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Influence Maximization Dynamics and Topological Order on Erdös-Rényi Networks

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Abstract: This paper concerns the study of the linear threshold model in random networks, specifically in Erdös-Rényi networks. In our approach, we consider an activation threshold defined by the expected value for the node degree and the associated influence activation mapping. According to these assumptions, we present a theoretical procedure for the linear threshold model, under fairly general conditions, regarding the topological structure of the networks and the activation threshold. Aiming at the dynamics of the influence maximization process, we analyze and discuss different choices for the seed set based on several centrality measures along with the state conditions for the procedure to trigger. The topological entropy established for Erdös-Rényi networks defines a topological order for this type of random networks. Sufficient conditions are presented for this topological entropy to be characterized by the spectral radius of the associated adjacency matrices. Consequently, a number of properties are proved. The threshold dynamics are analyzed through the relationship between the activation threshold and the topological entropy. Numerical studies are included to illustrate the theoretical results.

Keywords: linear threshold model; spread dynamics; Erdös-Rényi networks; topological entropy; activation threshold

MSC: 37N40; 91D30; 37B40; 05C90; 05C82

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1. Introduction and Motivation

The study of complex networks is an interdisciplinary topic that covers several areas of knowledge, such as computer science, mathematics, physics, biology, and sociology; see [1–4] and references therein. Concerning the emergence and importance of social networks in recent times, networks play a central role in representing the way ideas and information spread in modern society. A considerable part of our opinions and decisions are directly or indirectly determined by our social contacts. Social influence is a critical factor when we adopt a certain behavior, make a decision, adopt an innovation, or formalize our cultural, political, and religious ideologies; see [5].

The study of dynamic networks seeks to explain the evolution of propagation phenomena that are influenced by peers, such as the spread of epidemics, dissemination of information, and popularization of ideas and choices. The networks used to represent these connections can be created from real data or by using previously defined networks from the literature, such as classical random networks, small-world networks, and scale-free networks, among others; see [6]. Thus, it is possible to apply dynamic diffusion models to the chosen networks in order to understand these phenomena. These propagation processes are referred to as social contagion, as they are reminiscent of how a disease is transmitted between the individuals in a population. Several authors have presented different approaches to this theme over the past decades; see [7] for a complete and updated compilation of references in this field.

Influence maximization was first proposed in [8], which was later interpreted as a discrete optimization problem. This typification was a landmark in the research on influence maximization, and was soon considered an NP-hard problem (nondeterministic polynomial time hardness, which is a class of problems that have at least exponential complexity; see [9]) for the case when the given information propagation model is an independent cascade or a linear threshold model. On the one hand, a large number of publications on this topic have been presented, including two categories of classical influence maximization algorithms, namely, greedy algorithms and heuristic algorithms. While the greedy algorithm has high accuracy, it is computationally time-consuming; on the other hand, the heuristic algorithm is more efficient at the cost of accuracy. Most of the existing studies are based on static networks, although a few researchers have devoted themselves to the study of influence maximization in dynamic networks. Several of these works consider the dynamics in the information dissemination process, such as the dynamic activation probability, dynamic threshold, and dynamic perception, while others consider the dynamics of the network topology as links are added or removed over time. On this subject, see [7,10–12].

In the present work we study the linear threshold model in random networks, specifically in Erdös-Rényi networks; see [13,14]. The linear threshold model explains how an individual can be influenced depending on a previously established threshold value. This means that a node is only activated if the influence applied on it by its neighbours exceeds a certain value. Generally, the threshold value is randomly chosen by taking into account a certain range of values; see [15,16] and references therein. In our case, the threshold value is estimated taking into account the underlying network parameters, particularly from the expected value for the network nodes degree.

The rest of this work is organized as follows. In Section 2, the definitions of the activation threshold and influence activation mapping for Erdös-Rényi random networks are presented. The activation threshold expresses the network topology and its global dynamics. On the other hand, the influence activation mapping is defined by relying on the local dynamics of each node of the network under analysis. As a consequence of the previous definitions, the seed set for implementation of the diffusion process on Erdös-Rényi networks is characterized.

Section 3 elaborates on the establishment of the theoretical procedure for the linear threshold model over the Erdös-Rényi networks. This procedure mathematically formalizes the diffusion process of the linear threshold model for this type of random network; in this section, we highlight the generality of the conditions required for its implementation.

In Section 4, we analyze the influence maximization dynamics for the linear threshold model, i.e., we intend to state the conditions that will trigger this diffusion process. In particular, our focus is on the activation of the diffusion process considering the local and global dynamics of the networks (population dynamics). From these results, analytical properties for the probability of activating the influence diffusion process can be obtained for a randomly chosen seed, and these properties are related to the nodes with greater centrality degree in the Erdös-Rényi networks. The results revealed in this section highlight the complexity of the chosen approach as well as the richness and usefulness of the centrality measures.

Section 5 is devoted to the study of a topological order for the Erdös-Rényi networks, which is established at the expense of the definition of topological entropy for this type of random network. In this context, asymptotic behaviour of the spectral radius of the adjacency matrix of the Erdös-Rényi networks is used. This approach to the topological entropy, provided through the spectral radius of the corresponding adjacency matrices, characterizes the topological dynamics of Erdös-Rényi networks. Sufficient conditions are presented for this definition of topological entropy to be characterized by the expected value of the node degree of the considered random networks. Consequently, several properties are proved and a relation between the activation threshold and the topological entropy is established.

In Section 6, several numerical case studies are performed according to different choices for the seed set considering random or chosen seeds within the centrality measures, namely, degree, closeness, and betweenness. Finally, in Section 7 we discuss the work, provide our conclusions, and outline future research.

2. Influence Activation Mapping for the Linear Threshold Model

In this section we introduce an influence activation mapping for the linear threshold model which incorporates an activation threshold θ; see [15]. These characterizations allow us to present a definition for the seed set of the linear threshold model for the type of diffusion networks analyzed. Throughout this work, the considered type of diffusion networks are Erdös-Rényi random networks. An Erdös-Rényi random network consists of n nodes, where $n \in \mathbb{N}$ and each pair of nodes is linked by a connection or an edge with a certain probability $p \in [0,1]$. The edges between the nodes are independently and randomly generated with the same probability p; see [13,14]. Erdös-Rényi random networks are usually represented by $\mathcal{G}_{(n,p)} = (V,E)$, where $V = \{x_1, x_2, \ldots, x_n\}$ is the node set and represents the individuals, $|V| = n$ indicates that there are $n \in \mathbb{N}$ individuals, and the edge set E represents the connections between different individuals. Information propagates through the connections in the network. Throughout this paper, we denote by $N(x_i)$ the neighbourhood of x_i, i.e., the nodes of V that are linked to x_i, and by $\delta(x_i) = |N(x_i)|$ the degree of node x_i for $i = 1, \ldots, n$. For more details on Erdös-Rényi networks, see for example [1–4] and references therein.

Concerning the classical models of influence, we consider the linear threshold model, where every node at each instant has one of two possible states, namely, inactive or active; see [17,18]. We deem the network as unweighted, i.e., with a constant threshold value for every node. The choice of the threshold value θ is a very important feature in terms of linear threshold based influence maximization. This one of two vital points, along with the degree distribution of the network; see [16]. While there is no specific guideline or property for the choice of a given threshold, our purpose in this article is to define values that allow diffusion. These considerations lead us to reintroduce the following definition (for more, see [19]).

Definition 1. *Let $\mathcal{G}_{(n,p)}$ be an Erdös-Rényi random network. The activation threshold θ of the network $\mathcal{G}_{(n,p)}$ is defined by*

$$\theta = \frac{1}{\langle \delta(x_i) \rangle} = \frac{1}{p(n-1)}, \tag{1}$$

where $\langle \delta(x_i) \rangle$ is the expected value for the node degree of the network.

Note that this activation threshold is dependent on the network size n and the connection probability p; thus, this invariant reflects the network topology and its global dynamics.

