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Exact Null Controllability of a Wave Equation with Dirichlet–Neumann Boundary in a Non-Cylindrical Domain

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Abstract: In this paper, by applying the Hilbert Uniqueness Method in a non-cylindrical domain, we prove the exact null controllability of one wave equation with a moving boundary. The moving endpoint of this wave equation has a Neumann-type boundary condition, while the fixed endpoint has a Dirichlet boundary condition. We derived the exact null controllability and obtained an exact controllability time of the wave equation.

Keywords: wave equation; non-cylindrical domain; exact null controllability

MSC: 35L05

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1. Introduction

Notations:

The exact controllability of partial differential equations is a classical problem in cybernetics, and in particular the exact controllability of wave equations has been a very active area of research. A large number of research results have been achieved in cylindrical domains. Furthermore, applications of such equations in non-cylindrical domains are also very extensive. In the physical sense, many processes take place in domains with moving boundaries. A typical example is the interface of an ice–water mixture when the temperature rises. Therefore, it is necessary to study problems of the exact controllability of wave equations, which have moving or free boundaries.

Given $T > 0$. $\hat{Q}_T^k$ denotes a non-cylindrical domain in $\mathbb{R}^2$, defined by

$$\hat{Q}_T^k = \{(x,t) \in \mathbb{R}^2; 0 < x < \alpha_k(t),\ for\ all\ t \in (0,T)\},$$

where

$$\alpha_k(t) = 1 + kt. \tag{1}$$

let

$$V(0,\alpha_k(t)) = \left\{\varphi \in H^1(0,\alpha_k(t)); \varphi(0) = 0\right\}\ for\ t \in [0,T],$$

which is a subspace of $H^1(0,\alpha_k(t))$. $[V(0,\alpha_k(t))]'$ denotes its conjugate space.

Consider the motion of a string with one endpoint fixed and the other moving. It can be described by the wave equation in the non-cylindrical domain $\hat{Q}_T^k$, as follows:

$$\begin{cases} u_{tt} - u_{xx} = 0 & & in\ \hat{Q}_T^k, \\ u(0,t) = 0 & u_x(\alpha_k(t),t) = v(t) & on\ (0,T), \\ u(x,0) = u^0(x) & u_t(x,0) = u^1(x) & in\ (0,1), \end{cases} \tag{2}$$

where v is the control variable and u is the state variable. $(u^0, u^1) \in L^2(0,1) \times [V(0,1)]'$ is any given initial value. The constant k is called the speed of the moving endpoint. Using a similar method to that in [1,2], in the sense of a transposition, system (2) has a unique weak solution.

$$u \in C([0,T]; L^2(0,\alpha_k(t))) \cap C^1([0,T]; [V(0,\alpha_k(t))]').$$

Control problems can be seen everywhere in science, technology, and engineering practice. The theory of controllability of distributed parameter systems has become an important branch of modern mathematics. Control is categorized in different ways. According to the location of control in the system, control is categorized as boundary control and internal control; according to the relationship between the isochronous region and target, control is categorized as exact control, approximate control, and null control. In this paper, we mainly considered exact controllability and exact null controllability, which are equivalent in wave equations.

The controllability problem of wave equations in cylindrical domains has already been studied by different authors. However, in non-cylindrical domains, little work has been undertaken on wave equations (see [1–10]). The research in [1–3] dealt with the wave equation with Dirichlet boundary conditions. In [4], a globally distributed control was obtained through the stabilization of the wave equation. In [5,6], the wave equation was studied as follows:

$$\begin{cases} u_{tt} - u_{yy} = 0 & & in\ \hat{Q}_T^k, \\ u(0,t) = 0 & u(\alpha_k(t),t) = v(t) & on\ (0,T), \\ u(y,0) = u^0(y) & u_t(y,0) = u^1(y) & in\ (0,1), \end{cases}$$

in which [6] improved the exact controllability time of [5]. In [5], the exact controllability of system (2) was obtained by transforming the non-cylindrical domain into a cylindrical domain. In [9,10], I studied the internal exact controllability of wave equations in one dimension. In [11,12], null controllability of heat equations were discussed. In this article, we took a direct calculation in a non-cylindrical domain to obtain the exact null controllability of (2) when $k \in (0, \frac{\sqrt{3}}{2})$. But the exact controllability of (2) when $k \in (\frac{\sqrt{3}}{2}, 1)$ is still an open problem, and we shall try to address it in the future.

