mathematics

Article

Solving the Fredholm Integral Equation by Common Fixed Point Results in Bicomplex Valued Metric Spaces

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Abstract: The purpose of this research work is to explore the solution of the Fredholm integral equation by common fixed point results in bicomplex valued metric spaces. In this way, we develop some common fixed point theorems for generalized contractions containing point-dependent control functions in the context of bicomplex valued metric spaces. An illustrative and practical example is also given to show the novelty of the most important result.

Keywords: Fredholm integral equation; bicomplex valued metric space; common fixed point; point-dependent control functions

MSC: 46S40; 47H10; 54H25

Citation: Abdou, A.A.N. Solving the Fredholm Integral Equation by Common Fixed Point Results in Bicomplex Valued Metric Spaces. *Mathematics* **2023**, *11*, 3249. https://doi.org/10.3390/math11143249

Academic Editors: Mircea Balaj, Vasile Berinde and Massimiliano Giuli

Received: 10 June 2023
Revised: 12 July 2023
Accepted: 13 July 2023
Published: 24 July 2023

1. Introduction

The theory of bicomplex numbers was constructed by Segre [1] in which the elements or idempotents play a significant role. These bicomplex numbers lengthen complex numbers accurately to quaternions. For a more in-depth analysis of bicomplex numbers, we point out reference [2] to the readers. In 2007, Long-Guang et al. [3] presented the notion of cone metric spaces (CMSs) as an expansion of traditional metric space (MS) and determined fixed point results for contractive mappings. Later on, Azam et al. [4] introduced the conception of a complex valued metric space (CVMS) as a particular case of a CMS. The idea to initiate a CVMS was invented to construct rational expressions which cannot be given in CMSs and consequently numerous results of this theory cannot be obtained in CMSs; thus, CVMSs form a particular class of CMS. Indeed, the concept of a CMS starts to originate the notion of Banach space that is not a division ring. However, we can investigate the extensions of numerous theorems in the theory of fixed points including divisions in CVMSs. Furthermore, this concept is also utilized to introduce the notion of complex-valued Banach spaces [5], which provide a lot of areas for supplemental exploration.

Choi et al. [6] initiated the concept of bicomplex valued metric spaces (bi-CVMSs) by combining the ideas of bicomplex numbers and CVMSs. They proved some common fixed point theorems for weakly compatible mappings. Subsequently, Jebril et al. [7] used the idea of this novel space and presented theorems for two self-mappings in the framework of bi-CVMSs. In 2021, Beg et al. [8] reinforced the conception of bi-CVMSs and proved extrapolated fixed point results. Afterwards, Gnanaprakasam et al. [9] presented results for a contractive-type condition in the framework of bi-CVMSs and explored the solution to linear equations. Recently, Asifa et al. [10] obtained common fixed point results in a bi-CVMS for contractions containing control functions of two variables. For further details in this direction, we refer the reader to [11–31].

In this research article, we develop some common fixed point results in the context of bi-CVMSs for generalized contractions containing point-dependent control functions. We explore the solution of the Fredholm integral equation as an application.

2. Preliminaries

We represent $\mathbb{C}_0$ as a set of real numbers, $\mathbb{C}_1$ as a set of complex numbers and $\mathbb{C}_2$ as a set of bicomplex numbers. Segre [1] defined a bicomplex number in the following way

$$\varsigma = a_1 + a_2 i_1 + a_3 i_2 + a_4 i_1 i_2$$

where $a_1, a_2, a_3, a_4 \in \mathbb{C}_0$, the independent units i_1, i_2 are such that $i_1^2 = i_2^2 = -1$ and $i_1 i_2 = i_2 i_1,$ and $\mathbb{C}_2$ is given as

$$\mathbb{C}_2 = \{\varsigma : \varsigma = a_1 + a_2 i_1 + a_3 i_2 + a_4 i_1 i_2 : a_1, a_2, a_3, a_4 \in \mathbb{C}_0\}$$

that is

$$\mathbb{C}_2 = \{\varsigma : \varsigma = z_1 + i_2 z_2 : z_1, z_2 \in \mathbb{C}_1\}$$

where $z_1 = a_1 + a_2 i_1 \in \mathbb{C}_1$ and $z_2 = a_3 + a_4 i_1 \in \mathbb{C}_1$. If $\varsigma = z_1 + i_2 z_2$ and $\nu = \omega_1 + i_2 \omega_2$, then the sum is

$$\varsigma \pm \nu = (z_1 + i_2 z_2) \pm (\omega_1 + i_2 \omega_2) = (z_1 \pm \omega_1) + i_2(z_2 \pm \omega_2)$$

and the product is

$$\varsigma \cdot \nu = (z_1 + i_2 z_2) \cdot (\omega_1 + i_2 \omega_2) = (z_1 \omega_1 - z_2 \omega_2) + i_2(z_1 \omega_2 + z_2 \omega_1).$$

There are four idempotent members in $\mathbb{C}_2$, which are $0, 1, e_1 = \frac{1+i_1 i_2}{2}$ and $e_2 = \frac{1-i_1 i_2}{2}$, out of which e_1 and e_2 are nontrivial such that $e_1 + e_2 = 1$ and $e_1 e_2 = 0$. Every bicomplex number $z_1 + i_2 z_2$ can uniquely be demonstrated as the mixture of e_1 and e_2, namely

$$\varsigma = z_1 + i_2 z_2 = (z_1 - i_1 z_2)e_1 + (z_1 + i_1 z_2)e_2.$$

This description of ς is familiar, as the idempotent representation of ς and the complex coefficients $\varsigma_1 = (z_1 - i_1 z_2)$ and $\varsigma_2 = (z_1 + i_1 z_2)$ are called idempotent components of ς.

An element $\varsigma = z_1 + i_2 z_2 \in \mathbb{C}_2$ is called invertible if $\exists\, \nu \in \mathbb{C}_2$ in such a way that $\varsigma\nu = 1$, and ν is said to be the multiplicative inverse of ς. Hence, ς is said to be the multiplicative inverse of ν.

An element $\varsigma = z_1 + i_2 z_2 \in \mathbb{C}_2$ is nonsingular if $\left|z_1^2 + z_2^2\right| \neq 0$ and singular if $\left|z_1^2 + z_2^2\right| = 0$. The inverse of ς is defined as

$$\varsigma^{-1} = \nu = \frac{z_1 - i_2 z_2}{z_1^2 + z_2^2}.$$

Zero is the only member in $\mathbb{C}_0$ which does not have an inverse (multiplicative) and in $\mathbb{C}_1$, $0 = 0 + i0$ is the only member that does not have an inverse (multiplicative). We represent the set of singular members of $\mathbb{C}_0$ and $\mathbb{C}_1$ by $\aleph_0$ and $\aleph_1$, respectively. In $\mathbb{C}_2$, there are many elements which do not possess a multiplicative inverse. Let us denote the set of singular members of $\mathbb{C}_2$ by $\aleph_2$ and thus $\aleph_0 = \aleph_1 \subset \aleph_2$.

A bicomplex number $\varsigma = a_1 + a_2 i_1 + a_3 i_2 + a_4 i_1 i_2 \in \mathbb{C}_2$ is said to be degenerated if the matrix

$$\begin{pmatrix} a_1 & a_2 \\ a_3 & a_4 \end{pmatrix}_{2\times 2}$$

is degenerated. Thus, ς^{-1} exists and $\|\cdot\| : \mathbb{C}_2 \to \mathbb{C}_0^+$ is given as

$$
\begin{aligned}
\|\varsigma\| &= \|z_1 + i_2 z_2\| = \left\{|z_1|^2 + |z_2|^2\right\}^{\frac{1}{2}} \\
&= \left[\frac{|(z_1 - i_1 z_2)|^2 + |(z_1 + i_1 z_2)|^2}{2}\right]^{\frac{1}{2}} \\
&= \left(a_1^2 + a_2^2 + a_3^2 + a_4^2\right)^{\frac{1}{2}},
\end{aligned}
$$

where $\varsigma = a_1 + a_2 i_1 + a_3 i_2 + a_4 i_1 i_2 = z_1 + i_2 z_2 \in \mathbb{C}_2$.

The space $\mathbb{C}_2$ with regard to the norm $\|\cdot\| : \mathbb{C}_2 \to \mathbb{C}_0^+$. If $\varsigma, \nu \in \mathbb{C}_2$, then

$$\|\varsigma\nu\| \leq \sqrt{2}\|\varsigma\|\|\nu\|$$

holds instead of

$$\|\varsigma\nu\| \leq \|\varsigma\|\|\nu\|.$$

Hence, $\mathbb{C}_2$ is not Banach algebra. Let $\varsigma = z_1 + i_2 z_2$, $\nu = \omega_1 + i_2\omega_2 \in \mathbb{C}_2$, then we give

$$\varsigma \preceq_{i_2} \nu \text{ if and only if } \mathrm{Re}(z_1) \preceq \mathrm{Re}(\omega_1) \text{ and } \mathrm{Im}(z_2) \preceq \mathrm{Im}(\omega_2).$$

This implies

$$\varsigma \preceq_{i_2} \nu$$

if one of these assertions hold:

$$
\begin{aligned}
\text{(i) } z_1 &= \omega_1,\ z_2 \prec \omega_2; \\
\text{(ii) } z_1 &\prec \omega_1,\ z_2 = \omega_2; \\
\text{(iii) } z_1 &\prec \omega_1,\ z_2 \prec \omega_2; \\
\text{(iv) } z_1 &= \omega_1,\ z_2 = \omega_2.
\end{aligned}
$$

Specifically, $\varsigma \precnsim_{i_2} \nu$ if and only if $\varsigma \preceq_{i_2} \nu$ and $\varsigma \neq \nu$, i.e., one of the conditions (i), (ii) and (iii) are satisfied. Furthermore, $\varsigma \prec_{i_2} \nu$ if only condition (iii) holds. For $\varsigma, \nu \in \mathbb{C}_2$, the following conditions hold,

(i) $\varsigma \preceq_{i_2} \nu \Longrightarrow \|\varsigma\| \leq \|\nu\|$;
(ii) $\|\varsigma + \nu\| \leq \|\varsigma\| + \|\nu\|$;
(iii) $\|a\varsigma\| \leq a\|\nu\|$, where a is a non negative real number;
(iv) $\|\varsigma\nu\| \leq \sqrt{2}\|\varsigma\|\|\nu\|$;
(v) $\|\varsigma^{-1}\| = \|\varsigma\|^{-1}$;
(vi) $\|\frac{\varsigma}{\nu}\| = \frac{\|\varsigma\|}{\|\nu\|}$.

Azam et al. [4] defined the idea of a CVMS in this manner.

Definition 1 ([4])**.** *Let $\mathcal{Q} \neq \emptyset$, $\preceq$ be a partial order on $\mathbb{C}$ and $\tau : \mathcal{Q} \times \mathcal{Q} \to \mathbb{C}_1$ be a function such that*

(i) $0 \preceq \tau(\varsigma, \nu)$ *and* $\tau(\varsigma, \nu) = 0$ *if and only if* $\varsigma = \nu$;
(ii) $\tau(\varsigma, \nu) = \tau(\nu, \varsigma)$;
(iii) $\tau(\varsigma, \nu) \preceq \tau(\varsigma, \phi) + \tau(\phi, \nu)$,

for all $\varsigma, \nu, \phi \in \mathcal{Q}$. *Then,* $(\mathcal{Q}, \tau)$ *is a CVMS.*

Choi et al. [6] gave the bi-CVMS in this way.

Definition 2 ([6])**.** *Let $\mathcal{Q} \neq \emptyset$, $\preceq_{i_2}$ be a partial order on $\mathbb{C}_2$ and $\tau : \mathcal{Q} \times \mathcal{Q} \to \mathbb{C}_2$ be a function such that*

(i) $0 \preceq_{i_2} \tau(\varsigma, \nu)$ *and* $\tau(\varsigma, \nu) = 0$ *if and only if* $\varsigma = \nu$;
(ii) $\tau(\varsigma, \nu) = \tau(\nu, \varsigma)$;

(iii) $\tau(\varsigma,\nu) \preceq_{i_2} \tau(\varsigma,\phi)+\tau(\phi,\nu)$,

for all $\varsigma,\nu,\phi \in \mathcal{Q}$. *Then,* $(\mathcal{Q},\tau)$ *is a bi-CVMS.*

Example 1 ([8]). *Let* $\mathcal{Q}=\mathbb{C}_2$ *and* $\varsigma,\nu\in\mathcal{Q}$. *Define* $\tau:\mathcal{Q}\times\mathcal{Q}\to\mathbb{C}_2$ *by*

$$\tau(\varsigma,\nu)=|z_1-\omega_1|+i_2|z_2-\omega_2|$$

where $\varsigma=z_1+i_2z_2$ *and* $\nu=\omega_1+i_2\omega_2\in\mathbb{C}_2$. *Then,* $(\mathcal{Q},\tau)$ *is a bi-CVMS.*

Lemma 1 ([8]). *Let* $(\mathcal{Q},\tau)$ *be a bi-CVMS and let* $\{\varsigma_{\mathfrak{o}}\}\subseteq\mathcal{Q}$. *Then,* $\{\varsigma_{\mathfrak{o}}\}$ *converges to* ς *if and only if* $\|\tau(\varsigma_{\mathfrak{o}},\varsigma)\|\to 0$ *as* $\mathfrak{o}\to\infty$.

Lemma 2 ([8]). *Let* $(\mathcal{Q},\tau)$ *be a bi-CVMS and let* $\{\varsigma_{\mathfrak{o}}\}\subseteq\mathcal{Q}$. *Then,* $\{\varsigma_{\mathfrak{o}}\}$ *is a Cauchy sequence if and only if* $\|\tau(\varsigma_{\mathfrak{o}},\varsigma_{\mathfrak{o}+m})\|\to 0$ *as* $\mathfrak{o}\to\infty$, *where* $m\in\mathbb{N}$.

3. Main Result

Proposition 1. *Let* $(\mathcal{Q},\tau)$ *be a bi-CVMS and* $\mathfrak{I}_1,\mathfrak{I}_2:(\mathcal{Q},\tau)\to(\mathcal{Q},\tau)$. *Let* $\varsigma_0\in\mathcal{Q}$. *Define the sequence* $\{\varsigma_{\mathfrak{o}}\}$ *by*

$$\varsigma_{2\mathfrak{o}+1}=\mathfrak{I}_1\varsigma_{2\mathfrak{o}} \text{ and } \varsigma_{2\mathfrak{o}+2}=\mathfrak{I}_2\varsigma_{2\mathfrak{o}+1} \tag{1}$$

for all $\mathfrak{o}=0,1,2,\ldots$

Assume that there exists $\alpha:\mathcal{Q}\times\mathcal{Q}\times\mathcal{Q}\to[0,1)$ satisfying

$$\alpha(\mathfrak{I}_2\mathfrak{I}_1\varsigma,\nu,\theta)\le\alpha(\varsigma,\nu,\theta) \text{ and } \alpha(\varsigma,\mathfrak{I}_1\mathfrak{I}_2\nu,\theta)\le\alpha(\varsigma,\nu,\theta)$$

for all $\varsigma,\nu\in\mathcal{Q}$ and some fixed element $\theta\in\mathcal{Q}$. Then,

$$\alpha(\varsigma_{2\mathfrak{o}},\nu,\theta)\le\alpha(\varsigma_0,\nu,\theta) \text{ and } \alpha(\varsigma,\varsigma_{2\mathfrak{o}+1},\theta)\le\alpha(\varsigma,\varsigma_1,\theta)$$

for all $\varsigma,\nu\in\mathcal{Q}$ and $\mathfrak{o}=0,1,2,\ldots$

Proof. Let $\varsigma,\nu\in\mathcal{Q}$ and $\mathfrak{o}=0,1,2,\ldots$ Then, we obtain

$$\begin{aligned}\alpha(\varsigma_{2\mathfrak{o}},\nu,\theta) &= \alpha(\mathfrak{I}_2\mathfrak{I}_1\varsigma_{2\mathfrak{o}-2},\nu,\theta)\le\alpha(\varsigma_{2\mathfrak{o}-2},\nu,\theta)\\ &= \alpha(\mathfrak{I}_2\mathfrak{I}_1\varsigma_{2\mathfrak{o}-4},\nu,\theta)\le\alpha(\varsigma_{2\mathfrak{o}-4},\nu,\theta)\\ &\le \cdots\le\alpha(\varsigma_0,\nu,\theta).\end{aligned}$$

Similarly, we have

$$\begin{aligned}\alpha(\varsigma,\varsigma_{2\mathfrak{o}+1},\theta) &= \alpha(\varsigma,\mathfrak{I}_1\mathfrak{I}_2\varsigma_{2\mathfrak{o}-1},\theta)\le\alpha(\varsigma,\varsigma_{2\mathfrak{o}-1},\theta)\\ &= \alpha(\varsigma,\mathfrak{I}_1\mathfrak{I}_2\varsigma_{2\mathfrak{o}-3},\theta)\le\alpha(\varsigma,\varsigma_{2\mathfrak{o}-3},\theta)\\ &\le \cdots\le\alpha(\varsigma,\varsigma_1,\theta).\end{aligned}$$

□

Theorem 1. *Let* $(\mathcal{Q},\tau)$ *be a complete bi-CVMS and* $\mathfrak{I}_1,\mathfrak{I}_2:\mathcal{Q}\to\mathcal{Q}$. *If the functions* $\alpha,\pi,\kappa,\varpi,\varkappa:\mathcal{Q}^3\to[0,1)$ *satisfy the conditions*

(a)