In the linear threshold model, a node x_i becomes active if and only if the ratio between the active neighbours of x_i at time t (denoted by $\epsilon(x_i(t))$) and the degree of that node (denoted by $\delta(x_i)$) is higher than the threshold value θ. Thus, in the following definition we introduce the concept of influence activation mapping in a network, in particular for the Erdös-Rényi random network $\mathcal{G}_{(n,p)}$.

Definition 2. *Let $\mathcal{G}_{(n,p)}$ be an Erdös-Rényi random network, with $x_i(t) \in V \times \mathbb{N}_0$ being a node $x_i \in V$ at discrete time $t \in \mathbb{N}_0$ for $i = 1, 2, \ldots, n$ and θ the being the activation threshold provided by Equation (1). The influence activation mapping $f : V \times \mathbb{N}_0 \to \{0,1\}^n$, where*

$$f(x_1, x_2, \ldots, x_n)(t) = (f(x_1(t)), f(x_2(t)), \ldots, f(x_n(t))),$$

is defined as follows:

$$f(x_i(t)) = \begin{cases} 1, & if \ \frac{\epsilon(x_i(t-1))}{\delta(x_i)} \geq \theta \\ \\ 0, & if \ \frac{\epsilon(x_i(t-1))}{\delta(x_i)} < \theta \end{cases}. \tag{2}$$

We remark here that the influence activation mapping f is defined acoording to the local dynamics of each node $x_i \in V$ of the network $\mathcal{G}_{(n,p)}$ under analysis. Further, note that when considering the influence activation mapping f provided by Equation (2) we can recursively define the mapping of the active neighbours of a node x_i, $\epsilon : V \times \mathbb{N}_0 \to \mathbb{N}_0$, in terms of the influence activation mapping f in the following way:

$$\epsilon(x_i(t)) = \sum_{x_j \in N(x_i)} f(x_j(t)) \tag{3}$$

with $x_i \in V$ for $i = 1, 2, \ldots, n$ and for each discrete time $t \neq t_0$, where t_0 represents the initial instant of the diffusion process. It should be emphasized again that in the approach explored in this work, the topological structure of the network $\mathcal{G}_{(n,p)}$ is considered in the definitions of the influence activation mapping f and the active neighbours mapping ϵ provided by Equations (2) and (3), respectively.

In order to further explain this procedure, we can consider the seed set as the subset of V that specifies the initial conditions for the recursion mapping of the active neighbours, as provided by Equation (3). This determines which nodes are activated at the initial state of the diffusion process. It is our purpose to first choose these nodes according to different properties, then analyze and compare the diffusion effects of these choices as concerns the influence maximization problem. The concept of a seed set already exists; see for example [17,18] and references therein. However, here we present a formal definition for the seed set characterized according to the influence activation mapping f at the initial instant t_0.

Definition 3. *Let $\mathcal{G}_{(n,p)}$ be a contact Erdös-Rényi network and let $f(x_i(t_0))$ be the node state at the initial instant t_0, with $x_i \in V$. The seed set $\mathcal{H}$ of the network $\mathcal{G}_{(n,p)}$ is defined by*

$$\mathcal{H} = \{x_i \in V : f(x_i(t_0)) = 1\},$$

where $\emptyset \neq \mathcal{H} \subset V$.

3. Theoretical Procedure for the Linear Threshold Model

In this section, we introduce a theoretical procedure for the linear threshold model applied to diffusion networks of the Erdös-Rényi random network type. In the linear threshold model, a node is activated if and only if the influence exerted on it by its neighbours exceeds a certain value. This means that if the value exceeds the threshold θ for a specific node, the node becomes active. This indicates that it has adopted an idea, obtained information, changed its behavior, etc.; see [18] and references therein.

Procedure 1. *Let $\mathcal{G}_{(n,p)} = (V, E)$ be an Erdös-Rényi random network, let f be the influence activation mapping provided by Equation (2), and let $\mathcal{H}$ be the seed set provided by Definition 3. The theoretical procedure of the linear threshold model for the network $\mathcal{G}_{(n,p)}$ follows the next steps:*

1. *At the initial instant $t = t_0$, the nodes in the seed set $\mathcal{H} \subset V$ are all active, i.e.,*

$$f(x_i(t_0)) = 1, \ \forall x_i \in \mathcal{H};$$

2. *At instant $t^* \neq t_0$, if node $x_i \in V \backslash \mathcal{H}$ is activated, then node x_i remains activated, i.e.,*

$$if \ \exists \, t^* \neq t_0 : f(x_i(t^*)) = 1, \ then \ f(x_i(t)) = 1, \ \forall t \geq t^*;$$

3. *If at instant $t^* \neq t_0$ a node $x_i \in V \backslash \mathcal{H}$ is not activated, then the influence activation function is verified as provided in Equation (2);*
4. *At each instant $t \neq t_0$, processes 2 and 3 are repeated until reaching the following stopping criteria:*

$$(f(x_1(t)), f(x_2(t)), \ldots, f(x_n(t))) = (f(x_1(t-1)), f(x_2(t-1)), \ldots, f(x_n(t-1)))$$

i.e., at instant t the vector at iteration t is equal to the vector in the previous iteration.

The procedure formalized above corresponds to a finite number of steps, which are described as follows. In step 1, it is stated that the seed set $\mathcal{H}$ corresponds to the active nodes. Step 2 guarantees that an active node remains active throughout the procedure. Step 3 corresponds to the nodes that are not activated and can change their state. Finally, step 4 provides a stopping condition for the procedure.

Remark 1. *Note that if $t-1 = t_0$ in step 3 this situation corresponds to the case where the nodes in the seed set $\mathcal{H}$ are the only nodes that are active when the procedure ends.*

Remark 2. *The former theoretical procedure is presented with no restrictions on the values of the linear threshold model. Moreover, there are no restrictions on the underlying contact networks. Nevertheless, it is important to emphasize the following features:*

1. *The connectivity of Erdös-Rényi networks depends on the value of the connection probability p and the size of the network n. If $p < \frac{1}{n}$, then there is no giant component. Consequently, there will be no diffusion regardless of the propagation model considered.*
2. *Concerning the linear threshold model, the theoretical procedure applies to every value of the threshold θ.*

4. Influence Activation Dynamics for the Linear Threshold Model

Considering Remark 1 from the previous section, we want to analyze under which conditions there are active nodes other than the seed set $\mathcal{H}$. In another words, we intend to state the conditions that trigger the diffusion process for the linear threshold model. First, however, we need to recall certain features regarding connectivity of the Erdös-Rényi network $\mathcal{G}_{(n,p)}$ according to the values of its parameters n and p. For the extreme values of p, $p = 0$, and $p = 1$ we have a disconnected network and a complete network, respectively, while for $np > 1$ we have the emergence of a giant component, which is an example of a phase transition in Erdös-Rényi networks; see [20]. In this case, this rather abrupt change characterizes the transition between the most disconnected and most connected networks for the values of p; see [2].

More specifically, for an Erdös-Rényi random network with n nodes and connection probability p, the supercritical regime is defined as the range of p values such that the expected size of the largest connected component scales with n, meaning that there is a very high probability of there being no nodes outside the giant component. This regime is characterized by the fact that the probability of there being a giant component goes to one as the network size increases. In contrast, in the subcritical regime, i.e., $np < 1$, the probability of a giant component goes to zero as the network size increases, while in the critical regime, for $np = 1$ the probability of a giant component approaches a non-zero constant as the network size becomes large. For more details on topological and phase transitions in Erdös-Rényi networks, see for example [1,4].