We know the essence of the Hilbert Uniqueness Method: the exact controllability of the original system is equivalent to the observability of a certain dual system (for details, see [2]). This article is organized as follows: in Section 2, we introduce definitions of exact controllability and exact null controllability, and also show the main conclusion of this article. In Section 3, we obtain Lemmas 1 and 2 by using the multiplier method, and by combining the above two lemmas we prove Theorem 2 (observability). In Section 4, with the conclusion of observability of the dual system, we obtain the exact null controllability of the original system according to the Hilbert Uniqueness Method.

2. Preliminary Work and Main Results

First, we will give definitions of exact null controllability and exact controllability, as follows:

Definition 1. *Equation (2) is named exact null controllable at the time T, if for any given initial value*

$$(u^0, u^1) \in L^2(0,1) \times [V(0,1)]',$$

one can always find a control $v \in [H^1(0,T)]'$, such that the corresponding solution u of (2) in the sense of a transposition satisfies

$$u(T) = 0\,,\, u_t(T) = 0.$$

Definition 2. *Equation (2) is named exactly controllable at the time T, if for any given initial value*

$$(u^0, u^1) \in L^2(0,1) \times [V(0,1)]',$$

and any target function

$$(u_d^0, u_d^1) \in L^2(0, \alpha_k(T)) \times [V(0, \alpha_k(T))]',$$

one can always find a control $v_1 \in [H^1(0,T)]'$, *such that the corresponding solution* u *of (2) in the sense of a transposition satisfies*

$$u(T) = u_d^0 \, , u_t(T) = u_d^1.$$

Throughout this paper, we shall write

$$T_k^* = \frac{2}{1-k} \tag{3}$$

for the controllability time. The specific proof will be given later in this paper.

The following Theorem 1 is the focus of our proof in this paper.

Theorem 1. *For any given* $T > T_k^*$, *(2) is exactly null controllable at time* T *in the sense of Definition 1.*

From calculations, we know that the dual system of (2) is as follows:

$$\begin{cases} z_{tt} - z_{xx} = 0 & & \text{in } \hat{Q}_T^k, \\ z(0,t) = 0 & z_x(\alpha_k(t), t) + 2k z_t(\alpha_k(t), t) = 0 & \text{on } (0,T), \\ z(x,0) = z^0(x) & z_t(x,0) = z^1(x) & \text{in } (0,1), \end{cases} \tag{4}$$

where $(z^0, z^1) \in V(0,1) \times L^2(0,1)$ *is any given initial value. We learn that system (4) has a unique weak solution from [1].*

$$z \in C^1([0,T]; V(0, \alpha_k(t))) \cap C([0,T]; L^2(0, \alpha_k(t))).$$

The key to proving Theorem 1 lies in proving the observability of system (4), which is described as follows:

Theorem 2. *Let* $T > T_k^*$. *For any* $(z^0, z^1) \in V(0,1) \times L^2(0,1)$, *there exists a constant* $C > 0$, *such that the corresponding solution* z *of (4) satisfies*

$$\begin{aligned} &C(|z^0|^2_{V(0,1)} + |z^1|^2_{L^2(0,1)}) \\ &\leq \int_0^T |z_t(\alpha_k(t), t)|^2 dt \\ &\leq C(|z^0|^2_{V(0,1)} + |z^1|^2_{L^2(0,1)}). \end{aligned} \tag{5}$$

Remark 1. *It is easy to verify*

$$T_0 = \lim_{k \to 0} T_k^* = \lim_{k \to 0} \frac{2}{1-k} = 2.$$

The time $T_0 = 2$ *is in accord with the controllability time obtained in [2].*

Remark 2. *In fact, for a more general function* $\alpha_k(t)$, *where* $0 < {\alpha_k}'(t) < \frac{\sqrt{3}}{2}$, *we can obtain the same results as in this paper.*

