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Two-Person Stochastic Duel with Energy Fuel Constraint Ammo

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Abstract: This paper deals with a novel variation of the versatile stochastic duel game that incorporates an energy fuel constraint into a two-player duel game. The energy fuel not only measures the vitality of players but also determines the power of the shooting projectile. The game requires players to carefully balance their energy usage, while trying to outmaneuver their opponent. This unique theoretical framework for the stochastic game model provides a valuable method for understanding strategic behavior in competitive environments, particularly in decision-making scenarios with fluctuating processes. The proposed game provides players with the challenge of optimizing their energy fuel usage, while managing the risk of losing the game. This novel model has potential for implementation across diverse fields, as it allows for a versatile conception of energy fuel. These energy fuels may encompass conventional forms, such as natural gas, petroleum, and electrical power, and even financial budgets, human capital, and temporal resources. The unique rules and constraints of the game in this research are expected to contribute insights into the decision-making strategies and behaviors of players in a wide range of practical applications. This research primarily focuses on deriving compact closed-form solutions, utilizing transformation and flexible analysis techniques adapted to varying the concept of the energy fuel level. By presenting a comprehensive description of our novel analytical approach and its application to the proposed model, this study aims to elucidate the fundamental principles underlying the energy fuel constraint stochastic duel game model.

Keywords: duel game; energy level constraint; stochastic model; fluctuation theory; time domain game; backward induction; marked point process

MSC: 60C55; 60K10; 90B15; 90B50

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1. Introduction

Game theory is a powerful tool for modeling and analyzing strategic decision-making in a wide range of fields, from economics and political science to computer science and engineering [1–6]. Modern game theory has seen significant developments in recent years, particularly in the areas of repeated games, evolutionary game theory, and network games. One important recent development is the use of machine learning techniques in game theory, which has led to advances in predicting and optimizing outcomes in complex, multi-player games [7]. Another area of active research is the analysis of games in which players have incomplete information, which has applications in contract theory, mechanism design, and more [8]. Network games have also become an important area of research, as they model interactions among agents in a social or economic network, with implications for contagion and diffusion dynamics [9]. Another growing area of interest is the use of game theory in analyzing cybersecurity, particularly in the context of defending against cyber-attacks [3,5,10]. Overall, modern game theory has seen exciting developments and applications in a wide range of fields. As researchers continue to develop new methods and models, game theory is likely to remain a valuable tool for understanding strategic decision-making in complex, dynamic environments. Duel games have become the focus of study in game theory, due to their applicability to modeling competitive situations, ranging from military conflicts to economic decisions [11]. Recent research has explored variations of the duel game, such as the generalized stochastic duel game [12] and the duel game with multiple asymmetric players [13,14] and penalties. Other studies have examined the impact of different strategies on the outcome of duel games, including the use of mixed strategies [15] and time-dependent strategies with two players [16] or multiple players [17,18]. Several studies have also explored the application of duel games in specific contexts, such as decision-making in the presence of incomplete information [19], analysis of political competitions [20], and modeling of predator–prey interactions in ecological systems [21].

The versatile stochastic duel game is a game-theoretic model that incorporates elements of the fluctuation process [16]. This hybrid model has practical applications in a wide range of decision-making scenarios. Specifically, the model is designed to analyze a two-person duel-type game in the time domain. This unique theoretical framework for the stochastic game model allows the analysis of complex decision-making scenarios and provides a valuable tool for understanding strategic behavior in competitive environments. In OneToN stochastic duel games [13], a player who decides to shoot does so in an attempt to kill all other players simultaneously (i.e., one-shooting-to-kill-all). This is akin to using an automatic machine gun, as opposed to a single bullet gun. However, it is important to note that each player is only permitted to shoot once on their turn. This robust method was developed to identify the optimal strategies in a general antagonistic multi-person stochastic duel game [14]. This game consists of multiple stages, each with different pairs of players and corresponding optimal shooting moments. Despite the participation of multiple players across multiple battlefields, each player is limited to one bullet in the entire game. Although players may have a limited number of opportunities to shoot, they must carefully consider their options and wait for the optimal shooting opportunity. Once a fixed threshold is reached, players are allowed to take the best shooting opportunity at a random time with a random impact. This highly strategic game presents a challenge for identifying optimal strategies, which this robust method was designed to address.

