{bmatrix}. \quad (16)$$

Remark 3. *In this section (Section 5), and in this section only,* $r_{ij} = \sqrt{(x_i - x_j)^2 + (y_i - y_j)^2}$.

Let us try to find out, by analogy with Section 3, in which cases the matrix (16) is degenerate. For the determinant of a matrix to be equal to zero, it is sufficient that any two rows of this matrix are proportional to each other, for example, the i-th and j-th rows:

$$\begin{bmatrix} \cdots & \dfrac{1}{4(z_i+H)^2} & \cdots & \dfrac{z_i+z_j+2H}{((z_i+z_j+2H)^2+r_{ij}^2)^{3/2}} & \cdots \end{bmatrix}$$

and

$$\begin{bmatrix} \cdots & \dfrac{z_j+z_i+2H}{((z_j+z_i+2H)^2+r_{ji}^2)^{3/2}} & \cdots & \dfrac{1}{4(z_j+H)^2} & \cdots \end{bmatrix}.$$

Note that here: (1) in the row with index i, a diagonal element is written in the i-th column and another element in the j-th column; (2) in the row with index j, a diagonal element is written in the j-th column and another element in the i-th column.

Let us write the proportionality condition for the elements of the specified rows of matrix $\mathbf{A}$ (in this case, the index k corresponds to the element that is in the k-th column):

$$\begin{cases} \dfrac{(z_i+z_j+2H)^2}{((z_i+z_j+2H)^2+r_{ij}^2)^3} = \dfrac{1}{16(z_i+H)^2(z_j+H)^2}, \\ \dfrac{z_i+z_k+2H}{((z_i+z_k+2H)^2+r_{ik}^2)^{3/2}} = C\dfrac{z_k+z_j+2H}{((z_k+z_j+2H)^2+r_{kj}^2)^{3/2}}. \end{cases} \quad (17)$$

From the first equation of system (17), we can determine that

$$r_{ij}^2 = \left(16(z_i+H)^2(z_j+H)^2(z_i+z_j+2H)^2\right)^{1/3} - (z_i+z_j+2H)^2.$$

Taking into account the fact that $r_{ij}^2 \geq 0$, after simple transformations, we obtain the inequality

$$4(z_i+H)(z_j+H) \geq (z_i+z_j+2H)^2.$$

After opening the brackets and making simplifications, we obtain

$$(z_i-z_j)^2 \leq 0.$$

Thus, $z_i = z_j$ and $r_{ij}^2 = 0$.

Therefore, the diagonal elements of the two indicated rows cannot be proportional to the off-diagonal elements for any other values of the coordinates z of the observation points $\boldsymbol{r}_i = (x_i, y_i, z_i)$ and $\boldsymbol{r}_j = (x_j, y_j, z_j)$.

Thus, we can conclude that Theorem 6 is correct.

Theorem 6. *The system matrix (15), when representing the field using the single-layer potential, has a rank of at least two for two different observation points.*

Remark 4. *Note that in this study, we do not take into account the second equation from system (17). Thus, it is possible that the rank of system (15) is much greater than 2 for a larger number of observation points with different coordinates (though it cannot be lower).*

Let us now consider the case of representing the elements of the gravitational or magnetic fields in the form of a double-layer potential only.

5.2. Taking into Account a Double-Layer Potential Only

If any higher derivative of the magnetic or gravitational potential is approximated by the double-layer potential only, then the system matrix (15) takes the form

$$\mathbf{A} \equiv \begin{bmatrix} \dfrac{1}{16(z_1+H)^4} & \cdots & \dfrac{(z_1+z_N+2H)\left(6(z_1+z_N+2H)^2-9r_{1N}^2\right)}{((z_1+z_N+2H)^2+r_{1N}^2)^{7/2}} \\ \vdots & \ddots & \vdots \\ \dfrac{(z_N+z_1+2H)\left(6(z_N+z_1+2H)^2-9r_{N1}^2\right)}{((z_N+z_1+2H)^2+r_{N1}^2)^{7/2}} & \cdots & \dfrac{1}{16(z_N+H)^4} \end{bmatrix}. \quad (18)$$

Let us write, as in the case of a single layer (see the previous subsection), the proportionality condition for the diagonal and corresponding off-diagonal elements of two rows:

$$\frac{(z_i+z_j+2H)^2\left(6(z_i+z_j+2H)^2-9r_{ij}^2\right)^2}{((z_i+z_j+2H)^2+r_{ij}^2)^7} = \frac{1}{256(z_i+H)^4(z_j+H)^4}. \quad (19)$$

Our goal is to determine the conditions under which equality (19) holds. Let us show that this equality is true for two coinciding points, i.e., including the case $r_{ij} = 0$.