Remark 3. *We define* C *to be a positive constant related only to* T *and* k. C *may not be the same in different places.*

The weighted energy function for (4) can be defined:

$$E(t) = \frac{1}{2} \int_0^{\alpha_k(t)} [|z_t(x,t)|^2 + |z_x(x,t)|^2] dx \; for \; t \geq 0, \tag{6}$$

where z is the solution of (4). It is obvious that

$$E(0)=\frac{1}{2}\int_0^1[|z^1(x)|^2+|z_x^0(x)|^2]dx.$$

3. Proof of Theorem 2 (Observability)

To prove the observability of system (4), we would first take the multiplier method in the non-cylindrical domain $(0,\alpha_k(s))\times(0,t)$ for any $t\in[0,T]$ to obtain the following Lemmas 1 and 2. Then, combining the above Lemmas, we can obtain two important observability inequalities for system (4), which proves the observability of system (4). The specific proof process is as follows:

Lemma 1. *For any* $(z^0,z^1)\in V(0,1)\times L^2(0,1)$ and $t\in[0,T]$, *the corresponding solution z of (4) satisfies*

$$E(t)-E(0)=\frac{k(4k^2-3)}{2}\int_0^t|z_s(\alpha_k(s),s)|^2ds. \tag{7}$$

Proof. By multiplying the first equation in (4) by $z_s(x,s)$ while integrating on $(0,\alpha_k(s))\times(0,t)$, for any $t\in[0,T]$, we have the following equation:

$$\begin{aligned}0&=\int_0^t\int_0^{\alpha_k(s)}[z_{ss}(x,s)-z_{xx}(x,s)]z_s(x,s)dxds\\&=\tfrac{1}{2}\int_0^t\int_0^{\alpha_k(s)}[|z_s(x,s)|^2+|z_x(x,s)|^2]_sdxds\\&-\int_0^t\int_0^{\alpha_k(s)}(z_s(x,s)z_x(x,s))_xdxds.\end{aligned} \tag{8}$$

Since $\alpha_{k,s}(s)=k$, it follows from this above equality that

$$\begin{aligned}0&=\tfrac{1}{2}\int_0^t\tfrac{\partial}{\partial s}\int_0^{\alpha_k(s)}[|z_s(x,s)|^2+|z_x(x,s)|^2]dxds\\&-\tfrac{k}{2}\int_0^t[|z_s(\alpha_k(s),s)|^2+|z_x(\alpha_k(s),s)|^2]ds\\&-\int_0^t z_s(x,s)z_x(x,s)\Big|_0^{\alpha_k(s)}ds\\&=\tfrac{1}{2}\int_0^{\alpha_k(t)}[|z_t(x,t)|^2+|z_x(x,t)|^2]dx\\&-\tfrac{1}{2}\int_0^1[|z_t(x,0)|^2+|z_x(x,0)|^2]dx\\&-\tfrac{k}{2}\int_0^t[|z_s(\alpha_k(s),s)|^2+|z_x(\alpha_k(s),s)|^2]ds\\&-\int_0^t z_s(\alpha_k(s),s)z_x(\alpha_k(s),s)ds\\&+\int_0^t z_s(0,s)z_x(0,s)ds.\end{aligned} \tag{9}$$

From $z(0,s)=0$, we can find

$$z_s(0,s)=0. \tag{10}$$

Considering the definition of $E(t),\ E(0)$ and (10), it follows from (9) that

$$\begin{aligned}&E(t)-E(0)\\&=\tfrac{k}{2}\int_0^t[|z_s(\alpha_k(s),s)|^2+|z_x(\alpha_k(s),s)|^2]ds\\&+\int_0^t z_s(\alpha_k(s),s)z_x(\alpha_k(s),s)ds.\end{aligned} \tag{11}$$