This research proposes a novel version of the stochastic duel game, introducing a shooting projectile that is subject to energy fuel constraints. Energy fuel is the cost of sustaining player life and is slowly and randomly used up. This variant of a stochastic duel includes an expansion of describing various game situations with limited shooting resources. In practical applications of this new model, energy fuel (i.e., limited shooting resources) can take various forms, including energy sources, such as gas, oil, and coal; electrical power; and even financial assets, such as budgets, human resources, and time. Specifically, this game model is well-suited for analyzing funding strategies involving two competitive firms, thus rendering it a suitable framework for a two-player game. In this context, the financial budget may be perceived as the energy fuel, which is incrementally expended, while allocating the remaining budget for the development and launch of a new product or service. Notably, within the scope of the mathematical game model, a product launch parallels the concept of "shooting". In a scenario where the opposing player (i.e., player B) possesses insufficient remaining budget to fund a product launch, player A is declared the winner. Conversely, should player B maintain an adequate budget to counter the product launch by player A, player A loses the game. Basically, player A loses the game if their single shot totally misses the target or their shooting strength is less than the remaining energy fuel of the target player. Intriguingly, we can determine the result before even starting the game. Each player can find the best moment for shooting and the optimal energy fuel level.

This paper is constructed in the following manner: Section 2 presents the innovative stochastic duel game model, incorporating a unique variant that takes into account fuel-constrained ammo. This means that each player possesses a shooting power based on their individual energy level. Finally, the conclusions of this paper are provided in Section 3.

2. Stochastic Duel Game with Fuel Constraint Ammo

An antagonistic duel game of two players (called "A" and "B") in the time domain is considered and both players know the full information regarding the success probabilities based on the time domain [16]. Let us assigned $P_a(t)$ as the monotone increasing CDF (cumulative distribution function) for player A regarding hitting the opponent player (player B) at time t. Similarly, $P_b(t)$ is the probability of player B hitting the opponent player (player A) at time t. Each hitting probability (CDF) may be arbitrary chosen and this reaches 1 when the time t goes to the infinite ($<\infty$). The strategic decision of the shooting moment means finding the moment when a player will have the best chance to hit the other player. There is a certain point that maximizes the chance of succeeding in the shot, and this optimal point becomes the moment of success in the continuous time domain. This moment t^* is actually the same as the single bullet case and it is defined as follows [11,13]:

$$t^* = \inf\{t \geq 0 : P_a(t) + P_b(t) \geq 1\}, \tag{1}$$

which indicates that the probability of hitting the other player is higher than the probability of the other player missing. According to any conventional duel games [11], it is important to note that the individual who fires the initial shot does not necessarily possess superior or inferior shooting skills. Instead, success in these games is contingent upon surpassing a critical point, referred to as t^*, which is influenced by the collective abilities of both participants from (1). Additionally, each player has an energy fuel level which randomly drops, and the spending (or usage) of the energy fuel for each player is monotone and non-decreasing. Let (Ω, $\mathcal{F}(\Omega)$,P) be the probability space $\mathcal{F}_A$, $\mathcal{F}_B$, $\mathcal{F}_\tau \subseteq \mathcal{F}(\Omega)$ independent σ-subalgebras. Suppose

$$\mathcal{A} := \textstyle\sum_{k\geq 0} \varepsilon_{s_k}, s_0(=0) < s_1 < s_2 < \cdots, \text{a.s.} \tag{2}$$

$$\mathcal{B} := \textstyle\sum_{j\geq 0} \varepsilon_{t_j}, t_0(=0) < t_1 < t_2 < \cdots, \text{a.s.} \tag{3}$$

are $\mathcal{F}_A$-measurable and $\mathcal{F}_B$-measurable renewal point processes (ε_a is a point mass at a) with intensities λ_a and λ_b and position and point independent marking. The energy fuel level of each player becomes the shooting power to hit a opponent player. The energy fuel of player A drains at times $s_1, s_2, \ldots$ and the magnitudes of energy drains are formalized using process $\mathcal{A}$. The energy drain of player B is described by the process $\mathcal{B}$. The processes $\mathcal{A}$ and $\mathcal{B}$ are specified by their transforms:

$$\mathbb{E}\left[g^{\mathcal{A}(s)}\right] = e^{\lambda_a(s)(g-1)},\ \mathbb{E}\left[h^{\mathcal{B}(t)}\right] = e^{\lambda_b(t)(h-1)}. \tag{4}$$