For convenience, we introduce the following notation:

$$C \equiv \frac{(z_i + z_j + 2H)\left(6(z_i + z_j + 2H)^2 - 9r_{ij}^2\right)}{\left((z_i + z_j + 2H)^2 + r_{ij}^2\right)^{7/2}}, \tag{20}$$

$$C_1 \equiv z_i + z_j + 2H, \tag{21}$$

$$R \equiv \sqrt{(z_i + z_j + 2H)^2 + r_{ij}^2}. \tag{22}$$

Then, from (20) and (21), we obtain

$$C = \frac{C_1\left(6C_1^2 - 9r_{ij}^2\right)}{(C_1^2 + r_{ij}^2)^{7/2}}. \tag{23}$$

Taking into account (21) and (22), we can transform (23) to the form

$$\sqrt{R^2 - r_{ij}^2}\left(6R^2 - 15r_{ij}^2\right) = CR^7. \tag{24}$$

It follows from (24) that $r_{ij}^2 > 0$ only if $CR^6 < 6R^2$, which is the same as $R^4 < \dfrac{6}{C}$. However, it follows from (21) and (22) that $R \geq C_1$. Therefore, $C_1^4 \leq \dfrac{6}{C}$. We obtain a contradiction with (23), from which it follows that $C_1^4 > \dfrac{6}{C}$ for $r_{ij}^2 > 0$. Thus, $r_{ij}^2 \leq 0$, and this is possible only in the case $r_{ij} = 0$.

Thus, we can conclude that Theorem 7 is correct.

Theorem 7. *The system matrix (15), when representing the field using the double-layer potential, has a rank of at least two for two different observation points.*

5.3. Examples

Theorems 6 and 7 cannot be generalized to the case of an arbitrary number N of observation points. This is because the condition that no two rows of the matrix are linearly dependent is not sufficient for the system of linear algebraic equations to be nondegenerate.

However, we can provide two examples of the unique solvability of system (15) for $N = 3$ and $N = 4$.

5.3.1. Case of Three Points of Observation

If the number of observation points is three ($N = 3$), the coordinates in the variable z of all three observation points are the same, and the points themselves are located at the vertices of an equilateral triangle; then, the system matrix (15) for a single-layer potential takes the form

$$\mathbf{A} \equiv \begin{bmatrix} a & b & b \\ b & a & b \\ b & b & a \end{bmatrix}. \tag{25}$$

Here,

$$a = \frac{1}{4(z_1 + H)^2} = \frac{1}{4(z_2 + H)^2} = \frac{1}{4(z_3 + H)^2},$$
$$b = \frac{z_1 + z_2 + 2H}{((z_1 + z_2 + 2H)^2 + r_{12}^2)^{3/2}} = \frac{z_1 + z_3 + 2H}{((z_1 + z_3 + 2H)^2 + r_{13}^2)^{3/2}} = \frac{z_2 + z_3 + 2H}{((z_2 + z_3 + 2H)^2 + r_{23}^2)^{3/2}}.$$

Let us write an expression for the matrix determinant (25):

$$\det \mathbf{A} = a^3 + 2b^3 - 3ab^2.$$

Let us find out if it can be equal to zero. We have:

$$\det \mathbf{A} = a^3 + 2b^3 - 3ab^2 \Rightarrow a^3 - ab^2 = 2ab^2(a-b) \Rightarrow a(a-b)(a+b) = 2b^2(a-b).$$

The numbers a and b are always greater than zero. Therefore, the last equality can only hold if these numbers are equal. But they cannot be equal, according to Theorem 6, which was proved above. Therefore, in this particular case of a system of equations, the solution is uniquely determined.