Throughout this section, we consider a single seed, i.e., $|\mathcal{H}| = 1$, with the value of the connection probability p satisfying $p > \frac{1}{n}$, in order to ensure that the network has a giant component and that θ is the activation threshold provided by Equation (1). Suppose, without loss of generality, that $\mathcal{H} = \{x_i\}$ with $i \in \{1, 2, \ldots, n\}$ and that x_i lies in the giant component of the network $\mathcal{G}_{(n,p)}$. Let us further suppose that x_i has degree one, i.e., $\delta(x_i) = 1$. Let $x_{i_1} \in V$ be the only node in $N(x_i)$. According to the influence activation mapping f provided by Equation (2), at instant $t_0 + 1$ node x_{i_1} is active if and only if $\frac{1}{\delta(x_{i_1})} \geq \theta$. From Definition 1, we know that $\theta = \frac{1}{p(n-1)}$, meaning that node x_{i_1} is activated if and only if,

$$\delta(x_{i_1}) \leq p(n-1),$$

where $\langle \delta(x_i) \rangle = p(n-1)$ is the expected value for the node degree of the network $\mathcal{G}_{(n,p)}$. On the other hand, the degree distribution of the network $\mathcal{G}_{(n,p)}$ follows a binomial law with the same parameters; thus, we can write

$$\mathbb{P}[\delta(x_{i_1}) \leq p(n-1)] = 0.5.$$

In conclusion, if a seed corresponds to a pendent node on the giant component of the network $\mathcal{G}_{(n,p)}$, then the set of the active nodes through the diffusion procedure is equal to the seed set $\mathcal{H}$ with probability 0.5.

Let us now consider a more general case. Suppose that $\mathcal{H} = \{x_i\}$ with $i \in \{1, 2, \ldots, n\}$, where x_i is a node in the giant component of an Erdös-Rényi network $\mathcal{G}_{(n,p)}$, and further suppose that x_i has degree $d \neq 1$, i.e., $\delta(x_i) = d$, with $N(x_i) = \{x_{i_1}, x_{i_2}, \ldots, x_{i_d}\} \subset V$. For each $x_{i_j} \in N(x_i)$, to simplify the notation, we denote by δ_j the degree of node x_{i_j} for $j \in \{1, 2, \ldots, d\}$. It is our purpose here to analyze the case where x_i can activate other nodes in its neighbourhood. Using the same arguments as in the previous case, node x_i does not activate any neighbour if and only if the following condition holds,

$$\delta_1 \leq p(n-1) \wedge \delta_2 \leq p(n-1) \wedge \ldots \wedge \delta_d \leq p(n-1).$$

Let us now compute the probability of the previous event,

$$\mathbb{P}[\delta_1 \leq p(n-1) \wedge \delta_2 \leq p(n-1) \wedge \ldots \wedge \delta_d \leq p(n-1)] =$$

$$\mathbb{P}[\delta_1 \leq p(n-1)] \times \mathbb{P}[\delta_2 \leq p(n-1)] \times \ldots \times \mathbb{P}[\delta_d \leq p(n-1)] = (0.5)^d.$$

The first equality results from the fact that the degree distribution of each node is independent. Considering the above explanations, we are now in a condition to state the following result.

Proposition 1. *Let $\mathcal{G}_{(n,p)} = (V, E)$ be an Erdös-Rényi random network, f the activation mapping provided by Equation (2), t_0 the initial instant, and $\mathcal{H} = \{x_i\}$ the seed set provided by Definition 3, where $x_i \in V$ and $i \in \{1, 2, \ldots, n\}$. If $\delta(x_i) = d$ with $d \in \{1, 2, \ldots, n-1\}$, then*

$$\mathbb{P}[\exists\, j : f(x_{i_j}(t_0 + 1)) = 1 \mid x_{i_j} \in N(x_i)] = 1 - (0.5)^d$$

for $j \in \{1, 2, \ldots, d\}$.

In Proposition 1, we obtained the probability of initiating the diffusion process for a randomly chosen seed for the linear threshold model over Erdös-Rényi networks.

Now, let us consider $\mathcal{H} = \{x_i\}$, where x_i is the node with a greater degree of centrality in the network $\mathcal{G}_{(n,p)}$. Under these conditions, we need to introduce the following definition.

Definition 4. *Let $\mathcal{G}_{(n,p)} = (V, E)$ be an Erdös-Rényi random network, f the activation mapping provided by Equation (2), t_0 the initial instant, and $\mathcal{H} = \{x_i\}$ the seed set provided by Definition 3, where $x_i \in V$ and $i \in \{1, 2, \ldots, n\}$. The influence activation probability mapping $g : V \longrightarrow [0, 1]$ is defined as follows:*

$$g(v) = \mathbb{P}[\exists\, j : f(x_{i_j}(t_0 + 1)) = 1 \mid x_{i_j} \in N(x_i)] \tag{4}$$

for $j \in \{1, 2, \ldots, d\}$.

We are now in a position to establish the corollary that is a consequence of the previous proposition and definition.

Corollary 1. *Under the assumption in Proposition 1 and Definition 4, the influence activation probability mapping g attains its maximum if and only if v is the node with greater centrality degree in the Erdös-Rényi network $\mathcal{G}_{(n,p)}$.*

These remarks lead us to a very important and well-known topic, namely, the influence maximization problem, which asks for the best seed set to provide the maximum number of activated nodes at the final stage (see [9,18]). In this particular case, we can ask whether the best choice of a seed to initiate the diffusion process is also the best choice for having more active nodes at the final stage. In other words, is the best trigger also the best spreader? If not, what happens if we choose a seed considering other centrality measures for detecting influential nodes, such as the closeness or betweenness centrality, or even other measures?

5. Topological Order in Erdös-Rényi Random Networks and Activation Threshold

Spectral techniques have a very important role in recent graph theory, in particular concerning random graphs; see for example [1–4] and references therein. The results obtained from spectral graph theory, along with bounds or specific values for graph eigenvalues, provide tools for several graph algorithms. Erdös-Rényi networks, or random graphs $\mathcal{G}_{(n,p)}$ have an adjacency matrix A that can be analyzed as a random symmetric matrix with a diagonal equal to zero and the remaining entries being ones with probability p and zeroes with probability $1-p$. This allow us to relate the behaviour of these matrices using the well-known behaviour of symmetric matrices and their spectra.

Within this framework, we can study a topological invariant associated with the dynamics between the nodes of the Erdös-Rényi networks $\mathcal{G}_{(n,p)}$, which we refer to as the network topological entropy. Usually, the network entropy is the entropy of a stochastic matrix associated with the adjacency matrix A; see [21] and references therein. Considering $A=[a_{ij}]$, $n\times n$ as the adjacency matrix of the Erdös-Rényi network $\mathcal{G}_{(n,p)}$, where λ denotes the spectral radius of A, let (v_i) be the corresponding leading eigenvector. It has been shown that $\ln(\lambda)$ satisfies a variational principle and that the supremum over all possible stochastic matrices is attained for the unique stochastic matrix $P=[p_{ij}]$, $n\times n$, defined by

$$p_{ij}=\frac{a_{ij}\,v_j}{\lambda\,v_i},$$

with stationary distribution $\pi=\pi P$; see [21–25] and references therein. Under these conditions, the network entropy of this dynamical process is characterized as follows:

$$\ln(\lambda)=-\sum_{i,j=1}^{n}\pi_i\,p_{ij}\,\ln(p_{ij})+\sum_{i,j=1}^{n}\pi_i\,p_{ij}\,\ln(a_{ij}). \tag{5}$$

From the previous results, we establish the definition of network entropy for the networks $\mathcal{G}_{(n,p)}$.

Definition 5. *Let $\mathcal{G}_{(n,p)}$ be an Erdös-Rényi network and let A be the adjacency matrix associated with the spectral radius λ. The topological entropy of $\mathcal{G}_{(n,p)}$ is defined by*

$$h_{top}\left(\mathcal{G}_{(n,p)}\right)=\ln(\lambda). \tag{6}$$

Note that the approach taken in this work is the network topological entropy concept used in [25–28]. In this context, the topological entropy of the Erdös-Rényi network $\mathcal{G}_{(n,p)}$ characterizes the topological dynamics of this type of network through the spectral radius λ of the corresponding adjacency matrices A. The topological entropy established in Definition 5 defines a topological order on the Erdös-Rényi random network $\mathcal{G}_{(n,p)}$.