The game is observed at random times, in accordance with the point process

$$\mathcal{T} := \sum_{i\geq 0} \varepsilon_{\tau_i},\ \tau_0(>0)), \tau_1, \ldots, \tag{5}$$

which is assumed to be delayed renewal process.

$$(A(t), B(t)) :=\ \mathcal{A} \otimes \mathcal{B}([0, \tau_k]),\ k = 0, 1, \ldots, \tag{6}$$

forms an observation process upon $\mathcal{A} \otimes \mathcal{B}$ embedded over $\mathcal{T}$, with respective increments

$$(X_k, Y_k) := \mathcal{A} \otimes \mathcal{B}([\tau_{k-1}, \tau_k]),\ k = 1, 2, \ldots, \tag{7}$$

and

$$X_0 = A_0, Y_0 = B_0. \tag{8}$$

The observation process can be formalized as

$$\mathcal{A}_\tau \otimes \mathcal{B}_\tau := \sum_{k\geq 0} (X_k, Y_k)\varepsilon_{\tau_k}, \tag{9}$$

where

$$\mathcal{A}_\tau = \sum_{i\geq 0} X_i\varepsilon_{\tau_i}, \mathcal{B}_\tau = \sum_{i\geq 0} Y_i\varepsilon_{\tau_i}, \tag{10}$$

with position-dependent marking and with X_k and Y_k being dependent on the notation

$$\Delta_k := \tau_k - \tau_{k-1},\ k = 0, 1, \ldots,\ \tau_{-1} = 0, \tag{11}$$

and

$$\sigma(z, \theta) = \mathbb{E}\Big[z^{(X_i - Y_i)^+} e^{-\theta\Delta_i}\Big], z > 0. \tag{12}$$

By using the double expectation, we have

$$\sigma(z, \theta) = \sigma(\theta + (\lambda_a - \lambda_b)(1 - z)) \tag{13}$$

where

$$\sigma(\theta) = \mathbb{E}\Big[e^{-\theta\Delta_i}\Big], \sigma_0(\theta) = \mathbb{E}\Big[e^{-\theta\tau_0}\Big]. \tag{14}$$

Let us consider the maximum energy levels of players M_a and M_b. The energy level of player A after draining fuel is $M_a - \mathcal{A}_\tau$, with $M_b - \mathcal{B}_\tau$ for player B from (9). The stochastic processes for the energy level of each player are as follows:

$$\{M_a - \mathcal{A}_\tau\} \otimes \{M_b - \mathcal{B}_\tau\} := (M_a, M_b) - \sum_{k\geq 0} (X_k, Y_k)\varepsilon_{\tau_k}, \tag{15}$$

and the game is over when the k-th observation epoch τ_k, the shooting power of player A, which is equivalent to the remaining energy level at the moment of the shooting being greater than the energy fuel level of player B:

$$M_a - \mathcal{A}_{\tau_k} \geq M_b - \mathcal{B}_{\tau_k}. \tag{16}$$

For further formalization of the game, the exit index can be defined as follows:

$$\nu := \inf\{k : A_k = A_0 + X_1 + \cdots + X_k \leq \Sigma_a\}, \tag{17}$$

$$\mu := \inf\{j : B_j = H_0 + Y_1 + \cdots + Y_j \leq \Sigma_b\} \tag{18}$$

where

$$\Sigma_a = M_a - (M_b - B_j), \Sigma_b = M_b - (M_a - A_k)$$

Since player A is assumed to win the game at time τ_ν, we will target the confined game from the point of view of player A. The passage time τ_ν is associated exit time from the confined game and the formula (15) will be modified as

$$\{M_a - \overline{\mathcal{A}_\tau}\} \otimes \{M_b - \overline{\mathcal{B}_\tau}\} := (M_a, M_b) - \sum_{k\geq 0}^{\nu} (X_k, Y_k)\varepsilon_{\tau_k}, \tag{19}$$

where the path of the game from $\mathcal{F}(\Omega) \cap \{A_\nu - B_\nu \leq |M_a - M_b|\} \cap \{\mathcal{T} \geq t^*\}$, which gives an exact definition of the model observed until τ_ν. The joint functional of the stochastic duel with the fuel limited ammunition is as follows:

$$\begin{aligned} \Phi_{\{\nu,\mu\}} &= \Phi_{\{\nu(\Sigma_a),\mu(\Sigma_b)\}}(\zeta, z_0, z_1, \theta_0, \theta_1) \\ &= \mathbb{E}\left[\zeta^\nu z_0^{\left(A_{\nu-1} - B_{\mu-1}\right)} z_1^{\left(A_\nu - B_\mu\right)} e^{-\theta_0\tau_{\nu-1}} e^{-\theta_1\tau_\nu} \mathbf{1}_{\left\{\nu-\mu\leq\frac{M_{ab}}{\sigma}\right\}} \mathbf{1}_{\left\{\nu\geq\frac{t^*}{\sigma}\right\}}\right], \end{aligned} \tag{20}$$

where

$$\{A_\nu - B_\mu \le |M_a - M_b|\} \cap \{\mathcal{T} \ge t^*\} \equiv \left\{\nu - \mu \le \frac{\mathcal{M}_{ab}}{\overline{\sigma}}\right\} \cap \left\{\nu \ge \frac{t^*}{\overline{\sigma}}\right\}, \tag{21}$$

$$\mathcal{M}_{ab} = |M_a - M_b|, \overline{\sigma} = \mathbb{E}[\Delta_k]. \tag{22}$$

The functional $\Phi_{\{\nu,\mu\}}$ in this model shall represent the status of both players upon the exit time τ_ν and the pre-exit time $\tau_{\nu-1}$. The pre-exit time is of particular interest because player A wants to predict not only her time for the highest chance, but also the moment for the next highest chance prior to this. The **Theorem 1** establishes an explicit formula for $\mathit{\Phi}_{\{\mu,\nu\}}$ and we abbreviate with (23)–(27):

$$\gamma(z,\theta) = \sigma(z,\theta), \gamma_0(z,\theta) = \sigma_0(z,\theta), \tag{23}$$
$$\phi_A(x,\theta) = \mathbb{E}\left[x^{X_k} e^{-\theta\Delta_j}\right] = \sigma(\theta - \lambda_a(1-x)), \tag{24}$$
$$\phi_A^0(x,\theta) = \mathbb{E}\left[x^{A_0} e^{-\theta\tau_0}\right] = \sigma_0(\theta - \lambda_a(1-x)), \tag{25}$$
$$\phi_B(y,\theta) = \mathbb{E}\left[y^{Y_j} e^{-\theta\Delta_j}\right] = \sigma(\theta - \lambda_b(1-y)), \tag{26}$$
$$\phi_B^0(y,\theta) = \mathbb{E}\left[y^{B_0} e^{-\theta\tau_0}\right] = \sigma_0(\theta - \lambda_b(1-y)). \tag{27}$$

The linear operators are defined as follows:

$$\mathcal{D}_{(p,q)}\,[f(p,q)](x,y) := (1-x)(1-y) \sum_{p\ge 0}\sum_{q\ge 0} f(p,q) x^p y^q, \tag{28}$$

then

$$f(p,q) = \mathfrak{D}_{(x,y)}^{(p,q)}\left[\mathcal{D}_{(p,q)}\,\{f(p,q)\}\right], \tag{29}$$

where $\{f(p,q)\}$ is a sequence, with the inverse

$$\mathfrak{D}_{(x,y)}^{(p,q)}(\bullet) = \begin{cases} \left(\frac{1}{p!\cdot q!}\right) \lim_{(x,y)\to 0} \frac{\partial^p \partial^q}{\partial x^p \partial y^q} \frac{1}{(1-x)(1-y)}(\bullet), & p \ge 0, q \ge 0, \\ 0, & \text{otherwise}. \end{cases} \tag{30}$$