Remark 5. *In the case where all z_i, $i = \overline{1,3}$, are distinct, the matrix determinant*

$$\mathbf{A} \equiv \begin{bmatrix} a_1 & 1 & 1 \\ 1 & a_2 & 1 \\ 1 & 1 & a_3 \end{bmatrix}$$

can vanish (without a loss of generality, we denote the elements of b as 1). We will now determine when this is possible. The determinant of this matrix is $a_1a_2a_3 + 2 - a_1 - a_2 - a_3$. If all z_i, $i = \overline{1,3}$ satisfy the inequalities $1/2 \le z_i \le 1$, then this determinant is less than or equal to zero. If $\varepsilon \le z_i \le 1/2$, $0 < \varepsilon < 1/2$, then, conversely, the determinant is greater than or equal to zero. Therefore, it is possible for the determinant to be equal to zero and, therefore, the matrix of system (15) is degenerate.

5.3.2. Case of Four Observation Points

If the number of observation points is four ($N = 4$), then consider system matrix (15) of the following form:

$$\mathbf{A} \equiv \begin{bmatrix} a & c & d_1 & d_2 \\ c & b & d_3 & d_4 \\ d_1 & d_3 & a_1 & c_1 \\ d_2 & d_4 & c_1 & b_1 \end{bmatrix}. \tag{26}$$

Here,

$$\begin{aligned} a &= \frac{1}{4(z_1+H)^2}, \quad b = \frac{1}{4(z_2+H)^2}, \quad a_1 = \frac{1}{4(z_3+H)^2}, \quad b_1 = \frac{1}{4(z_4+H)^2}, \\ c &= \frac{z_1+z_2+2H}{((z_1+z_2+2H)^2+r_{12}^2)^{3/2}}, \quad d_1 = \frac{z_1+z_3+2H}{((z_1+z_3+2H)^2+r_{13}^2)^{3/2}}, \\ d_2 &= \frac{z_1+z_4+2H}{((z_1+z_4+2H)^2+r_{14}^2)^{3/2}}, \quad d_3 = \frac{z_2+z_3+2H}{((z_2+z_3+2H)^2+r_{23}^2)^{3/2}}, \\ d_4 &= \frac{z_2+z_4+2H}{((z_2+z_4+2H)^2+r_{24}^2)^{3/2}}, \quad c_1 = \frac{z_3+z_4+2H}{((z_2+z_4+2H)^2+r_{34}^2)^{3/2}}. \end{aligned}$$

We choose the coordinates of the observation points so that the elements of the matrix $\mathbf{A}$, denoted by d_i, $i = \overline{1,4}$, are the same. Let us show how this can be achieved.

Consider the surface $C = \dfrac{z}{(z^2+r^2)^{3/2}}$ in the space of coordinates x, y, z, where $r = \sqrt{x^2+y^2}$. Let us express r^2 in terms of the variable z explicitly:

$$r^2 = \frac{z^{2/3}}{C^{2/3}} - z^2 \equiv f(z).$$

The derivative of the function $f(z)$ is positive for $z < \frac{1}{(3C')^{3/4}}$, $C' = C^{2/3}$. The function $f(z)$ is a continuous function of its argument, and $f(z) \geq 0$ if $z \leq \frac{1}{(3C')^{3/4}}$. It follows from this that under the conditions

$$z_i < \frac{1}{\sqrt{C}} - \max(z_1, z_2) - 2H,$$

there exist r_{1i}, r_{2i} such that the points $\boldsymbol{r}_i$, $i = 3, 4$, lie on the above surface, and the matrix of system (15) has the form (26).

The following geometric interpretation is possible. One can imagine a convex quadrilateral with four observation points at its vertices. The coordinates in the variable z of these points are different, but the sums of the coordinates $z_i + z_j$ can be the same for some values of the indices (but not for all: four points on the plane cannot be located in pairs at the same distance from each other). We calculate the determinant of this matrix using the Laplace formula for the expansion in two-dimensional minors of the first two rows:

$$\det \mathbf{A} = \sum_{i=1}^{6} (-1)^{\varepsilon_i} M_i \bar{M}_i.$$

Here,

$$M_1 = \begin{vmatrix} a & c \\ c & b \end{vmatrix}, \quad M_2 = \begin{vmatrix} a & d \\ c & c \end{vmatrix}, \quad M_3 = \begin{vmatrix} a & d \\ c & d \end{vmatrix},$$
$$M_4 = M_5 = \begin{vmatrix} c & d \\ b & d \end{vmatrix}, \quad M_6 = \begin{vmatrix} d & d \\ d & d \end{vmatrix} = 0,$$

where ε_i is the permutation sign corresponding to the i-th minor, and $\bar{M}_i$ is an additional minor.

