mathematics

Review

A Review of Mathematical Models of Macroeconomics, Microeconomics, and Government Regulation of the Economy

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Abstract: This review analyzes articles on the mathematical modeling of economic facts and processes. Mathematical modeling of the economy has rapidly developed in the past and current centuries. This is explained by the fact that, firstly, economics does not tolerate full-scale experiments, secondly, mathematical modeling significantly improves the accuracy of research results, and, finally, thirdly, economics becomes a science only when it is based on mathematics. The article presents an overview of the main methods of economic modeling used in scientific research over the past twenty years. The review does not claim to cover all areas, methods, and models used in scientific research in the field of economics. This cannot be done in one article. Mathematical modeling of only three sections of economic theory is considered: macroeconomics, microeconomics, and state regulation of the economy. The review of research methods and models in the microeconomics section, which are available in the scientific research toolkit but have already been described in the macroeconomics section, has been omitted. Only effective, practice-tested models are used in the Review. We hope that this review will be useful to scientists involved in the indirect study of economic phenomena and processes.

Keywords: modeling; model; macroeconomics; microeconomics; sources; review

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1. Introduction

Improving the accuracy and reliability of the results of economic research is achieved on the basis of mathematics. In recent decades, mathematical modeling has become the main instrumental method for the indirect study of economic facts and processes.

A formalized mathematical description of stable quantitative laws of the economy has been actively conducted since the eighteenth century. Obviously, this review could not describe all the retrospective mathematical methods and models of the economy but it took the main ones, the most frequently used and tested by practice over the last two decades of the current century.

By analogy with the existing division of economic theory into macroeconomics, microeconomics, and state regulation of the economy, the Review provides an overview of the main methods of mathematical modeling in such a structural division.

2. Modeling of Macroeconomics

Table 1 presents the mathematical modeling of macroeconomics.

Table 1. Mathematical modeling of macroeconomics.

Method	Model	References
1. Statistical models of macroeconomics	Macroeconomic production functions	[1,2]
	Leontief model	[3,4]
2. Dynamic models of macroeconomics with discrete time	Dynamic Keynesian model	[5,6]
	Samuelson–Hicks model	[7,8]
	Dynamic Leontief model	[9,10]
	Neumann model	[11,12]
3. Linear dynamic systems	Linear dynamic element	[13,14]
	Multiplier	[15,16]
	Accelerator	[17,18]
	Inertial link	[19,20]
	Transmission function	[21,22]
	Oscillating link	[23,24]
4. Nonlinear dynamic systems	Dynamic element of a nonlinear dynamic system	[25,26]
	Nonlinear dynamic Keynesian model	[27,28]
	Market cycles in economy	[29,30]
	Optimal control of dynamic systems	[31,32]
	Pontryagin maximum principle	[33,34]
5. Small-sector nonlinear dynamic models of macroeconomics	Solow model	[35,36]
	The golden rule of accumulation	[37,38]
	One-sector model of optimal economic growth	[39,40]
	Three-sector model of the economy	[41,42]

Economics is an applied science; therefore, mathematical models of the economy arose as a result of the urgent need for social development. We agree that not all limitations reflecting the specifics of the objects of study are taken into account in the models. However, without models, as a rule, research is simply impossible.

1.1. Macroeconomic production functions. In macroeconomic production functions, the economy is considered as an unstructured unit, the input of which is resources and the output is the final product of the functioning of the economy. In this case, the resources are considered as arguments, and the final product is a function. In practice, the Cobb–Douglas production function is most often used:

$$F(K, L) = A{\cdot}K^{\alpha}{\cdot}L^{1-\alpha},$$

where $A > 0-$coefficient of neutral technical progress; $\alpha \in (0, 1)-$the coefficient of elasticity of output for capital K; and $(1 - \alpha)$ is for labor L.

With time-invariant parameters of the production function, the mathematical model is static [1,2].

1.2. Leontief model. In the Leontief model, the economy is structured and consists of a finite number of autonomous industries which produce one kind of product. For its production in a particular industry, its own product and products from other industries are used. The amount of products consumed for the production of a unit of product of the industry under study is taken into account by means of direct cost coefficients. These coefficients do not depend either on time or on the scale of production. Gross outputs of industries that ensure the production of the final product are determined according to the matrix of coefficients of direct costs for a given final product [3,4].

2.1. Dynamic Keynesian model. The gross output of the i-th product for the year x_i is divided into two parts: production consumption in all industries and nonproductive consumption. The net output of the i-th product is:

$$x_i - \sum_{j=1}^{n} a_{ij}x_j, \quad i = 1, \ldots, n,$$

where $\sum_{j=1}^{n} a_{ij}x_j$—production consumption of the i-th product by all industries.

If we equate the net output of each i-th product and the final demand for it y_i, then a system of equations is formed:

$$x_i - \sum_{j=1}^{n} a_{ij}x_j = y_i, \quad i = 1, \ldots, n,$$

which constitutes the static Leontief model.

Conclusion: Static macroeconomic models are still the best tool for system analysis of resource support for the production of products and services.

Linear dynamic models of macroeconomics with discrete time are represented by the dynamic Keynesian model, the dynamic Samuelson–Hicks model, the dynamic Leontief model, and the Neumann model. Here, mathematical methods and research models consider the economy as a simply connected system with discrete time.

In the dynamic Keynesian model, the economy is treated as a single dynamic element Y, and a time-varying endogenous variable is gross domestic product (GDP). GDP consists of four parts: nonproductive consumption fund C; gross private domestic investment I; government expenditures on the purchase of goods and services G, and net exports E. In the model, the economy is considered closed; thus, net exports equal zero and government expenditures are allocated to consumption and accumulation:

$$Y = C + I.$$

The model assumes that the demand for investment goods is constant and the demand for consumer goods in the next year is a linear function of the current year's GDP:

$$C_{t+1}^{D} = \underline{C} + cY_t,$$

where c is the lower limit of the nonproductive consumption fund; $0 < c < 1$—marginal propensity to consume. Dynamic Keynesian model arises if we equate the planned output of end-use goods with the projected demand for them:

$$Y_{t+1} = \underline{C} + cY_t + I.$$

This model can only be used for analysis and short-term forecasting of the economy. It is not suitable for long-term forecasting, since it does not reflect the reproduction process of capital. From a mathematical point of view, this model is a first-order linear finite-difference equation [5,6].

2.2. Samuelson–Hicks model. The Samuelson–Hicks model, at certain values of the parameters, is an oscillatory link and, in another case, it is represented by two first-order linear dynamic elements connected in series. The difference between the Samuelson–Hicks model and the dynamic Keynesian model is the rejection of the investments constancy and their introduction as a variable part, which is proportional to the GDP growth of the current year compared to the previous year:

$$Y_{t+1} = \underline{C} + cY_t + r(Y_t - Y_{t-1}) + I,$$

where $r-$acceleration coefficient, $0 < r < 1$.

From a mathematical point of view, the Samuelson–Hicks model is a second-order linear finite-difference equation. To find solutions of the dynamic model, finite-difference equations and Laurent transformations [7,8] are used.

2.3. Dynamic Leontief model. Leontief dynamic model of input–output balance reflects the reproduction process; thus, it is applicable to study the behavior of the economic system over sufficiently long time intervals while maintaining the technological structure [9,10].

2.4. Neumann model. The Neumann model is a generalization of the Leontief model, as it allows the production of one product in different ways. The model represents n products and m methods of their production, each j-th method is defined by the cost column vector a_j and the output column vector b_j per unit of process intensity:

$$a_j = \begin{pmatrix} a_{1j} \\ a_{2j} \\ \cdot \\ \cdot \\ \cdot \\ a_{nj} \end{pmatrix}, \; b_j = \begin{pmatrix} b_{1j} \\ b_{2j} \\ \cdot \\ \cdot \\ \cdot \\ b_{nj} \end{pmatrix}.$$

Cost and output matrices are formed from the input and output vectors:

$$A = (a_1,\ a_2, \ldots, a_m),\ B = (b_1, b_2, \ldots, b_m).$$

The input coefficients a_{ij} and output b_{ij} are non-negative. The implementation of any process requires the costs of at least one product, i.e., for each j there is at least one i such that $a_{ij} > 0$, and each product can be produced in at least one way, i.e., for each i there is some j such that $b_{ij} > 0$. Thus, each column of matrix A and each row of matrix B must have at least one positive element.

The Neumann model describes a closed economy in which the products produced in the previous production cycle (year $t-1$) are used to produce products in the next production cycle (year t):

$$Ax_t \le Bx_{t-1},\ x_t \ge 0,\ t = 1, 2, \ldots, T,$$

where $y_t = Ax_t-$cost vector for a given process intensity vector x_t:

$$x = \begin{pmatrix} x_1(t) \\ x_2(t) \\ \cdot \\ \cdot \\ \cdot \\ x_m(t) \end{pmatrix}$$

$z_t = Bx_{t-1}-$output vector.

It is assumed that the initial stock vector $Bx_0 > 0$ is given [11,12].

Neither the Leontief model nor the Neumann model is suitable for reflecting scientific and technological progress. In addition, these models do not reflect the reproductive process. Therefore, they can only be used for operational forecasting.

Conclusion: Dynamic models of macroeconomics with discrete time are not suitable for the analysis of reproduction and reflection of scientific and technological progress but can be successfully used for operational and short-term forecasting of economic processes and phenomena.

3.1. Linear dynamic element. The main results in the study of dynamic systems with continuous time were obtained in the study of technical systems within the framework of the theory of automatic control. The apparatus of differential equations was used as the main mathematical toolkit. The obtained scientific results of research are now successfully used in the economy.

A linear dynamic element of the n-th order is given by a linear differential equation [13,14]:

$$\sum_{j=0}^{n} a_j y^{(j)} = \sum_{i=0}^{n} b_i x^{(i)}.$$

3.2. Multiplier. Most often, there are elements of the zero order (multiplier, accelerator) and the first order (inertial link) seen in practice.

The multiplier is a linear static link given by the equation:

$$a_0 y = b_0 x \text{ or } y = \alpha x,\ \alpha = \frac{b_0}{a_0},$$

where α is the amplification factor (multiplier) [15,16].

3.3. Accelerator. The accelerator is a zero-order differentiator, the output of which is proportional to the input speed. For example, investment I can be expressed in terms of the rate of GDP change as follows:

$$I = r\frac{dY}{dt},$$

where r is the acceleration coefficient, i.e., an increase in the need for investment with an increase in GDP per unit [17,18].

3.4. Inertial link. The inertial link is given by a first-order differential equation [19,20]:

$$a_1 \frac{dy}{dt} + a_0 y = x(t).$$

3.5. Transmission function. The concept of the transfer function of a dynamic element is associated with the operator method for solving a differential equation.

The transfer function of series-connected elements is the relation of the output and input images:

$$G(s) = \frac{Y(s)}{X(s)} = \frac{Y_2(s)}{X_1(s)} = \frac{G_2(s)Y_1(s)}{X_1(s)} = G_1(s)G_2(s).$$

Thus, the transfer function of series-connected elements is equal to the product of their transfer functions.

The transfer function of parallel-connected elements with a summing link is equal to the sum (difference) of the transfer functions of the elements [21,22]:

$$G(s) = \frac{Y(s)}{X(s)} = \frac{Y_1(s) \pm Y_2(s)}{X(s)} = G_1(s) \pm G_2(s).$$

3.6. Oscillating link. An oscillatory link is used to model cyclical processes in the economy. The oscillatory link is given by a second-order differential equation:

$$a_2 \frac{d^2 y}{dt^2} + a_1 \frac{dy}{dt} + a_0 y = \sum_{i=0}^{n} b_i x^{(i)}(t)$$

with a negative discriminant made up of the coefficients on the left side of the equation [23,24].

$$a_1^2 - 4a_2 a_0 < 0$$

Conclusion: The apparatus of differential equations in the analysis of linear dynamic systems should be used to model cyclical processes in the economy.

4.1. Dynamic element of a nonlinear dynamic system. Dynamic systems are called nonlinear if they contain at least one nonlinear element.

The method of analysis of a nonlinear system depends on the type of nonlinearity. There are two main approaches: direct solution of nonlinear equations of a dynamic system by numerical integration on a computer and linearization of the system and the subsequent use of methods for studying linear dynamic systems.