Theorem 1. *The functional $\Phi_{\mu\nu}$ of the game on trace σ-algebra $\mathcal{F}(\Omega) \cap \{A_\nu - B_\nu \le |M_a - M_b|\} \cap \{\mathcal{T} \ge t^*\}$ satisfies the following formula:*

$$\mathit{\Phi}_{\{\mu,\nu\}} = \mathfrak{D}_{(x,y)}^{(M_a,M_b)}\left\{\frac{\zeta \mathit{\Gamma}_0 \gamma_1 (1-\phi_x)(\zeta\mathit{\Gamma})^{\left\lceil \frac{t^*}{\overline{\sigma}}\right\rceil}}{\phi_y^{\left\lfloor \frac{\mathcal{M}_{ab}}{\overline{\sigma}} \right\rfloor}(1-\zeta\mathit{\Gamma})}\right\}, \tag{31}$$

where

$$\mathit{\Gamma} = \gamma(z_0 z_1 x, \theta_0 + \theta_1), \tag{32}$$
$$\mathit{\Gamma}_0 = \gamma_0(z_0 z_1 x, \theta_0 + \theta_1), \tag{33}$$
$$\gamma_1 = \gamma(z_1, \theta_1), \tag{34}$$
$$\phi_x = \phi_A(x,0), \phi_y = \phi_B(y,0), \tag{35}$$

Proof. We find the explicit formula of the joint function $\mathit{\Phi}_{\{\mu,\nu\}}$ that starts from (19):

$$\Phi_{\{\nu,\mu\}} = \Phi_{\{\nu(\Sigma_a),\mu(\Sigma_b)\}}(\zeta, z_0, z_1, \theta_0, \theta_1)$$

$$= \sum_{j\ge 0} \sum_{k\ge 0} \zeta^j \mathbb{E}\Bigg[z_0^{(A_{\nu-1}-B_{\nu-1})} z_1^{(A_\nu - B_\nu)} e^{-\theta_0\tau_{\nu-1}} e^{-\theta_1\tau_\nu} \mathbf{1}_{\left\{j-k\le\left\lfloor\frac{\mathcal{M}_{ab}}{\overline{\sigma}}\right\rfloor\right\}} \mathbf{1}_{\left\{j\ge\left\lceil\frac{t^*}{\overline{\sigma}}\right\rceil\right\}} \mathbf{1}_{\{\nu=j,\mu=k\}}\Bigg]$$

and applying the operator $\mathcal{D}$ to random family $\left\{\mathbf{1}_{\{\nu(x)=p,\mu(y)=q\}} : x \geq 0\right\}$, we have

$$\mathcal{D}_{(p,q)}\left[\mathbf{1}_{\{\nu(x)=k,\mu(y)=j\}}\right](x,y) = \left(x^{A_{k-1}} - x^{A_k}\right)\left(y^{B_{j-1}} - y^{B_j}\right) \tag{36}$$

and, from previous research [1–5],

$$\begin{aligned}\Psi(x,y) &= \sum_{j\geq 0} \sum_{k\geq 0} \zeta^j \mathcal{D}_{(x,y)} \mathbb{E}\Big[z_0^{(A_{\nu-1}-B_{\nu-1})} z_1^{(A_\nu - B_\nu)} e^{-\theta_0\tau_{\nu-1}} e^{-\theta_1\tau_\nu} \\ &\qquad \mathbf{1}_{\left\{j-k\leq \left\lfloor \frac{\mathcal{M}_{ab}}{\sigma}\right\rfloor\right\}} \mathbf{1}_{\left\{j\geq \left\lceil \frac{t^*}{\sigma}\right\rceil\right\}} \mathbf{1}_{\{\nu=j,\mu=k\}}\Big] \\ &= \sum_{j\geq \left\lceil \frac{t^*}{\sigma}\right\rceil} \sum_{k\geq j-\left\lfloor \frac{\mathcal{M}_{ab}}{\sigma}\right\rfloor} \zeta^j \mathbb{E}\Big[(z_0 z_1)^{\left(A_{j-1}-B_{j-1}\right)} e^{-(\theta_0+\theta_1)\tau_{j-1}} z_1^{\left(X_j - Y_j\right)} e^{-\theta_1\Delta_j} \\ &\qquad x^{A_{j-1}}\left(1 - x^{X_j}\right) y^{B_{k-1}}\left(1 - y^{Y_k}\right)\mathbf{1}_{\{\nu=j,\mu=k\}}\Big] \\ &= \sum_{j\geq \left\lceil \frac{t^*}{\sigma}\right\rceil} L_{1j}L_{2j} \sum_{k\geq j-\left\lfloor \frac{\mathcal{M}_{ab}}{\sigma}\right\rfloor} L_{3jk}L_{4jk}.\end{aligned} \tag{37, 38}$$