Note that

$$z_x(\alpha_k(s),s)=-2kz_s(\alpha_k(s),s). \tag{12}$$

It is obvious that

$$
\begin{aligned}
&E(t)-E(0)\\
&=\tfrac{k}{2}\int_0^t[|z_s(\alpha_k(s),s)|^2+|-2kz_s(\alpha_k(s),s)|^2]ds\\
&-2k\int_0^t z_s(\alpha_k(s),s)z_s(\alpha_k(s),s)ds\\
&=\tfrac{k(4k^2-3)}{2}\int_0^t|z_s(\alpha_k(s),s)|^2ds.
\end{aligned}
\tag{13}
$$

□

Remark 4. *For $k\in(0,\frac{\sqrt{3}}{2})$, according to (7), it is not difficult to verify*

$$E'(t)=\frac{k(4k^2-3)}{2}|z_t(\alpha_k(t),t)|^2<0.$$

We can find that $E(t)$ is a monotonically decreasing function, and

$$E(t)<E(0). \tag{14}$$

Lemma 2. *For any $(z^0,z^1)\in V(0,1)\times L^2(0,1)$ and $t\in[0,T]$, the corresponding solution z of (4) satisfies*

$$
\begin{aligned}
&\int_0^T\alpha_k(t)|z_t(\alpha_k(t),t)|^2dt\\
&=2\int_0^T E(t)dt+2\int_0^{\alpha_k(T)}xz_t(x,T)z_x(x,T)dx-2\int_0^1 xz_t(x,0)z_x(x,0)dx.
\end{aligned}
\tag{15}
$$

Proof. By multiplying the first equation of (4) by $2xz_x(x,s)$ while integrating on $\hat{Q}_T^k$, we can find the following equation:

$$
\begin{aligned}
0&=\int_0^T\int_0^{\alpha_k(t)}[z_{tt}(x,t)-z_{xx}(x,t)]2xz_x(x,t)dxdt\\
&=2\int_0^T\int_0^{\alpha_k(t)}(xz_t(x,t)z_x(x,t))_t dxdt\\
&-\int_0^T\int_0^{\alpha_k(t)}[x|z_t(x,t)|^2+x|z_x(x,t)|^2]_x dxdt\\
&+\int_0^T\int_0^{\alpha_k(t)}[|z_t(x,t)|^2+|z_x(x,t)|^2]dxdt.
\end{aligned}
\tag{16}
$$

Since

$$\alpha_{k,t}(t)=k \tag{17}$$

and the definition of $E(t)$, we can conclude:

$$
\begin{aligned}
0&=2\int_0^T\frac{\partial}{\partial t}\int_0^{\alpha_k(t)}xz_t(x,t)z_x(x,t)dxdt\\
&-2k\int_0^T\alpha_k(t)z_t(\alpha_k(t),t)z_x(\alpha_k(t),t)dt\\
&-\int_0^T x[|z_t(x,t)|^2+|z_x(x,t)|^2]\Big|_0^{\alpha_k(t)}dt+2\int_0^T E(t)dt\\
&=2\int_0^{\alpha_k(T)}xz_t(x,T)z_x(x,T)dx\\
&-2\int_0^1 xz_t(x,0)z_x(x,0)dx\\
&-2k\int_0^T\alpha_k(t)z_t(\alpha_k(t),t)z_x(\alpha_k(t),t)dt\\
&-\int_0^T\alpha_k(t)[|z_t(\alpha_k(t),t)|^2+|z_x(\alpha_k(t),t)|^2]dt\\
&+2\int_0^T E(t)dt.
\end{aligned}
\tag{18}
$$

From $z_x(\alpha_k(t),t) = -2kz_t(\alpha_k(t),t)$, we can deduce that

$$\begin{aligned}
&\int_0^T \alpha_k(t)|z_t(\alpha_k(t),t)|^2 dt \\
&= 2\int_0^T E(t)dt + 2\int_0^{\alpha_k(T)} xz_t(x,T)z_x(x,T)dx \\
&-2\int_0^1 xz_t(x,0)z_x(x,0)dx.
\end{aligned} \tag{19}$$

Combining Lemmas 1 and 2, we can prove Theorem 2. The proof process is divided into two steps. □

Proof of Theorem 2.