We can say that a graph property holds *almost surely* (or a.s.) in $\mathcal{G}_{(n,p)}$ if the probability that $\mathcal{G}_{(n,p)}$ has $p \to 1$ is $n \to \infty$. In this section, we use the asymptotic behaviour of the spectral radius λ of the adjacency matrix A of $\mathcal{G}_{(n,p)}$. Let us denote by Δ the maximum degree in $\mathcal{G}_{(n,p)}$. There is a general result stating that the spectral radius of $\mathcal{G}_{(n,p)}$ satisfies

$$\lambda = (1 + o(1)) \max\{\sqrt{\Delta}, np\},$$

where $o(1) \to 0$, as $\max\{\sqrt{\Delta}, np\} \to \infty$. For more details on this subject, see [29] and references therein. This result was improved in [30], where several values were considered for the probability p. Suppose that we have $c_0 > 0$ and that the probability p satisfies the following condition:

$$c_0(\log n/n) \leq p \leq (\log n)^{5/3}\, n^{-2/3} \tag{7}$$

for a sufficiently large network size n; then, the spectral radius of the Erdös-Rényi network $\mathcal{G}_{(n,p)}$ is almost certain to be provided by

$$\lambda = np + O(\sqrt{np}). \tag{8}$$

Considering the results established in Definition 5 and Equation (8), we are now in a position to establish the following result.

Proposition 2. *Let $\mathcal{G}_{(n,p)}$ be an Erdös-Rényi network and let A be the adjacency matrix associated with spectral radius λ. If p satisfies Equation (7) and the network size n is sufficiently large, then the topological entropy of $\mathcal{G}_{(n,p)}$ is provided by*

$$h_{top}\left(\mathcal{G}_{(n,p)}\right) = \ln(np + O(\sqrt{np})). \tag{9}$$

Clearly, the topological entropy of the Erdös-Rényi network $\mathcal{G}_{(n,p)}$ is characterized by the expected value of the node degree $\langle \delta(x_i) \rangle \cong np$, with p satisfying Equation (7) and n being sufficiently large. Therefore, this topological invariant is dependent on the network size n and the connection probability p, which reflects the topology of the Erdös-Rényi networks under analysis. In the following results, several properties concerning the topological entropy of $\mathcal{G}_{(n,p)}$ are established.

Proposition 3. *Let $\mathcal{G}_{(n,p)}$ be an Erdös-Rényi network and let $h_{top}\left(\mathcal{G}_{(n,p)}\right)$ be the topological entropy of $\mathcal{G}_{(n,p)}$ provided by Equation (9). The following properties hold true:*

(P1) If the network size n is fixed and sufficiently large and if p satisfies Equation (7) with $c_0 = 1 \pm \epsilon > 0$ and $\epsilon > 0$, then the topological entropy $h_{top}\left(\mathcal{G}_{(n,p)}\right)$ increases with the increase of p and is verified:

$$\log(c_0 \log n + O(\sqrt{np})) \leq h_{top}\left(\mathcal{G}_{(n,p)}\right) \leq \log\left((\log n)^{5/3}\, n^{1/3} + O(\sqrt{np})\right). \tag{10}$$

(P2) If the network size $n \to \infty$ and if p satisfies Equation (7), then $h_{top}\left(\mathcal{G}_{(n,p)}\right) \to \infty$.

Proof. Considering the characterization of the topological entropy of $\mathcal{G}_{(n,p)}$ provided by Equation (9), and attending to the monotony of logarithmic function, we are able to conclude that $h_{top}\left(\mathcal{G}_{(n,p)}\right)$ increases with the growth of p, with p satisfying Equation (7) and the network size n being fixed and sufficiently large. On the other hand, again following the hypothesis that p satisfies Equation (7), the inequalities of Equation (14) are established using the equality provided by Equation (9). Thus, item *(P1)* is proved.

The proof of the result of item *(P2)* follows from Equation (9) and the behaviour of the logarithmic function when $n \to \infty$ and p satisfies Equation (7). This completes the proof of the proposition. $\square$

The above results provide properties of the topological entropy of the Erdös-Rényi network $\mathcal{G}_{(n,p)}$, which are typified as a dependence of the network parameters n and p, i.e., they reflect the topological structure of the network under study. Figure 1a illustrates the results of Proposition 3.

Because the expression for the topological entropy $h_{top}\left(\mathcal{G}_{(n,p)}\right)$ only holds when the values of p lie in the interval $[c_0(\log n/n), (\log n)^{5/3}\, n^{-2/3}]$ for $c_0 > 1$, it is our purpose to perform a deeper analysis concerning the possible variation in the values of the probability p as n becomes larger. Consider the following real functions defined in the positive real numbers:

$$g_1(x) = (\log x)/x \quad \text{and} \quad g_2(x) = (\log x)^{5/3} x^{-2/3}.$$

Computing the derivatives of each function g_1 and g_2, we obtain the following expressions:

$$g_1'(x) = (1 - \log x)/x^2 \quad \text{and} \quad g_2'(x) = \frac{5}{3}((\log x)^{2/3} - \frac{2}{3}(\log x)^{5/3})x^{-1}.$$

Through analysis of each derivative function g_1' and g_2', we can find that for $x = n > 13$, considering the natural numbers $n \in \mathbb{N}$, both sequences g_1 and g_2 decrease and both sequences tend to zero as $n \to +\infty$. In particular, we can establish the following relationship between the variation intervals of the connection probability p provided by Equation (7) between the network sizes n and $n + 1$.

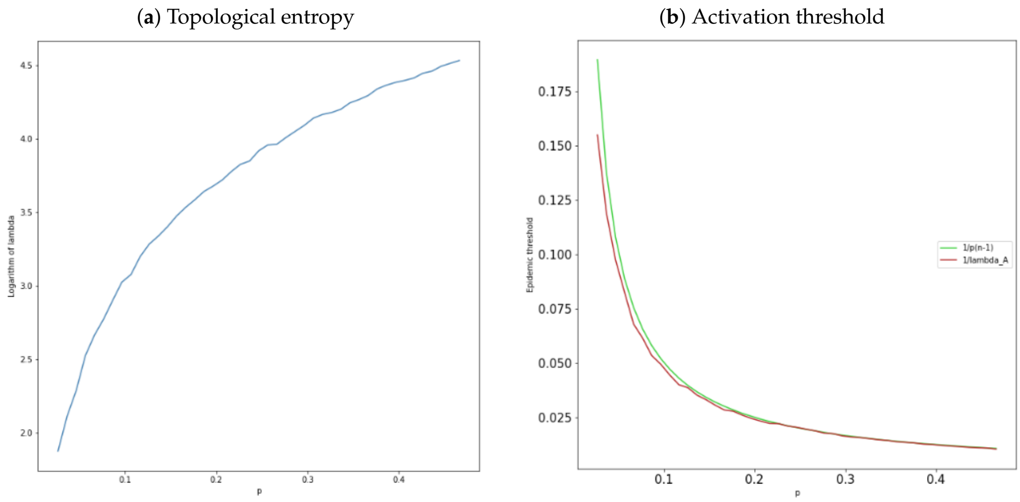

Figure 1. (**a**) Topological entropy of $\mathcal{G}_{(n,p)}$ with $n = 200$ and p satisfying Equation (7), exemplifying the results of Proposition 3; (**b**) comparison between Definition 1 and Definition 6 of the activation threshold θ of the network $\mathcal{G}_{(n,p)}$, with $n = 200$ and p satisfying Equation (7). The green line plots $\theta = \frac{1}{\langle \delta(x_i) \rangle} = \frac{1}{p(n-1)}$ and the red line plots $\theta = \frac{1}{np + O(\sqrt{np})} = \frac{1}{\lambda}$.