Schematically, using a dynamic element as an example, it looks as follows. The dynamic element equation has the form:

$$F\left(y, y^{'}, \ldots, y^{(n)}, x, x^{'}, \ldots, x^{(n)}\right) = 0.$$

This equation has a solution with respect to the highest derivative:

$$y^{(n)} = f\left(y, y^{'}, \ldots, y^{(n)}, x, x^{'}, \ldots, x^{(n)}\right),$$

followed by a transition to a system of differential equations with respect to variables $y_1, \ldots, y_n$:

$$\begin{gathered} y_1 = y, \\ \frac{dy_1}{dt} = y_2, \\ \vdots \\ \frac{dy_{n-1}}{dt} = y_n, \\ \frac{dy_n}{dt} = f\left(y, y^{'}, \ldots, y^{(n)}, x, x^{'}, \ldots, x^{(n)}\right). \end{gathered}$$

Next, it is necessary to obtain an analytical or numerical solution of an equation or a system of differential equations [25,26].

4.2. Nonlinear dynamic Keynesian model. The nonlinear Keynesian dynamic model can be represented as a first-order nonlinear dynamic link:

$$\frac{dy}{dt} = f(y, I),$$

that is, GDP growth rate (y) is a function of GDP and investment. In the linear case:

$$f(y, I) = \underline{C} - (I - c)y + I.$$

It is obvious that

$$\frac{\partial f}{\partial y} < 0,\ \frac{\partial f}{\partial I} > 0$$

therefore, the rate of GDP growth slows down with an increase in GDP and it increases with an increase in investment.

Let us suppose that at $t = 0$, investments were equal to I_0 and the system was in some equilibrium state (y_E^0, I_0), the first component of which is determined from the equation of a nonlinear dynamic link:

$$f\left(y_E^0, I_0\right) = 0.$$

With an increase in investment from I_0 to $I = I_0 + \Delta I,\ \Delta I > 0$, the system will satisfy the equation

$$\frac{dy}{dt} = f(y, I),\ y(0) = y_E^0.$$

Let us represent GDP as a sum of constant and variable parts:

$$y(t) = y_E^0 + \eta(t),\ \eta(t) > 0,\ \eta(0) = 0.$$

The variable part $\eta(t)$ satisfies the equation

$$\frac{dy}{dt} = f\left(y_E^0 + \eta,\ I_0 + \Delta I\right),\ \eta(0) = 0.$$

If the investment increment ΔI is relatively small, then, with the evolutionary nature of the function $f(y, I)$, the variable part $\eta(t)$ is also relatively small; so, the right side of the last equation can be expanded in the vicinity of the point (y_E^0, I_0) in a Taylor series, discarding terms of the second and higher orders:

$$\frac{dy}{dt} = \frac{\partial f}{\partial y}(y_E^0,\ I_0)\eta + \frac{\partial f}{\partial t}\left(y_E^0, I_0\right)\Delta I,\ \eta(0) = 0.$$

After transferring the term containing η to the left side and dividing both parts by $\frac{\partial f}{\partial y}(y_E^0,\ I_0)$, we get the equation of the inertial link:

$$T\frac{d\eta}{dt} + \eta = \alpha\Delta I,\ \eta(0) = 0,$$

where $\frac{1}{T} = -\frac{\partial f}{\partial y}\left(y_E^0, I_0\right)$—generalized propensity to accumulate in the initial state;

$$\alpha = -\frac{\frac{\partial f}{\partial I}(y_E^0,\ I_0)}{\frac{\partial f}{\partial y}(y_E^0,\ I_0)} > 0.$$

From the last equation, it follows that the variable part of GDP will be equal to:

$$\eta(\mathrm{t}) = \alpha\Delta I\left(1 - e^{\frac{1}{T}}\right),$$

and GDP in general will change following the dependence:

$$y(t) = y_E^0 + \alpha\Delta I\left(1 - e^{\frac{1}{T}}\right),$$

in this case, the new equilibrium state of GDP will be equal to [27,28]:

$$y_E = \lim_{t\to\infty} y(t) = y_E^0 + \alpha\Delta I = y_E^0 - \frac{\frac{\partial f}{\partial I}\left(y_E^0,\ I_0\right)}{\frac{\partial f}{\partial y}\left(y_E^0,\ I_0\right)}\Delta I$$

4.3. Market cycles in economy. Nonlinear multiply connected systems have seven types of stability and can have several equilibrium states. The state of equilibrium in such systems can be either a fixed point or a closed trajectory (limit cycle). In both cases, particular solutions of differential equations are used.

Market cycles in the economy are described by a second-order linear dynamic model and are studied using a continuous analogue of the Samuelson–Hicks model or a continuous analogue of the nonlinear Goodwin model.

The Goodwin model [29,30] assumes that capital intensity k, population growth rate n, and labor productivity γ remain constant:

$$k = \frac{K_t}{Y_t} = const,$$

where K_t—capital (fixed and current assets); $n = \frac{N_{t+1} - N_t}{N_t} = const$, where N_t—population in year t; $\gamma = \frac{y_{t+1} - y_t}{y_t} = const$, $y_t = \frac{Y_t}{L_t}$—labor productivity, $Y_t - GDP$, L_t—the number of employees.

The model has two endogenous variables, λ_t and δ_t : where $\lambda_t = \frac{L_t}{N_t}$ −the share of employed people in the total population; $\delta_t = \frac{w_t L_t}{Y_t} = \frac{w_t}{y_t}$ −the share of the consumption fund in GDP, w_t is the annual wage rate.

The continuous analogue of the nonlinear Goodwin model has the form:

$$\begin{cases} \frac{d\delta}{dt} = (a\lambda - a_0)\delta \\ \frac{d\lambda}{dt} = (-b\delta + b_0)\lambda' \end{cases}$$

where $a = \frac{\alpha}{1+\gamma} > 0$, $a_0 = \frac{\alpha_0}{1+\gamma} > 0$; $b = \frac{1}{k(1+\gamma)(1+n)} > 0$, $b_0 = \frac{1-k[\gamma-n(1+\gamma)]}{k(1+\gamma)(1+n)}$.

4.4. Optimal control of dynamic systems. The control of a dynamic system is understood as a direct impact on the system in order to achieve a given result. Optimal control is understood as a choice from a set of alternative options for such control, which, according to a given criterion, is optimal. As an optimality criterion, a certain functional of the phase and control trajectories is chosen, which is subject to maximization (minimization).

The behavior of any nonlinear multiply connected system is described by the following equations of motion:

$$\frac{dy_i}{dt} = f_i(y, x, t),\ y_i(0) = y_i^0,\ i = 1, \ldots, n,$$

where y−vector of phase coordinates that specifies the state of the system; x−vector of external (input) setting and (or) disturbing influences on the system; y_i^0−initial values of phase variables.

If the disturbing actions are negligible, some of the setting actions become control actions, and others are given known functions of time; then, we arrive at the following equations for the controlled dynamic system:

$$\frac{dy_i}{dt} = f_i(y, u, t),\ y_i(0) = y_i^0,\ i = 1, \ldots, n,$$

where u−vector of control parameters, $u \in U$; U¯area of acceptable values of control parameters.

The control trajectory (control) $u(t)$ is called admissible if it is piecewise continuous, continuous at the discontinuity points on the left:

$$u(\tau) = u(\tau - 0) = \lim_{\substack{i \to \tau \\ t < \tau}} u(t),$$

and, moreover, for any $tu(t) \in U$.

If the control law is given, i.e., an admissible control trajectory $u(t)$ is defined, then the equations for the phase variables take the form:

$$\frac{dy_i}{dt} = f_i(y, u(t), t),\ y_i(0) = y_i^0,\ i = 1, \ldots, n,$$

thus, for any initial conditions $y(0) = y^0$, the solution is uniquely determined.

As an optimality criterion, a certain functional of the phase and control trajectories is chosen, which is subject to maximization (minimization) [31,32].

4.5. Pontryagin maximum principle. The necessary conditions for solving such a problem are given by the Pontryagin maximum principle [33,34].

The Pontryagin maximum principle is applied to a general control problem of the form

$$\max_{u(t) \in U} \int_0^T f_0(y, u, t)dt + F\left(y^T, T\right),$$

$$\frac{dy}{dt} = f(y, u, t),\ y(0) = y^0,$$

where $y = \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix}$ −column vector of phase variables that determine the state of the dynamic system;

$f(y, u, t) = \begin{pmatrix} f_1(y, u, t) \\ \vdots \\ f_n(y, u, t) \end{pmatrix}$ −column vector of the right parts of the equations of the system;

$y^0,\ y^T$ −initial and final values of the state vector;

$u = \begin{pmatrix} u_1 \\ \vdots \\ u_n \end{pmatrix}$ −column vector of control parameters;

U−area of possible values of control parameters;

$f_1(y, u, t)$−integrand of the control criterion.

The functions $f_i(y, u, t),\ F\left(y^T, T\right)$ are continuous and differentiable with respect to each argument. If the equation $u(t)$ is defined, then the trajectory of the system $y(t)$ is uniquely defined for a given initial condition $y(0) = y^0$. The search for the trajectory of the system, corresponding to the optimal control, is reduced to finding the saddle point of the Lagrange function in a nonlinear programming problem.

Conclusion: Economic phenomena and processes are characterized by nonlinearity. However, the use of the mathematical method of system linearization makes it possible to apply methods for studying linear dynamic systems for the analysis of economic entities.

5.1. Solow model. Small-sector nonlinear dynamic models of macroeconomics.

Nonlinear small-sector models are used to study long-term trends, growth factors, and assess the consequences of options for macroeconomic decisions.

The base model is the one-sector Solow model [35,36]. In this model, the economic system is considered as a single unstructured whole that produces one universal product. In this case, the product can be both consumed and invested. The model in its most aggregated form reflects the process of reproduction and allows for the analyzing of the relationship between consumption and accumulation in general terms.

The state of the economy is given by five endogenous variables:

$X - GDP$;
C−nonproductive consumption fund;
I−investments;
L−the number of employees;
K−basic production assets.

The model uses three exogenous indicators:

ν−annual growth rate of the number of employees;
μ−the share of fixed production assets retired during the year;
ρ−the rate of accumulation (share of gross investment in GDP).

Exogenous indicators are within the following limits: $-1 < \nu < 1, 0 < \mu < 1, 0 < \rho < 1$.

It is assumed that endogenous variables change over time, while exogenous indicators are constant.

The Solow model in absolute terms would be:

$$L = L_0 e^{\nu t};\ \frac{dK}{dt} = -\mu K + \rho X; K(0) = K_0;$$

$$X = F(K, L); I = \rho X; C = (1 - \rho)X.$$

This model takes into account two aggregated products (means of production and commodities) and two sectors. The first sector produces means of production, the second, consumer goods.

5.2. The golden rule of accumulation. The Solow model solves the problem known as the "Golden Rule of Accumulation" [37,38]. Its essence boils down to the fact that in a stationary mode, with a proper choice of the rate of accumulation, in a relatively short period of time after the start of the transition process, it is possible to maximize per capita consumption. Genuinely:

$$c^E(p) = (1-p)A(k^E)^\alpha = (1-p)A\left[\frac{\rho A}{\lambda}\right]^{\frac{\alpha}{1-\alpha}} = B[g(\rho)]^{\frac{1}{1-\alpha}},$$

where $B = \left[\frac{A}{\lambda^\alpha}\right]^{\frac{1}{1-\alpha}}, g(\rho) = \rho^\alpha(1-\rho)^{1-\alpha}$.

Thus, per capita consumption c is entirely determined by the function $g(\rho)$.

We have

$$\frac{dg}{d\rho} = \left(\frac{\rho}{1-\rho}\right)^\alpha \frac{\alpha-\rho}{\rho},$$

Therefore $\frac{dc^E}{d\rho} > 0$ for $\rho < \alpha$, $\frac{dc^E}{d\rho} < 0$ for $\rho > \alpha$.

Thus, the highest average per capita consumption is achieved at $\rho^* = \alpha$, i.e., the rate of accumulation should be equal to the elasticity of output for funds.

5.3. One-sector model of optimal economic growth. With $\rho = const$ and current consumption per employee $c(t) = C(t)/L(t)$,e, the Solow model is transformed into a one-sector model of optimal economic growth [39,40]:

$$\frac{dk}{dt} = f(k) - (\mu+\nu)k - c,\ k(0) = k_0,$$

since the quantity $\rho f(k)$ in the Solow model is replaced by $f(k) - c(t)$. The last equation is the main equation of the controlled system.

Specific consumption $c(t)$ is considered as a control parameter. Its admissible trajectory, as it is customary in optimal control theory, can be any piecewise continuous trajectory that satisfies the boundary condition:

$$0 < \underline{c} \leq c(t) \leq f(k(t)),$$

where $\underline{c}$—maximum permissible lower limit of specific consumption.

The task of the governing body of the economic system is to choose the value of current consumption in such a way that, over a long period of time, the discounted utility from consumption would be maximum:

$$\int_0^\infty e^{-\delta t} u(c(t))dt \to max,$$

where δ—the discount rate by which future utilities are reduced to the present (assuming that immediate consumption is more important than distant consumption); $u(c)$ is the consumption utility function.