Step 1. We complete the proof of the first inequality of (5). From the Cauchy inequality, it is easy to deduce that:

$$\left|2\int_0^{\alpha_k(T)} xz_t(x,t)z_x(x,t)dx\right| \leq 2\alpha_k(T)E(T), \tag{20}$$

$$\left|2\int_0^1 xz_t(x,0)z_x(x,0)dx\right| \leq 2E(0). \tag{21}$$

Combining (14), (15), (20) and (21), it holds that

$$\begin{aligned}
&\int_0^T \alpha_k(t)|z_t(\alpha_k(t),t)|^2 dt \\
&= 2\int_0^T E(t)dt + 2\int_0^{\alpha_k(T)} xz_t(x,T)z_x(x,T)dx \\
&-2\int_0^1 xz_t(x,0)z_x(x,0)dx \\
&\geq 2\int_0^T E(t)dt - 2\alpha_k(T)E(T) - 2E(0) \\
&\geq 2\int_0^T E(t)dt - 2\alpha_k(T)E(0) - 2E(0).
\end{aligned} \tag{22}$$

From (7), we can deduce that

$$\begin{aligned}
&\int_0^T \alpha_k(t)|z_t(\alpha_k(t),t)|^2 dt \\
&\geq 2\int_0^T E(0) + \tfrac{k(4k^2-3)}{2}\int_0^t |z_s(\alpha_k(s),s)|^2 ds dt \\
&-2\alpha_k(T)E(0) - 2E(0).
\end{aligned} \tag{23}$$

From this, it follows that

$$\begin{aligned}
&[\alpha_k(T) - k(4k^2-3)T]\int_0^T |z_t(\alpha_k(t),t)|^2 dt \\
&\geq 2(T - \alpha_k(T) - 1)E(0).
\end{aligned} \tag{24}$$

Hence, if $T > T_k^*$ (see (3) for the definition of T_k^*), $2[(T - \alpha_k(T)) - 1] > 0$, and from this inequality and (24), we can obtain

$$\begin{aligned}
&\int_0^T |z_t(\alpha_k(t),t)|^2 dt \\
&\geq C[2(T - \alpha_k(T) - 1)]E(0) \\
&\geq C[2(T - \alpha_k(T) - 1)](|z^0|^2_{V(0,1)} + |z^1|^2_{L^2(0,1)}).
\end{aligned} \tag{25}$$

Step 2. We shall prove the second inequality of (5). From (14), (15), (20) and (21), we can obtain

$$\begin{aligned}
&\int_0^T \alpha_k(t)|z_t(\alpha_k(t),t)|^2dt \\
&= 2\int_0^T E(t)dt + 2\int_0^{\alpha_k(T)} xz_t(x,T)z_x(x,T)dx \\
&-2\int_0^1 xz_t(x,0)z_x(x,0)dx \\
&\leq 2\int_0^T E(t)dt + 2\alpha_k(T)E(T) + 2E(0) \\
&\leq 2(T+\alpha_k(T)+1)E(0).
\end{aligned} \tag{26}$$

Further, we have

$$\begin{aligned}
&\int_0^T |z_t(\alpha_k(t),t)|^2dt \\
&\leq C[2(T+\alpha_k(T)+1)]E(0) \\
&\leq C[2(T+\alpha_k(T)+1)](|z^0|^2_{V(0,1)} + |z^1|^2_{L^2(0,1)}).
\end{aligned} \tag{27}$$

Combining (25) and (27), we can conclude that

$$\begin{aligned}
&C(|z^0|^2_{V(0,1)} + |z^1|^2_{L^2(0,1)}) \\
&\leq \int_0^T |z_t(\alpha_k(t),t)|^2dt \\
&\leq C(|z^0|^2_{V(0,1)} + |z^1|^2_{L^2(0,1)}).
\end{aligned}$$

Hence, we can complete the proof of Theorem 2. □

4. Proof of Theorem 1 (Controllability)

From the proof of Section 3, we can obtain the observability of system (4). Based on the Hilbert Uniqueness Method, we can learn that the controllability of system (2) is equivalent to the observability of system (4).

