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Fejér-Type Inequalities for Some Classes of Differentiable Functions

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Abstract: We let υ be a convex function on an interval $[\iota_1, \iota_2] \subset \mathbb{R}$. If $\zeta \in C([\iota_1, \iota_2])$, $\zeta \geq 0$ and ζ is symmetric with respect to $\frac{\iota_1+\iota_2}{2}$, then $\upsilon\left(\frac{1}{2}\sum_{j=1}^{2}\iota_j\right)\int_{\iota_1}^{\iota_2}\zeta(s)\,ds \leq \int_{\iota_1}^{\iota_2}\upsilon(s)\zeta(s)\,ds \leq \frac{1}{2}\sum_{j=1}^{2}\upsilon(\iota_j)\int_{\iota_1}^{\iota_2}\zeta(s)\,ds$. The above estimates were obtained by Fejér in 1906 as a generalization of the Hermite–Hadamard inequality (the above inequality with $\zeta \equiv 1$). This work is focused on the study of right-side Fejér-type inequalities in one- and two-dimensional cases for new classes of differentiable functions υ. In the one-dimensional case, the obtained results hold without any symmetry condition imposed on the weight function ζ. In the two-dimensional case, the right side of Fejer's inequality is extended to the class of subharmonic functions υ on a disk.

Keywords: Fejér inequality; Hermite–Hadamard inequality; convex functions; differentiable functions; subharmonic functions

MSC: 26D15; 26A51; 26B25

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1. Introduction

Fejér's result can be stated as follows: If $\upsilon, \zeta : [\iota_1, \iota_2] \to \mathbb{R}$, $\iota_1 < \iota_2$, where υ is a convex function and ζ is a nonnegative, continuous and symmetric function w.r.t. $z = \frac{1}{2}\sum_{j=1}^{2}\iota_j$. Then,

$$\int_{\iota_1}^{\iota_2}\upsilon(s)\zeta(s)\,ds \leq \frac{1}{2}\sum_{j=1}^{2}\upsilon(\iota_j)\int_{\iota_1}^{\iota_2}\zeta(s)\,ds \tag{1}$$

and

$$\upsilon\left(\frac{1}{2}\sum_{j=1}^{2}\iota_j\right)\int_{\iota_1}^{\iota_2}\zeta(s)\,ds \leq \int_{\iota_1}^{\iota_2}\upsilon(s)\zeta(s)\,ds. \tag{2}$$

The above result was obtained by Fejér [1] in 1906 as a generalization of the Hermite–Hadamard inequality [2,3], which is a special case of (1) and (2) with $\zeta \equiv 1$. The literature contains various results related to inequalities of type (1) and (2) for different classes of functions. Due to the large number of contributions in this topic, we are not able to cite all the related references. We just refer to the monographs: Dragomir and Pearce [4], Niculescu and Persson [5], as well as papers [6–18]. In particular, Dragomir et al. [10] considered the class of functions $\upsilon : \mathbb{J} \subset \mathbb{R} \to \mathbb{R}$ ($\mathbb{J}$ is an interval of $\mathbb{R}$) satisfying

$$|\upsilon(\jmath) - \upsilon(\imath)| \leq \kappa|\jmath - \imath|$$

for all $\imath, \jmath \in \mathbb{J}$, where $\kappa > 0$ is a constant. It was proven that if υ satisfies the above property and $\iota_1, \iota_2 \in \mathbb{J}$ with $\iota_1 < \iota_2$, then

$$\left|\int_{\iota_1}^{\iota_2}\upsilon(s)\,ds - \frac{1}{2}\sum_{j=1}^{2}\upsilon(\iota_j)\right| \leq \frac{\kappa}{3}(\iota_2 - \iota_1). \tag{3}$$