Property 1. *Let $p \in [0,1]$ be a probability and $n \in \mathbb{N}$ a natural number such that $n > 13$. Under the assumption in Proposition 2, the following inequalities hold true:*

$$c_0(\log(n+1)/(n+1)) < c_0(\log n/n) \le p \le (\log(n+1))^{5/3}\,(n+1)^{-2/3} \le (\log n)^{5/3}\, n^{-2/3}. \tag{11}$$

Moreover, $(\log n)^{5/3}\, n^{-2/3} \le \frac{25}{6} e^{-\frac{5}{3}}$ for every $n \in \mathbb{N}$ such that $n > 13$, which corresponds to the maximum value for the probability p.

Activation Threshold Dynamics and Topological Entropy

The definition of the activation threshold θ in the Erdös-Rényi network $\mathcal{G}_{(n,p)}$ provided in Definition 1 has no restrictions on the network topology, i.e., on the n and p parameters, and as such can be improved by taking into account Equation (8). However, as set out in Section 5, the parameters n and p have restrictions: p must satisfy Equation (7), and the network size n must be sufficiently large. With these considerations, the definition of the activation threshold θ can be rewritten.

Definition 6. *Let $\mathcal{G}_{(n,p)}$ be an Erdös-Rényi network and let A be the adjacency matrix associated with the spectral radius λ. Considering a value of p that satisfies Equation (7) and sufficiently large network size n, the activation threshold θ of the network $\mathcal{G}_{(n,p)}$ is defined by*

$$\theta = \frac{1}{np + O\left(\sqrt{np}\right)} = \frac{1}{\lambda}. \tag{12}$$

Figure 1b illustrates the comparison between Definition 1 and Definition 6 of the activation threshold θ.

The characterization of the activation threshold of $\mathcal{G}_{(n,p)}$ provided by Equation (12) allows us to reinterpret the results set out in Proposition 3. Considering the definition of topological entropy provided by Proposition 2 and attending to the definition of the activation threshold θ for the network $\mathcal{G}_{(n,p)}$ provided by Equation (12), we are able to conclude that

$$\theta = e^{-h_{top}\left(\mathcal{G}_{(n,p)}\right)}. \tag{13}$$

Therefore, the activation threshold θ in the linear threshold model for $\mathcal{G}_{(n,p)}$ under the conditions in Definition 6 can be defined through the topological entropy of the network, as proven above.

Proposition 4. *Let $\mathcal{G}_{(n,p)}$ be an Erdös-Rényi network, θ the activation threshold of $\mathcal{G}_{(n,p)}$ provided by Equation (6), and $h_{top}\left(\mathcal{G}_{(n,p)}\right)$ the topological entropy of $\mathcal{G}_{(n,p)}$ provided by Equation (9). Then, the following properties hold true:*

(P1) If the network size n is fixed and sufficiently large and if p satisfies Equation (7) with $c_0 = 1 \pm \epsilon > 0$ and $\epsilon > 0$, then the activation threshold $\theta = e^{-h_{top}\left(\mathcal{G}_{(n,p)}\right)}$ decreases with the increase of p and is verified:

$$\frac{1}{(\log n)^{5/3}\, n^{1/3} + O\left(\sqrt{np}\right)} \le \theta \le \frac{1}{c_0\, \log n + O\left(\sqrt{np}\right)}. \tag{14}$$

(P2) If the network size $n \to \infty$ and if p satisfies Equation (7), then the activation threshold $\theta \to 0$.

The proof of the above results follows from Equation (13) and from arguments analogous to the proof of Proposition 3. Figure 1b illustrates the results of Proposition 4.

Remark 3. *The results proved in this section allow us to redefine the influence activation mapping f provided by Equation (2) in Definition 2 using the relationship between the local dynamics of the nodes (measured through the active neighbours of a node at time t along with the degree of the node) and the global dynamics of the network (measured by the topological entropy of $\mathcal{G}_{(n,p)}$), i.e.,*

$$f(x_i(t)) = \begin{cases} 1, & \text{if } \frac{\epsilon(x_i(t-1))}{\delta(x_i)} \ge e^{-h_{top}\left(\mathcal{G}_{(n,p)}\right)} \\ 0, & \text{if } \frac{\epsilon(x_i(t-1))}{\delta(x_i)} < e^{-h_{top}\left(\mathcal{G}_{(n,p)}\right)} \end{cases}. \tag{15}$$

6. Numerical Studies

In this section, it is our purpose to present numerical simulations concerning a number of different goals. On one hand, we intend to illustrate the results proved in the previous sections. On the other, we wish to analyze the results of our simulations, where it is not possible to state theoretical results, with the aim of addressing the influence maximization problem. We present four case studies that have the same parameters for the underlying network $\mathcal{G}_{(n,p)} = (V, E)$ and the linear threshold model, while differing in the choice of the seed between random, degree, closeness, and betweenness centrality measures. First, let us recall a few definitions.

Definition 7. *Let $\mathcal{G}_{(n,p)} = (V, E)$ be an Erdös-Rényi random network and let $x_k \in V$ with $k \in \{1, 2, \ldots, n\}$:*

1. *The degree centrality is $c_D(x_k) = \delta(x_k)$, where $\delta(x_k)$ is the degree of the node x_k*
2. *The closeness centrality is $c_C(x_k) = \dfrac{1}{\sum_{u \in V} d(u, x_k)}$, where $d(u, x_k)$ represents the distance from $u \in V$ to x_k*
3. *The betweenness centrality is $c_B(x_k) = \sum_{i<j\, k \neq i,j} b_{ij}(x_k)$, where*

$$b_{ij}(x_k) = \begin{cases} 0, & \text{no path between } x_i \text{ and } x_j \\ \frac{g_{ij}(x_k)}{g_{ij}}, & \text{otherwise} \end{cases},$$

g_{ij} is the number of paths between x_i and x_j, and $g_{ij}(x_k)$ is the number of paths between x_i and x_j that contain x_k.

The choice of the seed set $\mathcal{H} \subset V$ is related to the detection of influential nodes on a contact network. On one hand, the degree centrality concerns the network's local dynamics, while on the other different centrality measures such as closeness and betweenness take into account the network's global dynamics; see [31].

For each study case presented in the following sections, we intend to compare the effect of the seed choice on the dynamics of influence maximization. We want to illustrate two different scenarios from Proposition 1: first when the process is activated, and second when the theoretical procedure reaches step 4 at $t = t_0 + 1$. Because activation is a necessary condition that is not sufficient for diffusion, we simulate two different situations concerning the diffusion process.

Because the influence maximization problem is NP-hard (see [9]), numerical simulations are essential to comparing and discussing this phenomenon. In this section we present results for the linear threshold model by comparing the simulations in a family of forty Erdös-Rényi networks $\mathcal{G}_{(n,p)} = (V, E)$ for $n = 200$. Among other features, the diffusion relies on the network connectivity and the threshold value. Concerning the connectivity, we perform numerical studies considering the two most connected regimes in the Erdös-Rényi networks, i.e., where a giant connected component is present. These regimes are the supercritical regime for $\frac{1}{n} < p < \frac{\ln n}{n}$ and the connected regime for $\frac{\ln n}{n} < p < 1$. Recall that in the supercritical regime the giant component may have cycles and the other connected components are mainly trees, while in the connected regime the order of the giant component is almost n. All the simulations presented were generated in python.

6.1. Case Study 1: Random Seeds

In this section, we present the numerical simulations of two case studies with the seed set $\mathcal{H}$ random and $|\mathcal{H}| = 1$. For the implementation of Procedure 1, the theoretical procedure for the linear threshold model which is mathematically formalized in this work, we use an Erdös-Rényi network $\mathcal{G}_{(n,p)}$ with $n = 100$ nodes, connection probability $p = 0.05$, and activation threshold $\theta = 0.20$.

In Figure 2a, the Erdös-Rényi network $\mathcal{G}_{(n,p)}$ is plotted at the initial instant $t = t_0$; node x_{76} is the active node or seed, which was chosen randomly. In this case, the theoretical procedure is not activated, i.e., for every instant $t \neq t_0$ the procedure verifies step 4 (the stopping condition); see Figure 2b.