$\alpha(\mathfrak{I}_2\mathfrak{I}_1\varsigma,\nu,\theta)\le\alpha(\varsigma,\nu,\theta)$ *and* $\alpha(\varsigma,\mathfrak{I}_1\mathfrak{I}_2\nu,\theta)\le\alpha(\varsigma,\nu,\theta)$
$\pi(\mathfrak{I}_2\mathfrak{I}_1\varsigma,\nu,\theta)\le\pi(\varsigma,\nu,\theta)$ *and* $\pi(\varsigma,\mathfrak{I}_1\mathfrak{I}_2\nu,\theta)\le\pi(\varsigma,\nu,\theta)$
$\kappa(\mathfrak{I}_2\mathfrak{I}_1\varsigma,\nu,\theta)\le\kappa(\varsigma,\nu,\theta)$ *and* $\kappa(\varsigma,\mathfrak{I}_1\mathfrak{I}_2\nu,\theta)\le\kappa(\varsigma,\nu,\theta)$
$\varpi(\mathfrak{I}_2\mathfrak{I}_1\varsigma,\nu,\theta)\le\varpi(\varsigma,\nu,\theta)$ *and* $\varpi(\varsigma,\mathfrak{I}_1\mathfrak{I}_2\nu,\theta)\le\varpi(\varsigma,\nu,\theta)$
$\varkappa(\mathfrak{I}_2\mathfrak{I}_1\varsigma,\nu,\theta)\le\varkappa(\varsigma,\nu,\theta)$ *and* $\varkappa(\varsigma,\mathfrak{I}_1\mathfrak{I}_2\nu,\theta)\le\varkappa(\varsigma,\nu,\theta)$

(b) $\alpha(\varsigma,\nu,\theta)+2\pi(\varsigma,\nu,\theta)+\sqrt{2}\kappa(\varsigma,\nu,\theta)+\sqrt{2}\varpi(\varsigma,\nu,\theta)+\sqrt{2}\varkappa(\varsigma,\nu,\theta)<1$,

(c)

$$\begin{aligned}\tau(\mathfrak{I}_1\varsigma,\mathfrak{I}_2\nu) \preceq_{i_2} {} & \alpha(\varsigma,\nu,\theta)\tau(\varsigma,\nu) + \pi(\varsigma,\nu,\theta)(\tau(\varsigma,\mathfrak{I}_1\varsigma) + \tau(\nu,\mathfrak{I}_2\nu)) \\ & + \kappa(\varsigma,\nu,\theta)\frac{\tau(\varsigma,\mathfrak{I}_1\varsigma)\tau(\nu,\mathfrak{I}_2\nu)}{1+\tau(\varsigma,\nu)} \\ & + \varpi(\varsigma,\nu,\theta)\frac{\tau(\nu,\mathfrak{I}_1\varsigma)\tau(\varsigma,\mathfrak{I}_2\nu)}{1+\tau(\varsigma,\nu)} \\ & + \varkappa(\varsigma,\nu,\theta)\frac{\tau(\varsigma,\mathfrak{I}_1\varsigma)\tau(\varsigma,\mathfrak{I}_2\nu) + \tau(\nu,\mathfrak{I}_2\nu)\tau(\nu,\mathfrak{I}_1\varsigma)}{1+\tau(\varsigma,\mathfrak{I}_2\nu)+\tau(\nu,\mathfrak{I}_1\varsigma)}, \end{aligned} \tag{2}$$

for all $\varsigma,\nu \in \mathcal{Q}$ and for fixed element $\theta \in \mathcal{Q}$, then there exists a unique element $\varsigma^* \in \mathcal{Q}$ such that $\mathfrak{I}_1\varsigma^* = \mathfrak{I}_2\varsigma^* = \varsigma^*$.

Proof. Let $\varsigma,\nu \in \mathcal{Q}$. From (2), we have

$$\begin{aligned}\tau(\mathfrak{I}_1\varsigma,\mathfrak{I}_2\mathfrak{I}_1\varsigma) \preceq_{i_2} {} & \alpha(\varsigma,\mathfrak{I}_1\varsigma,\theta)\tau(\varsigma,\mathfrak{I}_1\varsigma) \\ & + \pi(\varsigma,\mathfrak{I}_1\varsigma,\theta)(\tau(\varsigma,\mathfrak{I}_1\varsigma) + \tau(\mathfrak{I}_1\varsigma,\mathfrak{I}_2\mathfrak{I}_1\varsigma)) \\ & + \kappa(\varsigma,\mathfrak{I}_1\varsigma,\theta)\frac{\tau(\varsigma,\mathfrak{I}_1\varsigma)\tau(\mathfrak{I}_1\varsigma,\mathfrak{I}_2\mathfrak{I}_1\varsigma)}{1+\tau(\varsigma,\mathfrak{I}_1\varsigma)} \\ & + \varpi(\varsigma,\mathfrak{I}_1\varsigma,\theta)\frac{\tau(\mathfrak{I}_1\varsigma,\mathfrak{I}_1\varsigma)\tau(\varsigma,\mathfrak{I}_2\mathfrak{I}_1\varsigma)}{1+\tau(\varsigma,\mathfrak{I}_1\varsigma)} \\ & + \varkappa(\varsigma,\mathfrak{I}_1\varsigma,\theta)\frac{\tau(\varsigma,\mathfrak{I}_1\varsigma)\tau(\varsigma,\mathfrak{I}_2\mathfrak{I}_1\varsigma) + \tau(\mathfrak{I}_1\varsigma,\mathfrak{I}_2\mathfrak{I}_1\varsigma)\tau(\mathfrak{I}_1\varsigma,\mathfrak{I}_1\varsigma)}{1+\tau(\varsigma,\mathfrak{I}_2\mathfrak{I}_1\varsigma)+\tau(\mathfrak{I}_1\varsigma,\mathfrak{I}_1\varsigma)} \end{aligned}$$

which implies

$$\begin{aligned}\|\tau(\mathfrak{I}_1\varsigma,\mathfrak{I}_2\mathfrak{I}_1\varsigma)\| \leq {} & \alpha(\varsigma,\mathfrak{I}_1\varsigma,\theta)\|\tau(\varsigma,\mathfrak{I}_1\varsigma)\| \\ & +\pi(\varsigma,\mathfrak{I}_1\varsigma,\theta)(\|\tau(\varsigma,\mathfrak{I}_1\varsigma)\| + \|\tau(\mathfrak{I}_1\varsigma,\mathfrak{I}_2\mathfrak{I}_1\varsigma)\|) \\ & +\sqrt{2}\kappa(\varsigma,\mathfrak{I}_1\varsigma,\theta)\frac{\|\tau(\varsigma,\mathfrak{I}_1\varsigma)\|}{\|1+\tau(\varsigma,\mathfrak{I}_1\varsigma)\|}\|\tau(\mathfrak{I}_1\varsigma,\mathfrak{I}_2\mathfrak{I}_1\varsigma)\| \\ & +\sqrt{2}\varkappa(\varsigma,\mathfrak{I}_1\varsigma,\theta)\|\tau(\varsigma,\mathfrak{I}_1\varsigma)\|\frac{\|\tau(\varsigma,\mathfrak{I}_2\mathfrak{I}_1\varsigma)\|}{\|1+\tau(\varsigma,\mathfrak{I}_2\mathfrak{I}_1\varsigma)\|}, \end{aligned}$$

since $\|\tau(\varsigma,\mathfrak{I}_1\varsigma)\| < \|1+\tau(\varsigma,\mathfrak{I}_1\varsigma)\|$, so $\frac{\|\tau(\varsigma,\mathfrak{I}_1\varsigma)\|}{\|1+\tau(\varsigma,\mathfrak{I}_1\varsigma)\|} < 1$. Similarly, $\|\tau(\varsigma,\mathfrak{I}_2\mathfrak{I}_1\varsigma)\| < \|1+\tau(\varsigma,\mathfrak{I}_2\mathfrak{I}_1\varsigma)\|$, and thus $\frac{\|\tau(\varsigma,\mathfrak{I}_2\mathfrak{I}_1\varsigma)\|}{\|1+\tau(\varsigma,\mathfrak{I}_2\mathfrak{I}_1\varsigma)\|} < 1$. Hence, we have

$$\begin{aligned}\|\tau(\mathfrak{I}_1\varsigma,\mathfrak{I}_2\mathfrak{I}_1\varsigma)\| \leq {} & \alpha(\varsigma,\mathfrak{I}_1\varsigma,\theta)\|\tau(\varsigma,\mathfrak{I}_1\varsigma)\| \\ & +\pi(\varsigma,\mathfrak{I}_1\varsigma,\theta)(\|\tau(\varsigma,\mathfrak{I}_1\varsigma)\| + \|\tau(\mathfrak{I}_1\varsigma,\mathfrak{I}_2\mathfrak{I}_1\varsigma)\|) \\ & +\sqrt{2}\kappa(\varsigma,\mathfrak{I}_1\varsigma,\theta)\|\tau(\mathfrak{I}_1\varsigma,\mathfrak{I}_2\mathfrak{I}_1\varsigma)\| \\ & +\sqrt{2}\varkappa(\varsigma,\mathfrak{I}_1\varsigma,\theta)\|\tau(\varsigma,\mathfrak{I}_1\varsigma)\|. \end{aligned} \tag{3}$$

Similarly,

$$\begin{aligned}\tau(\mathfrak{I}_1\mathfrak{I}_2\nu,\mathfrak{I}_2\nu) \preceq_{i_2} {} & \alpha(\mathfrak{I}_2\nu,\nu,\theta)\tau(\mathfrak{I}_2\nu,\nu) + \pi(\mathfrak{I}_2\nu,\nu,\theta)(\tau(\mathfrak{I}_2\nu,\mathfrak{I}_1\mathfrak{I}_2\nu) + \tau(\nu,\mathfrak{I}_2\nu)) \\ & +\kappa(\mathfrak{I}_2\nu,\nu,\theta)\frac{\tau(\mathfrak{I}_2\nu,\mathfrak{I}_1\mathfrak{I}_2\nu)\tau(\nu,\mathfrak{I}_2\nu)}{1+\tau(\mathfrak{I}_2\nu,\nu)} \\ & +\varpi(\mathfrak{I}_2\nu,\nu,\theta)\frac{\tau(\nu,\mathfrak{I}_1\mathfrak{I}_2\nu)\tau(\mathfrak{I}_2\nu,\mathfrak{I}_2\nu)}{1+\tau(\mathfrak{I}_2\nu,\nu)} \\ & + \varkappa(\mathfrak{I}_2\nu,\nu,\theta)\frac{\tau(\mathfrak{I}_2\nu,\mathfrak{I}_1\mathfrak{I}_2\nu)\tau(\mathfrak{I}_2\nu,\mathfrak{I}_2\nu) + \tau(\nu,\mathfrak{I}_2\nu)\tau(\nu,\mathfrak{I}_1\mathfrak{I}_2\nu)}{1+\tau(\mathfrak{I}_2\nu,\mathfrak{I}_2\nu)+\tau(\nu,\mathfrak{I}_1\mathfrak{I}_2\nu)} \end{aligned}$$

which implies

$$\begin{aligned}\|\tau(\mathfrak{J}_1\mathfrak{J}_2\nu,\mathfrak{J}_2\nu)\| &\leq \alpha(\mathfrak{J}_2\nu,\nu)\|\tau(\mathfrak{J}_2\nu,\nu)\| \\ &\quad +\pi(\mathfrak{J}_2\nu,\nu,\theta)(\|\tau(\mathfrak{J}_2\nu,\mathfrak{J}_1\mathfrak{J}_2\nu)\|+\|\tau(\nu,\mathfrak{J}_2\nu)\|) \\ &\quad +\sqrt{2}\kappa(\mathfrak{J}_2\nu,\nu)\|\tau(\mathfrak{J}_2\nu,\mathfrak{J}_1\mathfrak{J}_2\nu)\|\frac{\|\tau(\nu,\mathfrak{J}_2\nu)\|}{\|1+\tau(\mathfrak{J}_2\nu,\nu)\|} \\ &\quad +\sqrt{2}\varkappa(\mathfrak{J}_2\nu,\nu,\theta)\|\tau(\nu,\mathfrak{J}_2\nu)\|\frac{\|\tau(\nu,\mathfrak{J}_1\mathfrak{J}_2\nu)\|}{\|1+\tau(\nu,\mathfrak{J}_1\mathfrak{J}_2\nu)\|}.\end{aligned}$$

Since $\|\tau(\nu,\mathfrak{J}_2\nu)\|<\|1+\tau(\mathfrak{J}_2\nu,\nu)\|$, so $\frac{\|\tau(\nu,\mathfrak{J}_2\nu)\|}{\|1+\tau(\mathfrak{J}_2\nu,\nu)\|}<1$. Furthermore, $\|\tau(\nu,\mathfrak{J}_1\mathfrak{J}_2\nu)\|<\|1+\tau(\nu,\mathfrak{J}_1\mathfrak{J}_2\nu)\|$, and so $\frac{\|\tau(\nu,\mathfrak{J}_1\mathfrak{J}_2\nu)\|}{\|1+\tau(\nu,\mathfrak{J}_1\mathfrak{J}_2\nu)\|}<1$. Thus,

$$\begin{aligned}\|\tau(\mathfrak{J}_1\mathfrak{J}_2\nu,\mathfrak{J}_2\nu)\| &\leq \alpha(\mathfrak{J}_2\nu,\nu)\|\tau(\mathfrak{J}_2\nu,\nu)\| \\ &\quad +\pi(\mathfrak{J}_2\nu,\nu,\theta)(\|\tau(\mathfrak{J}_2\nu,\mathfrak{J}_1\mathfrak{J}_2\nu)\|+\|\tau(\nu,\mathfrak{J}_2\nu)\|) \\ &\quad +\sqrt{2}\kappa(\mathfrak{J}_2\nu,\nu)\|\tau(\mathfrak{J}_2\nu,\mathfrak{J}_1\mathfrak{J}_2\nu)\| \\ &\quad +\sqrt{2}\varkappa(\mathfrak{J}_2\nu,\nu,\theta)\|\tau(\nu,\mathfrak{J}_2\nu)\|. \end{aligned} \tag{4}$$

Let $\varsigma_0\in\mathcal{Q}$ and the sequence $\{\varsigma_{\mathfrak{o}}\}$ be defined by (1). By inequalities (3) and (4), we have

$$\begin{aligned}\|\tau(\varsigma_{2\mathfrak{o}+1},\varsigma_{2\mathfrak{o}})\| &= \|\tau(\mathfrak{J}_1\mathfrak{J}_2\varsigma_{2\mathfrak{o}-1},\mathfrak{J}_2\varsigma_{2\mathfrak{o}-1})\| \\ &\leq \alpha(\mathfrak{J}_2\varsigma_{2\mathfrak{o}-1},\varsigma_{2\mathfrak{o}-1},\theta)\|\tau(\mathfrak{J}_2\varsigma_{2\mathfrak{o}-1},\varsigma_{2\mathfrak{o}-1})\| \\ &\quad +\pi(\mathfrak{J}_2\varsigma_{2\mathfrak{o}-1},\varsigma_{2\mathfrak{o}-1},\theta)(\|\tau(\mathfrak{J}_2\varsigma_{2\mathfrak{o}-1},\mathfrak{J}_1\mathfrak{J}_2\varsigma_{2\mathfrak{o}-1})\|+\|\tau(\varsigma_{2\mathfrak{o}-1},\mathfrak{J}_2\varsigma_{2\mathfrak{o}-1})\|) \\ &\quad +\sqrt{2}\kappa(\mathfrak{J}_2\varsigma_{2\mathfrak{o}-1},\varsigma_{2\mathfrak{o}-1},\theta)\|\tau(\mathfrak{J}_2\varsigma_{2\mathfrak{o}-1},\mathfrak{J}_1\mathfrak{J}_2\varsigma_{2\mathfrak{o}-1})\| \\ &\quad +\sqrt{2}\varkappa(\mathfrak{J}_2\varsigma_{2\mathfrak{o}-1},\varsigma_{2\mathfrak{o}-1},\theta)\|\tau(\varsigma_{2\mathfrak{o}-1},\mathfrak{J}_2\varsigma_{2\mathfrak{o}-1})\| \\ &= \alpha(\varsigma_{2\mathfrak{o}},\varsigma_{2\mathfrak{o}-1},\theta)\|\tau(\varsigma_{2\mathfrak{o}},\varsigma_{2\mathfrak{o}-1})\| \\ &\quad +\pi(\varsigma_{2\mathfrak{o}},\varsigma_{2\mathfrak{o}-1},\theta)(\|\tau(\varsigma_{2\mathfrak{o}},\varsigma_{2\mathfrak{o}+1})\|+\|\tau(\varsigma_{2\mathfrak{o}-1},\varsigma_{2\mathfrak{o}})\|) \\ &\quad +\sqrt{2}\kappa(\varsigma_{2\mathfrak{o}},\varsigma_{2\mathfrak{o}-1},\theta)\|\tau(\varsigma_{2\mathfrak{o}},\varsigma_{2\mathfrak{o}+1})\| \\ &\quad +\sqrt{2}\varkappa(\varsigma_{2\mathfrak{o}},\varsigma_{2\mathfrak{o}-1},\theta)\|\tau(\varsigma_{2\mathfrak{o}-1},\varsigma_{2\mathfrak{o}})\|\end{aligned}$$