In [6], Abramovich and Persson established various Fejér-type inequalities for the class of k-quasiconvex functions, that is, the class of functions $\xi : \mathbb{J} \subset [0, +\infty[\to \mathbb{R}$ such that $\xi(s) = s^k v(s)$ and v is a convex function in $\mathbb{J}$. For instance, when $k = 1$, it was proven that, if $v, \zeta : [\iota_1, \iota_2] \to \mathbb{R}$, $0 \le \iota_1 < \iota_2$, where ζ is nonnegative, integrable, and symmetric w.r.t. $z = \frac{1}{2}\sum_{j=1}^{2} \iota_j$, and v is convex, then

$$\int_{\iota_1}^{\iota_2} \xi(s)\zeta(s)\, ds \le \frac{1}{2}\sum_{j=1}^{2} \xi(\iota_j) \int_{\iota_1}^{\iota_2} \zeta(s)\, ds - \int_{\iota_1}^{\iota_2} v'(s)(\iota_2 - s)(s - \iota_1)\zeta(s)\, ds, \tag{4}$$

where $\xi(s) = sv(s)$. We remark that if v is convex and increasing (so ξ is convex), then (4) is a refinement of (1).

A natural question is to ask whether it is possible to find other classes of functions for which inequalities (1), (3) and (4) hold without the symmetry condition imposed on the weight function ζ. Section 2 of this paper is devoted to the study of this question. Namely, we first introduce the set of functions $X_\zeta(\mathbb{J})$ so that, if $v \in X_\zeta(\mathbb{J})$, then (1) holds for all $\iota_1, \iota_2 \in I$ with $\iota_1 < \iota_2$. We also deduce some interesting consequences from the obtained result. Next, another set of functions $Y_\zeta(\mathbb{J})$ is introduced for which (4) holds for all $\iota_1, \iota_2 \in \mathbb{J}$ with $\iota_1 < \iota_2$. Finally, we introduce the set of functions $Z_\zeta^\alpha(\mathbb{J})$, $\alpha > 0$, for which a weighted version of (3) is established. In all the obtained results, no symmetry condition is imposed on the weight function ζ.

The Hermite–Hadamard inequality has also been studied in higher dimensions in various domains, see, e.g., [19–25]. For instance, Dragomir [20] considered the class of convex functions $v : \mathcal{D}_R \to \mathbb{R}$, where $R > 0$ and

$$\mathcal{D}_R = \left\{ z = (z_1, z_2) \in \mathbb{R}^2 : \sum_{j=1}^{2} z_j^2 \le R^2 \right\}. \tag{5}$$

For this class of functions, it was shown that

$$\int_{\mathcal{D}_R} v(x)\, dx \le \frac{R}{2} \int_{\partial\mathcal{D}_R} v(x)\, dS_x, \tag{6}$$

where

$$\partial\mathcal{D}_R = \left\{ z = (z_1, z_2) \in \mathbb{R}^2 : \sum_{j=1}^{2} z_j^2 = R^2 \right\}. \tag{7}$$

In Section 2 of this paper, a weighted version of (6) is obtained for the class of subharmonic functions $\Delta^+(\Omega)$.

We finish this section by fixing some notations that are used throughout this paper:

- $\mathbb{J}$: open interval of $\mathbb{R}$;
- $C(\mathbb{J})$: the space of (real-valued) continuous functions on $\mathbb{J}$;
- $C^1(\mathbb{J})$: the space of continuously differentiable functions on $\mathbb{J}$;
- $C^2(\mathbb{J})$: the space of twice continuously differentiable functions on $\mathbb{J}$;
- Ω: open subset of $\mathbb{R}^2$;
- $C^2(\Omega)$: the space of twice continuously differentiable functions on Ω;
- Δ: the Laplacian operator in $\mathbb{R}^2$;
- ∇: the gradient operator in $\mathbb{R}^2$;
- $(\cdot, \cdot)$: the inner product in $\mathbb{R}^2$;
- $\|\cdot\|$: the Euclidean norm in $\mathbb{R}^2$;
- $\mathcal{D}_R$, $R > 0$: see (5);
- $\partial\mathcal{D}_R$, $R > 0$: see (7).