(**a**) Initial instant $t = t_0$ (**b**) Instant $t \neq t_0$

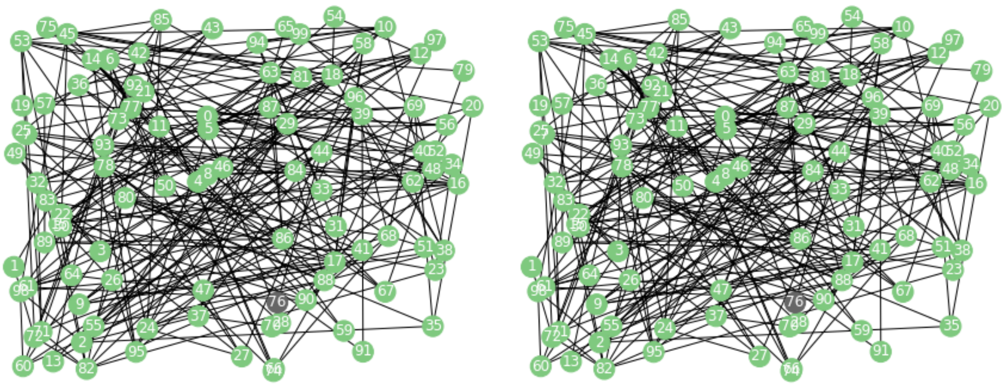

Figure 2. Implementation of Procedure 1: (**a**) Erdös-Rényi network $\mathcal{G}_{(n,p)}$ with n = 100 and p = 0.05 at instant $t = t_0$, where $\theta = 0.20$ is the activation threshold and the active node x_{76} is a random seed; (**b**) for every instant $t \neq t_0$, the procedure verifies step 4., i.e., the stopping condition, meaning that the activation process in network $\mathcal{G}_{(n,p)}$ does not start.

In Figure 3a, the same network is activated by node x_{70}, which again was randomly selected. There remains no diffusion of influence, and step 4 in the theoretical procedure is verified at instant $t = t_0 + 2$ with only two active nodes; see Figure 3b. In conclusion, in Figure 2 the theoretical procedure is not activated according to Proposition 1, while Figure 3 the procedure is activated. Neither simulation case involves diffusion of influence.

(**a**) Initial instant $t = t_0$ (**b**) Instant $t = t_0 + 2$

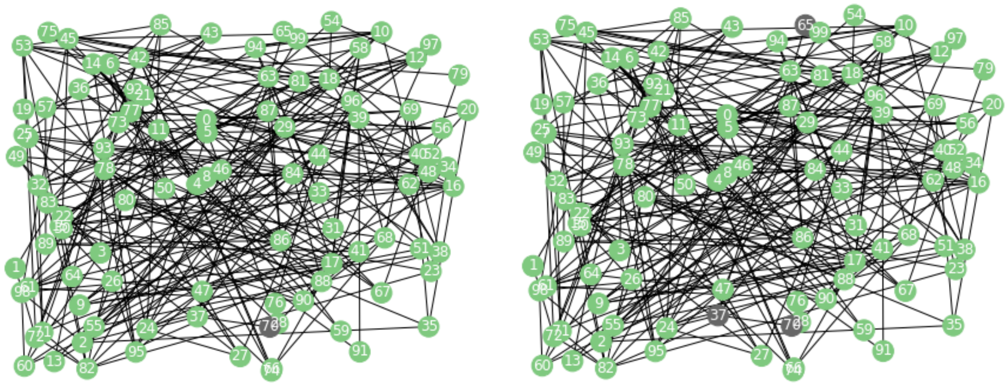

Figure 3. Implementation of Procedure 1: (**a**) Erdös-Rényi network $\mathcal{G}_{(n,p)}$ with $n = 100$ and $p = 0.05$ at instant $t = t_0$, where $\theta = 0.20$ is the activation threshold and the active node x_{70} is a random seed; (**b**) at instant $t = t_0 + 2$, the procedure verifies step 4, i.e., the stopping condition, and the diffusion process in the network $\mathcal{G}_{(n,p)}$ ends after two iterations.

When considering the numerical simulations of the 40 Erdös-Rényi networks, we chose a larger value for the network size at $n = 200$. In the supercritical regime, represented by Figure 4a, we have $p = 0.02$ and $\theta = 0.251$, while in the connected regime in Figure 4b we have $p = 0.03$ and $\theta = 0.168$. Note that we consider these values throughout all the subsequent subsections. In the first regime, most of the cases are not activated, and for those that are there is no diffusion. Conversely, in the connected regime, when activated we obtain influence maximization. In this case, the growth curve of the activated or influenced individuals is of the sigmoid type (the red graphics in Figure 4a), while the curve of nonactivated individuals is an inverse sigmoid function (the green graphics in Figure 4a).

(**a**) Supercritical regime (**b**) Connected regime

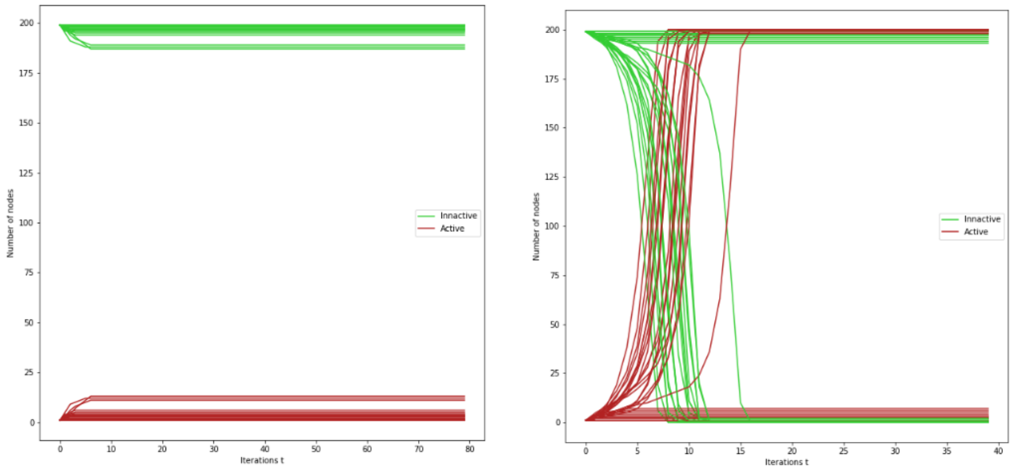

Figure 4. Representation of individuals activated, influenced, and not influenced for the linear threshold model over 40 Erdös-Rényi networks $\mathcal{G}_{(n,p)}$ with $n = 200$ and seed nodes randomly chosen: (**a**) supercritical regime with $p = 0.02$ and $\theta = 0.251$; (**b**) connected regime with $p = 0.03$ and $\theta = 0.168$.

6.2. Case Study 2: Seeds With Higher Degree Centrality

In order to compare the different choices for the only seed, throughout the cases studies we preserve the network topology and the activation threshold $\theta = 0.20$ established in case study 1. Figure 5a plots the Erdös-Rényi network $\mathcal{G}_{(n,p)}$ at the initial instant $t = t_0$, where node x_{17} is the node with higher degree. As can be observed in Figure 5b, not only is the theoretical procedure activated, the total maximum influence is obtained after nine iterations. Note that the only inactive node x_{19} is outside the giant component of the network.

For the degree centrality, it can be observed that a great many networks are activated in Figure 6a, even though the maximum of active nodes is approximately 25. In Figure 6b, when a network is activated maximum influence is obtained with the activation of almost all of the 200 nodes.