By Proposition 1, we have

$$\begin{aligned}\|\tau(\varsigma_{2\mathfrak{o}+1},\varsigma_{2\mathfrak{o}})\| &\leq \alpha(\varsigma_0,\varsigma_{2\mathfrak{o}-1},\theta)\|\tau(\varsigma_{2\mathfrak{o}},\varsigma_{2\mathfrak{o}-1})\| \\ &\quad +\pi(\varsigma_0,\varsigma_{2\mathfrak{o}-1},\theta)(\|\tau(\varsigma_{2\mathfrak{o}},\varsigma_{2\mathfrak{o}+1})\|+\|\tau(\varsigma_{2\mathfrak{o}-1},\varsigma_{2\mathfrak{o}})\|) \\ &\quad +\sqrt{2}\kappa(\varsigma_0,\varsigma_{2\mathfrak{o}-1},\theta)\|\tau(\varsigma_{2\mathfrak{o}},\varsigma_{2\mathfrak{o}+1})\| \\ &\quad +\sqrt{2}\varkappa(\varsigma_0,\varsigma_{2\mathfrak{o}-1},\theta)\|\tau(\varsigma_{2\mathfrak{o}-1},\varsigma_{2\mathfrak{o}})\| \\ &\leq \alpha(\varsigma_0,\varsigma_1,\theta)\|\tau(\varsigma_{2\mathfrak{o}},\varsigma_{2\mathfrak{o}-1})\| \\ &\quad +\pi(\varsigma_0,\varsigma_1,\theta)(\|\tau(\varsigma_{2\mathfrak{o}},\varsigma_{2\mathfrak{o}+1})\|+\|\tau(\varsigma_{2\mathfrak{o}-1},\varsigma_{2\mathfrak{o}})\|) \\ &\quad +\sqrt{2}\kappa(\varsigma_0,\varsigma_1,\theta)\|\tau(\varsigma_{2\mathfrak{o}},\varsigma_{2\mathfrak{o}+1})\| \\ &\quad +\sqrt{2}\varkappa(\varsigma_0,\varsigma_1,\theta)\|\tau(\varsigma_{2\mathfrak{o}-1},\varsigma_{2\mathfrak{o}})\|\end{aligned}$$

for all $\mathfrak{o}=0,1,2,\ldots$. This implies that

$$\|\tau(\varsigma_{2\mathfrak{o}+1},\varsigma_{2\mathfrak{o}})\|\leq\frac{\alpha(\varsigma_0,\varsigma_1,\theta)+\pi(\varsigma_0,\varsigma_1,\theta)+\sqrt{2}\varkappa(\varsigma_0,\varsigma_1,\theta)}{1-\pi(\varsigma_0,\varsigma_1,\theta)-\sqrt{2}\kappa(\varsigma_0,\varsigma_1,\theta)}\|\tau(\varsigma_{2\mathfrak{o}},\varsigma_{2\mathfrak{o}-1})\|. \tag{5}$$

Similarly, we have

$$
\begin{aligned}
\|\tau(\varsigma_{2\mathfrak{o}+2},\varsigma_{2\mathfrak{o}+1})\| &= \|\tau(\mathfrak{J}_2\mathfrak{J}_1\varsigma_{2\mathfrak{o}},\mathfrak{J}_1\varsigma_{2\mathfrak{o}})\| \\
&= \|\tau(\mathfrak{J}_1\varsigma_{2\mathfrak{o}},\mathfrak{J}_2\mathfrak{J}_1\varsigma_{2\mathfrak{o}})\| \\
&\leq \alpha(\varsigma_{2\mathfrak{o}},\mathfrak{J}_1\varsigma_{2\mathfrak{o}},\theta)\|\tau(\varsigma_{2\mathfrak{o}},\mathfrak{J}_1\varsigma_{2\mathfrak{o}})\| \\
&\quad +\pi(\varsigma_{2\mathfrak{o}},\mathfrak{J}_1\varsigma_{2\mathfrak{o}},\theta)(\|\tau(\varsigma_{2\mathfrak{o}},\mathfrak{J}_1\varsigma_{2\mathfrak{o}})\| + \|\tau(\mathfrak{J}_1\varsigma_{2\mathfrak{o}},\mathfrak{J}_2\mathfrak{J}_1\varsigma_{2\mathfrak{o}})\|) \\
&\quad +\sqrt{2}\kappa(\varsigma_{2\mathfrak{o}},\mathfrak{J}_1\varsigma_{2\mathfrak{o}},\theta)\|\tau(\mathfrak{J}_1\varsigma_{2\mathfrak{o}},\mathfrak{J}_2\mathfrak{J}_1\varsigma_{2\mathfrak{o}})\| \\
&\quad +\sqrt{2}\varkappa(\varsigma_{2\mathfrak{o}},\mathfrak{J}_1\varsigma_{2\mathfrak{o}},\theta)\|\tau(\varsigma_{2\mathfrak{o}},\mathfrak{J}_1\varsigma_{2\mathfrak{o}})\| \\
&= \alpha(\varsigma_{2\mathfrak{o}},\varsigma_{2\mathfrak{o}+1},\theta)\|\tau(\varsigma_{2\mathfrak{o}},\varsigma_{2\mathfrak{o}+1})\| \\
&\quad +\pi(\varsigma_{2\mathfrak{o}},\varsigma_{2\mathfrak{o}+1},\theta)(\|\tau(\varsigma_{2\mathfrak{o}},\varsigma_{2\mathfrak{o}+1})\| + \|\tau(\varsigma_{2\mathfrak{o}+1},\varsigma_{2\mathfrak{o}+2})\|) \\
&\quad +\sqrt{2}\kappa(\varsigma_{2\mathfrak{o}},\varsigma_{2\mathfrak{o}+1},\theta)\|\tau(\varsigma_{2\mathfrak{o}+1},\varsigma_{2\mathfrak{o}+2})\| \\
&\quad +\sqrt{2}\varkappa(\varsigma_{2\mathfrak{o}},\varsigma_{2\mathfrak{o}+1},\theta)\|\tau(\varsigma_{2\mathfrak{o}},\varsigma_{2\mathfrak{o}+1})\| \\
&\leq \alpha(\varsigma_0,\varsigma_1)\|\tau(\varsigma_{2\mathfrak{o}},\varsigma_{2\mathfrak{o}+1})\| + \sqrt{2}\kappa(\varsigma_0,\varsigma_1)\|\tau(\varsigma_{2\mathfrak{o}+1},\varsigma_{2\mathfrak{o}+2})\|
\end{aligned}
$$

By Proposition 1, we have

$$
\begin{aligned}
\|\tau(\varsigma_{2\mathfrak{o}+2},\varsigma_{2\mathfrak{o}+1})\| &\leq \alpha(\varsigma_0,\varsigma_{2\mathfrak{o}+1},\theta)\|\tau(\varsigma_{2\mathfrak{o}},\varsigma_{2\mathfrak{o}+1})\| \\
&\quad +\pi(\varsigma_0,\varsigma_{2\mathfrak{o}+1},\theta)(\|\tau(\varsigma_{2\mathfrak{o}},\varsigma_{2\mathfrak{o}+1})\| + \|\tau(\varsigma_{2\mathfrak{o}+1},\varsigma_{2\mathfrak{o}+2})\|) \\
&\quad +\sqrt{2}\kappa(\varsigma_0,\varsigma_{2\mathfrak{o}+1},\theta)\|\tau(\varsigma_{2\mathfrak{o}+1},\varsigma_{2\mathfrak{o}+2})\| \\
&\quad +\sqrt{2}\varkappa(\varsigma_0,\varsigma_{2\mathfrak{o}+1},\theta)\|\tau(\varsigma_{2\mathfrak{o}},\varsigma_{2\mathfrak{o}+1})\| \\
&\leq \alpha(\varsigma_0,\varsigma_1,\theta)\|\tau(\varsigma_{2\mathfrak{o}},\varsigma_{2\mathfrak{o}+1})\| \\
&\quad +\pi(\varsigma_0,\varsigma_1,\theta)(\|\tau(\varsigma_{2\mathfrak{o}},\varsigma_{2\mathfrak{o}+1})\| + \|\tau(\varsigma_{2\mathfrak{o}+1},\varsigma_{2\mathfrak{o}+2})\|) \\
&\quad +\sqrt{2}\kappa(\varsigma_0,\varsigma_1,\theta)\|\tau(\varsigma_{2\mathfrak{o}+1},\varsigma_{2\mathfrak{o}+2})\| \\
&\quad +\sqrt{2}\varkappa(\varsigma_0,\varsigma_1,\theta)\|\tau(\varsigma_{2\mathfrak{o}},\varsigma_{2\mathfrak{o}+1})\|
\end{aligned}
$$

which implies that

$$
\begin{aligned}
\|\tau(\varsigma_{2\mathfrak{o}+2},\varsigma_{2\mathfrak{o}+1})\| &\leq \frac{\alpha(\varsigma_0,\varsigma_1,\theta)+\pi(\varsigma_0,\varsigma_1,\theta)+\sqrt{2}\varkappa(\varsigma_0,\varsigma_1,\theta)}{1-\pi(\varsigma_0,\varsigma_1,\theta)-\sqrt{2}\kappa(\varsigma_0,\varsigma_1,\theta)}\|\tau(\varsigma_{2\mathfrak{o}},\varsigma_{2\mathfrak{o}+1})\| \\
&= \frac{\alpha(\varsigma_0,\varsigma_1,\theta)+\pi(\varsigma_0,\varsigma_1,\theta)+\sqrt{2}\varkappa(\varsigma_0,\varsigma_1,\theta)}{1-\pi(\varsigma_0,\varsigma_1,\theta)-\sqrt{2}\kappa(\varsigma_0,\varsigma_1,\theta)}\|\tau(\varsigma_{2\mathfrak{o}+1},\varsigma_{2\mathfrak{o}})\|. \qquad (6)
\end{aligned}
$$

Let $\lambda = \frac{\alpha(\varsigma_0,\varsigma_1,\theta)+\pi(\varsigma_0,\varsigma_1,\theta)+\sqrt{2}\varkappa(\varsigma_0,\varsigma_1,\theta)}{1-\pi(\varsigma_0,\varsigma_1,\theta)-\sqrt{2}\kappa(\varsigma_0,\varsigma_1,\theta)} < 1$. Then, from (5) and (6), we have

$$\|\tau(\varsigma_{\mathfrak{o}+1},\varsigma_{\mathfrak{o}})\| \leq \lambda\|\tau(\varsigma_{\mathfrak{o}},\varsigma_{\mathfrak{o}-1})\|$$

for all $\mathfrak{o} \in \mathbb{N}$. Thus, deductively, we can set up a sequence $\{\varsigma_{\mathfrak{o}}\}$ in $\mathcal{Q}$ such that

$$
\begin{aligned}
\|\tau(\varsigma_{\mathfrak{o}+1},\varsigma_{\mathfrak{o}})\| &\leq \lambda\|\tau(\varsigma_{\mathfrak{o}},\varsigma_{\mathfrak{o}-1})\| \\
&\leq \lambda^2\|\tau(\varsigma_{\mathfrak{o}-1},\varsigma_{\mathfrak{o}-2})\| \cdots \leq \lambda^{\mathfrak{o}}\|\tau(\varsigma_1,\varsigma_0)\| = \lambda^{\mathfrak{o}}\|\tau(\varsigma_0,\varsigma_1)\|
\end{aligned}
$$

for all $\mathfrak{o} \in \mathbb{N}$. Now, for $m > \mathfrak{o}$, we obtain

$$
\begin{aligned}
\|\tau(\varsigma_{\mathfrak{o}},\varsigma_m)\| &\leq \lambda^{\mathfrak{o}}\|\tau(\varsigma_0,\varsigma_1)\| \\
&\quad +\lambda^{\mathfrak{o}+1}\|\tau(\varsigma_0,\varsigma_1)\| \\
&\quad +\cdots+ \\
&\quad \lambda^{m-1}\|\tau(\varsigma_0,\varsigma_1)\| \\
&\leq \frac{\lambda^{\mathfrak{o}}}{1-\lambda}\|\tau(\varsigma_0,\varsigma_1)\|.
\end{aligned}
$$

Now, by taking $\mathfrak{o}, m \to \infty$, we obtain

$$\|\tau(\varsigma_{\mathfrak{o}}, \varsigma_m)\| \to 0.$$

Thus, the sequence $\{\varsigma_{\mathfrak{o}}\}$ is Cauchy by Lemma 2. Since $\mathcal{Q}$ is complete, then $\exists\, \varsigma^* \in \mathcal{Q}$ such that $\varsigma_{\mathfrak{o}} \to \varsigma^*$ as $\mathfrak{o} \to \infty$. $\square$

Now, from (2), we have

$$\begin{aligned}
\tau(\varsigma^*, \mathfrak{J}_1\varsigma^*) &\preceq_{i_2} \tau(\varsigma^*, \mathfrak{J}_2\varsigma_{2\mathfrak{o}+1}) + \tau(\mathfrak{J}_2\varsigma_{2\mathfrak{o}+1}, \mathfrak{J}_1\varsigma^*) \\
&= \tau(\varsigma^*, \mathfrak{J}_2\varsigma_{2\mathfrak{o}+1}) + \tau(\mathfrak{J}_1\varsigma^*, \mathfrak{J}_2\varsigma_{2\mathfrak{o}+1}) \\
&\preceq_{i_2} \begin{pmatrix}
\tau(\varsigma^*, \varsigma_{2\mathfrak{o}+2}) + \alpha(\varsigma^*, \varsigma_{2\mathfrak{o}+1}, \theta)\tau(\varsigma^*, \varsigma_{2\mathfrak{o}+1}) \\
+\pi(\varsigma^*, \varsigma_{2\mathfrak{o}+1}, \theta)(\tau(\varsigma^*, \mathfrak{J}_1\varsigma^*) + \tau(\varsigma_{2\mathfrak{o}+1}, \mathfrak{J}_2\varsigma_{2\mathfrak{o}+1})) \\
+\kappa(\varsigma^*, \varsigma_{2\mathfrak{o}+1}, \theta)\frac{\tau(\varsigma^*, \mathfrak{J}_1\varsigma^*)\tau(\varsigma_{2\mathfrak{o}+1}, \mathfrak{J}_2\varsigma_{2\mathfrak{o}+1})}{1+\tau(\varsigma^*, \varsigma_{2\mathfrak{o}+1})} \\
+\varpi(\varsigma^*, \varsigma_{2\mathfrak{o}+1}, \theta)\frac{\tau(\varsigma_{2\mathfrak{o}+1}, \mathfrak{J}_1\varsigma^*)\tau(\varsigma^*, \mathfrak{J}_2\varsigma_{2\mathfrak{o}+1})}{1+\tau(\varsigma^*, \varsigma_{2\mathfrak{o}+1})} \\
+\varkappa(\varsigma^*, \varsigma_{2\mathfrak{o}+1}, \theta)\frac{\tau(\varsigma^*, \mathfrak{J}_1\varsigma^*)\tau(\varsigma^*, \mathfrak{J}_2\varsigma_{2\mathfrak{o}+1}) + \tau(\varsigma_{2\mathfrak{o}+1}, \mathfrak{J}_2\varsigma_{2\mathfrak{o}+1})\tau(\varsigma_{2\mathfrak{o}+1}, \mathfrak{J}_1\varsigma^*)}{1+\tau(\varsigma^*, \mathfrak{J}_2\varsigma_{2\mathfrak{o}+1}) + \tau(\varsigma_{2\mathfrak{o}+1}, \mathfrak{J}_1\varsigma^*)}
\end{pmatrix} \\
&\preceq_{i_2} \begin{pmatrix}
\tau(\varsigma^*, \varsigma_{2\mathfrak{o}+2}) + \alpha(\varsigma^*, \varsigma_{2\mathfrak{o}+1}, \theta)\tau(\varsigma^*, \varsigma_{2\mathfrak{o}+1}) \\
+\pi(\varsigma^*, \varsigma_{2\mathfrak{o}+1}, \theta)(\tau(\varsigma^*, \mathfrak{J}_1\varsigma^*) + \tau(\varsigma_{2\mathfrak{o}+1}, \varsigma_{2\mathfrak{o}+2})) \\
+\kappa(\varsigma^*, \varsigma_{2\mathfrak{o}+1}, \theta)\frac{\tau(\varsigma^*, \mathfrak{J}_1\varsigma^*)\tau(\varsigma_{2\mathfrak{o}+1}, \varsigma_{2\mathfrak{o}+2})}{1+\tau(\varsigma^*, \varsigma_{2\mathfrak{o}+1})} \\
+\varpi(\varsigma^*, \varsigma_{2\mathfrak{o}+1}, \theta)\frac{\tau(\varsigma_{2\mathfrak{o}+1}, \mathfrak{J}_1\varsigma^*)\tau(\varsigma^*, \varsigma_{2\mathfrak{o}+2})}{1+\tau(\varsigma^*, \varsigma_{2\mathfrak{o}+1})} \\
+\varkappa(\varsigma^*, \varsigma_{2\mathfrak{o}+1}, \theta)\frac{\tau(\varsigma^*, \mathfrak{J}_1\varsigma^*)\tau(\varsigma^*, \varsigma_{2\mathfrak{o}+2}) + \tau(\varsigma_{2\mathfrak{o}+1}, \varsigma_{2\mathfrak{o}+2})\tau(\varsigma_{2\mathfrak{o}+1}, \mathfrak{J}_1\varsigma^*)}{1+\tau(\varsigma^*, \varsigma_{2\mathfrak{o}+2}) + \tau(\varsigma_{2\mathfrak{o}+1}, \mathfrak{J}_1\varsigma^*)}
\end{pmatrix}.
\end{aligned}$$