In fact, Theorem 1 is equivalent to showing that, given the initial data $(u^0,u^1) \in L^2(0,1)\times[V(0,1)]'$, we can find a control $v(t) \in [H^1(0,T)]'$ such that the solution of system (2) satisfies

$$u(T) = 0 \; and \; u_t(T) = 0.$$

Proof of Theorem 1. We can complete the proof in the following three steps.

Step 1. We define the linear operator as $\Gamma : V(0,1)\times L^2(0,1) \to [V(0,1)]'\times L^2(0,1)$.

For any $(z^0,z^1) \in V(0,1)\times L^2(0,1)$, we denote z as the corresponding solution of (4). Now, we consider the wave equation:

$$\begin{cases}
\xi_{tt} - \xi_{xx} = 0 & & in \; \hat{Q}^k_T, \\
\xi(0,t) = 0 & \xi_x(\alpha_k(t),t) = G_{z(\alpha_k(t),t)} & on \; (0,T), \\
\xi(x,T) = 0 & \xi_t(x,T) = 0 & in \; (0,1).
\end{cases} \tag{28}$$

It is worth noting that here $G_{z(\alpha_k(t),t)}$ is defined as follows:

$$\left\langle G_{z(\alpha_k(t),t)}, \phi \right\rangle_{((H^1(0,T))',H^1(0,T))} = \int_0^T z_t(\alpha_k(t),t)\phi_t(t)dt, \quad for\ any \quad \phi \in H^1(0,T). \tag{29}$$

From [1], we know that (28), in the sense of a transposition, has a unique weak solution ξ. We set the following:

$$(\xi^0,\xi^1) \triangleq (\xi(x,0),\ \xi_t(x,0)) \in L^2(0,1)\times[V(0,1)]'.$$

Now, we define the operator:

$$\Gamma : V(0,1) \times L^2(0,1) \to L^2(0,1) \times [V(0,1)]',$$
$$(z^0, z^1) \to (\xi^0, -\xi^1).$$

Therefore,

$$\left\langle \Gamma(z^0, z^1), (z^0, z^1) \right\rangle = \int_0^1 z^1 \xi^0 - z^0 \xi^1 dx.$$

Step 2. By multiplying the first equation of (28) by $z(x,t)$ while integrating on $\hat{Q}_T^k$, we can obtain

$$\begin{aligned} 0 &= \int_0^T \int_0^{\alpha_k(t)} [\xi_{tt}(x,t) - \xi_{xx}(x,t)] z(x,t) dx dt \\ &= \int_0^T \int_0^{\alpha_k(t)} [\xi_t(x,t) z(x,t) - \xi(x,t) z_t(x,t)]_t dx dt \\ &- \int_0^T \int_0^{\alpha_k(t)} [\xi_x(x,t) z(x,t) - \xi(x,t) z_x(x,t)]_x dx dt. \end{aligned} \tag{30}$$

According to (17), it follows from (30) that

$$\begin{aligned} 0 &= [\int_0^{\alpha_k(t)} \xi_t(x,t) z(x,t) - \xi(x,t) z_t(x,t) dx]\big|_0^T \\ &- k\int_0^T [\xi_t(\alpha_k(t),t) z(\alpha_k(t),t) - \xi(\alpha_k(t),t) z_t(\alpha_k(t),t)] dt \\ &- \int_0^T [\xi_x(x,t) z(x,t) - \xi(x,t) z_x(x,t)]\Big|_0^{\alpha_k(t)} dt \\ &= \int_0^{\alpha_k(T)} \xi_t(x,T) z(x,T) - \xi(x,T) z_t(x,T) dx \\ &- \int_0^1 \xi_t(x,0) z(x,0) - \xi(x,0) z_t(x,0) dx \\ &- k\int_0^T [\xi_t(\alpha_k(t),t) z(\alpha_k(t),t) - \xi(\alpha_k(t),t) z_t(\alpha_k(t),t)] dt \\ &- \int_0^T [\xi_x(\alpha_k(t),t) z(\alpha_k(t),t) - \xi(\alpha_k(t),t) z_x(\alpha_k(t),t)] dt \\ &+ \int_0^T [\xi_x(0,t) z(0,t) - \xi(0,t) z_x(0,t)] dt. \end{aligned} \tag{31}$$