(**a**) Initial instant $t = t_0$ (**b**) Instant $t = t_0 + 9$

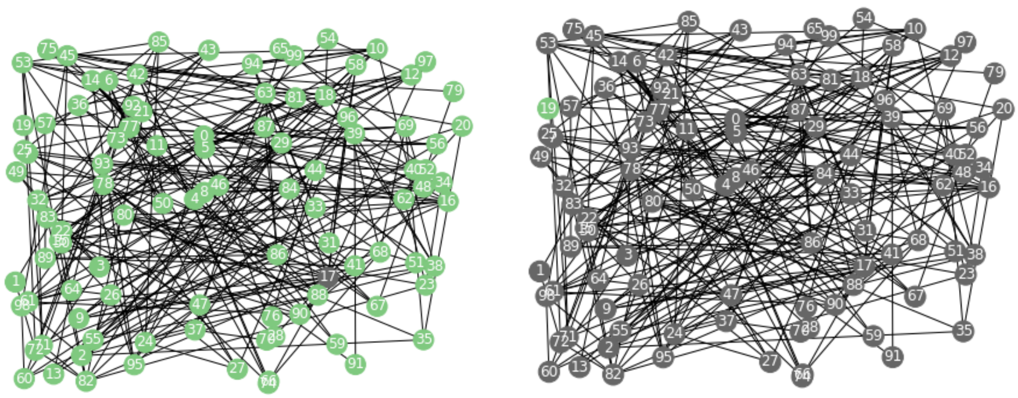

Figure 5. Implementation of Procedure 1: (**a**) Erdös-Rényi network $\mathcal{G}_{(n,p)}$ with $n = 100$ and $p = 0.05$ at instant $t = t_0$, where $\theta = 0.20$ is the activation threshold and the active node x_{17} is a seed with higher degree centrality; (**b**) at instant $t = t_0 + 9$, the procedure verifies step 4, i.e., the stopping condition; the diffusion process in network $\mathcal{G}_{(n,p)}$ ends after nine iterations.

(**a**) Supercritical regime (**b**) Connected regime

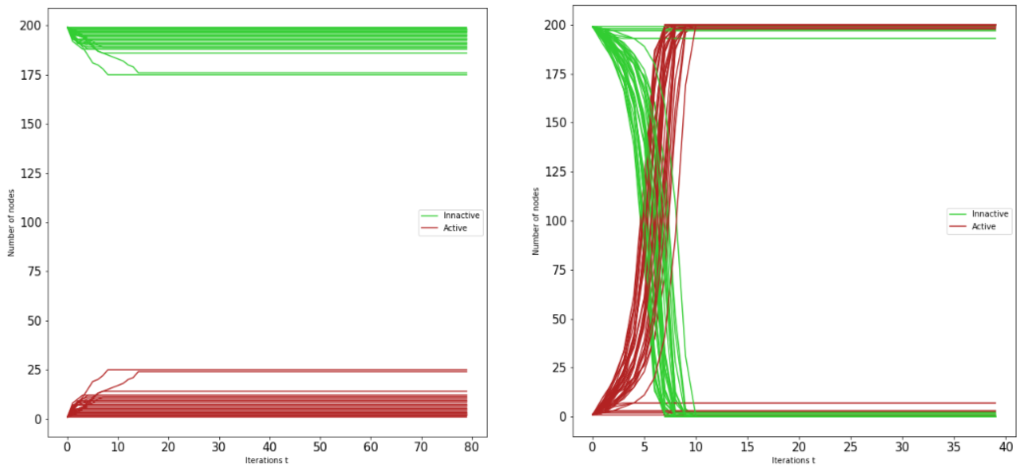

Figure 6. Representation of individuals activated, influenced, and not influenced for the linear threshold model over 40 Erdös-Rényi networks $\mathcal{G}_{(n,p)}$ with $n = 200$ and seed nodes with higher degree centrality: (**a**) supercritical regime with $p = 0.02$ and $\theta = 0.251$; (**b**) connected regime with $p = 0.03$ and $\theta = 0.168$.

6.3. Case Study 3: Seeds With Higher Closeness Centrality

Figure 7a plots the Erdös-Rényi network $\mathcal{G}_{(n,p)}$ at the initial instant $t = t_0$, where node x_{47} is the node with higher closeness centrality. As in the previous case study, it can be verified that the theoretical procedure is activated in Figure 7b and that influence maximization is fulfilled. The only inactive node, x_{19}, is outside the giant component of the network. The stopping condition is achieved rather quickly at instant $t = t_0 + 12$. Under these conditions, influence maximization is achieved over the entire connected component of the network $\mathcal{G}_{(n,p)}$, to which the seed node belongs.

(**a**) Initial instant $t = t_0$ (**b**) Instant $t = t_0 + 12$

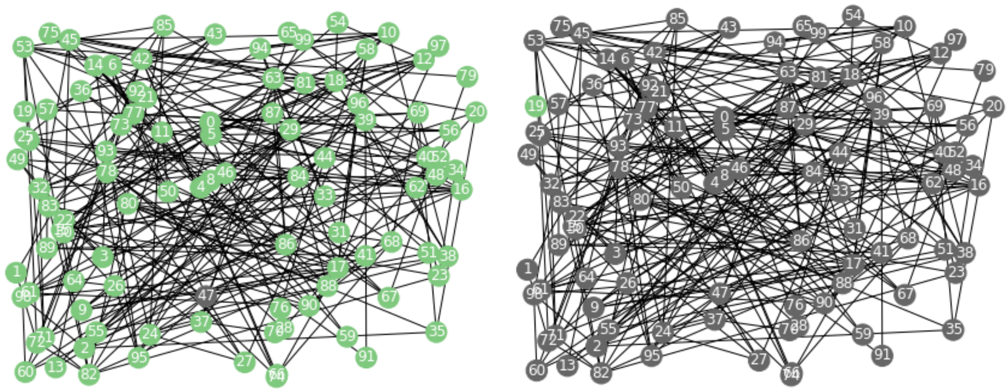

Figure 7. Implementation of Procedure 1: (**a**) Erdös-Rényi network $\mathcal{G}_{(n,p)}$ with $n = 100$ and $p = 0.05$ at instant $t = t_0$, where $\theta = 0.20$ is the activation threshold and active node x_{47} is the seed with higher closeness centrality; (**b**) at instant $t = t_0 + 12$, the procedure verifies step 4, i.e., the stopping condition; the diffusion process in the network $\mathcal{G}_{(n,p)}$ ends after twelve iterations.

Again, similar to the previous subsection, there is no influence maximization for the network topology defined by Figure 8a, while either almost complete activation or no activation is obtained in the connected regime in Figure 8b. The influence spreads somewhat slower than in the previous case.

(**a**) Supercritical regime (**b**) Connected regime

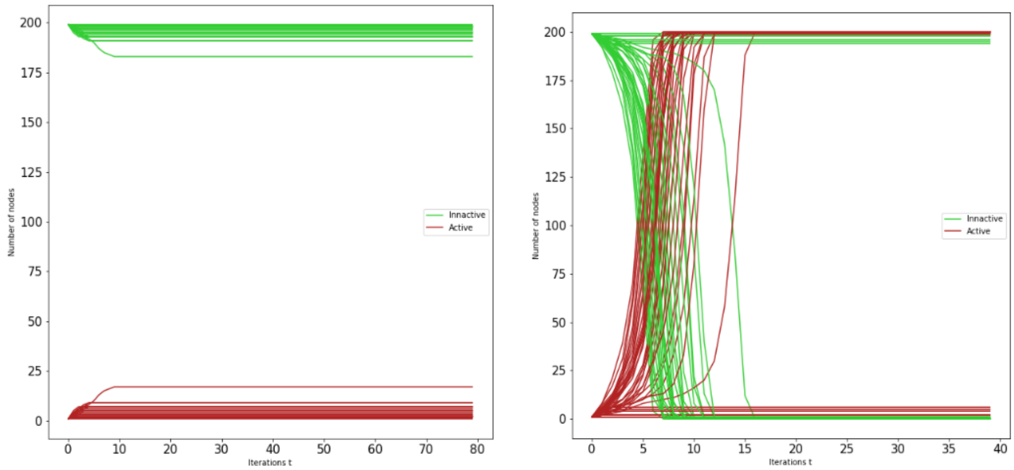

Figure 8. Representation of individuals activated, influenced, and not influenced for the linear threshold model over 40 Erdös-Rényi networks $\mathcal{G}_{(n,p)}$ with $n = 200$ and seed nodes with higher closeness centrality: (**a**) supercritical regime with $p = 0.02$ and $\theta = 0.251$; (**b**) connected regime with $p = 0.03$ and $\theta = 0.168$.