This implies that

$$\|\tau(\varsigma^*, \mathfrak{J}_1\varsigma^*)\| \leq \begin{pmatrix}
\|\tau(\varsigma^*, \varsigma_{2\mathfrak{o}+2})\| + \alpha(\varsigma^*, \varsigma_{2\mathfrak{o}+1}, \theta)\|\tau(\varsigma^*, \varsigma_{2\mathfrak{o}+1})\| \\
+\pi(\varsigma^*, \varsigma_{2\mathfrak{o}+1}, \theta)(\|\tau(\varsigma^*, \mathfrak{J}_1\varsigma^*)\| + \|\tau(\varsigma_{2\mathfrak{o}+1}, \varsigma_{2\mathfrak{o}+2})\|) \\
+\sqrt{2}\kappa(\varsigma^*, \varsigma_{2\mathfrak{o}+1}, \theta)\frac{\|\tau(\varsigma^*, \mathfrak{J}_1\varsigma^*)\|\|\tau(\varsigma_{2\mathfrak{o}+1}, \varsigma_{2\mathfrak{o}+2})\|}{\|1+\tau(\varsigma^*, \varsigma_{2\mathfrak{o}+1})\|} \\
+\sqrt{2}\varpi(\varsigma^*, \varsigma_{2\mathfrak{o}+1}, \theta)\frac{\|\tau(\varsigma_{2\mathfrak{o}+1}, \mathfrak{J}_1\varsigma^*)\|\|\tau(\varsigma^*, \varsigma_{2\mathfrak{o}+2})\|}{\|1+\tau(\varsigma^*, \varsigma_{2\mathfrak{o}+1})\|} \\
+\sqrt{2}\varkappa(\varsigma^*, \varsigma_{2\mathfrak{o}+1}, \theta)\frac{\|\tau(\varsigma^*, \mathfrak{J}_1\varsigma^*)\|\|\tau(\varsigma^*, \varsigma_{2\mathfrak{o}+2})\| + \|\tau(\varsigma_{2\mathfrak{o}+1}, \varsigma_{2\mathfrak{o}+2})\|\|\tau(\varsigma_{2\mathfrak{o}+1}, \mathfrak{J}_1\varsigma^*)\|}{\|1+\tau(\varsigma^*, \varsigma_{2\mathfrak{o}+2}) + \tau(\varsigma_{2\mathfrak{o}+1}, \mathfrak{J}_1\varsigma^*)\|}
\end{pmatrix}.$$

Letting $\mathfrak{o} \to \infty$, we have

$$\frac{1}{\pi(\varsigma^*, \varsigma_{2\mathfrak{o}+1}, \theta)}\|\tau(\varsigma^*, \mathfrak{J}_1\varsigma^*)\| \leq 0.$$

Since $\frac{1}{\pi(\varsigma^*, \varsigma_{2\mathfrak{o}+1}, \theta)} \neq 0$, then $\|\tau(\varsigma^*, \mathfrak{J}_1\varsigma^*)\| = 0$. Thus, $\varsigma^* = \mathfrak{J}_1\varsigma^*$. Now, we show that ς^* is a fixed point of $\mathfrak{J}_2$. By (2), we have

$$\begin{aligned}
\tau(\varsigma^*, \mathfrak{J}_2\varsigma^*) &\preceq_{i_2} (\tau(\varsigma^*, \mathfrak{J}_1\varsigma_{2\mathfrak{o}}) + \tau(\mathfrak{J}_1\varsigma_{2\mathfrak{o}}, \mathfrak{J}_2\varsigma^*)) \\
&\preceq_{i_2} \begin{pmatrix}
\tau(\varsigma^*, \mathfrak{J}_1\varsigma_{2\mathfrak{o}}) + \alpha(\varsigma_{2\mathfrak{o}}, \varsigma^*, \theta)\tau(\varsigma_{2\mathfrak{o}}, \varsigma^*) \\
+\pi(\varsigma_{2\mathfrak{o}}, \varsigma^*, \theta)(\tau(\varsigma_{2\mathfrak{o}}, \mathfrak{J}_1\varsigma_{2\mathfrak{o}}) + \tau(\varsigma^*, \mathfrak{J}_2\varsigma^*)) \\
+\kappa(\varsigma_{2\mathfrak{o}}, \varsigma^*, \theta)\frac{\tau(\varsigma_{2\mathfrak{o}}, \mathfrak{J}_1\varsigma_{2\mathfrak{o}})\tau(\varsigma^*, \mathfrak{J}_2\varsigma^*)}{1+\tau(\varsigma_{2\mathfrak{o}}, \varsigma^*)} \\
+\varpi(\varsigma_{2\mathfrak{o}}, \varsigma^*, \theta)\frac{\tau(\varsigma^*, \mathfrak{J}_1\varsigma_{2\mathfrak{o}})\tau(\varsigma_{2\mathfrak{o}}, \mathfrak{J}_2\varsigma^*)}{1+\tau(\varsigma_{2\mathfrak{o}}, \varsigma^*)} \\
+\varkappa(\varsigma_{2\mathfrak{o}}, \varsigma^*, \theta)\frac{\tau(\varsigma_{2\mathfrak{o}}, \mathfrak{J}_1\varsigma_{2\mathfrak{o}})\tau(\varsigma_{2\mathfrak{o}}, \mathfrak{J}_2\varsigma^*) + \tau(\varsigma^*, \mathfrak{J}_2\varsigma^*)\tau(\varsigma^*, \mathfrak{J}_1\varsigma_{2\mathfrak{o}})}{1+\tau(\varsigma_{2\mathfrak{o}}, \mathfrak{J}_2\varsigma^*) + \tau(\varsigma^*, \mathfrak{J}_1\varsigma_{2\mathfrak{o}})}
\end{pmatrix} \\
&\preceq_{i_2} \begin{pmatrix}
\tau(\varsigma^*, \varsigma_{2\mathfrak{o}+1}) + \alpha(\varsigma_{2\mathfrak{o}}, \varsigma^*, \theta)\tau(\varsigma_{2\mathfrak{o}}, \varsigma^*) \\
+\kappa(\varsigma_{2\mathfrak{o}}, \varsigma^*, \theta)\frac{\tau(\varsigma_{2\mathfrak{o}}, \varsigma_{2\mathfrak{o}+1})\tau(\varsigma^*, \mathfrak{J}_2\varsigma^*)}{1+\tau(\varsigma_{2\mathfrak{o}}, \varsigma^*)} \\
+\pi(\varsigma_{2\mathfrak{o}}, \varsigma^*, \theta)(\tau(\varsigma_{2\mathfrak{o}}, \mathfrak{J}_1\varsigma_{2\mathfrak{o}}) + \tau(\varsigma^*, \mathfrak{J}_2\varsigma^*)) \\
+\varpi(\varsigma_{2\mathfrak{o}}, \varsigma^*, \theta)\frac{\tau(\varsigma^*, \varsigma_{2\mathfrak{o}+1})\tau(\varsigma_{2\mathfrak{o}}, \mathfrak{J}_2\varsigma^*)}{1+\tau(\varsigma_{2\mathfrak{o}}, \varsigma^*)} \\
+\varkappa(\varsigma_{2\mathfrak{o}}, \varsigma^*, \theta)\frac{\tau(\varsigma_{2\mathfrak{o}}, \varsigma_{2\mathfrak{o}+1})\tau(\varsigma_{2\mathfrak{o}}, \mathfrak{J}_2\varsigma^*) + \tau(\varsigma^*, \mathfrak{J}_2\varsigma^*)\tau(\varsigma^*, \varsigma_{2\mathfrak{o}+1})}{1+\tau(\varsigma_{2\mathfrak{o}}, \mathfrak{J}_2\varsigma^*) + \tau(\varsigma^*, \varsigma_{2\mathfrak{o}+1})}
\end{pmatrix}.
\end{aligned}$$

This implies that

$$\|\tau(\varsigma^*,\mathfrak{J}_2\varsigma^*)\| \leq \left(\begin{array}{c} \|\tau(\varsigma^*,\varsigma_{2\mathfrak{o}+1})\| + \alpha(\varsigma_{2\mathfrak{o}},\varsigma^*,\theta)\|\tau(\varsigma_{2\mathfrak{o}},\varsigma^*)\| \\ +\pi(\varsigma_{2\mathfrak{o}},\varsigma^*,\theta)(\|\tau(\varsigma_{2\mathfrak{o}},\mathfrak{J}_1\varsigma_{2\mathfrak{o}})\| + \|\tau(\varsigma^*,\mathfrak{J}_2\varsigma^*)\|) \\ +\sqrt{2}\kappa(\varsigma_{2\mathfrak{o}},\varsigma^*,\theta)\frac{\|\tau(\varsigma_{2\mathfrak{o}},\varsigma_{2\mathfrak{o}+1})\|\|\tau(\varsigma^*,\mathfrak{J}_2\varsigma^*)\|}{\|1+\tau(\varsigma_{2\mathfrak{o}},\varsigma^*)\|} \\ +\sqrt{2}\varpi(\varsigma_{2\mathfrak{o}},\varsigma^*,\theta)\frac{\|\tau(\varsigma^*,\varsigma_{2\mathfrak{o}+1})\|\|\tau(\varsigma_{2\mathfrak{o}},\mathfrak{J}_2\varsigma^*)\|}{\|1+\tau(\varsigma_{2\mathfrak{o}},\varsigma^*)\|} \\ +\sqrt{2}\varkappa(\varsigma_{2\mathfrak{o}},\varsigma^*,\theta)\frac{\|\tau(\varsigma_{2\mathfrak{o}},\varsigma_{2\mathfrak{o}+1})\|\|\tau(\varsigma_{2\mathfrak{o}},\mathfrak{J}_2\varsigma^*)\|+\|\tau(\varsigma^*,\mathfrak{J}_2\varsigma^*)\|\|\tau(\varsigma^*,\varsigma_{2\mathfrak{o}+1})\|}{\|1+\tau(\varsigma_{2\mathfrak{o}},\mathfrak{J}_2\varsigma^*)+\tau(\varsigma^*,\varsigma_{2\mathfrak{o}+1})\|} \end{array} \right).$$

Letting $\mathfrak{o} \to \infty$, we have $\frac{1}{\pi(\varsigma_{2\mathfrak{o}},\varsigma^*,\theta)}\|\tau(\varsigma^*,\mathfrak{J}_2\varsigma^*)\| \leq 0$ since $\frac{1}{\pi(\varsigma_{2\mathfrak{o}},\varsigma^*,\theta)} \neq 0$. Hence, $\varsigma^* = \mathfrak{J}_2\varsigma^*$. Thus, ς^* is a common fixed point of $\mathfrak{J}_1$ and $\mathfrak{J}_2$. We assume that there exists another common fixed point of $\mathfrak{J}_1$ and $\mathfrak{J}_2$, that is,

$$\varsigma^{/} = \mathfrak{J}_1\varsigma^{/} = \mathfrak{J}_2\varsigma^{/}$$

but $\varsigma^* \neq \varsigma^{/}$. Now, from (2), we have

$$\begin{aligned} \tau\left(\varsigma^*,\varsigma^{/}\right) = \tau\left(\mathfrak{J}_1\varsigma^*,\mathfrak{J}_2\varsigma^{/}\right) &\preceq_{i_2} \alpha\left(\varsigma^*,\varsigma^{/},\theta\right)\tau\left(\varsigma^*,\varsigma^{/}\right) \\ &+ \pi\left(\varsigma^*,\varsigma^{/},\theta\right)\left(\tau(\varsigma^*,\mathfrak{J}_1\varsigma^*) + \tau\left(\varsigma^{/},\mathfrak{J}_2\varsigma^{/}\right)\right) \\ &+ \kappa\left(\varsigma^*,\varsigma^{/},\theta\right)\frac{\tau(\varsigma^*,\mathfrak{J}\varsigma^*)\tau\left(\varsigma^{/},\mathfrak{J}_2\varsigma^{/}\right)}{1+\tau(\varsigma^*,\varsigma^{/})} \\ &+ \varpi\left(\varsigma^*,\varsigma^{/},\theta\right)\frac{\tau\left(\varsigma^{/},\mathfrak{J}_1\varsigma^*\right)\tau\left(\varsigma^*,\mathfrak{J}_2\varsigma^{/}\right)}{1+\tau(\varsigma^*,\varsigma^{/})} \\ &+ \varkappa\left(\varsigma^*,\varsigma^{/},\theta\right)\frac{\tau(\varsigma^*,\mathfrak{J}_1\varsigma^*)\tau\left(\varsigma^*,\mathfrak{J}_2\varsigma^{/}\right) + \tau\left(\varsigma^{/},\mathfrak{J}_2\varsigma^{/}\right)\tau\left(\varsigma^{/},\mathfrak{J}_1\varsigma^*\right)}{1+\tau(\varsigma^*,\mathfrak{J}_2\varsigma^{/})+\tau(\varsigma^{/},\mathfrak{J}_1\varsigma^*)} \end{aligned}$$

which implies that

$$\begin{aligned} \tau\left(\varsigma^*,\varsigma^{/}\right) &\preceq_{i_2} \alpha\left(\varsigma^*,\varsigma^{/},\theta\right)\tau\left(\varsigma^*,\varsigma^{/}\right) \\ &+ \varpi\left(\varsigma^*,\varsigma^{/},\theta\right)\frac{\tau\left(\varsigma^{/},\varsigma^*\right)\tau\left(\varsigma^*,\varsigma^{/}\right)}{1+\tau(\varsigma^*,\varsigma^{/})}. \end{aligned}$$

This yields that

$$\begin{aligned} \left\|\tau\left(\varsigma^*,\varsigma^{/}\right)\right\| &\leq \alpha\left(\varsigma^*,\varsigma^{/},\theta\right)\left\|\tau\left(\varsigma^*,\varsigma^{/}\right)\right\| \\ &\quad +\sqrt{2}\varpi\left(\varsigma^*,\varsigma^{/},\theta\right)\left\|\tau\left(\varsigma^*,\varsigma^{/}\right)\right\|\left\|\frac{\tau\left(\varsigma^*,\varsigma^{/}\right)}{1+\tau(\varsigma^*,\varsigma^{/})}\right\| \\ &\leq \alpha\left(\varsigma^*,\varsigma^{/},\theta\right)\left\|\tau\left(\varsigma^*,\varsigma^{/}\right)\right\| + \sqrt{2}\varpi\left(\varsigma^*,\varsigma^{/},\theta\right)\left\|\tau\left(\varsigma^*,\varsigma^{/}\right)\right\| \\ &= \left(\alpha\left(\varsigma^*,\varsigma^{/},\theta\right) + \sqrt{2}\varpi\left(\varsigma^*,\varsigma^{/},\theta\right)\right)\left\|\tau\left(\varsigma^*,\varsigma^{/}\right)\right\|, \end{aligned}$$

that is,

$$\frac{1}{\alpha(\varsigma^*,\varsigma^{/},\theta)+\sqrt{2}\varpi(\varsigma^*,\varsigma^{/},\theta)}\left\|\tau\left(\varsigma^*,\varsigma^{/}\right)\right\| \leq 0$$

As $\frac{1}{\alpha(\varsigma^*,\varsigma^{/},\theta)+\sqrt{2}\varpi(\varsigma^*,\varsigma^{/},\theta)} \neq 0$, we have

$$\left\|\tau\left(\varsigma^*,\varsigma^{/}\right)\right\| = 0.$$

Thus, $\varsigma^* = \varsigma^{/}$.

Note: From now onwards, we consider $(\mathcal{Q}, \tau)$ as a complete bi-CVMS.

Corollary 1. *Let $\mathfrak{I}_1, \mathfrak{I}_2 : (\mathcal{Q}, \tau) \to (\mathcal{Q}, \tau)$ be self-mappings. If the functions $\alpha, \pi, \kappa, \varpi : \mathcal{Q}^3 \to [0,1)$ satisfy the conditions*

(a)

$\alpha(\mathfrak{I}_2\mathfrak{I}_1\varsigma, \nu, \theta) \leq \alpha(\varsigma, \nu, \theta)$ *and* $\alpha(\varsigma, \mathfrak{I}_1\mathfrak{I}_2\nu, \theta) \leq \alpha(\varsigma, \nu, \theta)$
$\pi(\mathfrak{I}_2\mathfrak{I}_1\varsigma, \nu, \theta) \leq \pi(\varsigma, \nu, \theta)$ *and* $\pi(\varsigma, \mathfrak{I}_1\mathfrak{I}_2\nu, \theta) \leq \pi(\varsigma, \nu, \theta)$
$\kappa(\mathfrak{I}_2\mathfrak{I}_1\varsigma, \nu, \theta) \leq \kappa(\varsigma, \nu, \theta)$ *and* $\kappa(\varsigma, \mathfrak{I}_1\mathfrak{I}_2\nu, \theta) \leq \kappa(\varsigma, \nu, \theta)$
$\varpi(\mathfrak{I}_2\mathfrak{I}_1\varsigma, \nu, \theta) \leq \varpi(\varsigma, \nu, \theta)$ *and* $\varpi(\varsigma, \mathfrak{I}_1\mathfrak{I}_2\nu, \theta) \leq \varpi(\varsigma, \nu, \theta)$;

(b) $\alpha(\varsigma, \nu, \theta) + 2\pi(\varsigma, \nu, \theta) + \sqrt{2}\kappa(\varsigma, \nu, \theta) + \sqrt{2}\varpi(\varsigma, \nu, \theta) < 1$;

(c)

$$\begin{aligned} \tau(\mathfrak{I}_1\varsigma, \mathfrak{I}_2\nu) \preceq_{i_2} & \alpha(\varsigma, \nu, \theta)\tau(\varsigma, \nu) + \pi(\varsigma, \nu, \theta)(\tau(\varsigma, \mathfrak{I}_1\varsigma) + \tau(\nu, \mathfrak{I}_2\nu)) \\ & + \kappa(\varsigma, \nu, \theta)\frac{\tau(\varsigma, \mathfrak{I}_1\varsigma)\tau(\nu, \mathfrak{I}_2\nu)}{1 + \tau(\varsigma, \nu)} \\ & + \varpi(\varsigma, \nu, \theta)\frac{\tau(\nu, \mathfrak{I}_1\varsigma)\tau(\varsigma, \mathfrak{I}_2\nu)}{1 + \tau(\varsigma, \nu)} \end{aligned}$$

for all $\varsigma, \nu \in \mathcal{Q}$ and for fixed element $\theta \in \mathcal{Q}$, then there exists a unique element $\varsigma^* \in \mathcal{Q}$ such that $\mathfrak{I}_1\varsigma^* = \mathfrak{I}_2\varsigma^* = \varsigma^*$.