5.4. Three-sector model of the economy. The process of reproduction is reflected in more details by a three-sector model of the economy, in which there are three aggregated products (objects of labor, means of labor, and consumer goods), and each of the three sectors produces its own product; namely, the material sector produces objects of labor, capital creating produces means of labor, and the consumer produces consumer goods [41,42].

When constructing a three-sector model of the economy, the following assumptions were made:

1. The technological structure is considered constant and is set using linearly homogeneous neoclassical production functions

$$X_i = F_i(K_i, L_i),$$

where $X_i,\ K_i, L_i$—output, fixed production assets and the number of people employed in the i-th sector;
2. The total number of employed L in the manufacturing sector changes with a constant growth rate ν;
3. There is no investment lag;
4. The depreciation coefficients of fixed production assets μ_i and direct material costs α_i of the sectors are constant;
5. The economy is closed, i.e., foreign trade is not considered;
6. The time t changes continuously.

Assumption (2) in discrete time has the following form (t is the number of the year):

$$\frac{L(t+1) - L(t)}{L(T)} = \nu,$$

And, upon transition to continuous time, it takes the form of a differential equation

$$\frac{dL}{dt} = \nu L,\ L(0) = L^0,$$

which has a solution:

$$L = L^0 e^{\nu t}.$$

From assumptions (3, 4), it follows that the change over the year of the fixed production assets of the i-th sector consists of two parts: depreciation ($\mu_i K_i$) and growth due to gross capital investments ($+I_i$), i.e.,

$$K_i(t+1) - K_i(t) = -\mu K_i(t) + I_i(t),\ i = 0,1,2,$$

or in continuous time

$$K_i(t+\Delta t) - K_i(t) = -[\mu K_i(t) + I_i(t)]\Delta t,$$

with $\Delta t \to 0$ we obtain differential equations for the main production assets of the sectors

$$\frac{dK_i}{dt} = -\mu K_i + I_i,\ K_i(0) = K_i^0,\ i = 0,1,2.$$

Thus, under the assumptions made, the three-sector model of the economy (with the signs of time omitted) in absolute terms takes the form:

$L = L(O)e^{\nu t}$—number of employees;
$L_0 + L_1 + L_2 = L$—distribution of the employed by sectors;
$\frac{dK_i}{dt} = -\mu K_i + I_i,\ K_i(0) = K_i^0,\ i = 0,1,2$—dynamics of funds by sectors;
$X_i = F_i(K_i, L_i),\ i = 0,1,2$—output by sectors;
$X_1 = I_0 + I_1 + I_2$—distribution of products of the fund-creating sector;
$X_0 = a_0 X_0 + a_1 X_1 + a_2 X_2$—distribution of products of the material sector.

With the help of a three-sector model, conditions are identified under which the economy falls into stagnation or balanced economic growth. It is proved that in a stationary state a three-sector economy has a technological optimum. Therefore, any change in the stationary state of the economy as a result of an external influence or a control decision can be assessed as positive if there has been a movement towards the optimum point and negative otherwise.

Conclusion: Based on the basic one-sector Solow model, an arsenal of models has been developed and recommended for a small-sector study of the state of the economy and determining its technological optimum.

3. Modeling of Microeconomics

Table 2 presents mathematical modeling of macroeconomics.

Table 2. Mathematical modeling of microeconomics.

Method	Model	References
6. Consumer behavior models	Consumer preferences and utility function	[43,44]
	Consumer behavior model	[45,46]
7. Producer behavior models	Firm model	[47,48]
	Duopoly model	[49,50]
8. Models of interaction between consumers and producers	Equilibrium price model	[51,52]
	Walrasian model	[53,54]

6.1. Consumer preferences and utility function. The household (consumer) is an important concept in microeconomics. The main problem in the study of consumer behavior is to establish the magnitude of his demand for purchased goods and services at given prices and his income.

A consumer's decision to buy a certain set of goods can be mathematically represented as a choice of a specific point in the space of goods. Let n be a finite number of goods under consideration; $x = (x_i, \ldots, x_n)^i -$column vector of volumes of goods purchased by the consumer for a certain period at given prices and income for the same period. The space of goods is the set of possible sets of goods x with non-negative coordinates:

$$C = \{x : x \geq 0\}.$$

In consumer choice theory [43,44], it is assumed that each consumer initially has his own preferences on some subset of the product space $X \subset \{x : x \geq 0\}$. This means that, for every pair $x \ni X$, $y \ni Y$, one of three relations takes place:

$x \succ y -$set x is preferred over y;
$x \prec y -$set x is less preferred than y.

$x \sim y -$for the consumer, both sets have the same degree of preference. Preference relations have the following properties:

- if $x \succ y, y \succ z$, then $x \succ z$ (transitivity);
- if $x > y$, then $x \succ y$ (unsaturation: a larger set is always preferable to a smaller one).

The preference relations of each consumer can be represented in the form of a preference indicator, i.e., a utility function $u(x)$ such that $x \succ y$ implies $u(x) > u(y)$ and $x \sim y$ implies $u(x) = u(y)$. For each consumer, such a representation is multivariate. The introduction of a utility function makes it possible to replace preference relations with the usual relations between numbers: greater than, less than, and equal to.

6.2. Consumer behavior model. In the model of consumer behavior [45,46], it is assumed that the consumer always seeks to maximize his utility and is constrained only by limited income:

$$\max_{x \in \delta \cap X} u(x) = \max_{px \in M} u(x)$$

This conditional extremum problem reduces to finding the unconditional extremum of the Lagrange function:

$$L(x) = u(x) + \lambda(M - px).$$

Necessary conditions for a local extremum:

$$\sum_{j=1}^{n} p_j x_j^* = M,$$

$$\frac{\partial L}{\partial x_i} = \frac{\partial u}{\partial x_i}(x_i^*) - \lambda^* p_i = 0,\ i = 1, \ldots, n.$$

These conditions really determine the maximum point, since the matrix U is negative definite.

Conclusion: The developed mathematical arsenal allows for the determination of the behavior of the consumer—his preferences and usefulness in the face of budget constraints—not on a qualitative but on a quantitative level.

7.1. Firm model. When studying the behavior of a manufacturer in the firm model [47,48], it is assumed that a manufacturing firm produces one type of product or several types but with a constant structure; X is the number of units of one type of product or the number of multiproduct units.

Each of the three aggregated types of resources (labor L, funds K, and materials M) has a certain number of varieties.

The technology of a firm is determined by its production function, which expresses the relationship between resource inputs and output:

$$X = F(x),$$

where $x = (x_1, \ldots, x_n)$—a column vector of possible costs for various types of resources.

It is assumed that $F(x)$ is twice continuously differentiable and neoclassical. Moreover, the matrix of its second derivatives is negative definite.

If the price of a unit of production is equal to p, and the price of a unit of a resource of the j-th type is $w_j,\ j = 1, \ldots, n$, then each cost vector x corresponds to a profit

$$\Pi(x) = pF(x) - wx,$$

where $w = (w_1, w_2 \ldots, w_n)$—row vector of resource prices.

In the presence of a natural restriction on nonnegativity of the sizes of resources involved in production, the problem of maximizing profit takes the form:

$$\max_{(x \geq 0)} [pF(x) - wx].$$

This is a nonlinear programming problem with n non-negativity conditions $x \geq 0$. The necessary conditions for its solution are the Kuhn–Tucker conditions:

$$\frac{\partial \Pi}{\partial x} = p\frac{\partial F}{\partial x} - w \leq 0,$$

$$\frac{\partial \Pi}{\partial x}x = \left(p\frac{\partial F}{\partial x} - w\right)x = 0,$$

$$x \geq 0.$$

If all types of resources are used in the optimal solution, i.e., $x^* > 0$, then the solution of the problem takes the form:

$$p\frac{\partial F(x^*)}{\partial x} = w \ or \ p\frac{\partial F(x^*)}{\partial x_j} = w_j,\ j = 1, \ldots, n,$$

that is, the optimal point, the value of the marginal product of a given resource, must be equal to its price.

7.2. Duopoly model. In the most general case of the duopoly model [49,50], two competitors produce one type of product in accordance with their production function

$$X_i = F_i\left(x^i\right),\ i = 1, 2.$$

The price of products depends on both issues:

$$p = p(X_1, X_2),$$

and, as output increases, the price falls:

$$\frac{\partial p}{\partial X_1} < 0,\ \frac{\partial p}{\partial X_2} < 0.$$

The resource price also depends on the volume of its purchases x_j^1, x_j^2 by the first and second firms:

$$w_j = w_j\left(x_j^1, x_j^2\right),\ j = 1, \ldots, n,$$

where prices rise as demand increases:

$$\frac{\partial w_j}{\partial x_j^1} > 0,\ \frac{\partial w_j}{\partial x_j^2} > 0.$$

Every firm seeks to maximize its profits. For example, the first firm should act as follows:

$$\max_{(X_1, x_1^1, \ldots, x_n^1)} \left[p(X_1, X_2)X_1 - \sum_{j=1}^{n} w_i\left(x_j^1, x_j^2\right)x_j^1 \right]$$

on condition $X_1 = F_1\left(x_1^1, \ldots, x_n^1\right)$.

The Lagrange function for this problem has the form:

$$\begin{gathered}
L\left(X_1, x^1, \lambda\right) = p(X_1, X_2)X_1 - \sum_{j=1}^{n} w_i\left(x_j^1, x_j^2\right)x_j^1 + \lambda\left(F_1\left(x_1^1, \ldots, x_n^1\right) - X_1\right), \\
\frac{\partial L}{\partial X_1} = p(X_1, X_2) + X_1\frac{\partial p}{\partial X_1} + X_1\frac{\partial p}{\partial X_2}\frac{\partial X_2}{\partial X_1} - \lambda = 0, \\
\frac{\partial L}{\partial x_j^{(1)}} = -w_j\left(x_j^1, x_j^2\right) - x_j^1\frac{\partial w_j}{\partial x_j^1} - x_j^1\frac{\partial w_j}{\partial x_j^2}\frac{\partial x_j^2}{\partial x_j^1} + \lambda\frac{\partial F_1}{\partial x_j^1} = 0,\ j = 1, \ldots, n, \\
\frac{\partial L}{\partial \lambda} = F_1\left(x_1^1, \ldots, x_n^1\right) - X_1 = 0.
\end{gathered}$$

Eliminating λ, we get the $(n+1)$ equation for determining the strategy $X_1, x_1^1, \ldots, x_n^1$ of the first firm:

$$\begin{gathered}
\left[p(X_1, X_2) + \left(X_1\frac{\partial p}{\partial X_1} + X_1\frac{\partial p}{\partial X_2}\frac{\partial X_2}{\partial X_1}\right)\right]\frac{\partial F_1}{\partial x_j^1} = w_j + x_j^{(1)}\left(\frac{\partial w_j}{\partial x_j^1} + \frac{\partial w_j}{\partial x_j^2}\frac{\partial x_j^2}{\partial x_j^1}\right),\ j = 1, \ldots, n, \\
X_1 = F_1\left(x_1^1, \ldots, x_n^1\right)
\end{gathered}$$

The solution of these equations depends on $\frac{\partial X_2}{\partial X_1}$ and $\frac{\partial x_j^2}{\partial x_j^1}$, $j = 1, \ldots, n$. The latter equations represent the expected response of the second firm to the strategy $X_1, x_1^1, \ldots, x_n^1$ of the first firm. Under different assumptions about this response, different solutions to the competitors' problem will be obtained in the duopoly model.

Conclusion: Based on the production functions, the above models allow for the determination of the optimal strategy of the company in a competitive environment.

8.1. Equilibrium price model. Models for establishing an equilibrium price in the processes of interaction between consumers and producers are based on the assumption that price changes depend on the difference between supply and demand: if demand is higher than supply, then the price increases; otherwise, it decreases.

The most well-known model for establishing an equilibrium price in the market for one product is the "cobweb" one [51]. In this model, demand is characterized by a decreasing aggregate demand function $\Phi(p)$, while supply is characterized by an increasing aggregate supply function $\psi(p)$. These functions are defined and continuous for all $p > 0$. Moreover,

$$\lim_{p\to 0}\Phi(p) = \infty, \quad \lim_{p\to\infty}\Phi(p) = 0,$$

$$\lim_{p\to 0}\psi(p) = 0, \quad \lim_{p\to\infty}\psi(p) = \infty.$$

The state of equilibrium is characterized by the equality of supply and demand:

$$\Phi(p) = \psi(p),$$

notably, by virtue of the assumptions made, the last equation has a unique solution p^E, so that the equilibrium state

$$\Phi\left(p^E\right) = \psi\left(p^E\right) = x^E$$

is unique.