2. Fejér-Type Inequalities on an Interval

2.1. The Set of Functions $X_\zeta(\mathbb{J})$

We define the set of functions $X_\zeta(\mathbb{J})$ as follows:

Definition 1. *Let $\zeta \in C(\mathbb{J})$. Function $v \in X_\zeta(\mathbb{J})$, if the following conditions hold:*

(i) $v \in C^1(\mathbb{J})$;

(ii) *for all $\iota_1, \iota_2 \in \mathbb{J}$ with $\iota_1 < \iota_2$, we have*

$$\zeta(\iota_2)(v(\iota_1) - v(\iota_2)) \geq -v'(\iota_2) \int_{\iota_1}^{\iota_2} \zeta(s)\, ds. \tag{8}$$

We provide below some examples of functions v and ζ such that $v \in X_\zeta(\mathbb{J})$.

Example 1. *Let $v \in C^1(\mathbb{J})$ be a convex function in $\mathbb{J}$. Then, by the characterization of convex functions (see, e.g., [5]), we have*

$$v(\iota_1) - v(\iota_2) \geq (\iota_1 - \iota_2)v'(\iota_2)$$

for every $\iota_1, \iota_2 \in \mathbb{J}$. This shows that v satisfies (8) with $\zeta \equiv 1$. Then, $v \in X_1(\mathbb{J})$.

Example 2. *Let $v \in C^1(\mathbb{J})$ and $\zeta = v'$. For all $\iota_1, \iota_2 \in \mathbb{J}$, we have*

$$\begin{aligned} \zeta(\iota_2)(v(\iota_1) - v(\iota_2)) &= -v'(\iota_2)(v(\iota_2) - v(\iota_1)) \\ &= -v'(\iota_2) \int_{\iota_1}^{\iota_2} v'(s)\, ds \\ &= -v'(\iota_2) \int_{\iota_1}^{\iota_2} \zeta(s)\, ds, \end{aligned}$$

which shows that $v \in X_{v'}(\mathbb{J})$.

Example 3. *Let $\zeta \in C(\mathbb{J})$ be a nonnegative and decreasing function. If $v \in C^1(\mathbb{J})$ is convex and nondecreasing, then $v \in X_\zeta(\mathbb{J})$. Namely, for all $\iota_1, \iota_2 \in \mathbb{J}$ with $\iota_1 < \iota_2$, we have*

$$\int_{\iota_1}^{\iota_2} \zeta(s)\, ds \geq \zeta(\iota_2)(\iota_2 - \iota_1),$$

which yields (since $v' \geq 0$)

$$v'(\iota_2) \int_{\iota_1}^{\iota_2} \zeta(s)\, ds \geq \zeta(\iota_2)(\iota_2 - \iota_1)v'(\iota_2),$$

that is,

$$-v'(\iota_2) \int_{\iota_1}^{\iota_2} \zeta(s)\, ds \leq \zeta(\iota_2)(\iota_1 - \iota_2)v'(\iota_2). \tag{9}$$

On the other hand, due to the convexity of v, we have

$$v(\iota_1) - v(\iota_2) \geq (\iota_1 - \iota_2)v'(\iota_2),$$

which yields (since $\zeta \geq 0$)

$$\zeta(\iota_2)(v(\iota_1) - v(\iota_2)) \geq \zeta(\iota_2)(\iota_1 - \iota_2)v'(\iota_2). \tag{10}$$

Thus, in view of (9) and (10), (8) is satisfied.

Example 4. *Let $v \in C^1(\mathbb{J})$ and $\zeta \in C(\mathbb{J})$. Assume that*

$$\zeta(\iota_2)v'(s) \leq v'(\iota_2)\zeta(s) \tag{11}$$

for all $\iota_2, s \in I$ with $s < \iota_2$. Then, $v \in X_\zeta(\mathbb{J})$. To show this, taking $\iota_1 \in \mathbb{J}$ with $\iota_1 < \iota_2$ and integrating (11) w.r.t. $s \in]\iota_1, \iota_2[$, we obtain

$$\zeta(\iota_2) \int_{\iota_1}^{\iota_2} v'(s)\, ds \leq v'(\iota_2) \int_{\iota_1}^{\iota_2} \zeta(s)\, ds,$$

that is,

$$\zeta(\iota_2)(v(\iota_2) - v(\iota_1)) \leq v'(\iota_2) \int_{\iota_1}^{\iota_2} \zeta(s)\, ds,$$

which shows that the pair of functions (v, ζ) satisfies (8).