Proof. Take $\varkappa : \mathcal{Q} \times \mathcal{Q} \times \mathcal{Q} \to [0,1)$ by $\varpi(\varsigma, \nu, \theta) = 0$ in Theorem 1. □

Corollary 2. *Let $\mathfrak{I}_1, \mathfrak{I}_2 : (\mathcal{Q}, \tau) \to (\mathcal{Q}, \tau)$ be self-mappings. If the functions $\alpha, \pi, \kappa, \varkappa : \mathcal{Q}^3 \to [0,1)$ satisfy the conditions*

(a)

$\alpha(\mathfrak{I}_2\mathfrak{I}_1\varsigma, \nu, \theta) \leq \alpha(\varsigma, \nu, \theta)$ *and* $\alpha(\varsigma, \mathfrak{I}_1\mathfrak{I}_2\nu, \theta) \leq \alpha(\varsigma, \nu, \theta)$
$\pi(\mathfrak{I}_2\mathfrak{I}_1\varsigma, \nu, \theta) \leq \pi(\varsigma, \nu, \theta)$ *and* $\pi(\varsigma, \mathfrak{I}_1\mathfrak{I}_2\nu, \theta) \leq \pi(\varsigma, \nu, \theta)$
$\kappa(\mathfrak{I}_2\mathfrak{I}_1\varsigma, \nu, \theta) \leq \kappa(\varsigma, \nu, \theta)$ *and* $\kappa(\varsigma, \mathfrak{I}_1\mathfrak{I}_2\nu, \theta) \leq \kappa(\varsigma, \nu, \theta)$
$\varkappa(\mathfrak{I}_2\mathfrak{I}_1\varsigma, \nu, \theta) \leq \varkappa(\varsigma, \nu, \theta)$ *and* $\varkappa(\varsigma, \mathfrak{I}_1\mathfrak{I}_2\nu, \theta) \leq \varkappa(\varsigma, \nu, \theta)$;

(b) $\alpha(\varsigma, \nu, \theta) + 2\pi(\varsigma, \nu, \theta) + \sqrt{2}\kappa(\varsigma, \nu, \theta) + \sqrt{2}\varkappa(\varsigma, \nu, \theta) < 1$;

(c)

$$\begin{aligned} \tau(\mathfrak{I}_1\varsigma, \mathfrak{I}_2\nu) \preceq_{i_2} & \alpha(\varsigma, \nu, \theta)\tau(\varsigma, \nu) + \pi(\varsigma, \nu, \theta)(\tau(\varsigma, \mathfrak{I}_1\varsigma) + \tau(\nu, \mathfrak{I}_2\nu)) \\ & + \kappa(\varsigma, \nu, \theta)\frac{\tau(\varsigma, \mathfrak{I}_1\varsigma)\tau(\nu, \mathfrak{I}_2\nu)}{1 + \tau(\varsigma, \nu)} \\ & + \varkappa(\varsigma, \nu, \theta)\frac{\tau(\varsigma, \mathfrak{I}_1\varsigma)\tau(\varsigma, \mathfrak{I}_2\nu) + \tau(\nu, \mathfrak{I}_2\nu)\tau(\nu, \mathfrak{I}_1\varsigma)}{1 + \tau(\varsigma, \mathfrak{I}_2\nu) + \tau(\nu, \mathfrak{I}_1\varsigma)}, \end{aligned}$$

for all $\varsigma, \nu \in \mathcal{Q}$ and for fixed element $\theta \in \mathcal{Q}$, then there exists a unique element $\varsigma^* \in \mathcal{Q}$ such that $\mathfrak{I}_1\varsigma^* = \mathfrak{I}_2\varsigma^* = \varsigma^*$.

Proof. Take $\varpi : \mathcal{Q} \times \mathcal{Q} \times \mathcal{Q} \to [0,1)$ by $\varpi(\varsigma, \nu, \theta) = 0$ in Theorem 1. □

Corollary 3. *Let $\mathfrak{I}_1, \mathfrak{I}_2 : (\mathcal{Q}, \tau) \to (\mathcal{Q}, \tau)$ be self-mappings. If the functions $\alpha, \pi, \varpi, \varkappa : \mathcal{Q}^3 \to [0,1)$ satisfy the conditions*

(a)

$\alpha(\mathfrak{I}_2\mathfrak{I}_1\varsigma, \nu, \theta) \leq \alpha(\varsigma, \nu, \theta)$ *and* $\alpha(\varsigma, \mathfrak{I}_1\mathfrak{I}_2\nu, \theta) \leq \alpha(\varsigma, \nu, \theta)$
$\pi(\mathfrak{I}_2\mathfrak{I}_1\varsigma, \nu, \theta) \leq \pi(\varsigma, \nu, \theta)$ *and* $\pi(\varsigma, \mathfrak{I}_1\mathfrak{I}_2\nu, \theta) \leq \pi(\varsigma, \nu, \theta)$
$\varpi(\mathfrak{I}_2\mathfrak{I}_1\varsigma, \nu, \theta) \leq \varpi(\varsigma, \nu, \theta)$ *and* $\varpi(\varsigma, \mathfrak{I}_1\mathfrak{I}_2\nu, \theta) \leq \varpi(\varsigma, \nu, \theta)$
$\varkappa(\mathfrak{I}_2\mathfrak{I}_1\varsigma, \nu, \theta) \leq \varkappa(\varsigma, \nu, \theta)$ *and* $\varkappa(\varsigma, \mathfrak{I}_1\mathfrak{I}_2\nu, \theta) \leq \varkappa(\varsigma, \nu, \theta)$;

(b) $\alpha(\varsigma, \nu, \theta) + 2\pi(\varsigma, \nu, \theta) + \sqrt{2}\kappa(\varsigma, \nu, \theta) + \sqrt{2}\varpi(\varsigma, \nu, \theta) + \sqrt{2}\varkappa(\varsigma, \nu, \theta) < 1$;

(c)

$$\tau(\mathfrak{I}_1\varsigma, \mathfrak{I}_2\nu) \preceq_{i_2} \alpha(\varsigma, \nu, \theta)\tau(\varsigma, \nu) + \pi(\varsigma, \nu, \theta)(\tau(\varsigma, \mathfrak{I}_1\varsigma) + \tau(\nu, \mathfrak{I}_2\nu))$$

$$+\varpi(\varsigma,\nu,\theta)\frac{\tau(\nu,\Im_1\varsigma)\tau(\varsigma,\Im_2\nu)}{1+\tau(\varsigma,\nu)}$$

$$+\varkappa(\varsigma,\nu,\theta)\frac{\tau(\varsigma,\Im_1\varsigma)\tau(\varsigma,\Im_2\nu)+\tau(\nu,\Im_2\nu)\tau(\nu,\Im_1\varsigma)}{1+\tau(\varsigma,\Im_2\nu)+\tau(\nu,\Im_1\varsigma)},$$

for all $\varsigma,\nu \in \mathcal{Q}$ and for fixed element $\theta \in \mathcal{Q}$, then there exists a unique element $\varsigma^* \in \mathcal{Q}$ such that $\Im_1\varsigma^* = \Im_2\varsigma^* = \varsigma^*$.

Proof. Take $\kappa : \mathcal{Q}\times\mathcal{Q}\times\mathcal{Q}\to[0,1)$ by $\kappa(\varsigma,\nu,\theta) = 0$ in Theorem 1. □

Corollary 4. *Let $\Im_1, \Im_2 : (\mathcal{Q},\tau) \to (\mathcal{Q},\tau)$ be self-mappings. If the functions $\alpha,\kappa,\varpi,\varkappa : \mathcal{Q}^3\to[0,1)$ satisfy the conditions*

(a)

$\alpha(\Im_2\Im_1\varsigma,\nu,\theta) \leq \alpha(\varsigma,\nu,\theta)$ *and* $\alpha(\varsigma,\Im_1\Im_2\nu,\theta) \leq \alpha(\varsigma,\nu,\theta)$

$\kappa(\Im_2\Im_1\varsigma,\nu,\theta) \leq \kappa(\varsigma,\nu,\theta)$ *and* $\kappa(\varsigma,\Im_1\Im_2\nu,\theta) \leq \kappa(\varsigma,\nu,\theta)$

$\varpi(\Im_2\Im_1\varsigma,\nu,\theta) \leq \varpi(\varsigma,\nu,\theta)$ *and* $\varpi(\varsigma,\Im_1\Im_2\nu,\theta) \leq \varpi(\varsigma,\nu,\theta)$

$\varkappa(\Im_2\Im_1\varsigma,\nu,\theta) \leq \varkappa(\varsigma,\nu,\theta)$ *and* $\varkappa(\varsigma,\Im_1\Im_2\nu,\theta) \leq \varkappa(\varsigma,\nu,\theta)$;

(b) $\alpha(\varsigma,\nu,\theta) + \sqrt{2}\kappa(\varsigma,\nu,\theta) + \sqrt{2}\varpi(\varsigma,\nu,\theta) + \sqrt{2}\varkappa(\varsigma,\nu,\theta) < 1$;

(c)

$$\begin{aligned}\tau(\Im_1\varsigma,\Im_2\nu) \preceq_{i_2} {} & \alpha(\varsigma,\nu,\theta)\tau(\varsigma,\nu)\\ & + \kappa(\varsigma,\nu,\theta)\frac{\tau(\varsigma,\Im_1\varsigma)\tau(\nu,\Im_2\nu)}{1+\tau(\varsigma,\nu)}\\ & + \varpi(\varsigma,\nu,\theta)\frac{\tau(\nu,\Im_1\varsigma)\tau(\varsigma,\Im_2\nu)}{1+\tau(\varsigma,\nu)}\\ & + \varkappa(\varsigma,\nu,\theta)\frac{\tau(\varsigma,\Im_1\varsigma)\tau(\varsigma,\Im_2\nu)+\tau(\nu,\Im_2\nu)\tau(\nu,\Im_1\varsigma)}{1+\tau(\varsigma,\Im_2\nu)+\tau(\nu,\Im_1\varsigma)},\end{aligned}$$

for all $\varsigma,\nu \in \mathcal{Q}$ and for fixed element $\theta \in \mathcal{Q}$, then there exists a unique element $\varsigma^* \in \mathcal{Q}$ such that $\Im_1\varsigma^* = \Im_2\varsigma^* = \varsigma^*$.

Proof. Take $\pi : \mathcal{Q}\times\mathcal{Q}\times\mathcal{Q}\to[0,1)$ by $\pi(\varsigma,\nu,\theta) = 0$ in Theorem 1. □

Corollary 5. *Let $\Im_1, \Im_2 : (\mathcal{Q},\tau) \to (\mathcal{Q},\tau)$ be self-mappings. If the functions $\pi,\kappa,\varpi,\varkappa : \mathcal{Q}^3\to[0,1)$ satisfy the conditions*

(a)

$\pi(\Im_2\Im_1\varsigma,\nu,\theta) \leq \pi(\varsigma,\nu,\theta)$ *and* $\pi(\varsigma,\Im_1\Im_2\nu,\theta) \leq \pi(\varsigma,\nu,\theta)$

$\kappa(\Im_2\Im_1\varsigma,\nu,\theta) \leq \kappa(\varsigma,\nu,\theta)$ *and* $\kappa(\varsigma,\Im_1\Im_2\nu,\theta) \leq \kappa(\varsigma,\nu,\theta)$

$\varpi(\Im_2\Im_1\varsigma,\nu,\theta) \leq \varpi(\varsigma,\nu,\theta)$ *and* $\varpi(\varsigma,\Im_1\Im_2\nu,\theta) \leq \varpi(\varsigma,\nu,\theta)$

$\varkappa(\Im_2\Im_1\varsigma,\nu,\theta) \leq \varkappa(\varsigma,\nu,\theta)$ *and* $\varkappa(\varsigma,\Im_1\Im_2\nu,\theta) \leq \varkappa(\varsigma,\nu,\theta)$;

(b) $2\pi(\varsigma,\nu,\theta) + \sqrt{2}\kappa(\varsigma,\nu,\theta) + \sqrt{2}\varpi(\varsigma,\nu,\theta) + \sqrt{2}\varkappa(\varsigma,\nu,\theta) < 1$;

(c)

$$\begin{aligned}\tau(\Im_1\varsigma,\Im_2\nu) \preceq_{i_2} {} & \pi(\varsigma,\nu,\theta)(\tau(\varsigma,\Im_1\varsigma)+\tau(\nu,\Im_2\nu))\\ & + \kappa(\varsigma,\nu,\theta)\frac{\tau(\varsigma,\Im_1\varsigma)\tau(\nu,\Im_2\nu)}{1+\tau(\varsigma,\nu)}\\ & + \varpi(\varsigma,\nu,\theta)\frac{\tau(\nu,\Im_1\varsigma)\tau(\varsigma,\Im_2\nu)}{1+\tau(\varsigma,\nu)}\\ & + \varkappa(\varsigma,\nu,\theta)\frac{\tau(\varsigma,\Im_1\varsigma)\tau(\varsigma,\Im_2\nu)+\tau(\nu,\Im_2\nu)\tau(\nu,\Im_1\varsigma)}{1+\tau(\varsigma,\Im_2\nu)+\tau(\nu,\Im_1\varsigma)},\end{aligned}$$

for all $\varsigma,\nu \in \mathcal{Q}$ and for fixed element $\theta \in \mathcal{Q}$, then there exists a unique element $\varsigma^* \in \mathcal{Q}$ such that $\Im_1\varsigma^* = \Im_2\varsigma^* = \varsigma^*$.

Proof. Take $\alpha : \mathcal{Q}\times\mathcal{Q}\times\mathcal{Q}\to[0,1)$ by $\alpha(\varsigma,\nu,\theta) = 0$ in Theorem 1. □

Example 2. *Let $\mathcal{Q} = [0,1]$ and $\tau : \mathcal{Q} \times \mathcal{Q} \to \mathbb{C}_2$ be defined by*

$$\tau(\varsigma, \nu) = (1 + i_2)|\varsigma - \nu|$$

for all $\varsigma, \nu \in \mathcal{Q}$. Then, $(\mathcal{Q}, \tau)$ is a complete bi-CVMS. Define $\mathfrak{J}_1, \mathfrak{J}_2 : \mathcal{Q} \to \mathcal{Q}$ by

$$\mathfrak{J}_1\varsigma = \frac{\varsigma}{4} \quad \text{and } \mathfrak{J}_2\varsigma = \frac{\varsigma}{3}.$$

Consider

$$\alpha, \pi, \kappa, \varpi, \varkappa : \mathcal{Q} \times \mathcal{Q} \times \mathcal{Q} \to [0,1)$$

by

$$\alpha(\varsigma, \nu, \theta) = \frac{\varsigma}{4} + \frac{\nu}{5} + \theta,$$

$$\pi(\varsigma, \nu, \theta) = \frac{\varsigma^3\theta^3}{64} + \frac{\nu^3}{125},$$

$$\kappa(\varsigma, \nu, \theta) = \frac{\varsigma^2\nu^2\theta^2}{45},$$

$$\varpi(\varsigma, \nu, \theta) = \frac{\varsigma^2\theta^2}{9} + \frac{\nu^2\theta^2}{16},$$

$$\varkappa(\varsigma, \nu, \theta) = \frac{\varsigma}{25} + \frac{\nu}{36} + \frac{\theta}{2}$$

for all $\varsigma, \nu \in \mathcal{Q}$ and for fixed element $\theta \in \mathcal{Q}$. Then, evidently,

$$\alpha(\varsigma, \nu, \theta) + 2\pi(\varsigma, \nu, \theta) + \sqrt{2}\kappa(\varsigma, \nu, \theta) + \sqrt{2}\varpi(\varsigma, \nu, \theta) + \sqrt{2}\varkappa(\varsigma, \nu, \theta) < 1.$$