The "cobweb" model makes it possible to implement the process of iterative approximation to the equilibrium price. Let us assume that at the initial moment of time the price p_0 is set, while the demand turned out to be less than the supply:

$$\Phi(p_0) < \psi(p_0),$$

then, in the model, we lower the price to a level at which demand will be equal to supply at the initial price:

$$\Phi(p_1) = \psi(p_0).$$

At the new price p_1, demand exceeds supply:

$$\Phi(p_1) > \psi(p_1),$$

therefore, we raise the price to the level p_2, at which

$$\Phi(p_2) = \psi(p_1),$$

and so on. Thus, the process described by the recurrent relation $\Phi(p_1) = \psi(p_1)$, $i = 1, 2, \ldots,$ converges.

To determine the forecast values of the equilibrium price in the world's youngest energy market in Russia, according to the data of 2004–2017, a recurrent neural network was built and tested in seven federal districts [52]. The simulation results for two districts are shown in Figures 1 and 2. Simulation made it possible to obtain a high accuracy of the forecast. The error for all districts was less than 2%. High-precision forecasting ensures that energy consumers operate in an equilibrium market with relatively low tariffs.

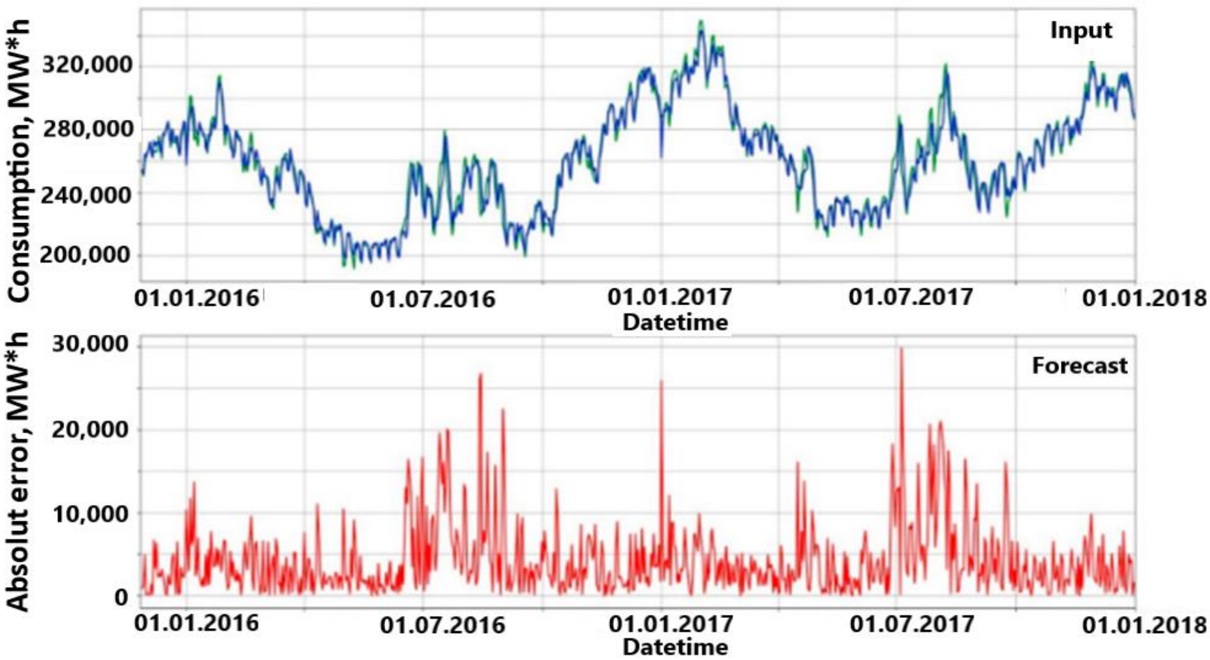

Figure 1. The results of the run on the test dataset for the Southern Federal District.

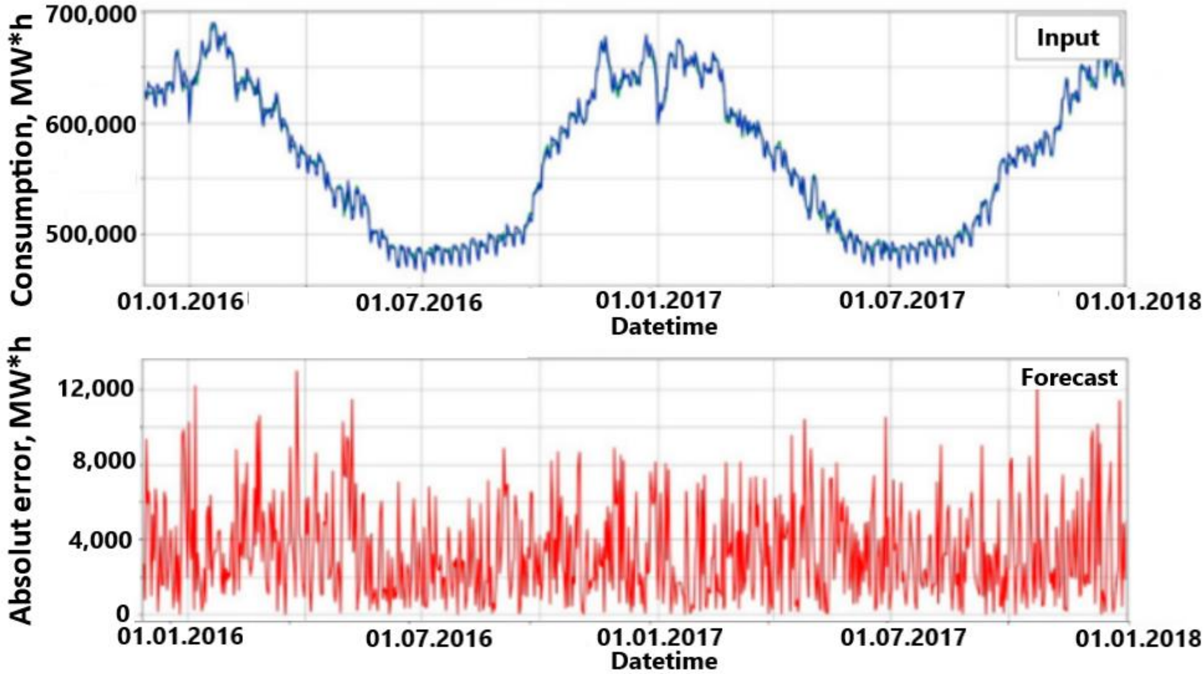

Figure 2. The results of the run on the test dataset for the Siberian Federal District.

8.2. Walrasian model. The Walrasian model [53,54] considers an economy with I consumers ($i = 1, \ldots, I$), m producers ($k = 1, \ldots, m$), and n types of goods ($j = 1, \ldots, n$). The row vector of prices will be denoted through $p = (p_1, \ldots, p_n)$ and the column vector of goods—through $x = (x_1, \ldots, x_n)$.

Each consumer has income $K(p)$ and has his own preference field for goods, which can be specified as a utility function $u(x)$. If we denote the set of possible sets of goods available to the consumer at prices p by $X(p) = \{x^* : x \in X,\ px \leq K(p)\}$, X is the domain of definition of $u(x)$; then, the consumer-demand function is given by the following way:

$$\Phi(p) = \begin{cases} x^* : x \in X(p),\ u(x^*) = \max\limits_{x \in X(p)} u(p) \\ 0,\ u(x^*) \neq \max\limits_{x \in X(p)} u(p), \end{cases}$$

i.e., the demand function is the set of available goods, each of which maximizes consumer utility at given prices p. It is assumed that the income of each consumer consists of two parts: the income pb_i from the sale of the initial stock of goods b_i and the income $I_i p$ as a result of the consumer's participation in production, i.e., $K_i(p) = pb_i + I_i p$.

Each manufacturer (firm) is set by its technological capabilities. Let us denote the input–output column vector of the k-th producer by $= (y_{k1}, \ldots, y_{kn})$: the positive components of this vector define the firm's output; the negative components define the costs. Therefore, the dot product py_k represents the profit of the firm. The technological capabilities of a firm are defined as the set of admissible input–output vectors Y_k. This set is called the production possibilities set.

The distribution of production is carried out by choosing the input–output vector y_k from the technological set of production possibilities Y_k for each producer $k = 1, \ldots, m$. The sum $Y = \sum\limits_{k=1}^{m} y_k$ represents the overall production process. Distribution of consumption is carried out by choosing a consumption menu $x_i \in X_i,\ i = 1, \ldots, l$ by each consumer. The sum $x = \sum\limits_{i=1}^{l} x_i$ is a vector of aggregate demand, some components of which may be negative if they represent supply (for example, labor).

The joint distribution of production and consumption is understood as such a set of consumption vectors and input–output vectors $(x_1, \ldots, x_i, \ldots, x_l,\ y_1, \ldots, y_k, \ldots, y_m)$, $x_i \in X_i,\ y_k \in Y_k$,, for which the aggregate demand matches the total offer:

$$x = \sum_{i=1}^{l} x_i = b + \sum_{k=1}^{m} y_k = b + y.$$

The set $\left(x_1^*, \ldots, x_i^*, \ldots, x_l^*, y_1^*, \ldots, y_k^*, \ldots, y_m^*, p^*\right)$ defines a competitive equilibrium in the Walrasian model if

$$x_i^* \in \Phi_{\mathrm{i}}\left(p^*\right),\ i = 1, \ldots, l,\ y_k^* \in \psi_k\left(p^*\right), k = 1, \ldots, m,$$

$$\sum_{k=1}^{m} y_k^* + b \geq \sum_{i=1}^{l} x_i^*,$$

$$p\left(\sum_{k=1}^{m} y_k^* + b\right) = p^* \sum_{i=1}^{l} x_i^*.$$

In this case, p^* is called the vector of competitive prices, and the last two equations are called the Walras's law.

Conclusion: The above mathematical models are of great practical importance in the processes of establishing an equilibrium price in the interactions of consumers and producers.

Despite the fact that most of the considered mathematical models of macroeconomics and microeconomics were developed relatively long ago, at the beginning and in the middle of the last century, they have not lost their relevance to the present day and are widely demanded by practitioners in the study of economic phenomena and processes. This is evidenced by a far from complete list of references cited.

4. Modeling the Regulation of the Economy

Table 3 presents mathematical modeling of regulation of the economy.

Table 3. Mathematical modeling of regulation of the economy.

Method	Model	References
9. Mathematical models of market economy	Modeling the labor market	[55,56]
	Modeling the money market	[57,58]
	Financial market models	[59,60]
	Forecasting currency crises and financial risks	[61,62]
10. Modeling inflation	Modeling inflation	[63,64]
	Impact of inflation on production	[65,66]
11. Mathematical models of state regulation of the economy	Taxes in a three-sector economy	[67,68]
	Impact of higher taxes on production and consumption	[69,70]
12. Modeling foreign trade	Model of an open three-sector economy	[41,71]
	Conditions for entering the foreign market	[72,73]
	The golden rule of foreign trade	[74,75]
	The impact of foreign trade on the national economy	[76,77]
13. Modeling the goal of social development	Mathematical theory of public choice	[78,79]
	Models of cooperation and competition	[80,81]
	Simulation of scientific and technological progress	[82,83]

9.1. Modeling the labor market. The labor market is described using three dependencies: demand functions, supply functions, and equilibrium conditions [55,56]. In equilibrium, the marginal product of labor in value terms is equal to the wage rate:

$$p\frac{\partial F}{\partial L} = w,$$

where $p-$product price; $F = F(K, L)$, wherein $K-$funds, and $L-$number of employees.

Assuming that all factors of production, except labor, are fixed, we obtain the necessary condition for the maximum profit:

$$\frac{\partial \Pi}{\partial L} = p\frac{\partial F}{\partial L} - w = 0.$$

9.2. Modeling the money market. The theory of demand for money in the classical model is based on the hypothesis that the total demand for money M^D) (is a function directly proportional to money income):

$$M^D = kY_p,$$

Y_p-gross domestic product. Money offer (M^S) considered as a fixed, exogenously given, quantity [57].