Example 5. *Let $v \in C^1(\mathbb{J})$ and $\zeta \in C(\mathbb{J})$. Assume that $v' > 0$ in $\mathbb{J}$ and $\frac{\zeta}{v'}$ is a decreasing function in $\mathbb{J}$. Then, for all $\iota_2, s \in \mathbb{J}$ with $s < \iota_2$, we have*

$$\frac{\zeta(\iota_2)}{v'(\iota_2)} \leq \frac{\zeta(s)}{v'(s)},$$

that is,

$$\zeta(\iota_2)v'(s) \leq v'(\iota_2)\zeta(s).$$

Hence, by Example 4, it holds that $v \in X_\zeta(\mathbb{J})$.

We have the following Fejér-type inequality for the class of functions $v \in X_\zeta(\mathbb{J})$.

Theorem 1. *Let $\zeta \in C(\mathbb{J})$. For all $v \in X_\zeta(\mathbb{J})$, it holds that*

$$\int_{\iota_1}^{\iota_2} v(s)\zeta(s)\, ds \leq \frac{1}{2} \sum_{j=1}^{2} v(\iota_j) \int_{\iota_1}^{\iota_2} \zeta(s)\, ds, \quad (\iota_1, \iota_2) \in \mathbb{J}^2,\ \iota_1 < \iota_2. \tag{12}$$

Proof. Let $v \in X_\zeta(\mathbb{J})$ and $\iota_1, \iota_2 \in \mathbb{J}$ with $\iota_1 < \iota_2$. Then, for all $\iota_1 < t < \iota_2$, we have

$$\zeta(t)(v(\iota_1) - v(t)) \geq -v'(t) \int_{\iota_1}^{t} \zeta(s)\, ds.$$

Integrating w.r.t. $t \in]\iota_1, \iota_2[$, we obtain

$$v(\iota_1) \int_{\iota_1}^{\iota_2} \zeta(t)\, dt - \int_{\iota_1}^{\iota_2} v(t)\zeta(t)\, dt \geq -\int_{\iota_1}^{\iota_2} v'(t) \int_{\iota_1}^{t} \zeta(s)\, ds\, dt. \tag{13}$$

On the other hand, integrating by parts, we obtain

$$\begin{aligned} -\int_{\iota_1}^{\iota_2} v'(t) \int_{\iota_1}^{t} \zeta(s)\, ds\, dt &= -\left[v(t) \int_{\iota_1}^{t} \zeta(s)\, ds \right]_{t=\iota_1}^{\iota_2} + \int_{\iota_1}^{\iota_2} v(t)\zeta(t)\, dt \\ &= -v(\iota_2) \int_{\iota_1}^{\iota_2} \zeta(s)\, ds + \int_{\iota_1}^{\iota_2} v(t)\zeta(t)\, dt. \end{aligned}$$

Hence, in view of (13), we have

$$v(\iota_1) \int_{\iota_1}^{\iota_2} \zeta(t)\, dt - \int_{\iota_1}^{\iota_2} v(t)\zeta(t)\, dt \geq -v(\iota_2) \int_{\iota_1}^{\iota_2} \zeta(s)\, ds + \int_{\iota_1}^{\iota_2} v(t)\zeta(t)\, dt,$$

that is,

$$2\int_{\iota_1}^{\iota_2} \upsilon(t)\zeta(t)\,dt \le (\upsilon(\iota_1)+\upsilon(\iota_2))\int_{\iota_1}^{\iota_2} \zeta(t)\,dt,$$

which yields (12). □

Remark 1. *Let $\upsilon \in C^1(\mathbb{J})$ be a convex function in $\mathbb{J}$. Then, from Example 1, $\upsilon \in X_1(\mathbb{J})$. Hence, taking $\zeta \equiv 1$ in (12), we obtain the standard Hermite–Hadamard inequality.*