As a result, we obtain

$$\det \mathbf{A} = (ab - c^2)(a_1 b_1 - c_1^2) + d^2(a + b - 2c)(a_1 + b_1 - 2c_1). \tag{27}$$

According to Theorem 6, the first term is always positive. The second term is also positive, since $(a^2 + b^2)(a + b)^2 \geq 2ab(a + b)^2 \geq 8a^2 b^2$ for any $a, b > 0$.

Thus, in this particular case of four observation points, the solution of the system (15) is uniquely determined.

Let us now assume that four points in three-dimensional space are located in such a way that the matrix of system (15) in the case of a single-layer potential is Toeplitz:

$$\mathbf{A} \equiv \begin{bmatrix} a & b & c & d \\ b & a & b & c \\ c & b & a & b \\ d & c & b & a \end{bmatrix}. \tag{28}$$

Here,

$$a = \frac{1}{4(z_1 + H)^2}, \quad b = \frac{z_1 + z_2 + 2H}{((z_1 + z_2 + 2H)^2 + r_{12}^2)^{3/2}},$$
$$c = \frac{z_1 + z_3 + 2H}{((z_1 + z_3 + 2H)^2 + r_{13}^2)^{3/2}}, \quad d = \frac{z_1 + z_4 + 2H}{((z_1 + z_4 + 2H)^2 + r_{14}^2)^{3/2}}.$$

This type of matrix is possible when $z_1 = z_2 = z_3 = z_4$, $r_{12} = r_{23} = r_{34}$, $r_{13} = r_{24}$. That is, one can see that the observation points are located at the vertices of a rhombus with an angle of $60°$. Then, the determinant (27) takes the form

$$\det \mathbf{A} = (b^2 - ac)^2 - b^2(a - c).$$

We assume that $a > b > c$ (this condition can be satisfied using the arrangement of observation points indicated above). Then, $b^2(b^2 - a^2) > c^2(b^2 - a^2)$. Thus, we show that in the considered case, the matrix of the system (15) is not degenerate, and, consequently, the system has a unique solution.

5.3.3. The Case of N Observation Points Lying on the Same Straight Line in Three-Dimensional Space

If we represent a magnetic or gravitational field as a single-layer potential only, then the matrix of the system of linear algebraic equations to which the inverse problem is reduced has the form (16). We showed above that in particular cases (for $N = 3$ and $N = 4$), the SLAE is nondegenerate. However, for an arbitrary number of observation points, it is very difficult to analyze the unique solvability of system (15). Let us try to achieve this for the case when all points lie on the same straight line parallel to the Oz axis. Then, all r_{ij}^2 are equal to zero, and matrix (16) becomes

$$\mathbf{A} \equiv \begin{bmatrix} \frac{1}{4(z_1+H)^2} & \frac{1}{(z_1+z_2+2H)^2} & \cdots & \frac{1}{(z_1+z_N+2H)^2} \\ \frac{1}{(z_2+z_1+2H)^2} & \frac{1}{4(z_2+H)^2} & \cdots & \frac{1}{(z_2+z_N+2H)^2} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{1}{(z_N+z_1+2H)^2} & \frac{1}{(z_N+z_2+2H)^2} & \cdots & \frac{1}{4(z_N+H)^2} \end{bmatrix}.$$