In this way, the further analysis of a portfolio should be moved from the rather obvious two-dimensional "profitability-risk" analysis to three-dimensional "profitability-reliability—riskiness" analysis. Thus, analyzing the surface from one side, there is a set of guarantees, from another, a set of survival functions. As a result, this surface provides all the information for decisions' possibility, profitability, reliability, and riskiness levels evaluation. Thus, it becomes clear that an investor is directly interested in two investment features. This is the profitability possibilities and guarantee, or reliability of each possibility, which is measured in probability p, that possible profitability ξ will be not smaller than our selected profitability x.

$$\overline{F}(x) = P\{\xi \geq x\}\ p$$

Thus, we can see that an investor, in principle, should fully know the probability distribution of profitability possibilities

$$F(x) = P\{\xi < x\}$$

Often, if the mean value of possibilities and possibilities' variance is given, the probability distribution of these possibilities is also known. However, it is not always the case. Usually, knowing mean value and variance does not allow for describing fully the probability distribution and, in turn, the reliability and survival function

$$\overline{F}(x\ ) = 1 - F(x)$$

Also, and what is especially important, investor's risk usually goes beyond assets and portfolio riskiness and this riskiness is only one of the factors influencing the extent of investor's risk. At that time, reliability of outcome entirely rests on the profitability possibilities distribution function [58].

9.3. Financial market models. Suppose that a loan is provided in the amount of $S(0)$ with the condition that the amount of $S(T)$ be returned in time T. Per unit of loan, the lender will receive a profit:

$$r_T = \frac{S(T) - S(0)}{S(0)}.$$

The value of r_T is called the efficiency of a financial transaction or the interest rate.

Another indicator of the effectiveness of a financial transaction is the discount—the ratio of profit to the amount returned:

$$d_T = \frac{S(T) - S(0)}{S(T)}.$$

These values are in the following ratios:

$$r_T = \frac{d_T}{1 - d_T}.$$

If we consider the flow of payments from the standpoint of one of the participants, then the result of such a distributed operation can be measured by bringing all payments to the initial point in time. This value is called net present value:

$$NPV = \sum_{k=1}^{N} S_k \frac{1}{(1+r)^{t_k}},$$

where $S_k -$payments on the interval [1,N]; $r-$discount rate; and $t_k -$time of payment.

Consider the problem of optimizing a portfolio of securities. Let there be n types of securities from which the investor forms a portfolio. These papers are characterized by efficiency $R_1,\ R_2, \ldots, R_n$, which are random variables with known mathematical expectations $MR = m_i$ and the known covariance matrix $B = \left\| cov\left(R_i, R_j\right) \right\|$, in particular $cov\left(R_i, R_j\right) = DR_i = \sigma_i^2$.

If the investor has distributed his capital in shares $\Theta_i,\ 0 \leq \Theta_i \leq 1,\ \sum_{i=1}^{n} \Theta_i = 1$, into different securities, then the efficiency of the formed portfolio

$$R_p = \sum_{i=1}^{n} \Theta_{\rm i} {\rm R}_{\rm i},$$

Moreover, this random efficiency has the following mathematical expectation and variance:

$$MR_p = M\left(\sum_{i=1}^{n} \Theta_{\mathrm{i}}\mathrm{R}_{\mathrm{i}}\right) = \sum_{i=1}^{n} \Theta_{\mathrm{i}}\mathrm{MR}_{\mathrm{i}} = \sum_{i=1}^{n} \Theta_{\mathrm{i}}\mathrm{m}_{\mathrm{i}},$$

$$\sigma_p^2 = DR_p = D\left(\sum_{i=1}^{n} \Theta_{\mathrm{i}}\mathrm{R}_{\mathrm{i}}\right) = cov\left(\sum_{i=1}^{n} \Theta_{\mathrm{i}}\mathrm{R}_{\mathrm{i}}, \sum_{j=1}^{n} \Theta_{\mathrm{j}}R_j\right) = \sum_{i=1}^{n}\sum_{j=1}^{n} \Theta_{\mathrm{i}}\Theta_{\mathrm{j}}\mathrm{cov}(R_i, R_j).$$

Distribution $(\Theta_1, \ldots, \Theta_{\mathrm{n}}), 0 \leq \Theta_{\mathrm{i}} \leq 1, \sum_{\mathrm{i}=1}^{\mathrm{n}} \Theta_{\mathrm{i}} = 1$ is called the portfolio structure. As a result, we obtain the following task of optimizing a portfolio of securities:

$$min \sum_{i=1}^{n} b_{ij}\Theta_{\mathrm{i}}\Theta_{\mathrm{j}}, b_{ij} = \mathrm{cov}(R_i, R_j),$$
$$\sum_{i=1}^{n} \Theta_{\mathrm{i}} = 1,$$
$$\sum_{i=1}^{n} m_i\Theta_{\mathrm{i}} = \mathrm{m_p},$$
$$\Theta_1 \geq 0, \ldots, \Theta_{\mathrm{n}} \geq 0,$$

where m_p—the value of the average portfolio efficiency chosen by the investor.

This is the problem of minimizing a quadratic form in n variables $\Theta_1, \ldots, \Theta_{\mathrm{n}}$, related by two relations

$$\sum_{i=1}^{n} \Theta_{\mathrm{i}} = 1, \sum_{i=1}^{n} m_i\Theta_{\mathrm{i}} = \mathrm{m_p},$$

as well as conditions $\Theta_i \geq 0,\ i = 1, \ldots, n$, i.e., a Markowitz quadratic programming problem. Its solution using the Lagrange function in mathematical economics is known [59,60].

9.4. Forecasting currency crises and financial risks. The following probabilistic model can be used to predict the logarithmic gain (per day) of financial assets:

$$\delta_t = \xi_t + \xi_t^n,\ t = 0, 1, \ldots,$$

$$\xi_t = \mu + \sigma_t\varepsilon_t,\ \xi_t^n = \eta_t J_t,$$

$$ln\sigma_t^2 = a_0 + a_1 ln\sigma_{t-1}^2 + a_2\left|\frac{\varepsilon_{t-1}}{\sigma_{t-1}}\right| + a_3\frac{\varepsilon_{t-1}}{\sigma_{t-1}} + \xi_t,$$

$$\varepsilon_t - N(0,1),\ cov(\varepsilon_t, \varepsilon_{t'}) = 0 \text{ при } t \neq t',\ \eta_t = \begin{pmatrix} 0 & 1 \\ 1-p & p \end{pmatrix},$$

$$J_t - N(y, \gamma),\ M\xi_t = 0.$$

According to this model, the logarithmic profit has two components: regular ξ_t, generated by "long money", and jumping ξ_t^n, generated by "short money".

In the regular component, the average is close to zero $\mu \approx 0$; the standard deviation σ_t is determined from the statistical ratio established from the past data, while $\varepsilon_t,\ t = 0, 1, 2, \ldots$, is a sequence of uncorrelated standard normal values. Thus, the regular component is a mixture of normal distributions.

At any time t, with a probability of $1 - p$, the jump component does not appear and, with a probability p, there is a jump in the average value y with a standard deviation γ.

Model parameters $\mu, a_0, a_1, a_2, a_3, p, y, \gamma$ are determined using the maximum likelihood method and other mathematical and statistical methods based on actual data:

μ, a_0, a_1, a_2, a_3 – by time series $\gamma_t, \sigma_t,\ t = 1, T;\ p, \Theta, \gamma$ – over a subset of time points $\hat{T}$, in which the jumps took place, according to the following formulas:

$$\hat{p} = \frac{T^t}{T},\ \hat{y} = \frac{1}{T^t}\sum_{i \ni T}\Theta_{\mathrm{i}}, \hat{y}^2 = \frac{1}{\mathrm{T}^{\mathrm{t}} - 1}\sum_{i \ni T}\left(\Theta_{\mathrm{i}} - \hat{\Theta}\right)^2,$$

where T^t – the number of time points at which the jumps took place. If, at time t, the portfolio is formed $V = V_t$ out of n assets $\left(\Theta_i = \frac{V_i}{V},\ \sum\limits_{i=1}^{n} V_i = V\right)$, then the logarithmic profit of the portfolio at the next moment of time will be equal to

$$\hat{\delta}_{i+1} = \sum_{i=1}^{n}\Theta_{\mathrm{i}}\hat{\delta}^i_{i+1},$$

where $\hat{\delta}^i_{i+1}$ – predicted value of the logarithmic profit of the i-th asset.

The potential amount of loss is considered as a risk L_{i+1}, the corresponding quantile K_q, corresponding to the probability q :

$$\begin{gathered} L_{i+1} = -V_i K_q\sqrt{\Theta\sum\Theta'} > 0, \\ P\left\{\hat{\delta}_{i+1} < K_q\right\} = q,\ K_q < 0, \end{gathered}$$

where $\Theta = (\Theta_0, \ldots, \Theta_n)$ – portfolio structure vector; $\sum$ – covariance matrix of logarithmic asset returns [61,62].

Conclusion: At the conceptual level, a comparison was made between the monetarist and Keynesian approaches to forecasting and regulating the market economy in the context of the segments of the economy: labor, monetary and financial, and credit. The model for forecasting financial risks and currency crises is given.

10.1. Modeling inflation. Consider three common generalized autoregressive conditional heteroscedasticity (GARCH) models that are employed to model inflation uncertainty. First, consider a standard one, namely the GARCH model (referred to as GARCH hereinafter):

$$\pi_t = \alpha + \epsilon_t,\ \epsilon_t \sim N\left(0, \sigma_t^2\right)$$

$$\sigma_t^2 = \beta + \gamma\sigma_{t-1}^2 + \delta\epsilon_{t-1}^2,$$

where π_t is the inflation rate, σ_0^2 is constant, and $\epsilon_t = 0$. To make sure the variance process is always stationary, we impose the restriction $\gamma + \delta < 1$. It can be clearly seen that the conditional variance σ_t^2 representing a proxy for the inflation volatility is determined by past data and the model parameters.

Another common GARCH model that is widely used in modelling inflation uncertainty is the GARCH-GJR model. The GARCH-GJR model accounts for asymmetric (leverage) effects of positive and negative disturbances on the conditional variance. The conditional variance equation is defined as follows:

$$\sigma_t^2 = \beta + \gamma\sigma_{t-1}^2 + [\delta + \theta 1(\epsilon_{t-1} < 0)]\epsilon_{t-1}^2,$$

where 1 $(\cdot)$ denotes an indicator function. The parameter θ captures the asymmetric effect: if $\theta > 0$, a negative shock would have a greater impact on inflation uncertainty; if $\theta < 0$, a negative shock would lower inflation uncertainty; and if $\theta = 0$, there is no asymmetric effect documented and, thus, this specification becomes the standard GARCH model.

The last one we consider is the GARCH in mean model (referred to as GARCH-M) which accounts for potential volatility feedback on the inflation rates:

$$\pi_t = \alpha + \lambda\sigma_t^2 + \epsilon_t;\ \epsilon_t \sim N\left(0, \sigma_t^2\right),$$

$$\sigma_t^2 = \beta + \gamma\sigma_{t-1}^2 + \delta\Big(\pi_{t-1} - \alpha - \lambda\sigma_{t-1}^2\big)^2.$$

The effect of inflation volatility on inflation itself is captured by the parameter λ: when $\lambda > 0$, inflation uncertainty has a positive impact on the inflation rate; when $\lambda < 0$, inflation uncertainty has a negative impact on the inflation rate; and when $\lambda = 0$, inflation uncertainty has no impact on the inflation rate and, thus, this specification reduces to the standard GARCH model [63,64].

10.2. Impact of inflation on production. There are two points of view regarding the impact of inflation on production. Keynesians believe that controlled inflation is the source of growth. Monetarists believe that controlled inflation causes a short-term increase in production, which then stops. Both approaches are based on the premise that the behavior of prices lags somewhat behind changes in the money supply. Keynesians argue their position from the condition of maximum profit at the national level:

$$p\frac{\partial F}{\partial K} = r,$$

where $p-$price level;

$F(K, L)-$production function of the national economy;
$r-$a rate of return roughly equal to the interest rate.

If there is more money, then the interest rate must decrease; therefore, the marginal product of capital must decrease $\frac{\partial F}{\partial K}$, what is observed with the growth of capital. Thus, a fall in the rate of profit leads to a decrease in the marginal product of capital, which causes an increase in the demand for investment goods and, as a result, production increases and unemployment decreases.

The reasoning of the monetarists is based on the main macroeconomic equation and the pricing equation:

$$\pi - \pi_{-1} = \lambda\Big(y_{-1} - y^E\Big),$$

where $\pi, \pi_{-1}-$growth rates of prices (inflation rate) at the current and past points in time;

$$y = logY,\ y^E = logY^E,$$

where $Y^E Y,\ Y^E-$current and established volumes of GDP.

Let us introduce the notation:

$p = logP,\ m = logM,\ e = logY$. Then, the main macroeconomic equation can be written as:

$$p = m - y + const.$$

Taking the difference of the last equations at adjacent times, we get:

$$p - p_{-1} = m - m_{-1} - (y - y_{-1})$$

or

$$\pi = \dot{m} - (y - y_{-1}),$$

where $\pi = (p - p_{-1})-$the rate of price growth or the rate of inflation;

$m = (m - m_{-1})-$the growth rate of the money supply.