Now,

$$\alpha(\mathfrak{J}_2\mathfrak{J}_1\varsigma, \nu, \theta) = \alpha\Big(\mathfrak{J}_2(\frac{\varsigma}{4}), \nu, \theta\Big) = \alpha\Big(\frac{\varsigma}{12}, \nu, \theta\Big) = \frac{\varsigma}{48} + \frac{\nu}{5} + \theta \le \frac{\varsigma}{4} + \frac{\nu}{5} + \theta = \alpha(\varsigma, \nu, \theta)$$

$$\alpha(\varsigma, \mathfrak{J}_1\mathfrak{J}_2\nu, \theta) = \alpha\Big(\varsigma, \mathfrak{J}_1(\frac{\nu}{3}), \theta\Big) = \alpha\Big(\varsigma, \frac{\nu}{12}, \theta\Big) = \frac{\varsigma}{4} + \frac{\nu}{60} + \theta \le \frac{\varsigma}{4} + \frac{\nu}{5} + \theta = \alpha(\varsigma, \nu, \theta)$$

$$\pi(\mathfrak{J}_2\mathfrak{J}_1\varsigma, \nu, \theta) = \pi\Big(\mathfrak{J}_2(\frac{\varsigma}{4}), \nu, \theta\Big) = \pi\Big(\frac{\varsigma}{12}, \nu, \theta\Big) = \frac{\varsigma^3\theta^3}{110592} + \frac{\nu^3}{125} \le \frac{\varsigma^3\theta^3}{64} + \frac{\nu^3}{125} = \pi(\varsigma, \nu, \theta)$$

$$\pi(\varsigma, \mathfrak{J}_1\mathfrak{J}_2\nu, \theta) = \pi\Big(\varsigma, \mathfrak{J}_1(\frac{\nu}{3}), \theta\Big) = \pi\Big(\varsigma, \frac{\nu}{12}, \theta\Big) = \frac{\varsigma^3\theta^3}{64} + \frac{\nu^3}{216000} \le \frac{\varsigma^3\theta^3}{64} + \frac{\nu^3}{125} = \pi(\varsigma, \nu, \theta)$$

$$\kappa(\mathfrak{J}_2\mathfrak{J}_1\varsigma, \nu, \theta) = \kappa\Big(\mathfrak{J}_2(\frac{\varsigma}{4}), \nu, \theta\Big) = \kappa\Big(\frac{\varsigma}{12}, \nu, \theta\Big) = \frac{\varsigma^2\nu^2\theta^2}{54540} \le \frac{\varsigma^2\nu^2\theta^2}{45} = \kappa(\varsigma, \nu, \theta)$$

$$\kappa(\varsigma, \mathfrak{J}_1\mathfrak{J}_2\nu, \theta) = \kappa\Big(\varsigma, \mathfrak{J}_1(\frac{\nu}{3}), \theta\Big) = \kappa\Big(\varsigma, \frac{\nu}{12}, \theta\Big) = \frac{\varsigma^2\nu^2\theta^2}{54540} \le \frac{\varsigma^2\nu^2\theta^2}{45} = \kappa(\varsigma, \nu, \theta)$$

$$\varpi(\mathfrak{J}_2\mathfrak{J}_1\varsigma, \nu, \theta) = \varpi\Big(\mathfrak{J}_2(\frac{\varsigma}{4}), \nu, \theta\Big) = \varpi\Big(\frac{\varsigma}{12}, \nu, \theta\Big) = \frac{\varsigma^2\theta^2}{1296} + \frac{\nu^2\theta^2}{16} \le \frac{\varsigma^2\theta^2}{9} + \frac{\nu^2\theta^2}{16} = \varpi(\varsigma, \nu, \theta)$$

$$\varpi(\varsigma, \mathfrak{J}_1\mathfrak{J}_2\nu, \theta) = \varpi\Big(\varsigma, \mathfrak{J}_1(\frac{\nu}{3}), \theta\Big) = \varpi\Big(\varsigma, \frac{\nu}{12}, \theta\Big) = \frac{\varsigma^2\theta^2}{9} + \frac{\nu^2\theta^2}{2304} \le \frac{\varsigma^2\theta^2}{9} + \frac{\nu^2\theta^2}{16} = \varpi(\varsigma, \nu, \theta).$$

Furthermore,

$$\varkappa(\mathfrak{J}_2\mathfrak{J}_1\varsigma, \nu, \theta) = \varkappa\Big(\mathfrak{J}_2(\frac{\varsigma}{4}), \nu, \theta\Big) = \varkappa\Big(\frac{\varsigma}{12}, \nu, \theta\Big) = \frac{\varsigma}{300} + \frac{\nu}{36} + \frac{\theta}{2} \le \frac{\varsigma}{25} + \frac{\nu}{36} + \frac{\theta}{2} = \varkappa(\varsigma, \nu, \theta)$$

$$\varkappa(\varsigma, \mathfrak{J}_1\mathfrak{J}_2\nu, \theta) = \varkappa\Big(\varsigma, \mathfrak{J}_1(\frac{\nu}{3}), \theta\Big) = \varkappa\Big(\varsigma, \frac{\nu}{12}, \theta\Big) = \frac{\varsigma}{25} + \frac{\nu}{432} + \frac{\theta}{2} \le \frac{\varsigma}{25} + \frac{\nu}{36} + \frac{\theta}{2} = \varkappa(\varsigma, \nu, \theta).$$

Now, we prove the contractive condition in this way

$$\begin{aligned}
\tau(\mathfrak{J}_1\varsigma, \mathfrak{J}_2\nu) &= \tau(\frac{\varsigma}{4}, \frac{\nu}{3}) = (1+i_2)\Big|\frac{\varsigma}{4} - \frac{\nu}{3}\Big| \\
&= (1+i_2)\Big|\frac{3\varsigma - 4\nu}{12}\Big| \\
&\preceq_{i_2} (1+i_2)\Big|\frac{3\varsigma - 3\nu}{12}\Big| \\
&= \frac{1}{4}(1+i_2)|\varsigma - \nu| \\
&\preceq_{i_2} \frac{13}{20}(1+i_2)|\varsigma - \nu| \\
&\preceq_{i_2} \alpha(\varsigma,\nu,\theta)\tau(\varsigma,\nu) \\
&+ \pi(\varsigma,\nu,\theta)(\tau(\varsigma,\mathfrak{J}_1\varsigma) + \tau(\nu,\mathfrak{J}_2\nu)) \\
&+ \kappa(\varsigma,\nu,\theta)\frac{\tau(\varsigma,\mathfrak{J}_1\varsigma)\tau(\nu,\mathfrak{J}_2\nu)}{1+\tau(\varsigma,\nu)} \\
&+ \varpi(\varsigma,\nu,\theta)\frac{\tau(\nu,\mathfrak{J}_1\varsigma)\tau(\varsigma,\mathfrak{J}_2\nu)}{1+\tau(\varsigma,\nu)} \\
&+ \varkappa(\varsigma,\nu,\theta)\frac{\tau(\varsigma,\mathfrak{J}_1\varsigma)\tau(\varsigma,\mathfrak{J}_2\nu) + \tau(\nu,\mathfrak{J}_2\nu)\tau(\nu,\mathfrak{J}_1\varsigma)}{1+\tau(\varsigma,\mathfrak{J}_2\nu)+\tau(\nu,\mathfrak{J}_1\varsigma)}.
\end{aligned}$$

Hence, all the conditions of Theorem 1 are satisfied and $0 = \mathfrak{J}_1 0 = \mathfrak{J}_2 0$.

Remark 1. *If we replace* $\alpha, \pi, \kappa, \varpi, \varkappa : \mathcal{Q}\times\mathcal{Q}\times\mathcal{Q}\to[0,1)$ *with* $\alpha, \pi, \kappa, \varpi, \varkappa : \mathcal{Q}\times\mathcal{Q}\to[0,1)$ *by*

$$\alpha(\varsigma,\nu,\theta) = \alpha(\varsigma,\nu),$$

$$\pi(\varsigma,\nu,\theta) = \pi(\varsigma,\nu),$$

$$\kappa(\varsigma,\nu,\theta) = \kappa(\varsigma,\nu),$$

$$\varpi(\varsigma,\nu,\theta) = \varpi(\varsigma,\nu),$$

$$\varkappa(\varsigma,\nu,\theta) = \varkappa(\varsigma,\nu),$$

then we have following result.

Corollary 6. *Let* $\mathfrak{J}_1, \mathfrak{J}_2 : (\mathcal{Q},\tau) \to (\mathcal{Q},\tau)$ *be self-mappings. If the functions* $\alpha, \pi, \kappa, \varpi, \varkappa : \mathcal{Q}^2\to[0,1)$ *satisfy the conditions*

(a) $\alpha(\mathfrak{J}_2\mathfrak{J}_1\varsigma,\nu) \leq \alpha(\varsigma,\nu)$ *and* $\alpha(\varsigma,\mathfrak{J}_1\mathfrak{J}_2\nu) \leq \alpha(\varsigma,\nu)$
$\pi(\mathfrak{J}_2\mathfrak{J}_1\varsigma,\nu) \leq \pi(\varsigma,\nu)$ *and* $\pi(\varsigma,\mathfrak{J}_1\mathfrak{J}_2\nu) \leq \pi(\varsigma,\nu)$
$\kappa(\mathfrak{J}_2\mathfrak{J}_1\varsigma,\nu) \leq \kappa(\varsigma,\nu)$ *and* $\kappa(\varsigma,\mathfrak{J}_1\mathfrak{J}_2\nu) \leq \kappa(\varsigma,\nu)$
$\varpi(\mathfrak{J}_2\mathfrak{J}_1\varsigma,\nu) \leq \varpi(\varsigma,\nu)$ *and* $\varpi(\varsigma,\mathfrak{J}_1\mathfrak{J}_2\nu) \leq \varpi(\varsigma,\nu)$
$\varkappa(\mathfrak{J}_2\mathfrak{J}_1\varsigma,\nu) \leq \varkappa(\varsigma,\nu)$ *and* $\varkappa(\varsigma,\mathfrak{J}_1\mathfrak{J}_2\nu) \leq \varkappa(\varsigma,\nu)$;

(b) $\alpha(\varsigma,\nu) + 2\pi(\varsigma,\nu) + \sqrt{2}\kappa(\varsigma,\nu) + \sqrt{2}\varpi(\varsigma,\nu) + \sqrt{2}\varkappa(\varsigma,\nu) < 1$;

(c)

$$\begin{aligned}
\tau(\mathfrak{J}_1\varsigma, \mathfrak{J}_2\nu) &\preceq_{i_2} \alpha(\varsigma,\nu)\tau(\varsigma,\nu) + \pi(\varsigma,\nu)(\tau(\varsigma,\mathfrak{J}_1\varsigma) + \tau(\nu,\mathfrak{J}_2\nu)) \\
&+\kappa(\varsigma,\nu)\frac{\tau(\varsigma,\mathfrak{J}_1\varsigma)\tau(\nu,\mathfrak{J}_2\nu)}{1+\tau(\varsigma,\nu)} \\
&+\varpi(\varsigma,\nu)\frac{\tau(\nu,\mathfrak{J}_1\varsigma)\tau(\varsigma,\mathfrak{J}_2\nu)}{1+\tau(\varsigma,\nu)} \\
&+ \varkappa(\varsigma,\nu)\frac{\tau(\varsigma,\mathfrak{J}_1\varsigma)\tau(\varsigma,\mathfrak{J}_2\nu) + \tau(\nu,\mathfrak{J}_2\nu)\tau(\nu,\mathfrak{J}_1\varsigma)}{1+\tau(\varsigma,\mathfrak{J}_2\nu)+\tau(\nu,\mathfrak{J}_1\varsigma)},
\end{aligned}$$

for all $\varsigma, \nu \in \mathcal{Q}$, then there exists a unique element $\varsigma^* \in \mathcal{Q}$ such that $\mathfrak{J}_1\varsigma^* = \mathfrak{J}_2\varsigma^* = \varsigma^*$.

If we define $\pi, \varkappa : \mathcal{Q}\times\mathcal{Q}\to[0,1)$ by $\pi(\varsigma,\nu) = \varkappa(\varsigma,\nu) = 0$, then we achieve the key result presented by Tassaddiq et al. [10].

Corollary 7 ([10]). *Let $\mathfrak{J}_1, \mathfrak{J}_2 : (\mathcal{Q},\tau) \to (\mathcal{Q},\tau)$ be self-mappings. If the functions $\alpha, \kappa, \varpi : \mathcal{Q}^2\to[0,1)$ satisfy the conditions*

(a) $\alpha(\mathfrak{J}_2\mathfrak{J}_1\varsigma,\nu) \le \alpha(\varsigma,\nu)$ and $\alpha(\varsigma,\mathfrak{J}_1\mathfrak{J}_2\nu) \le \alpha(\varsigma,\nu)$
$\kappa(\mathfrak{J}_2\mathfrak{J}_1\varsigma,\nu) \le \kappa(\varsigma,\nu)$ and $\kappa(\varsigma,\mathfrak{J}_1\mathfrak{J}_2\nu) \le \kappa(\varsigma,\nu)$
$\varpi(\mathfrak{J}_2\mathfrak{J}_1\varsigma,\nu) \le \varpi(\varsigma,\nu)$ and $\varpi(\varsigma,\mathfrak{J}_1\mathfrak{J}_2\nu) \le \varpi(\varsigma,\nu)$;
(b) $\alpha(\varsigma,\nu) + \sqrt{2}\kappa(\varsigma,\nu) + \sqrt{2}\varpi(\varsigma,\nu) < 1$;
(c)

$$\begin{aligned}\tau(\mathfrak{J}_1\varsigma,\mathfrak{J}_2\nu) \preceq_{i_2} \alpha(\varsigma,\nu)\tau(\varsigma,\nu)& \\ +\kappa(\varsigma,\nu)\frac{\tau(\varsigma,\mathfrak{J}_1\varsigma)\tau(\nu,\mathfrak{J}_2\nu)}{1+\tau(\varsigma,\nu)}& \\ +\varpi(\varsigma,\nu)\frac{\tau(\nu,\mathfrak{J}_1\varsigma)\tau(\varsigma,\mathfrak{J}_2\nu)}{1+\tau(\varsigma,\nu)}&\end{aligned}$$

for all $\varsigma, \nu \in \mathcal{Q}$, then there exists a unique $\varsigma^* \in \mathcal{Q}$ such that $\mathfrak{J}_1\varsigma^* = \mathfrak{J}_2\varsigma^* = \varsigma^*$.

Remark 2. *By defining $\alpha, \pi, \kappa, \varpi, \varkappa : \mathcal{Q}\times\mathcal{Q}\to[0,1)$ as 0 in all possible combinations, one can obtain all the corollaries presented by Tassaddiq et al. [10] and a host of corollaries including the Banach contraction principle and Kannan's fixed point theorem in the setting of a complete bi-CVMS.*

Corollary 8. *Let $\mathfrak{J}_1, \mathfrak{J}_2 : (\mathcal{Q},\tau) \to (\mathcal{Q},\tau)$ be self-mappings. If the functions $\alpha, \pi, \kappa, \varpi, \varkappa : \mathcal{Q}\to[0,1)$ satisfy the conditions*

(a) $\alpha(\mathfrak{J}_1\varsigma) \le \alpha(\varsigma)$ and $\alpha(\mathfrak{J}_2\varsigma) \le \alpha(\varsigma)$
$\pi(\mathfrak{J}_1\varsigma) \le \pi(\varsigma)$ and $\pi(\mathfrak{J}_2\varsigma) \le \pi(\varsigma)$
$\kappa(\mathfrak{J}_1\varsigma) \le \kappa(\varsigma)$ and $\kappa(\mathfrak{J}_2\varsigma) \le \kappa(\varsigma)$
$\varpi(\mathfrak{J}_1\varsigma) \le \varpi(\varsigma)$ and $\varpi(\mathfrak{J}_2\varsigma) \le \varpi(\varsigma)$
$\varkappa(\mathfrak{J}_1\varsigma) \le \varkappa(\varsigma)$ and $\varkappa(\mathfrak{J}_2\varsigma) \le \varkappa(\varsigma)$;
(b) $\alpha(\varsigma) + 2\pi(\varsigma) + \sqrt{2}\kappa(\varsigma) + \sqrt{2}\varpi(\varsigma) + \sqrt{2}\varkappa(\varsigma) < 1$;
(c)

$$\begin{aligned}\tau(\mathfrak{J}_1\varsigma,\mathfrak{J}_2\nu) \preceq_{i_2} &\alpha(\varsigma)\tau(\varsigma,\nu) + \pi(\varsigma)(\tau(\varsigma,\mathfrak{J}_1\varsigma) + \tau(\nu,\mathfrak{J}_2\nu)) \\ &+\kappa(\varsigma)\frac{\tau(\varsigma,\mathfrak{J}_1\varsigma)\tau(\nu,\mathfrak{J}_2\nu)}{1+\tau(\varsigma,\nu)} \\ &+\varpi(\varsigma)\frac{\tau(\nu,\mathfrak{J}_1\varsigma)\tau(\varsigma,\mathfrak{J}_2\nu)}{1+\tau(\varsigma,\nu)} \\ &+\varkappa(\varsigma)\frac{\tau(\varsigma,\mathfrak{J}_1\varsigma)\tau(\varsigma,\mathfrak{J}_2\nu) + \tau(\nu,\mathfrak{J}_2\nu)\tau(\nu,\mathfrak{J}_1\varsigma)}{1+\tau(\varsigma,\mathfrak{J}_2\nu) + \tau(\nu,\mathfrak{J}_1\varsigma)},\end{aligned}$$

for all $\varsigma, \nu \in \mathcal{Q}$, then there exists a unique element $\varsigma^* \in \mathcal{Q}$ such that $\mathfrak{J}_1\varsigma^* = \mathfrak{J}_2\varsigma^* = \varsigma^*$.