Using the following conditions,

$$\xi_t(T) = \xi(T) = z(0,t) = \xi(0,t) = 0,$$
$$z_x(\alpha_k(t),t) + 2k z_t(\alpha_k(t),t) = 0.$$

From (31), we can conclude that

$$\int_0^T G_{z(\alpha_k(t),t)} z(\alpha_k(t),t) dt = \int_0^1 z^1 \xi^0 - z^0 \xi^1 dx. \tag{32}$$

From (29), it holds that

$$\int_0^T |z_t(\alpha_k(t),t)|^2 dt = \int_0^1 z^1 \xi^0 - z^0 \xi^1 dx. \tag{33}$$

From Theorem 2, we can deduce that Γ is bounded and coercive. Hence, we can conclude that Γ is an isomorphism using the Lax–Milgram Theorem.

Step 3. We can prove the exact null controllability of (2). Indeed, for any given initial value

$$(u^0, u^1) \in L^2(0,1) \times [V(0,1)]',$$

we choose

$$v(\cdot) = G_{z(\cdot,t)} \in (H^1(0,T))',$$

where z is the solution of (4) associated with $\Gamma(z^0, z^1) = (u^0, -u^1)$.

From the definition of Γ, we can deduce that

$$\Gamma(z^0, z^1) = (\xi^0, -\xi^1), \tag{34}$$

where ξ is the solution of (28). Then, ξ satisfies

$$(\xi^0, -\xi^1) = (u^0, -u^1).$$

Considering the uniqueness of (28), u satisfies

$$(u(x,T), u_t(x,T)) = (0,0).$$

Hence, we can obtain the exact null controllability of (2). □

Remark 5. *The exact null controllability of the wave equation is equivalent to its exact controllability. The specific proof process is as follows:*

For any $(u^0, u^1) \in L^2(0,1) \times [V(0,1)]'$, u denotes the solution of system (2). Consider the system as follows:

$$\begin{cases} \eta_{tt} - \eta_{xx} = 0 & & in\ \hat{Q}_T^k, \\ \eta(0,t) = 0 & \eta_x(\alpha_k(t), t) = 0 & on\ (0,T), \\ \eta(x,T) = u_d^0(x) & \eta_t(x,T) = u_d^1(x) & in\ (0,1). \end{cases} \tag{35}$$

Since

$$\Gamma : V(0,1) \times L^2(0,1) \to L^2(0,1) \times [V(0,1)]',$$

we can derive

$$(u^0 - \eta^0, u^1 - \eta^1) \in L^2(0,1) \times [V(0,1)]'.$$

From the definition of Γ, we can find z_0, z_1 such that the following equation holds:

$$\Gamma(z^0, z^1) = (u^0 - \eta^0, u^1 - \eta^1).$$

Therefore, combining (34), we can see

$$(\xi^0, -\xi^1) = (u^0 - \eta^0, \eta^1 - u^1).$$

This allows the conclusion that $u = \xi + \eta$ satisfied both (2) and (28).
We can therefore complete the proof of Remark 5.

5. Conclusions

According to the essence of the Hilbert Uniqueness Method, in order to obtain the exact null controllability of the original system, we need to obtain the observability of a certain dual system. Thus, the main points of this article are as follows: Step 1. We proved the observability of the dual system by proving two key inequalities in Section 3. Step 2. The controllability of the original system was obtained based on the Hilbert Uniqueness Method in Section 4. In the future, we will consider controllability problems of wave equations with more complex conditions.

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