6.4. Case Study 4: Seeds with Higher Betweenness Centrality

In our last case study, the seed is chosen according to the betweenness centrality attained in node x_{82}. In Figure 9a, the Erdös-Rényi network $\mathcal{G}_{(n,p)}$ is plotted at the initial instant $t = t_0$, while in Figure 9b the theoretical procedure leads to the situation of maximum influence. Again, regarding the network topology, the only remaining inactive node x_{19} is disconnected from all of the other nodes. In this case, the influence maximization is faster than in case study 3.

(**a**) Initial instant $t = t_0$ (**b**) Instant $t = t_0 + 10$

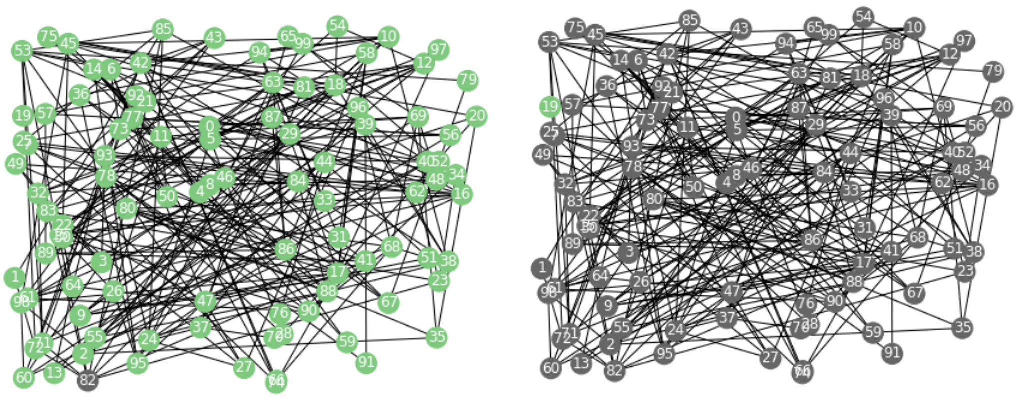

Figure 9. Implementation of Procedure 1: (**a**) Erdös-Rényi network $\mathcal{G}_{(n,p)}$ with $n = 100$ and $p = 0.05$ at instant $t = t_0$, where $\theta = 0.20$ is the activation threshold and the active node x_{82} is the seed with higher betweenness centrality; (**b**) at instant $t = t_0 + 10$, the procedure verifies step 4, i.e., the stopping condition; the diffusion process in the network $\mathcal{G}_{(n,p)}$ ends after ten iterations.

The differences between the two regimes are very deep. In these simulations, when the seed is chosen considering the betweenness centrality, as in Figure 10a, we obtain the highest number of active nodes in the supercritical regime. In Figure 10b it is possible to identify two limit situations: the procedure stops at instant $t = t_0 + 1$, or the number of active nodes is almost $n = 200$.

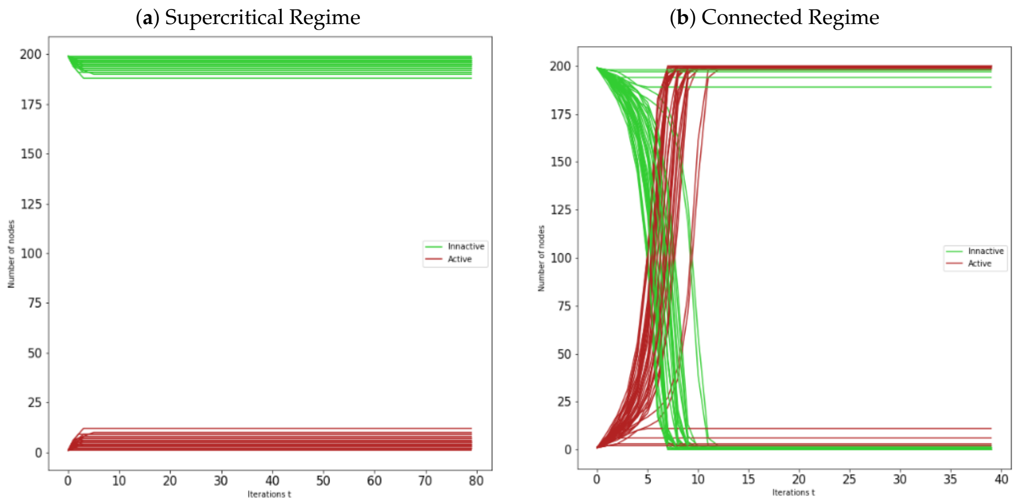

Figure 10. Representation of individuals activated, influenced, and not influenced for the linear threshold model over 40 Erdös-Rényi networks $\mathcal{G}_{(n,p)}$ with $n = 200$ and seed nodes with higher betweenness centrality: (**a**) supercritical regime with $p = 0.02$ and $\theta = 0.251$; (**b**) connected regime with $p = 0.03$ and $\theta = 0.168$.

In conclusion, as referred to previously, in general the influence maximization problem has no theoretical solution even when numerical approaches are considered. In this section we have investigated different choices of seeds, which provide very similar simulation results; however, the network topology is preserved, and it may be the case that this situation leads to the obtained results.

7. Discussion and Conclusions

In this work, we have considered the linear threshold model over the family of Erdös-Rényi random networks $\mathcal{G}_{(n,p)}$, where n is the network size and p is the probability of two nodes being connected. Throughout this paper, it has been our intention to analyze and discuss the influence, activation, and maximization dynamics in a specific population, as well as to identify and characterize certain properties of the underlying network. Our starting point was the mathematical formalization of the linear threshold diffusion model by reference to the definition of the influence activation mapping in Definition 2. This formalization led us to the establishment of a theoretical procedure for the linear threshold model, provided by Procedure 1. In particular, we have highlighted the general conditions under which this procedure is established, as explained in Remark 2. As a consequence, an expression for the activation probability can be obtained under this model considering the case of a unique seed, as provided Proposition 1. At this point, it is possible to discuss the maximum value of the influence activation probability mapping depending on the centrality degree of the network, raising the question of whether other centrality measures can be used as the criterion of choice for seed or activation nodes.

Regarding the topology of Erdös-Rényi random networks, by using the spectral radius of the asymptotically defined associated adjacency matrices, we obtained an explicit expression for the topological entropy for this type of random network by considering different values of p provided by Definition 5. In particular, this definition allowed us to characterize the topological entropy through the expected value of the network node degree. This result, provided by Proposition 2, determines the topological invariant under study via the behaviour of the global dynamics of the network. According to this definition, a topological order can be presented for Erdös-Rényi random networks. By taking a dynamical systems point of view, the approach to the topic of influence maximization discussed in this paper allowed us to prove topological entropy properties in Erdös-Rényi random networks as provided by Proposition 3. Furthermore, a relation between the activation threshold and the topological entropy was derived considering Definition 6. Consequently, it was possible for us to redefine the influence activation mapping in Equation (2) provided by Definition 2 through the relation between the local dynamics of the nodes (measured through the active neighbours of a node at time t and the degree of the node) and the global dynamics of the network (measured by the topological entropy of $\mathcal{G}_{(n,p)}$). The culmination of this result is certainly the most original contribution of the present work, relating discrete dynamical systems theory and influence maximization in Erdös-Rényi networks via the linear threshold model.

In general, the influence maximization problem is theoretically difficult. Moreover, it is known that the spread of influence is computationally complex under any diffusion model for a given seed set. In particular, the linear threshold model is in fact an NP-hard problem; see [9]. Because there are no analytical results that provide a good seed set, we have performed numerical case studies in order to analyze different choices for the activation nodes, which are exhibited and discussed in Section 6.

In fact, the subject under discussion is highly complex and interdisciplinary, and the approaches and topics to be analyzed can be very diverse. As a future perspective, network stability analysis is very important in many practical applications; for an example, see the approach presented in [32].

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