where

$$L_{1j} = \zeta^j \mathbb{E}\left[(z_0 z_1)^{\left(A_{j-1}-B_{j-1}\right)}(xy)^{B_{j-1}} e^{-(\theta_0+\theta_1)\tau_{j-1}}\right] = \zeta^j \Gamma_0 \Gamma^{j-1}, \tag{39}$$

$$L_{2j} = \mathbb{E}\left[z_1^{\left(X_j-Y_j\right)} e^{-\theta_1\Delta_j}\left(1 - x^{X_j}\right)\right] = \gamma_1(1-\phi_x), \tag{40}$$

$$L_{3jk} = \mathbb{E}\left[y^{Y_j+Y_{j+1}+\cdots+Y_{k-1}}\right] = \phi_y^{k-j}, \tag{41}$$

$$L_{4jk} = \mathbb{E}\left[1 - y^{Y_k}\right] = 1 - \phi_y. \tag{42}$$

From (39)–(42), we have

$$\Psi(x,y) = \sum_{j\geq \left\lceil \frac{t^*}{\sigma}\right\rceil} \left(\zeta^j \Gamma_0 \Gamma^{j-1}\right)\left(\gamma_1(1-\phi_x)\right)\left(\phi_y^{-\left\lfloor \frac{\mathcal{M}_{ab}}{\sigma}\right\rfloor}\right). \tag{43}$$

Therefore,

$$\Psi(x,y) = \frac{\zeta\Gamma_0\gamma_1(1-\phi_x)(\zeta\Gamma)^{\left\lceil \frac{t^*}{\sigma}\right\rceil}}{\phi_y^{\left\lfloor \frac{\mathcal{M}_{ab}}{\sigma}\right\rfloor}(1-\zeta\Gamma)}. \tag{44}$$

From (29) and (44), finally, we have:

$$\Phi_{\{\mu,\nu\}} = \mathfrak{D}_{(x,y)}^{(M_a,M_b)}\left\{\frac{\zeta\Gamma_0\gamma_1(1-\phi_x)(\zeta\Gamma)^{\left\lceil \frac{t^*}{\sigma}\right\rceil}}{\phi_y^{\left\lfloor \frac{\mathcal{M}_{ab}}{\sigma}\right\rfloor}(1-\zeta\Gamma)}\right\}. \tag{31}$$

□

The functional $\Phi_{\{\nu,\mu\}}$ contains all decision making parameters regarding this standard stopping game. The information includes the optimal number of iterations of players (i.e., ν and μ), the best moments for shooting (τ_ν, τ_μ; *exit time*) and step prior to the best shooting times ($\tau_{\nu-1}$, $\tau_{\mu-1}$; *pre-exit* time). The information for player A from the closed functional is as follows:

$$\mathbb{E}[\nu] = \lim_{\zeta\to 1}\left(\frac{\partial}{\partial\zeta}\right)\Phi_{\{\nu,\mu\}}(\zeta,1,1,0,0), \tag{45}$$

$$\mathbb{E}[\tau_{\nu-1}] = \lim_{\theta\to 0}\left(-\frac{\partial}{\partial\theta}\right)\Phi_{\{\nu,\mu\}}(1,1,1,\theta,0). \tag{46}$$

Additionally, there are some special cases that can be considered. The first case is where the assets of both players are the same (i.e., $\mathcal{M}_{ab} = 0$). From (31), the formula is changed as follows:

$$\Phi_{\{\mu,\nu\}} = \mathfrak{D}_{(x,y)}^{(M_a,M_b)}\left\{\frac{\zeta\Gamma_0\gamma_1(1-\phi_x)(\zeta\Gamma)^{\left\lceil \frac{t^*}{\sigma}\right\rceil}}{1-\zeta\Gamma}\right\}, \tag{47}$$

where $\mathcal{M}_{ab} = 0$. This implication means that both players have the same energy fuel level and the winning strategy for this game would be to take the shot as soon as passing the threshold from (1). The other case would be where the initial energy level of player A is smaller than that of player B (i.e., $M_a < M_b$). The best strategy for player A would be to wait until player B shoots (and fails), because player A has no chance of winning, even if he hits the target correctly.