Corollary 1. *Let $\zeta \in C(\mathbb{J})$. For all $\upsilon \in C^1(\mathbb{J})$ satisfying*

$$\zeta(\iota_2)\upsilon'(\iota_1) \le \upsilon'(\iota_2)\zeta(\iota_1)$$

for all $\iota_1, \iota_2 \in \mathbb{J}$ with $\iota_1 < \iota_2$, (12) holds.

Proof. The result follows from Example 4 and Theorem 1. □

Corollary 2. *Let $\zeta \in C(\mathbb{J})$. For all $\upsilon \in C^1(\mathbb{J})$ satisfying*

(i) $\upsilon'(x) \neq 0, \quad x \in \mathbb{J}$,

(ii) $\dfrac{\zeta}{\upsilon'}$ *is a decreasing function in $\mathbb{J}$,*

(12) holds.

Proof. The result follows from Example 5 and Theorem 1. □

2.2. The Set of Functions $Y_\zeta(\mathbb{J})$

We define the set of functions $Y_\zeta(\mathbb{J})$ as follows:

Definition 2. *Let $\zeta \in C(\mathbb{J})$. Function $\upsilon \in Y_\zeta(\mathbb{J})$ if the following conditions hold:*

(i) $\upsilon \in C^1(\mathbb{J})$;

(ii) *for all $\iota_1, \iota_2 \in \mathbb{J}$ with $\iota_1 < \iota_2$, we have*

$$\zeta(\iota_1)\xi'(\iota_2) - \zeta(\iota_2)\xi'(\iota_1) \ge 2\int_{\iota_1}^{\iota_2} \upsilon'(s)\zeta(s)\,ds, \tag{14}$$

where $\xi(s) = s\upsilon(s)$.

Some examples of functions υ and ζ such that $\upsilon \in Y_\zeta(\mathbb{J})$ are given below.

Example 6. *Let $\mathbb{J} =]0, +\infty[$, $\zeta(s) = 1$. If $\upsilon \in C^1(\mathbb{J})$ is convex, then $\upsilon \in Y_\zeta(\mathbb{J})$. Namely, for all $\iota_1, \iota_2 \in \mathbb{J}$ with $\iota_1 < \iota_2$, we have*

$$\zeta(\iota_1)\xi'(\iota_2) - \zeta(\iota_2)\xi'(\iota_1) = \xi'(\iota_2) - \xi'(\iota_1),$$

that is,

$$\zeta(\iota_1)\xi'(\iota_2) - \zeta(\iota_2)\xi'(\iota_1) = \upsilon(\iota_2) - \upsilon(\iota_1) + \iota_2\upsilon'(\iota_2) - \iota_1\upsilon'(\iota_1). \tag{15}$$

On the other hand, we have

$$2\int_{\iota_1}^{\iota_2} \upsilon'(s)\zeta(s)\,ds = -2(\upsilon(\iota_1) - \upsilon(\iota_2)).$$

Next, by (15), we obtain

$$\zeta(\iota_1)\xi'(\iota_2) - \zeta(\iota_2)\xi'(\iota_1) - 2\int_{\iota_1}^{\iota_2} \upsilon'(s)\zeta(s)\,ds = \upsilon(\iota_1) - \upsilon(\iota_2) + \iota_2\upsilon'(\iota_2) - \iota_1\upsilon'(\iota_1),$$

which implies by the convexity of υ that

$$\begin{aligned}\zeta(\iota_1)\xi'(\iota_2) - \zeta(\iota_2)\xi'(\iota_1) - 2\int_{\iota_1}^{\iota_2} \upsilon'(s)\zeta(s)\,ds &\geq (\iota_1 - \iota_2)\upsilon'(\iota_2) + \iota_2\upsilon'(\iota_2) - \iota_1\upsilon'(\iota_1) \\ &= \iota_1(\upsilon'(\iota_2) - \upsilon'(\iota_1)) \\ &\geq 0.\end{aligned}$$