To study the effect of inflation on production, the following system of equations is considered:

$$\begin{cases} \pi = \pi_{-1} + y - y^E, \\ \pi = \frac{1}{2}\big(\dot{m} + \pi_{-1} + y - y^E\big). \end{cases}$$

Steady state $\dot{m} = 0, y = y_{-1} = y^E,\ \pi = 0$ [65,66].

Conclusion: The mathematical study of the mechanism of the emergence and self-sustaining of inflation using a three-sector model of the economy is given. The study of inflation is based on the main macroeconomic equation, according to which the supply of money and the demand for them are in dynamic equilibrium.

11.1. Taxes in a three-sector economy. Since the three-sector economy is considered as closed, the gross income of each sector is spent in the following four main areas: for the purchase of materials; for the purchase of investment goods; for the payment of salaries and bonuses; for the payment of taxes. Therefore, the balances of income and expenditure of the sectors will be written as follows:

$$\begin{aligned} p_0X_0 &= p_0a_0X_0 + p_1s_0X_1 + w_0L_0 + t_0X_0, \\ p_1X_1 &= p_0a_1X_1 + p_1s_1X_1 + w_1L_1 + t_1X_1, \\ p_2X_2 &= p_0a_2X_2 + p_1s_2X_1 + w_2L_2 + t_2X_2, \end{aligned}$$

where p_i—product price i—th sector;

w_i—wages with bonuses per one employed in the i-th sector;
s_i—the share of the i-th sector in the distribution of products of the fund-creating sector;
t_i—tax rate per unit of output of the i-th sector.

Using the commodity output of the sectors, we transform the cost balances to the form:

$$\begin{aligned} p_0(1-a_0)X_0 &= p_1s_0X_1 + w_0L_0 + t_0X_0, \\ p_1(1-s_1)X_1 &= p_1a_1X_1 + w_1L_1 + t_1X_1, \\ p_2X_2 &= p_0a_2X_2 + p_1s_2X_1 + w_2L_2 + t_2X_2. \end{aligned}$$

Let us add up these balances and move to the left side all the terms containing the prices for the products of the sectors as a multiplier:

$$p_0[(1-a_0)X_0 - a_1X_1 - a_2X_2] + p_1X_1(1-s_0-s_1-s_2) + p_2X_2 = \sum_{i=0}^{2} w_iL_i + \sum_{i=0}^{2} t_iX_i.$$

Since there are material and investment balances:

$$(1-a_0)X_0 = a_1X_1 + a_2X_2,$$

$$s_0 + s_1 + s_2 = 1,$$

then the coefficients at prices $p_0,\ p_1$ are equal to zero; therefore, as a result, we obtain a balance of supply and demand for commodities [67,68]:

$$p_2X_2 = \sum_{i=0}^{2} w_iL_i + \sum_{i=0}^{2} t_iX_i.$$

11.2. Impact of higher taxes on production and consumption. In a closed economy, the only source of consumption is the own production of consumer goods. Therefore, it is the behavior of specific outputs of sectors that determines consumption.

The regulatory impact of the state is to change tax rates in all three sectors of the economy: $dt_0,\ dt_1, dt_2$. The direct impact of this impact is presented in the form of a pseudo-increment in the tax burden:

$$d\hat{t}^p = \sum_{i=0}^{2} x_i dt_i.$$

There will be an increase in taxes if the pseudo increment is positive: $d\hat{t}^p > 0$. In particular, this situation includes the usual increase in taxes in the sectors of the economy: $dt_0 > 0,\ dt_1 > 0, dt_2 > 0$.

Since $v > 0$, $d\hat{t}^p > 0$, then the sign of ds_2 is determined by the sign of the expression

$$\hat{v}(s_2) = \sum_{i=0}^{2} a_i v_i(s_2) t_i x_i(s_2),$$

wherein $\hat{v}(0) = 0, \hat{v}(1) = 0$ and, of the three terms of the last sum, only the last can be negative for $s_2 < s_2^*$, so for $s_2 > s_2^*$ $\hat{v}(s_2) > 0$.

When studying the sign $\hat{v}(s_2)$ for $0 < s_2 < s_2^*$, the following circumstances must be taken into account: s_2 all functions $v_i(s_2)$ are growing and $v_2(s_2) < 0$, $s_2 < s_2^*$, $v_2(s_2^*) = 0$; specific output $x_0(s_2)$ at $s_1^0 > a_1$ first increases, then reaches a maximum at $s_2 = \hat{s}_2^*$, after which it decreases and $x_2(s_2)$ is growing. Therefore, near $s_2 = 0$, generally speaking, it is possible $\hat{v}(s_2) < 0$. However, in most cases of practical interest, s_2 differs significantly from zero, since the situation $s_2 = 0$ is "production for production"; so $\hat{v}(s_2) > 0$ at $s_2 > \underline{s_2}$ $\left(\hat{v}(\underline{s_2}) = 0\right)$.

At $\hat{v} > 0$, tax increase $\left(\sum\limits_{i=0}^{2} x_i dt_i > 0\right)$ leads to the overflow of investment resources into the consumer sector $(ds_2 > 0)$; so, when $s_2 < \hat{s}_2^*$, this has the effect of increasing the production of consumer goods and at $s_2 > \hat{s}_2^*$—reduction in production.

The production of investment goods is reduced with an increase in taxes since the overflow of investment resources into the consumer sector $(ds_2 > 0)$ occurs primarily at the expense of the fund-creating sector $\left(\frac{ds_1}{s_1} = -\frac{q_1}{q_2} \cdot \frac{ds_2}{s_2};\ q_1 > 0,\ q_2 > 0\right)$. Production of materials at $s_2 > \hat{s}_2^*$ is also declining, although the tax burden on capital-goods sectors may be reduced.

The situation of "redistribution of the tax burden" is characterized by equality:

$$(x_0 dt_0 + t_0 dx_0) + (x_1 dt_1 + t_1 dx_1) + (x_2 dt_2 + t_2 dx_2) = 0,$$

in which each of the brackets is the actual change in the tax burden on the corresponding sector [69,70].

Conclusion: The fiscal function of taxes consists in the tax burden, determined by the state's expenses for its functions. Mathematical models provide a search for the optimal tax burden for business entities.

12.1. Model of an open three-sector economy. When forming the model of an open three-sector economy, the following changes are introduced into the model of a closed three-sector model of the economy:

- in the income part of the investment balance, the term Y_1 is added—the import of investment goods;
- in the expenditure part of the material balance, the term Y_0 is added—the export of materials;
- on the consumer market, along with its own production X_2, imports of consumer goods Y_2 are added;
- finally, the foreign trade balance is added.

As a result, the model of an open three-sector economy takes the following form.

The technological structure is in the form of linearly homogeneous production functions:

$$X_i = F_i(K_i, L_i),\ \ i = 0, 1, 2.$$

Dynamics of the total number of employees:

$$L = L(0)e^{vt}.$$

Dynamics of sectors of fixed production assets:

$$\frac{dK_i}{dt} = -\mu K_i + I_i,\ \ K_i = K_i(0),\ i = 0, 1, 2.$$

Labor balance:

$$L = L_0 + L_1 + L_2.$$

Investment balance:

$$X_1 + Y_1 = I_0 + I_1 + I_2.$$

Material balance:

$$(1 - a_0)X_0 = a_1 X_1 + a_2 X_2 + Y_0.$$

Foreign trade balance:

$$q_0 Y_0 = q_1 Y_1 + q_2 Y_2,$$

where q_0, q_1, q_2—world prices for products of the material, capital creating, and consumer sectors.

We introduce the following relative indicators:

$\theta_i = \frac{L_i}{L}$, $s_i = \frac{I_i}{X_i + Y_i}$—shares of the i-th sector in the distribution of labor and investment resources;
$f_i(k_i) = \frac{F_i(K_i, L_i)}{L_i}$—industry productivity of the i-th sector;
$x_i = \frac{X_i}{L}$—economic productivity of the i-th sector;
$k_i = \frac{K_i}{L_i}$—capital–labor ratio of one employed in the i-th sector;
$y_0 = \frac{Y_0}{L}$—net export of materials per employee;
$y_1 = \frac{Y_1}{L}$—net import of investment goods per employee;
$y_2 = \frac{Y_2}{L}$—net imports of consumer goods per employee.

Then the model of an open three-sector economy in relative terms will be written as follows:

$$\begin{gathered} x_i = \theta_i f_i(k_i),\ i = 0,1,2; \\ \frac{dk_i}{dt} = -\lambda_i k_i + \frac{s_i}{\theta_i}(x_1 + y_1),\ \lambda_i = \mu_i + v; \\ k_i(0) = \frac{K_i(0)}{\theta_i L(0)},\ i = 0,1,2; \\ \theta_0 + \theta_1 + \theta_2 = 1,\ \theta_i \geq 0,\ i = 0,1,2; \\ s_0 + s_1 + s_2 = 1,\ s_i \geq 0,\ i = 0,1,2; \\ (1 - a_0)x_0 = a_1 x_1 + a_2 x_2 + y_0,\ y_0 \geq 0; \\ q_0 y_0 = q_1 y_1 + q_2 y_2,\ y_1 \geq 0,\ y_2 \geq 0. \end{gathered}$$

In the given notation of the model, internal cost balances are not considered, since their form depends on the type of behavior of the sectors, whether they act in cooperation or compete with each other [41,71].

12.2. Conditions for entering the foreign market. The expediency of the entry of the national economy into the world market is possible under the following options: without changing the existing distribution of resources, i.e., only by regulating the components of foreign trade and with a change in the existing distribution of resources. Both options are possible if the economy is in a state of autarky, i.e., when the volume of foreign trade is small. From a mathematical point of view, this means that it is possible to linearize nonlinear dependencies by discarding quadratic terms and terms of a higher order (with respect to $y_{i,}\ \ i = 0,1,2$).

Since $y_0,\ y_2$ enter the model in a linear way, it is necessary to linearize only the specific outputs of sectors that depend on y_i nonlinearly.

When the national economy enters the world market without changing the existing distribution of resources, i.e., with constant $\theta_i,\ s_i$, the specific outputs of the sectors will be:

$$x_1 = x_1^0 + \frac{\alpha_1 y_1}{(1 - \alpha_1)},\ x_i = x_i^0 + \frac{\alpha_i x_i^0 y_1}{(1 - \alpha_1) x_1^0},\ \ i = 0,2.$$

Let us substitute the last expression into the material balance equation:

$$(1-a_0)\left[x_0^0+\frac{\alpha_0 x_0^0 y_1}{(1-\alpha_1)x_1^0}\right]=a_1\left[x_1^0+\frac{\alpha_1 y_1}{(1-\alpha_1)}\right]+a_2\left[x_2^0+\frac{\alpha_2 x_2^0 y_1}{(1-\alpha_1)x_1^0}\right]+y_0,$$

but in a state of autarky $(1-a_0)x_0^0=a_1x_1^0+a_2x_2^0$, therefore,

$$y_0=\frac{y_1}{(1-\alpha_1)x_1^0}\left[\alpha_0(1-a_0)x_0^0-\alpha_1a_1x_1^0-\alpha_2a_2x_2^0\right].$$

Since $\alpha_0<\alpha_1$, $\alpha_0<\alpha_2$, then the expression in square brackets is negative, so $y_0<0$, i.e., a small import of machinery and equipment in the amount y_i cannot be compensated by the corresponding export of materials. Thus, the entry of a resource-based national economy into the world market strengthens its focus on raw materials, since it requires the transfer of additional resources to the material sector.

For the expediency of foreign trade according to the second option, the following two conditions must be met:

$a_1q_0\geq(1+\gamma)q_1$ (the cost of selling materials needed to produce a unit of investment goods is not less than the cost of acquiring such a unit with a load γ);

$$\frac{\alpha_0+{}^{b_0}/_{b_1}}{\alpha_1}>\delta_2.$$

These two conditions, which ensure $dx_1+dy_1>0$, are quite strict, especially the second condition, according to which the shares of labor and investment resources $(\theta_1,\ s_1)$ directed to the fund-creating sector should be as follows:

$$(1-\alpha_0)\theta_1\geq\alpha_0\theta_0,\ (1-\alpha_1)s_1\geq\alpha_1 s_0,$$

i.e., much higher than observed values in the real economy. Therefore, the first option is more appropriate [72,73].

12.3. The golden rule of foreign trade. The "golden rule of foreign trade" is understood as such a choice of structural and foreign trade parameters (θ,s,y), in which all balances and specific consumption are met to the maximum. Specific consumption is formed as the sum of domestic production and imports of consumer goods per employee. The problem is posed and solved in a stationary state and in specific indicators.

The model of an open three-sector economy in a stationary state, with Cobb–Douglas production functions and in specific indicators, is written as follows.