Proof. Define $\alpha, \pi, \kappa, \varpi, \varkappa : \mathcal{Q}\times\mathcal{Q}\times\mathcal{Q}\to[0,1)$ by

$$\alpha(\varsigma,\nu,\theta) = \alpha(\varsigma),$$

$$\pi(\varsigma,\nu,\theta) = \pi(\varsigma),$$

$$\kappa(\varsigma,\nu,\theta) = \kappa(\varsigma),$$

$$\varpi(\varsigma,\nu,\theta) = \varpi(\varsigma),$$

$$\varkappa(\varsigma,\nu,\theta) = \varkappa(\varsigma).$$

Then, for all $\varsigma, \nu \in \mathcal{Q}$ and for a fixed element $\theta \in \mathcal{Q}$, we have

(a)
$\alpha(\mathfrak{J}_2\mathfrak{J}_1\varsigma, \nu, \theta) = \alpha(\mathfrak{J}_2\mathfrak{J}_1\varsigma) \leq \alpha(\mathfrak{J}_1\varsigma) \leq \alpha(\varsigma) = \alpha(\varsigma, \nu, \theta)$ and $\alpha(\varsigma, \mathfrak{J}_1\mathfrak{J}_2\nu, \theta) = \alpha(\varsigma) = \alpha(\varsigma, \nu, \theta)$
$\pi(\mathfrak{J}_2\mathfrak{J}_1\varsigma, \nu, \theta) = \pi(\mathfrak{J}_2\mathfrak{J}_1\varsigma) \leq \pi(\mathfrak{J}_1\varsigma) \leq \pi(\varsigma) = \pi(\varsigma, \nu, \theta)$ and $\pi(\varsigma, \mathfrak{J}_1\mathfrak{J}_2\nu, \theta) = \pi(\varsigma) = \pi(\varsigma, \nu, \theta)$
$\kappa(\mathfrak{J}_2\mathfrak{J}_1\varsigma, \nu, \theta) = \kappa(\mathfrak{J}_2\mathfrak{J}_1\varsigma) \leq \kappa(\mathfrak{J}_1\varsigma) \leq \kappa(\varsigma) = \kappa(\varsigma, \nu, \theta)$ and $\kappa(\varsigma, \mathfrak{J}_1\mathfrak{J}_2\nu, \theta) = \kappa(\varsigma) = \kappa(\varsigma, \nu, \theta)$
$\varpi(\mathfrak{J}_2\mathfrak{J}_1\varsigma, \nu, \theta) = \varpi(\mathfrak{J}_2\mathfrak{J}_1\varsigma) \leq \varpi(\mathfrak{J}_1\varsigma) \leq \varpi(\varsigma) = \varpi(\varsigma, \nu, \theta)$ and $\varpi(\varsigma, \mathfrak{J}_1\mathfrak{J}_2\nu, \theta) = \varpi(\varsigma) = \varpi(\varsigma, \nu, \theta)$
$\varkappa(\mathfrak{J}_2\mathfrak{J}_1\varsigma, \nu, \theta) = \varkappa(\mathfrak{J}_2\mathfrak{J}_1\varsigma) \leq \varkappa(\mathfrak{J}_1\varsigma) \leq \varkappa(\varsigma) = \varkappa(\varsigma, \nu, \theta)$ and $\varkappa(\varsigma, \mathfrak{J}_1\mathfrak{J}_2\nu, \theta) = \varkappa(\varsigma) = \varkappa(\varsigma, \nu, \theta)$;
(b)

$$
\begin{aligned}
& \alpha(\varsigma, \nu, \theta) + 2\pi(\varsigma, \nu, \theta) + \sqrt{2}\kappa(\varsigma, \nu, \theta) + \sqrt{2}\varpi(\varsigma, \nu, \theta) + \sqrt{2}\varkappa(\varsigma, \nu, \theta) \\
= \; & \alpha(\varsigma) + 2\pi(\varsigma) + \sqrt{2}\kappa(\varsigma) + \sqrt{2}\varpi(\varsigma) + \sqrt{2}\varkappa(\varsigma) < 1;
\end{aligned}
$$

(c)

$$
\begin{aligned}
\tau(\mathfrak{J}_1\varsigma, \mathfrak{J}_2\nu) \preceq_{i_2} & \; \alpha(\varsigma)\tau(\varsigma, \nu) + \pi(\varsigma)(\tau(\varsigma, \mathfrak{J}_1\varsigma) + \tau(\nu, \mathfrak{J}_2\nu)) \\
& + \kappa(\varsigma)\frac{\tau(\varsigma, \mathfrak{J}_1\varsigma)\tau(\nu, \mathfrak{J}_2\nu)}{1 + \tau(\varsigma, \nu)} \\
& + \varpi(\varsigma)\frac{\tau(\nu, \mathfrak{J}_1\varsigma)\tau(\varsigma, \mathfrak{J}_2\nu)}{1 + \tau(\varsigma, \nu)} \\
& + \varkappa(\varsigma)\frac{\tau(\varsigma, \mathfrak{J}_1\varsigma)\tau(\varsigma, \mathfrak{J}_2\nu) + \tau(\nu, \mathfrak{J}_2\nu)\tau(\nu, \mathfrak{J}_1\varsigma)}{1 + \tau(\varsigma, \mathfrak{J}_2\nu) + \tau(\nu, \mathfrak{J}_1\varsigma)}. \\
= & \; \alpha(\varsigma, \nu, \theta)\tau(\varsigma, \nu) \\
& + \pi(\varsigma, \nu, \theta)(\tau(\varsigma, \mathfrak{J}_1\varsigma) + \tau(\nu, \mathfrak{J}_2\nu)) \\
& + \kappa(\varsigma, \nu, \theta)\frac{\tau(\varsigma, \mathfrak{J}_1\varsigma)\tau(\nu, \mathfrak{J}_2\nu)}{1 + \tau(\varsigma, \nu)} \\
& + \varpi(\varsigma, \nu, \theta)\frac{\tau(\nu, \mathfrak{J}_1\varsigma)\tau(\varsigma, \mathfrak{J}_2\nu)}{1 + \tau(\varsigma, \nu)} \\
& + \varkappa(\varsigma, \nu, \theta)\frac{\tau(\varsigma, \mathfrak{J}_1\varsigma)\tau(\varsigma, \mathfrak{J}_2\nu) + \tau(\nu, \mathfrak{J}_2\nu)\tau(\nu, \mathfrak{J}_1\varsigma)}{1 + \tau(\varsigma, \mathfrak{J}_2\nu) + \tau(\nu, \mathfrak{J}_1\varsigma)}.
\end{aligned}
$$

Then, by Theorem 1, there exists $\varsigma^* \in \mathcal{Q}$ such that $\mathfrak{J}_1\varsigma^* = \mathfrak{J}_2\varsigma^* = \varsigma^*$. $\square$

Corollary 9. *Let $\mathfrak{J}_1, \mathfrak{J}_2 : (\mathcal{Q}, \tau) \to (\mathcal{Q}, \tau)$ be self-mappings. If there exist constants $\alpha, \pi, \kappa, \varpi, \varkappa \in [0, 1)$ such that*
$\alpha + 2\pi + \sqrt{2}\kappa + \sqrt{2}\varpi + \sqrt{2}\varkappa < 1$ *and*

$$
\begin{aligned}
\tau(\mathfrak{J}_1\varsigma, \mathfrak{J}_2\nu) \preceq_{i_2} & \; \alpha\tau(\varsigma, \nu) + \pi(\tau(\varsigma, \mathfrak{J}_1\varsigma) + \tau(\nu, \mathfrak{J}_2\nu)) \\
& + \kappa\frac{\tau(\varsigma, \mathfrak{J}_1\varsigma)\tau(\nu, \mathfrak{J}_2\nu)}{1 + \tau(\varsigma, \nu)} \\
& + \varpi\frac{\tau(\nu, \mathfrak{J}_1\varsigma)\tau(\varsigma, \mathfrak{J}_2\nu)}{1 + \tau(\varsigma, \nu)} \\
& + \varkappa\frac{\tau(\varsigma, \mathfrak{J}_1\varsigma)\tau(\varsigma, \mathfrak{J}_2\nu) + \tau(\nu, \mathfrak{J}_2\nu)\tau(\nu, \mathfrak{J}_1\varsigma)}{1 + \tau(\varsigma, \mathfrak{J}_2\nu) + \tau(\nu, \mathfrak{J}_1\varsigma)},
\end{aligned}
$$

for all $\varsigma, \nu \in \mathcal{Q}$, then there exists a unique $\varsigma^* \in \mathcal{Q}$ such that $\mathfrak{J}_1\varsigma^* = \mathfrak{J}_2\varsigma^* = \varsigma^*$.

Proof. Define $\alpha, \pi, \kappa, \varpi, \varkappa : \mathcal{Q} \to [0,1)$ by $\alpha(\cdot) = \alpha, \pi(\cdot) = \pi, \kappa(\cdot) = \kappa, \varpi(\cdot)$ and $\varkappa(\cdot) = \varkappa$ in Corollary 8. □

If we consider $\pi = \varkappa = 0$ in Corollary 9, then we obtain the key result of Gnanaprakasam et al. [9] in this manner.

Corollary 10 ([9])**.** *Let $\mathfrak{J}_1, \mathfrak{J}_2 : (\mathcal{Q}, \tau) \to (\mathcal{Q}, \tau)$ be self-mappings. If there exist $\alpha, \kappa, \varpi \in [0,1)$ such that $\alpha + \sqrt{2}\kappa + \sqrt{2}\varpi < 1$ and*

$$\tau(\mathfrak{J}_1\varsigma, \mathfrak{J}_2\nu) \preceq_{i_2} \alpha\tau(\varsigma,\nu) + \kappa\frac{\tau(\varsigma,\mathfrak{J}_1\varsigma)\tau(\nu,\mathfrak{J}_2\nu)}{1+\tau(\varsigma,\nu)} + \varpi\frac{\tau(\nu,\mathfrak{J}_1\varsigma)\tau(\varsigma,\mathfrak{J}_2\nu)}{1+\tau(\varsigma,\nu)},$$

for all $\varsigma, \nu \in \mathcal{Q}$, then there exists a unique element $\varsigma^* \in \mathcal{Q}$ such that $\mathfrak{J}_1\varsigma^* = \mathfrak{J}_2\varsigma^* = \varsigma^*$.

Corollary 11. *Let $\mathfrak{J} : (\mathcal{Q}, \tau) \to (\mathcal{Q}, \tau)$ be a self-mapping. If the functions $\alpha, \pi, \kappa, \varpi, \varkappa : \mathcal{Q} \to [0,1)$ satisfy the conditions*

(a) $\alpha(\mathfrak{J}\varsigma) \leq \alpha(\varsigma)$,
$\pi(\mathfrak{J}\varsigma) \leq \pi(\varsigma)$,
$\kappa(\mathfrak{J}\varsigma) \leq \kappa(\varsigma)$,
$\varpi(\mathfrak{J}\varsigma) \leq \varpi(\varsigma)$,
$\varkappa(\mathfrak{J}\varsigma) \leq \varkappa(\varsigma)$;

(b) $\alpha(\varsigma) + 2\pi(\varsigma) + \sqrt{2}\kappa(\varsigma) + \sqrt{2}\varpi(\varsigma) + \sqrt{2}\varkappa(\varsigma) < 1$;

(c)

$$\begin{aligned}\tau(\mathfrak{J}\varsigma, \mathfrak{J}\nu) \preceq_{i_2} \alpha(\varsigma)\tau(\varsigma,\nu) &+ \pi(\varsigma)(\tau(\varsigma,\mathfrak{J}\varsigma) + \tau(\nu,\mathfrak{J}\nu)) \\ &+\kappa(\varsigma)\frac{\tau(\varsigma,\mathfrak{J}\varsigma)\tau(\nu,\mathfrak{J}\nu)}{1+\tau(\varsigma,\nu)} \\ &+\varpi(\varsigma)\frac{\tau(\nu,\mathfrak{J}\varsigma)\tau(\varsigma,\mathfrak{J}\nu)}{1+\tau(\varsigma,\nu)} \\ &+ \varkappa(\varsigma)\frac{\tau(\varsigma,\mathfrak{J}\varsigma)\tau(\varsigma,\mathfrak{J}\nu) + \tau(\nu,\mathfrak{J}\nu)\tau(\nu,\mathfrak{J}\varsigma)}{1+\tau(\varsigma,\mathfrak{J}\nu)+\tau(\nu,\mathfrak{J}\varsigma)},\end{aligned}$$

for all $\varsigma, \nu \in \mathcal{Q}$, then there exists a unique element $\varsigma^* \in \mathcal{Q}$ such that $\mathfrak{J}\varsigma^* = \varsigma^*$.

Proof. Take $\mathfrak{J}_1 = \mathfrak{J}_2 = \mathfrak{J}$ in Corollary 8. □

Corollary 12. *Let $\mathfrak{J} : (\mathcal{Q}, \tau) \to (\mathcal{Q}, \tau)$ be a self-mapping. If there exist $\alpha, \pi, \kappa, \varpi, \varkappa \in [0,1)$ such that $\alpha + 2\pi + \sqrt{2}\kappa + \sqrt{2}\varpi + \sqrt{2}\varkappa < 1$ and*

$$\begin{aligned}\tau(\mathfrak{J}\varsigma, \mathfrak{J}\nu) \preceq_{i_2} \alpha\tau(\varsigma,\nu) &+ \pi(\tau(\varsigma,\mathfrak{J}\varsigma) + \tau(\nu,\mathfrak{J}\nu)) \\ &+\kappa\frac{\tau(\varsigma,\mathfrak{J}\varsigma)\tau(\nu,\mathfrak{J}\nu)}{1+\tau(\varsigma,\nu)} \\ &+\varpi\frac{\tau(\nu,\mathfrak{J}\varsigma)\tau(\varsigma,\mathfrak{J}\nu)}{1+\tau(\varsigma,\nu)} \\ &+ \varkappa\frac{\tau(\varsigma,\mathfrak{J}\varsigma)\tau(\varsigma,\mathfrak{J}\nu) + \tau(\nu,\mathfrak{J}\nu)\tau(\nu,\mathfrak{J}\varsigma)}{1+\tau(\varsigma,\mathfrak{J}\nu)+\tau(\nu,\mathfrak{J}\varsigma)},\end{aligned}$$

for all $\varsigma, \nu \in \mathcal{Q}$, then there exists a unique element $\varsigma^* \in \mathcal{Q}$ such that $\mathfrak{J}\varsigma^* = \varsigma^*$.

Proof. Define $\alpha, \pi, \kappa, \varpi, \varkappa : \mathcal{Q} \to [0,1)$ by $\alpha(\cdot) = \alpha, \pi(\cdot) = \pi, \kappa(\cdot) = \kappa, \varpi(\cdot)$ and $\varkappa(\cdot) = \varkappa$ in Corollary 11. □

Corollary 13. *Let $\mathfrak{J} : (\mathcal{Q}, \tau) \to (\mathcal{Q}, \tau)$ be a self-mapping. If there exist $\alpha, \pi, \kappa, \varpi, \varkappa \in [0,1)$ such that $\alpha + 2\pi + \sqrt{2}\kappa + \sqrt{2}\varpi + \sqrt{2}\varkappa < 1$ and*

$$\begin{aligned}
\tau(\mathfrak{J}^n\varsigma,\mathfrak{J}^n\nu) \preceq_{i_2} {} & \alpha\tau(\varsigma,\nu)+\pi(\tau(\varsigma,\mathfrak{J}^n\varsigma)+\tau(\nu,\mathfrak{J}^n\nu)) \\
& +\kappa\frac{\tau(\varsigma,\mathfrak{J}^n\varsigma)\tau(\nu,\mathfrak{J}^n\nu)}{1+\tau(\varsigma,\nu)} \\
& +\varpi\frac{\tau(\nu,\mathfrak{J}^n\varsigma)\tau(\varsigma,\mathfrak{J}^n\nu)}{1+\tau(\varsigma,\nu)} \\
& +\varkappa\frac{\tau(\varsigma,\mathfrak{J}^n\varsigma)\tau(\varsigma,\mathfrak{J}^n\nu)+\tau(\nu,\mathfrak{J}^n\nu)\tau(\nu,\mathfrak{J}^n\varsigma)}{1+\tau(\varsigma,\mathfrak{J}^n\nu)+\tau(\nu,\mathfrak{J}^n\varsigma)},
\end{aligned}$$

for all $\varsigma,\nu\in\mathcal{Q}$, then there exists a unique element $\varsigma^*\in\mathcal{Q}$ such that $\mathfrak{J}\varsigma^*=\varsigma^*$.