3. Conclusions

A new successor to the antagonistic stochastic duel game was studied. This research primarily focused on deriving compact closed-form solutions utilizing transformation and flexible analysis techniques, which were adapted by varying the concept of the energy fuel level. In this innovative duel game, a player can win the game only if their bullets hit the target player and if their shooting power exceeds the remaining energy fuel level of the target player at the moment of shooting. A joint functional of the standard stopping game was constructed to analyze the strategic decision parameters, which indicated the best moment for shooting in the time domain stochastic game. This study provides a thorough account of the innovative analytic approach, shedding light on the fundamental principles that underlie the stochastic duel game model with energy fuel constraints.

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References

1. Kim, S.-K. Blockchain Governance Game. *Comp. Indust. Eng.* **2019**, *136*, 373–380. [CrossRef]
2. Kim, S.-K. Strategic Alliance for Blockchain Governance Game. *Probab. Eng. Inf. Sci.* **2022**, *36*, 184–200. [CrossRef]
3. Kim, S.-K. Enhanced IoV Security Network by Using Blockchain Governance Game. *Mathematics* **2021**, *9*, 109. [CrossRef]
4. Kim, S.-K. Multi-Layered Blockchain Governance Game. *Axioms* **2022**, *11*, 27. [CrossRef]
5. Kim, S.-K. Advanced Drone Swarm Security by Using Blockchain Governance Game. *Mathematics* **2022**, *10*, 3338. [CrossRef]
6. Moschini, G. Nash equilibrium in strictly competitive games: Live play in soccer. *Econ. Lett.* **2004**, *85*, 365–371. [CrossRef]
7. Piergigli, D.; Ripamonti, L.A.; Maggiorini, D.; Gadia, D. Gadia, Deep Reinforcement Learning to train agents in a multiplayer First Person Shooter: Some preliminary results. In Proceedings of the 2019 IEEE Conference on Games, London, UK, 20–23 August 2019; pp. 1–8.
8. Peters, H. *Finite Games with Incomplete Information, Game Theory*; Springer: Berlin/Heidelberg, Germany, 2008; pp. 59–71.
9. Charilas, D.E.; Panagopoulos, A.D. A survey on game theory applications in wireless networks. *Comput. Netw.* **2010**, *54*, 3421–3430. [CrossRef]
10. Do, C.T.; Tran, N.H.; Hong, C.; Kamhoua, C.A.; Kwiat, K.A.; Blasch, E.; Ren, S.; Pissinou, N.; Iyengar, S.S. Game Theory for Cyber Security and Privacy. *ACM Comput. Surv.* **2017**, *50*, 1–37. [CrossRef]
11. Polak, B. Discussion of Duel, Open Yale Courses. 2008. Available online: http://oyc.yale.edu/economics/econ-159/lecture-16 (accessed on 1 March 2023).
12. Cardaliaguet, P.; Rainer, C. Stochastic Differential Games with Asymmetric Information. *Appl. Math. Optim.* **2009**, *59*, 1–36. [CrossRef]
13. Kim, S.-K. Antagonistic One-To-N Stochastic Duel Game. *Mathematics* **2020**, *8*, 1114. [CrossRef]
14. Kim, S.-K. Robust Pairwise n-Person Stochastic Duel Game. *Mathematics* **2021**, *9*, 825. [CrossRef]
15. Reny, P.J. On the Existence of Pure and Mixed Strategy Nash Equilibria in Discontinuous Games. *Econometrica* **1999**, *67*, 1029–1056. [CrossRef]
16. Kim, S.-K. A Versatile Stochastic Duel Game. *Mathematics* **2020**, *8*, 678. [CrossRef]
17. Merlevede, J.; Johnson, B.; Grossklags, J.; Holvoet, T. Time-Dependent Strategies in Games of Timing. *Lect. Notes Comput. Sci.* **2019**, *11836*, 310–330.
18. Park, A.; Smith, L. Caller Number Five and related timing games. *Theor. Econ.* **2008**, *3*, 231–256.
19. Reif, J.H. The complexity of two-player games of incomplete information. *J. Comput. Syst. Sci.* **1984**, *29*, 274–301. [CrossRef]

20. McCarty, N.; Meirowitz, A. *A Political Game Theory: An Introduction*; Cambridge University Press: Cambridge, MA, USA, 2007.
21. Tolon, V.; Martin, J.; Dray, S.; Loison, A.; Fischer, C.; Baubet, E. Predator-prey spatial game as a tool to understand the effects of protected areas on harvester-wildlife interactions. *Ecol. Appl.* **2012**, *22*, 648–657. [CrossRef] [PubMed]

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