Economic productivity of sectors:

$$x_i=B_i\theta_i^{1-\alpha_i}s_i^{\alpha_i}s_i^{a_i}(x_1+y_1)^{a_i},\ B_i=A_i\lambda_i^{-a_i},\ i=0,1,2.$$

Labor balance:

$$\theta_0+\theta_1+\theta_2=1,\ \theta_i\geq 0.$$

Investment balance:

$$s_0+s_1+s_2=1,\ s_i\geq 0.$$

Material balance:

$$(1-a_0)x_0=a_1x_1+a_2x_2+y_0,\ y_0\geq 0.$$

Foreign trade balance:

$$q_0y_0=q_1y_1+q_2y_2,$$

where y_0-specific export of materials;

$y_1,\ y_2-$specific import of investment and consumer goods;

q_0, q_1, q_2—world market prices for materials, investment, and consumer goods;
a_0, a_1, a_2—coefficients of direct material costs of the material, fund creating, and consumer sectors.

In this model, endogenous variables (i.e., determined by means of the model) are $(\theta, s, y) = (\theta_0,\ \theta_1, \theta_2, s_0, s_1, s_2, y_0, y_1, y_2)$. These nine endogenous variables are linked by four balance ratios (labor, investment, material, and foreign trade), so there are five degrees of freedom in their change. These degrees of freedom can be used to select the endogenous variables to maximize the specific consumption $c(\theta, s, y) = x_2(\theta, s, y) + y_2$.

Thus, we arrive at the following nonlinear programming problem:

$$\max_{(\theta,s,y)} \left[B_2 \theta_2^{1-a_2} s_2^{a_2} (x_1 + y_1)^{a_2} + y_2 \right]$$

when fulfilling the restrictions set by labor, investment, material, and foreign trade, in which $x_i = x_i(\theta, s, y)$ are given by the ratios indicated in the national economic productivity of the sectors.

By introducing five free variables (according to the number of degrees of freedom), the problem of nonlinear programming is reduced to the problem of finding the unconditional maximum of a function of five variables. In connection with the leading role of investments in the development of the economy, it is reasonable to include θ_i, s_i, and y_i among the free variables.

If the shares of resources θ_1, s_1 directed to the fund-creating sector are set, then when the specified restrictions are met, the material and consumer sectors are left with $(1 - \theta_1)$ labor and $(1 - s_1)$ investment resources. Let us introduce variables h, i, characterizing the distribution of these residual resources between the material and consumer sectors:

$$\theta_0 = (1 - lh)(1 - \theta_1),\ \theta_2 = lh(1 - \theta_1),$$
$$s_0 = (1 - h)(1 - s_1),\ s_2 = h(1 - s_1),$$

where $lh,\ h$ are the share of the consumer sector in the distribution of labor and investment resources inherited by the material and consumer sectors and l can be interpreted as the relative labor supply of investment resources directed to the consumer sector.

For any $l, h(0 \le h \le 1)$, the distribution of resources determined by them satisfies the labor and investment balances, while the specific outputs of the material and consumer sectors are transformed to the form [74,75]:

$$x_0 = B_0 \Big(1 - lh)^{1-a_0} \Big(1 - h)^{a_0} (1 - \theta_1)^{1-a_0} (1 - s_1)^{a_0} (x_1 + y_1)^{a_0},$$
$$x_2 = B_2 l^{1-a_2} h (1 - \theta_1)^{1-a_2} (1 - s_1)^{a_2} (x_1 + y_1)^{a_2}.$$

12.4. The impact of foreign trade on the national economy. Consider a situation where the national economy is already integrated into the world market. With $\theta_1 = s_1$, the specific outputs of the sectors will take the form:

$$x_0 = B_0(1 - h)(1 - s_1)(x_1 + y_1)^{a_0},$$
$$x_1 = B_1 x_1 (x_1 + y_1)^{a_1},$$
$$x_2 = B_2 h (1 - s_1)(x_1 + y_1)^{a_2}.$$

Let the specific import of machinery and equipment increase by dy_1; then (with the same import of consumer goods), the specific export of raw materials and other materials must increase by a certain amount dy_0 in order to compensate for the increase in import, while, according to the foreign trade balance:

$$q_0 dy_0 = q_1 dy_1.$$

If the country adheres to the policy of industrial security, then an increase in the output of materials can only be achieved by transferring resources from the consumer sector to the material sector while maintaining the share of the fund-creating sector in the distribution of resources (the first option). If the economy is sufficiently industrialized, then the growth in the output of materials can also be achieved by reducing the share of the fund-creating sector in the distribution of resources (second option). A combination of these two structural policy options is also possible [76,77].

Conclusion: With a given technological mode, based on the criterion of maximizing stationary specific consumption, it is advisable for countries with an underdeveloped manufacturing industry to export as much raw materials and materials as the technological capabilities of the material sector allow. For countries with a sufficiently developed manufacturing industry, there is a critical level of export of raw materials, which it is not advisable to exceed.

13.1. Mathematical theory of public choice. There are many approaches to establishing a public-choice criterion. One of these is the determination of the economic optimum according to Pareto. The Pareto optimal is understood as such a state of the economy in which an acceptable redistribution of products and costs is impossible, leading to an increase in the utility of some without reducing the utility of others.

Consider the public-choice problem in the following aggregated form: there are two types of resources, two types of goods, two types of consumers. For definiteness, by resources we will understand the main production assets K and the number of employees L, by goods: food products X_i and nonfood products Y_i; by consumers, two large social divisions that together make up the whole society. These divisions differ in their product preferences. Each of the two goods is produced by its own sector of the economy with its own neoclassical production function:

$$X_i = F_i(K_i, L_i),\ i = 1, 2,$$

it is assumed that the sector of the economy that produces this type of product includes all industries, subsectors, and industries that not only produce goods of this type but also ensure their release.

It is assumed that funds and labor can move freely between the food and nonfood sectors, in accordance with the available resources K, L:

$$\begin{gathered} K_1 + K_2 = K, \\ L_1 + L_2 = L. \end{gathered}$$

The overall picture of the output of goods and the distribution of resources can be depicted on the production Edgeworth–Bowley diagram (Figure 3). The rectangle $O_1 L O_2 K$, whose side lengths correspond to the availability of resources, can be called a resource allocation rectangle for this reason. Each of its points has coordinates (K_1, L_1) in the first coordinate system and coordinates (K_2, L_2) in the second coordinate system. Since the lengths of the sides of the rectangle are $K,\ L$, then, at any of its points, balances of resource consumption are performed.

Thus, each point of the resource allocation rectangle is characterized by six indicators:

$$(K_1, L_1, K_2, L_2, X_1, X_2),$$

and

$$K_1 + K_2 = K,\ \ L_1 + L_2 = L,\ \ X_1 = F_1(K_1, L_1),\ \ X_2 = F_2(K_2, L_2).$$

Since the production functions of the sectors are neoclassical, their isoquants are convex functions (each in its own coordinate system). When viewed on an Edgeworth–Bowley diagram, the isoquants of the first function remain convex, while the isoquants of the second function in the O_1 coordinate system become concave. Therefore, each specific isoquant of the first function can be in the following relationship with a specific isoquant of the second function:

– do not intersect at any point;

- intersect at two points;
- touch.

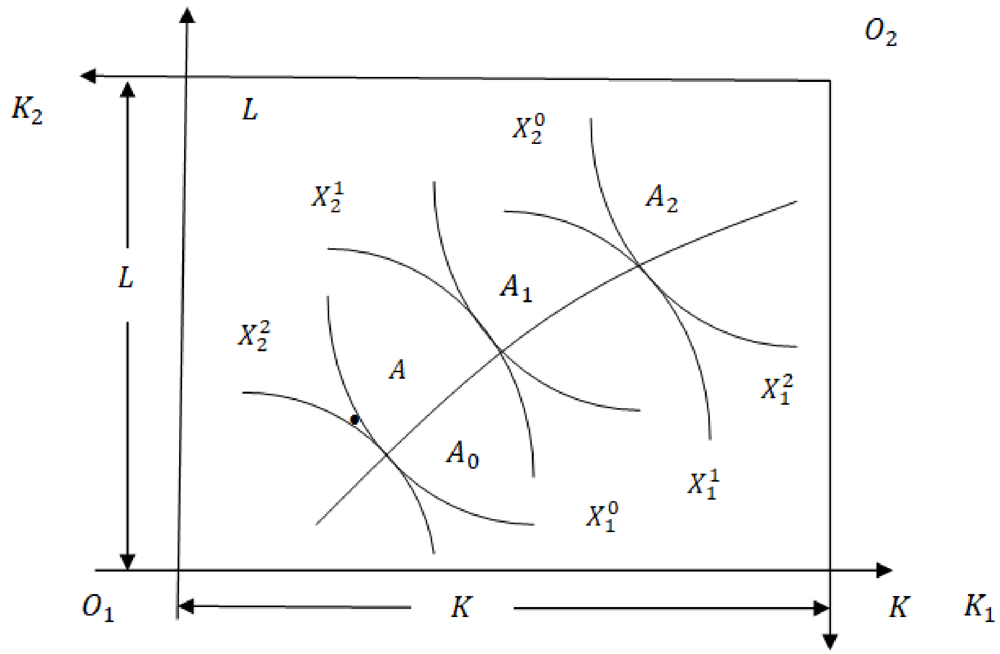

Figure 3. Resource allocation rectangle.

Curve $A_0A_1A_2$, composed of points of contact isoquant sectors, is called the production curve. All points of this curve characterize such states of the economy when more than any one product cannot be produced without reducing the production of another product. Thus, all points on this curve are Pareto optimal.

From the condition of contact of isoquants at some point $(K_1, L_1, K_2, L_2, X_1, X_2)$, it follows that $gradF_1(K_1, L_1)$ is collinear with $gradF_2(K_2, L_2)$:

$$S_K^1(L_1, K_1) = \frac{\frac{\partial F_1}{\partial L_1}(K_1, L_1)}{\frac{\partial F_1}{\partial K_1}(K_1, L_1)} = \frac{\frac{\partial F_2}{\partial L_2}(K_2, L_2)}{\frac{\partial F_2}{\partial K_2}(K_2, L_2)} = S_K^2(L_2, K_2),$$

i.e., on the production curve, the marginal rates of replacement of sectoral resources are equal [78,79].

13.2. Models of cooperation and competition. Consider, for simplicity, a cooperation and competition model for only two persons with a finite number of strategies for each player: $G = \{M, N; A, B\}$ where $M = \{1, \ldots, m\}; N = \{1, \ldots, n\}$ are sets of strategies for the first and second players, $A = \|a_{ij}\|; B = \|b_{ij}\|$ are payoff functions of the first and second players.

The basis of such a bimatrix (including cooperative) game is the bimatrix:

$$(A, B) = (a_{ij}, b_{ij}),$$

where a_{ij}—the payoff of the first player, when the first player uses the pure strategy i, and the second player uses the pure strategy j, $i = 1, \ldots, m,\ j = 1, \ldots, n;\ b_{ij}$—the payoff of the second player under the same conditions.

If $b_{ij} = -a_{ij}$, then we come to the usual game of two persons with zero sum.

In addition to pure strategies, mixed ones are considered:

$$p = (p_1, \ldots, p_m),\ q = (q_1, \ldots, q_n),$$

where p_i—the probability of the first player using the i-th strategy, $i = 1, \ldots, m; q_j$—the probability of the second player using the j-th strategy, $j = 1, \ldots, n$.

When players use mixed strategies p, q their average payoffs are respectively equal to

$$u_1(p,q) = \sum\nolimits_{i,j} a_{ij} p_i q_i, \ \ u_2(p,q) = \sum\nolimits_{i,j} b_{ij} p_i q_i.$$

Exodus (p^*, q^*) is called Pareto optimal if for any p, q

$$u_1(p^*, q^*) \geq u_1(p, q^*), \ \ u_2(p^*, q^*) \geq u_2(p^*, q).$$

Exodus (p, q) dominates the outcome $\left(p', q'\right)$ if

$$u_1\left(p', q'\right) \leq u_1(p,q), \ \ i = 1,2.$$

A strategy is called maximin if it provides the player with the maximum of the minimum payoffs (guaranteed payoff)

$$v_1 = \max_p \min_q u_1(p,q),$$
$$v_2 = \max_q \min_p u_2(p,q).$$

A bimatrix game in which negotiations between participants are not allowed is called noncooperative. In such a game, it is preferable for each player to adhere to a cautious (maximum) strategy that provides a guaranteed win. On the contrary, a matrix game is called cooperative if negotiations and joint actions of participants are allowed in it.