The determinant of such a system can be calculated using the formula for finding the values of the Cauchy determinant. Let us first set $\tilde{z}_i = z_i + H$, $\hat{z}_j = z_j + H$ (that is, "remove" the symmetry of the SLAE matrix). Thus, the elements a_{ij} of the matrix $\mathbf{A}$ take the following form:

$$a_{ij} = \frac{1}{\tilde{z}_i + \hat{z}_j}.$$

Then,

$$\begin{aligned} \det \mathbf{A} &= \begin{vmatrix} \frac{1}{(\tilde{z}_1+\hat{z}_1)^2} & \cdots & \frac{1}{(\tilde{z}_1+\hat{z}_N)^2} \\ \vdots & \ddots & \vdots \\ \frac{1}{(\tilde{z}_N+\hat{z}_1)^2} & \cdots & \frac{1}{(\tilde{z}_N+\hat{z}_N)^2} \end{vmatrix} = \\ &= (-1)^N \frac{d}{d\hat{z}_N} \frac{d}{d\hat{z}_{N-1}} \cdots \frac{d}{d\hat{z}_1} \begin{vmatrix} \frac{1}{\tilde{z}_1+\hat{z}_1} & \cdots & \frac{1}{\tilde{z}_1+\hat{z}_N} \\ \vdots & \ddots & \vdots \\ \frac{1}{\tilde{z}_N+\hat{z}_1} & \cdots & \frac{1}{\tilde{z}_N+\hat{z}_N} \end{vmatrix} = \\ &= \frac{1}{\tilde{z}_1\tilde{z}_2\ldots\tilde{z}_N}(-1)^N \frac{d}{d\hat{z}_N} \frac{d}{d\hat{z}_{N-1}} \cdots \frac{d}{d\hat{z}_1} \begin{vmatrix} \frac{1}{1+\frac{\hat{z}_1}{\tilde{z}_1}} & \cdots & \frac{1}{1+\frac{\hat{z}_N}{\tilde{z}_1}} \\ \vdots & \ddots & \vdots \\ \frac{1}{1+\frac{\hat{z}_1}{\tilde{z}_N}} & \cdots & \frac{1}{1+\frac{\hat{z}_N}{\tilde{z}_N}} \end{vmatrix}. \end{aligned} \tag{29}$$

It can be shown by direct calculations that in the case $N = 3$ and under the conditions $\tilde{z}_i = z_i + H > 0$. $\tilde{z}_i \neq \tilde{z}_j$ if $i \neq j$, Theorem 8 is true.

Theorem 8. *If three observation points located on the same straight line parallel to the Oz axis are different, then the system (15) has a unique solution.*

To prove a similar theorem in the case of an arbitrary value N, further studies are needed. It can be assumed (based on (29)) that the N-fold derivative of the Cauchy determinant does not vanish if $\tilde{z}_i$ is positive and they are not equal to each other. But for now, this is only a hypothesis.

5.4. Building an Optimal Network of Observation Points

Consider the elements of matrix (16), requiring that the observation points are on the level surface of the harmonic function:

$$\frac{(z_i + z_j + 2H)^2}{\left((x_i - x_j)^2 + (y_i - y_j)^2 + (z_i + z_j + 2H)^2\right)^3} = \frac{1}{C}. \tag{30}$$

For convenience, we continue to use the notation $r_{ij}^2 \equiv (x_i - x_j)^2 + (y_i - y_j)^2$. Then, Equation (30) can be transformed to the following:

$$r_{ij}^2 = \left(C(z_i + z_j + 2H)^2\right)^{1/3} - (z_i + z_j + 2H)^2. \tag{31}$$

In Equation (31), the constant C should be chosen in such a way that the diagonal in matrix (16) contains elements that significantly exceed the value of $\frac{1}{C}$. Next, we introduce some new notation and write an equation that must be satisfied by the powers of the sums of the z-coordinates for each two observation points.

If we introduce the notation $\tilde{y}_{ij} = (z_i + z_j + 2H)^{2/3}$ and $C' \equiv C^{1/3}$, then Equation (31) becomes

$$r_{ij}^2 = C'\tilde{y}_{ij} - \tilde{y}_{ij}^3.$$

We transform the resulting equation into the form $\tilde{y}_{ij}^3 - p\tilde{y}_{ij} + q = 0$, where $p = C'$, $q = r_{ij}^2$, $D = 4p^3 - 27q^2$. Then,

$$\tilde{y}_{ij} = \left(-\frac{q}{2} \pm \frac{\sqrt{-3D}}{18}\right)^{1/3} + \left(-\frac{q}{2} \mp \frac{\sqrt{-3D}}{18}\right)^{1/3}. \tag{32}$$