Proof. By Corollary 12, we can obtain $\varsigma\in\mathcal{Q}$ such that $\mathfrak{J}^n\varsigma=\varsigma$. Now,

$$\begin{aligned}
\tau(\mathfrak{J}\varsigma,\varsigma) &= \tau(\mathfrak{J}\mathfrak{J}^n\varsigma,\mathfrak{J}^n\varsigma) \\
&= \tau(\mathfrak{J}^n\mathfrak{J}\varsigma,\mathfrak{J}^n\varsigma) \\
&\preceq_{i_2} \alpha\tau(\mathfrak{J}\varsigma,\varsigma)+\pi(\tau(\mathfrak{J}\varsigma,\mathfrak{J}^n\mathfrak{J}\varsigma)+\tau(\varsigma,\mathfrak{J}^n\varsigma)) \\
&\quad +\kappa\frac{\tau(\mathfrak{J}\varsigma,\mathfrak{J}^n\mathfrak{J}\varsigma)\tau(\varsigma,\mathfrak{J}^n\varsigma)}{1+\tau(\mathfrak{J}\varsigma,\varsigma)} \\
&\quad +\varpi\frac{\tau(\varsigma,\mathfrak{J}^n\mathfrak{J}\varsigma)\tau(\mathfrak{J}\varsigma,\mathfrak{J}^n\varsigma)}{1+\tau(\mathfrak{J}\varsigma,\varsigma)} \\
&\quad +\varkappa\frac{\tau(\mathfrak{J}\varsigma,\mathfrak{J}^n\mathfrak{J}\varsigma)\tau(\mathfrak{J}\varsigma,\mathfrak{J}^n\varsigma)+\tau(\varsigma,\mathfrak{J}^n\varsigma)\tau(\varsigma,\mathfrak{J}^n\mathfrak{J}\varsigma)}{1+\tau(\mathfrak{J}\varsigma,\mathfrak{J}^n\varsigma)+\tau(\varsigma,\mathfrak{J}^n\mathfrak{J}\varsigma)} \\
&\preceq_{i_2} \alpha\tau(\mathfrak{J}\varsigma,\varsigma)+\pi(\tau(\mathfrak{J}\varsigma,\mathfrak{J}\varsigma)+\tau(\varsigma,\varsigma)) \\
&\quad +\kappa\frac{\tau(\mathfrak{J}\varsigma,\mathfrak{J}\varsigma)\tau(\varsigma,\varsigma)}{1+\tau(\mathfrak{J}\varsigma,\varsigma)} \\
&\quad +\varpi\frac{\tau(\varsigma,\mathfrak{J}\varsigma)\tau(\mathfrak{J}\varsigma,\varsigma)}{1+\tau(\mathfrak{J}\varsigma,\varsigma)} \\
&\quad +\varkappa\frac{\tau(\mathfrak{J}\varsigma,\mathfrak{J}\varsigma)\tau(\mathfrak{J}\varsigma,\varsigma)+\tau(\varsigma,\varsigma)\tau(\varsigma,\mathfrak{J}\varsigma)}{1+\tau(\mathfrak{J}\varsigma,\varsigma)+\tau(\varsigma,\mathfrak{J}\varsigma)} \\
&= \alpha\tau(\mathfrak{J}\varsigma,\varsigma)+\varpi\frac{\tau(\varsigma,\mathfrak{J}\varsigma)\tau(\mathfrak{J}\varsigma,\varsigma)}{1+\tau(\mathfrak{J}\varsigma,\varsigma)}
\end{aligned}$$

which implies that

$$\begin{aligned}
\|\tau(\mathfrak{J}\varsigma,\varsigma)\| &\leq \alpha\|\tau(\mathfrak{J}\varsigma,\varsigma)\|+\sqrt{2}\varpi\|\tau(\varsigma,\mathfrak{J}\varsigma)\|\left\|\frac{\tau(\mathfrak{J}\varsigma,\varsigma)}{1+\tau(\mathfrak{J}\varsigma,\varsigma)}\right\| \\
&\leq \alpha\|\tau(\mathfrak{J}\varsigma,\varsigma)\|+\sqrt{2}\varpi\|\tau(\varsigma,\mathfrak{J}\varsigma)\| \\
&= \left(\alpha+\sqrt{2}\varpi\right)\|\tau(\varsigma,\mathfrak{J}\varsigma)\|
\end{aligned}$$

which is possible only whenever $\|\tau(\mathfrak{J}\varsigma,\varsigma)\|=0$. Thus, $\mathfrak{J}\varsigma=\varsigma$. □

Corollary 14 ([8])**.** *Let $\mathfrak{J}:(\mathcal{Q},\tau)\to(\mathcal{Q},\tau)$ be a self-mapping. If there exist $\alpha,\kappa\in[0,1)$ such that $\alpha+\sqrt{2}\kappa<1$ and for all $\varsigma,\nu\in\mathcal{Q}$,*

$$\tau(\mathfrak{J}\varsigma,\mathfrak{J}\nu)\preceq_{i_2}\alpha\tau(\varsigma,\nu)+\kappa\frac{\tau(\varsigma,\mathfrak{J}\varsigma)\tau(\nu,\mathfrak{J}\nu)}{1+\tau(\varsigma,\nu)}$$

then there exists a unique element $\varsigma^*\in\mathcal{Q}$ such that $\mathfrak{J}\varsigma^*=\varsigma^*$.

Proof. Take $\pi = \varpi = \varkappa = 0$ in Corollary 12. $\square$

Remark 3. *It is notable that (a) and (b) of Theorem 1 above can be weakened by the condition*

$$\alpha(\mathfrak{J}_2\mathfrak{J}_1\varsigma) \leq \alpha(\varsigma)$$

$$\pi(\mathfrak{J}_2\mathfrak{J}_1\varsigma) \leq \pi(\varsigma)$$

$$\kappa(\mathfrak{J}_2\mathfrak{J}_1\varsigma) \leq \kappa(\varsigma)$$

$$\varpi(\mathfrak{J}_2\mathfrak{J}_1\varsigma) \leq \varpi(\varsigma)$$

$$\varkappa(\mathfrak{J}_2\mathfrak{J}_1\varsigma) \leq \varkappa(\varsigma)$$

for all $\varsigma \in \mathcal{Q}$.

Corollary 15. *Let* $\mathfrak{J}_1, \mathfrak{J}_2 : (\mathcal{Q}, \tau) \to (\mathcal{Q}, \tau)$ *be self-mappings. If there exist* $\alpha, \pi, \kappa, \varpi, \varkappa : \mathcal{Q} \to [0,1)$ *such that for all* $\varsigma, \nu \in \mathcal{Q}$,

(a) $\alpha(\mathfrak{J}_2\mathfrak{J}_1\varsigma) \leq \alpha(\varsigma)$,
$\pi(\mathfrak{J}_2\mathfrak{J}_1\varsigma) \leq \pi(\varsigma)$,
$\kappa(\mathfrak{J}_2\mathfrak{J}_1\varsigma) \leq \kappa(\varsigma)$,
$\varpi(\mathfrak{J}_2\mathfrak{J}_1\varsigma) \leq \varpi(\varsigma)$,
$\varkappa(\mathfrak{J}_2\mathfrak{J}_1\varsigma) \leq \varkappa(\varsigma)$;

(b) $\alpha(\varsigma) + 2\pi(\varsigma) + \sqrt{2}\kappa(\varsigma) + \sqrt{2}\varpi(\varsigma) + \sqrt{2}\varkappa(\varsigma) < 1$;

(c)

$$\begin{aligned}
\tau(\mathfrak{J}_1\varsigma, \mathfrak{J}_2\nu) \preceq_{i_2} & \alpha(\varsigma)\tau(\varsigma,\nu) + \pi(\varsigma)(\tau(\varsigma, \mathfrak{J}_1\varsigma) + \tau(\nu, \mathfrak{J}_2\nu)) \\
& + \kappa(\varsigma)\frac{\tau(\varsigma, \mathfrak{J}_1\varsigma)\tau(\nu, \mathfrak{J}_2\nu)}{1+\tau(\varsigma,\nu)} \\
& + \varpi(\varsigma)\frac{\tau(\nu, \mathfrak{J}_1\varsigma)\tau(\varsigma, \mathfrak{J}_2\nu)}{1+\tau(\varsigma,\nu)} \\
& + \varkappa(\varsigma)\frac{\tau(\varsigma, \mathfrak{J}_1\varsigma)\tau(\varsigma, \mathfrak{J}_2\nu) + \tau(\nu, \mathfrak{J}_2\nu)\tau(\nu, \mathfrak{J}_1\varsigma)}{1+\tau(\varsigma, \mathfrak{J}_2\nu) + \tau(\nu, \mathfrak{J}_1\varsigma)},
\end{aligned}$$

then there exists a unique element $\varsigma^* \in \mathcal{Q}$ *such that* $\mathfrak{J}_1\varsigma^* = \mathfrak{J}_2\varsigma^* = \varsigma^*$.

Proof. Define $\alpha, \pi, \kappa, \varpi, \varkappa : \mathcal{Q} \times \mathcal{Q} \times \mathcal{Q} \to [0,1)$ by

$$\alpha(\varsigma, \nu, \theta) = \alpha(\varsigma),$$

$$\pi(\varsigma, \nu, \theta) = \pi(\varsigma),$$

$$\kappa(\varsigma, \nu, \theta) = \kappa(\varsigma),$$

$$\varpi(\varsigma, \nu, \theta) = \varpi(\varsigma),$$

$$\varkappa(\varsigma, \nu, \theta) = \varkappa(\varsigma).$$

Then, for all $\varsigma, \nu \in \mathcal{Q}$, we have

(a)

$\alpha(\mathfrak{J}_2\mathfrak{J}_1\varsigma, \nu, \theta) = \alpha(\mathfrak{J}_2\mathfrak{J}_1\varsigma) \leq \alpha(\mathfrak{J}_1\varsigma) \leq \alpha(\varsigma) = \alpha(\varsigma, \nu, \theta)$ and $\alpha(\varsigma, \mathfrak{J}_1\mathfrak{J}_2\nu, \theta) = \alpha(\varsigma) = \alpha(\varsigma, \nu, \theta)$

$\pi(\mathfrak{J}_2\mathfrak{J}_1\varsigma, \nu, \theta) = \pi(\mathfrak{J}_2\mathfrak{J}_1\varsigma) \leq \pi(\mathfrak{J}_1\varsigma) \leq \pi(\varsigma) = \pi(\varsigma, \nu, \theta)$ and $\pi(\varsigma, \mathfrak{J}_1\mathfrak{J}_2\nu, \theta) = \pi(\varsigma) = \pi(\varsigma, \nu, \theta)$

$\kappa(\mathfrak{J}_2\mathfrak{J}_1\varsigma, \nu, \theta) = \kappa(\mathfrak{J}_2\mathfrak{J}_1\varsigma) \leq \kappa(\mathfrak{J}_1\varsigma) \leq \kappa(\varsigma) = \kappa(\varsigma, \nu, \theta)$ and $\kappa(\varsigma, \mathfrak{J}_1\mathfrak{J}_2\nu, \theta) = \kappa(\varsigma) = \kappa(\varsigma, \nu, \theta)$

$\varpi(\mathfrak{J}_2\mathfrak{J}_1\varsigma, \nu, \theta) = \varpi(\mathfrak{J}_2\mathfrak{J}_1\varsigma) \leq \varpi(\mathfrak{J}_1\varsigma) \leq \varpi(\varsigma) = \varpi(\varsigma, \nu, \theta)$ and $\varpi(\varsigma, \mathfrak{J}_1\mathfrak{J}_2\nu, \theta) = \varpi(\varsigma) = \varpi(\varsigma, \nu, \theta)$

$\varkappa(\Im_2\Im_1\varsigma,\nu,\theta) = \varkappa(\Im_2\Im_1\varsigma) \leq \varkappa(\Im_1\varsigma) \leq \varkappa(\varsigma) = \varkappa(\varsigma,\nu,\theta)$ and $\varkappa(\varsigma,\Im_1\Im_2\nu,\theta) = \varkappa(\varsigma) = \varkappa(\varsigma,\nu,\theta)$;

(b)

$$\begin{aligned} & \alpha(\varsigma,\nu,\theta) + 2\pi(\varsigma,\nu,\theta) + \sqrt{2}\kappa(\varsigma,\nu,\theta) + \sqrt{2}\varpi(\varsigma,\nu,\theta) + \sqrt{2}\varkappa(\varsigma,\nu,\theta) \\ = \; & \alpha(\varsigma) + 2\pi(\varsigma) + \sqrt{2}\kappa(\varsigma) + \sqrt{2}\varpi(\varsigma) + \sqrt{2}\varkappa(\varsigma) < 1; \end{aligned}$$

(c)

$$\begin{aligned} \tau(\Im_1\varsigma,\Im_2\nu) \preceq_{i_2} & \alpha(\varsigma)\tau(\varsigma,\nu) + \pi(\varsigma)(\tau(\varsigma,\Im_1\varsigma) + \tau(\nu,\Im_2\nu)) \\ & +\kappa(\varsigma)\frac{\tau(\varsigma,\Im_1\varsigma)\tau(\nu,\Im_2\nu)}{1+\tau(\varsigma,\nu)} \\ & +\varpi(\varsigma)\frac{\tau(\nu,\Im_1\varsigma)\tau(\varsigma,\Im_2\nu)}{1+\tau(\varsigma,\nu)} \\ & +\varkappa(\varsigma)\frac{\tau(\varsigma,\Im_1\varsigma)\tau(\varsigma,\Im_2\nu)+\tau(\nu,\Im_2\nu)\tau(\nu,\Im_1\varsigma)}{1+\tau(\varsigma,\Im_2\nu)+\tau(\nu,\Im_1\varsigma)} \\ = \; & \alpha(\varsigma,\nu,\theta)\tau(\varsigma,\nu) \\ & +\pi(\varsigma,\nu,\theta)(\tau(\varsigma,\Im_1\varsigma) + \tau(\nu,\Im_2\nu)) \\ & +\kappa(\varsigma,\nu,\theta)\frac{\tau(\varsigma,\Im_1\varsigma)\tau(\nu,\Im_2\nu)}{1+\tau(\varsigma,\nu)} \\ & +\varpi(\varsigma,\nu,\theta)\frac{\tau(\nu,\Im_1\varsigma)\tau(\varsigma,\Im_2\nu)}{1+\tau(\varsigma,\nu)} \\ & +\varkappa(\varsigma,\nu,\theta)\frac{\tau(\varsigma,\Im_1\varsigma)\tau(\varsigma,\Im_2\nu)+\tau(\nu,\Im_2\nu)\tau(\nu,\Im_1\varsigma)}{1+\tau(\varsigma,\Im_2\nu)+\tau(\nu,\Im_1\varsigma)}. \end{aligned}$$

Then, by Theorem 1, there exists a unique $\varsigma^* \in \mathcal{Q}$ such that $\Im_1\varsigma^* = \Im_2\varsigma^* = \varsigma^*$. □

4. Applications

Let $C[a,b]$ represent the class of all real continuous functions defined on $[a,b]$ and $\tau : C([a,b]) \times C([a,b]) \to \mathbb{C}_2$ be defined as follows

$$\tau(\varsigma,\nu) = \max_{t\in[a,b]}(1+i)(|\varsigma(t)-\nu(t)|)$$

for all $\varsigma,\nu \in C([a,b])$ and $t \in [a,b]$. Then, $(C([a,b],\mathbb{R}),\tau)$ is a complete bi-CVMS. Take the integral equations

$$\varsigma(t) = \int_a^b K_1(t,s,\varsigma(s))\tau s + g(t), \tag{7}$$

$$\varsigma(t) = \int_a^b K_2(t,s,\varsigma(s))\tau s + g(t), \tag{8}$$

where $g : [a,b] \to \mathbb{R}$ and $K_1, K_2 : [a,b] \times [a,b] \times \mathbb{R} \to \mathbb{R}$ are continuous for $t \in [a,b]$. In $\mathbb{C}_2$, we define $\preceq_{i_2}$ in this way

$$\varsigma(t) \preceq_{i_2} \nu(t) \Longleftrightarrow \varsigma \leq \nu.$$

Theorem 2. *Suppose there exists some fixed element $\theta \in \mathcal{Q}$ such that the following condition*

$$|K_1(t,s,\varsigma(s)) - K_2(t,s,\nu(s))| \leq \alpha(\varsigma,\nu,\theta)|\varsigma(s)-\nu(s)|$$

holds for all $\varsigma,\nu \in \mathcal{Q}$ with $\varsigma \neq \nu$ and $\alpha : \mathcal{Q}\times\mathcal{Q}\times\mathcal{Q}\to[0,1)$. Then, (7) and (8) have a unique common solution.

Proof. Define $\mathfrak{J}_1, \mathfrak{J}_2 : \mathcal{Q} \to \mathcal{Q}$ by

$$\mathfrak{J}_1\varsigma(t) = \frac{1}{b-a}\int_a^b K_1(t,s,\varsigma(s))\tau s + g(t),$$

$$\mathfrak{J}_2\varsigma(t) = \frac{1}{b-a}\int_a^b K_2(t,s,\varsigma(s))\tau s + g(t),$$

for all $t \in [a,b]$. Consider

$$\begin{aligned}
\tau(\mathfrak{J}_1\varsigma, \mathfrak{J}_2\nu) &= \max_{t\in[a,b]}(1+i_2)|\mathfrak{J}_1\varsigma(t) - \mathfrak{J}_2 h(t)| \\
&= \max_{t\in[a,b]}(1+i_2)\left(\frac{1}{b-a}\left|\int_a^b K_1(t,s,\varsigma(s))\tau s - \int_a^b K_2(t,s,h(s))\tau s\right|\right) \\
&\preceq_{i_2} \max_{t\in[a,b]}(1+i_2)\left(\frac{1}{b-a}\int_a^b |K_1(t,s,\varsigma(s)) - K_2(t,s,h(s))|\tau s\right) \\
&\preceq_{i_2} \max_{t\in[a,b]}(1+i_2)\left(\frac{\alpha(\varsigma,\nu,\theta)}{b-a}\int_a^b |\varsigma(s) - \nu(s)|\tau s\right).
\end{aligned}$$

Thus,

$$\tau(\mathfrak{J}_1\varsigma, \mathfrak{J}_2\nu) \preceq_{i_2} \alpha(\varsigma,\nu,\theta)\tau(\varsigma,\nu).$$

Now, with $\pi, \kappa, \varpi, \varkappa : \mathcal{Q}\times\mathcal{Q}\times\mathcal{Q}\to[0,1)$ defined by

$$\pi(\varsigma,\nu,\theta) = \kappa(\varsigma,\nu,\theta) = \varpi(\varsigma,\nu,\theta) = \varkappa(\varsigma,\nu,\theta) = 0$$

for every $\varsigma, \nu \in \mathcal{Q}$, all the hypotheses of Theorem 1 are fulfilled and the integral Equations (7) and (8) have a unique common solution. □

5. Conclusions

Complex-valued metric spaces and their several generalizations allow us to consider the distances between points in a set, either classically or non-classically. In this draft, we have obtained common fixed-point results for rational contractions involving point-dependent control functions in bi-CVMSs. In this way, we have derived the key results of Beg et al. [8], Gnanaprakasam et al. [9] and Tassaddiq et al. [10] from our results. We apply our result to solve the Fredholm integral equation as an application.

For future work, one can expand the notion of bi-CVMSs to hypercomplex-valued metric spaces. Moreover, the results established in this paper can be lengthened to set-valued mappings. Some integral and differential inclusions can be explored to apply fixed-point theorems for set-valued mappings in the framework of bi-CVMSs.

Funding: This work was funded by the University of Jeddah, Jeddah, Saudi Arabia, under grant No. (UJ-23-DR-5). The author, therefore, thanks the University of Jeddah for its technical and financial support.

Data Availability Statement: Not applicable.

Conflicts of Interest: The author declares no conflict of interest.

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