The solution of a cooperative game is reduced to finding the optimal, in a certain sense, joint strategy P^* among joint strategies:

$$P = \|p_{ij}\|, \sum_{i,j} p_{ij} = 1, \, p_{ij} \geq 0,$$
$$u_1(P) = \sum_{i,j} a_{ij} p_{ij}, \ \ u_2(P) = \sum_{i,j} b_{ij} p_{ij}.$$

It should be noted that the desired joint strategy $P = \|p_{ij}\|, \sum\limits_{i,j} p_{ij} = 1$ can be obtained, for example, as a result of the following joint actions of players:

$$p_i = \sum_{j=1}^{a} p_{ij}, \ \ i = 1, \ldots, m, \ \ q_j(i) = \frac{P_{ij}}{p_i}, \ \ j = 1, \ldots, n.$$

The whole set of joint strategies P forms a convex hull of points (a_{ij}, b_{ij}), $i = 1, \ldots, n$ on surface (u_1, u_2).A point (u_1, u_2) dominates a point $\left(\widetilde{u}_1, \widetilde{u}_2\right)$ if $u_1 \geq \widetilde{u}_1$, $u_2 \geq \widetilde{u}_2$. A subset of points in a convex hull is Pareto optimal if none of them dominates the other. This subset is called the negotiation set.

The choice of a certain point among the points of the negotiation set is the result of negotiations and compromise between the two players. One way to find such a compromise is the Nash algorithm. According to this algorithm, the optimal point is the point of the negotiation set at which the product of the gain increments of the first and second players reaches its maximum. This product is called the Nash function:

$$\max(u_1 - v_1)(u_2 - v_2),$$
$$(u_1, u_2) \ni T.$$

This or that compromise choice of a specific point of the negotiation set determines the optimal solution of the corporate game [80,81]

$$(u_1^*, u_2^*), u_1^* = \sum_{i,j} p_{ij}^* a_{ij}, \quad u_2^* = \sum_{i,j} p_{ij}^* b_{ij}.$$

13.3. Simulation of scientific and technological progress. In evolutionary models of scientific and technological progress (STP), the economy is considered as one unstructured whole and is described by a production function with coefficients drifting in time. At the same time, a slow increase in resource productivity as a result of scientific and technical progress is reflected by including an exponent in the coefficient of neutral technical progress:

$$X_i = A_0 e^{\lambda t} K_t^{\alpha_K} L_t^{\alpha_L}, \quad A(t) = A_0 e^{\lambda t},$$

where $\lambda -$measure of NTP.

This approach has been developed and differentiated:

1. Labor-increasing progress:

   $X_t = F(K_t, L_t^*)$, $L_t^* = A_L(t) L_t -$how many units of labor would be required if there were no scientific and technical progress;

2. Capital-increasing progress:

   $X_t = F(K_t^*, L_t)$, $K_t^* = A_K(t) K_t -$how many units of funds would be required if there were no scientific and technological progress;

3. Resource-increasing progress:

$$X_t = F(K_t^*, L_t^*), \ K_t^* = A_K(t) K_t, \ L_t^* = A_L(t) L_t.$$

4. Product-increasing progress:

   $X_t = A(t) F(K_t, L_t)$,

   where $A(t)$, $A_L(t), A_K(t) -$some growing functions of time are generally exponents $(A(t) = e^{\lambda t})$.

The use of exponentials when changing the NTP is advisable when the corresponding function $A(t)$ grows with an approximately constant growth rate λ, then $A(t) = (1+\lambda) \approx e^{\lambda t}$, the latter is true for a small value of λ. This is the case when progress is evolutionary.

Technical progress is called neutral if it does not change the ratio of the values of certain parameters. There is neutrality according to Hicks, Harrod, and Solow.

Progress is Hicks neutral if, for a given capital–labor ratio, the marginal rate of replacement of labor by funds is constant for any output:

$$s_0 = \frac{\frac{\partial X}{\partial L}}{\frac{\partial X}{\partial K}} = \frac{\frac{\partial F}{\partial L^*}}{\frac{\partial F}{\partial K^*}} \cdot \frac{e^{\lambda_L t}}{e^{\lambda_K t}} = \frac{\frac{\partial F}{\partial L^*}}{\frac{\partial F}{\partial K^*}} \text{ at } \lambda_L = \lambda_K;$$

Therefore, Hicks neutrality means that progress is resource increasing with $\lambda_L = \lambda_K$ or that progress is product increasing.

Progress is Harrod neutral if the marginal product of funds does not change:

$$\frac{\partial X}{\partial K} = \frac{\partial F}{\partial K},$$

therefore, Harrod's neutrality means that progress is labor increasing:

$$X = F\left(K, e^{\lambda_L t} L\right).$$

Progress is Solow neutral if the marginal product of labor does not change:

$$\frac{\partial X}{\partial L} = \frac{\partial F}{\partial L},$$

therefore, Solow neutrality means that progress is capital increasing:

$$X = F\left(e^{\lambda_K t}K, L\right).$$

The model for changing the technological structure takes into account that the rearmament processes in different production subsystems can take place asynchronously. Let the production functions of the old and new production methods be given:

$$F_0(K,L) = A_0 K_0^{\alpha_0} L_0^{1-\alpha_0},\ F_1(K,L) = A_1 K_1^{\alpha_1} L_1^{1-\alpha_1};$$

moreover, at the same costs, the output of the new method is significantly larger than the old one, i.e.,

$$F_1(K,L) \gg F_0(K,L).$$

Let us assume that the retirement rates are the same for the old and new methods, i.e., $\mu_0 = \mu_1 = \mu$. In addition, let us assume that labor resources are also constant, i.e., $L(t) = L = const$, there are no investment lags within each method.

Since the old method has exhausted itself, by the beginning of the rearmament it was already in stationary mode; therefore:

$$k_0 = \left(A_0\rho_0/\mu_0\right)^{t(1-\alpha_0)},\ x_0 = A_0(k_0)^{\alpha_0},\ i_0 = \rho_0 x_0,\ c_0 = (1-\rho_0)x_0.$$

We will assume that the investment of the old method in the creation of a new method occurs with a fixed lag τ. If the specific consumption can be reduced to the level $\underline{c}$, $\underline{c} < c_0$, then the released capacities can be used to produce the means of labor for the new method and, due to the presence of a lag, investments are made at the time $t-\tau$; funds are deposited at time t, i.e., $V(t) = I(t-\tau)$.

During time τ, the total investment will be $L(c_0 - \underline{c})t,\ t < \tau$.

The transitional period $0 < t < T$ is divided into three stages.

Accumulation stage $(0 < t < \tau)$. Accumulation occurs due to the reduction of specific consumption to the minimum allowable level. There is no return on investment in the new method yet, so only the old method works:

$$k(t) = k_0,\ x(t) = x_0,\ c(t) = \underline{c}, i(t) = c_0 - \underline{c},\ I(t) = (c_0 - \underline{c})Lt,\ V(t) = 0.$$

Stage of return of savings $(\tau < t < 2\tau)$. Accumulations of the old method into the new one begin to return and the old method stops accumulating for the new one; therefore $c_0(\tau) = c_0$; in addition, the new method accumulates for itself (without lag):

$$k(t) = \theta_0 k_0 + \theta_1 k_1,\ x(t) = \theta_0 k_0 + \theta_1 A_1 k_1^{\alpha_1}, c(t) = \theta_0 c_0 + \theta_1(1-\rho_1)A_1 k_1^{\alpha_1},$$
$$c(t) = \theta_0 c_0 + \theta_1(1-\rho_1)A_1 k_1^{\alpha_1},$$

where k_1-capital–labor ratio adopted for the transitional period of the new method

$$k_0 \le k_1 \le k_1^0, k_1^0 = \left[\frac{\rho_1 A_1}{\mu}\right]^{\frac{1}{1-\alpha_1}}$$

stationary capital–labor ratio of the new method at the rate of accumulation ρ_1; $\theta_1 = \frac{L_1}{L},\ i = 0,1-$the share of the i-th method in the use of labor resources.

New mode funds satisfy the differential equation

$$\frac{\partial K_1}{\partial t} = -\mu K_1 + \rho_1 A_1 k_1^{\alpha_1} L_1^{1-\alpha_1} + (c_0 - \underline{c})L, \;\; K_1(\tau) = 0.$$

Since $K_1 = k_1 L_1$, then the last equation goes into the equation

$$\frac{dL_1}{dt} = -\mu L_1 + \rho_1 A_1 k_1^{*} L_1 + \frac{(c_0 - \underline{c})L}{k_1}, \;\; L_1(t) = 0,$$

or

$$\frac{dL_1}{dt} = bL_1 + d, \;\; L_1(t) = 0,$$

where $b = \mu\left[\left(k_1^0/k_1\right)^{1-\alpha_1} - 1\right], \quad d = \frac{(c_0-\underline{c})L}{k_1}$.

The share of the new way in the use of ore resources grows exponentially, starting from $\theta_1(t) = 0$. The moment of the end of the transition process T is determined from the equation $\theta_1(t) = 1$, which means the end of the overflow of labor resources into a new way.

At $T < 2\tau$, an accelerated transient process takes place, which ends already at the second stage, while:

$$\frac{(c_0 - \underline{c})\left[e^{b(T-\tau)} - 1\right]}{\mu k_1\left[(k_1^0/k_1)^{1-\alpha_1} - 1\right]} = 1,$$

where

$$T = \tau + \frac{ln\left\{1 + \frac{\mu k_1\left[(k_1^0/\,k_1)^{1-\alpha_1} - 1\right]}{c_0 - \underline{c}}\right\}}{\mu\left[(k_1^0/\,k_1)^{1-\alpha_1} - 1\right]}.$$

If the capital–labor ratio k_1 is close to k_1^0, or the difference between the initial and the minimum allowable specific consumption $c_0 - \underline{c}$ is large enough, then the last condition will be written as follows:

$$k_1 < \tau(c_0 - \underline{c}).$$

Otherwise, a slow transient process takes place, which ends at $T > 2\tau$, i.e., ends in the third stage.

The equation for funds in this case will take the form:

$$\frac{dK_1}{dt} = -\mu K_1 + \rho_1 A_1 k_1^{\alpha_1} L_1^{1-\alpha_1}, \;\; K_1(2\tau) = dk_1 \frac{e^{bt} - 1}{b},$$

or for new way workforce

$$\frac{dL_1}{dt} = bL_1, \;\; L_1(2\tau) = d\frac{e^{bt} - 1}{b}.$$

The last equation has the following solution:

$$L_1(t) = d\frac{e^{bt} - 1}{b} e^{(b-2\tau)} = \frac{(c_0 - \underline{c})L\left(e^{bt} - 1\right)e^{(b-2\tau)}}{\mu k_1\left[(k_1^0/\,k_1)^{1-\alpha_1} - 1\right]},$$

where

$$\theta_1(t) = \frac{(c_0 - \underline{c})\left(e^{bt} - 1\right)}{\mu k_1\left[(k_1^0/\,k_1)^{1-\alpha_1} - 1\right]} e^{(b-2\tau)}.$$

The condition $\theta_1(t) = 1$ gives the following expression for the end time of the transient:

$$T = 2\tau + \frac{1}{\delta} ln\left\{1 - \frac{\mu_1 k_1\left[(k_1^0/\,k_1)^{1-\alpha_1} - 1\right]}{(c_0 - \underline{c})\left(e^{bt} - 1\right)}\right\}.$$

After the old method is completely replaced from the time $t = T$, the usual transition process begins in the Solow model for the new method from the capital–labor ratio $k_1(T) = k_1$ to the stationary capital–labor ratio k_1^E [82,83].

Conclusion: The formalized expression of the goal of social development at the conceptual level comes down to improving the wellbeing of members of society, creating comfortable and safe living conditions, and increasing life expectancy. The socially oriented policy of the state ensures an increase in the birth rate and a decrease in the death rate. In models of optimal economic growth, the Pareto optimality criterion is most often used.

5. Conclusions

To date, the most effective methods for studying the economy are methods based on the mathematical apparatus. This is explained by the fact that the economy does not tolerate full-scale experiments and the best methods of indirect study of economic phenomena and processes are mathematical models. Starting with the world's first economic and mathematical model of the social economy, Francois Quesnay, which was published in 1758, the arsenal of mathematical models of the economy has become extremely extensive. Obviously, this review could not describe all the retrospective mathematical methods and models of the economy, but took the main ones, the most frequently used and tested by practice over the last two decades of the current century. The review contains an analysis of articles and a description of some studies in the field of mathematical modeling of macroeconomics, microeconomics, and state regulation of the economy. We hope that this article will be useful to undergraduates, specialists, and graduate students who study economic phenomena using mathematical methods and models.

It is currently difficult to accurately determine the prospect of using the above models. Apparently, in the near future there will be a change in the paradigm of the economy, which will objectively require the development of new conceptual models.

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