Given N points on the plane, between which the squared distances are denoted as $r_{ij}^2 = (x_i - x_j)^2 + (y_i - y_j)^2$, the sums of the i-th and j-th z-coordinates are determined from (32) by solving a cubic equation. We emphasize that in (32), a non-negative expression must be placed under the square-root sign. If the roots of the cubic equation defined by (31) are complex (in this case, they are complex conjugate), then we choose the third real root of the cubic equation, which is equal to the sum of the other two, taken with the opposite sign. The number of possible values r_{ij}^2 is equal to $N(N-1)/2$, and the number of unknown z-coordinates is equal to N. As a result, we obtain an overdetermined system of linear algebraic equations. When constructing, in a certain sense, an optimal network of observation points, one can proceed as follows:

1. Define three points, P_1, P_2, and P_3, on the plane. Next, define $\tilde{y}_{ij}$ according to (32) for the three pairs of specified points. For the case of three points, the number of unknowns in (32) is equal to the number of equations.
2. Set the fourth point P_4 on the plane in such a way that the deviation

$$\Delta_{14} \equiv \left| r_{14} - \sqrt{\left(C(z_1 + z_4 + 2H)^2\right)^{1/3} - (z_1 + z_4 + 2H)^2} \right.$$

is minimal. In this case, the z_4 coordinate is determined by the distances from the fourth point to the second and third points according to (31).

We can extend the described algorithm to all subsequent points: we place points on the plane, starting from the fourth point, so that the distances from these points, determined by their z-coordinates according to (31), deviate as little as possible from the actual values. This way of specifying a network of observation points allows one to select sensors "approximately" on the level surface of the harmonic function.

If the diagonal elements of matrix (16) are much larger than the value of $\frac{1}{C}$, then the SLAE condition number is not too large. Additionally, as one can see from (16), the diagonal elements depend only on the z-coordinates of the points.

6. Discussion

1. We considered the matrices of systems to which the inverse problems of magnetometry and gravimetry are reduced. However, matrices of the form (25) and (28) also arise when solving problems of combinatorics, the optimal control of various processes, etc. In addition, it should be emphasized that, for example, the elements of matrices (16) and (18) are derivatives of the potential of a point source with respect to the Cartesian coordinate z. Thus, the elements of these matrices are harmonic functions in some region of space. In the future, we plan to consider the issues of the unique solvability of SLAEs whose elements are harmonic functions of the corresponding variables (both in Cartesian and spherical coordinate systems). If the field elements are studied over large areas, then the potential and its higher derivatives should be considered as functions of spherical coordinates. It is possible, of course, to take into account the ellipticity of the planets, but this is still impractical: the field values differ insignificantly when correcting for ellipticity, and the mathematical formulation of the problem becomes much more complicated.
2. Several new methods, such as "multigrid homotopy", "multigrid with constraint data", and "constrained homotopy", have been used for the inverse problems of magnetometry and gravimetry (see, for example, [28–32]). These methods have produced good results for the corresponding problems in two-dimensional formulations. In this regard, we would like to note that all the problems that we have attempted to solve over the past 20 years have been considered exclusively in three-dimensional formulations, which is adequate for real geophysical practice. Two-dimensional problems can be studied, but in this case a lot of a priori information about the physical fields of the Earth and planets is lost, and the possibility of restoring small inhomogeneities, especially deep ones, is also lost.
3. In the future, we intend to study the issues of constructing optimal observation networks using the method of linear integral representations based on Formulas (30)–(32). We plan to conduct a mathematical experiment with a different number of observation points. We also hope to consider a regional version of the method of linear integral representations. In this case, the elements of the matrix will be elliptic integrals of the first kind. In addition, we believe that the question of the parametrization of the optimal network and the connection of this problem with the deformation of a family of matrices that can be simultaneously reduced to a diagonal form is important.

7. Conclusions

When processing experimental data from measurements of magnetic and/or gravitational fields, various approaches can be used to reduce the problem being solved to the solution of systems of linear algebraic equations. The method of integral representations (the method of S-approximations) makes it possible to construct analytical approximations of the elements of anomalous potential fields in the "first approximation" in terms of accuracy. Such an approximation makes it possible to form a general idea of the nature of the sources, the nature of the anomaly, and so on. The theorems presented in this article allow one to choose the most effective algorithms for constructing networks of observation points (measuring experimental data) in each particular case, for which the vectors of unknown parameters can be uniquely determined.

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