Example 7. *Let $\mathbb{J} =]0,1[$, $\zeta(s) = (1-s)e^{-s}$ and $\upsilon(s) = e^{-s}$. For all $\iota_1, \iota_2 \in \mathbb{J}$ with $\iota_1 < \iota_2$, we have*

$$\begin{aligned}\zeta(\iota_1)\xi'(\iota_2) - \zeta(\iota_2)\xi'(\iota_1) &= (1-\iota_1)e^{-\iota_1}\left(e^{-\iota_2} - \iota_2 e^{-\iota_2}\right) - (1-\iota_2)e^{-\iota_2}\left(e^{-\iota_1} - \iota_1 e^{-\iota_1}\right) \\ &= (1-\iota_1)(1-\iota_2)e^{-\iota_1}e^{-\iota_2} - (1-\iota_2)(1-\iota_1)e^{-\iota_2}e^{-\iota_1} \\ &= 0 \\ &\geq -2\int_{\iota_1}^{\iota_2} (1-s)e^{-2s}\upsilon'(s)\,ds \\ &= 2\int_{\iota_1}^{\iota_2} \upsilon'(s)\zeta(s)\,ds.\end{aligned}$$

This shows that $\upsilon \in Y_\zeta(\mathbb{J})$.

We have the following Fejér-type inequality for the class of functions $\upsilon \in Y_\zeta(\mathbb{J})$.

Theorem 2. *Let $\zeta \in C(\mathbb{J})$. For all $\upsilon \in Y_\zeta(\mathbb{J})$, it holds that*

$$\int_{\iota_1}^{\iota_2} \xi(s)\zeta(s)\,ds \leq \frac{1}{2}\sum_{j=1}^{2} \xi(\iota_j)\int_{\iota_1}^{\iota_2} \zeta(s)\,ds - \int_{\iota_1}^{\iota_2} \upsilon'(s)(\iota_2 - s)(s - \iota_1)\zeta(s)\,ds, \quad (\iota_1, \iota_2) \in \mathbb{J}^2,\ \iota_1 < \iota_2, \tag{16}$$

where $\xi(s) = s\upsilon(s)$.

Proof. Let $\upsilon \in Y_\zeta(\mathbb{J})$ and $\iota_1, \iota_2 \in \mathbb{J}$ with $\iota_1 < \iota_2$. Then, for all $\iota_1 < t < \iota_2$, we have

$$\zeta(t)\xi'(\iota_2) - \zeta(\iota_2)\xi'(t) \geq 2\int_t^{\iota_2} \upsilon'(s)\zeta(s)\,ds.$$

Integrating w.r.t. $t \in]\iota_1, \iota_2[$, we obtain

$$\xi'(\iota_2)\int_{\iota_1}^{\iota_2} \zeta(t)\,dt - \zeta(\iota_2)\int_{\iota_1}^{\iota_2} \xi'(t)\,dt \geq 2\int_{\iota_1}^{\iota_2}\int_t^{\iota_2} \upsilon'(s)\zeta(s)\,ds\,dt,$$

that is,

$$\xi'(\iota_2)\int_{\iota_1}^{\iota_2} \zeta(t)\,dt - \zeta(\iota_2)(\xi(\iota_2) - \xi(\iota_1)) \geq 2\int_{\iota_1}^{\iota_2}\int_t^{\iota_2} \upsilon'(s)\zeta(s)\,ds\,dt. \tag{17}$$

Integrating by parts, we obtain

$$\begin{aligned}2\int_{\iota_1}^{\iota_2}\int_t^{\iota_2} \upsilon'(s)\zeta(s)\,ds\,dt &= 2\left[t\int_t^{\iota_2} \upsilon'(s)\zeta(s)\,ds\right]_{t=\iota_1}^{\iota_2} + 2\int_{\iota_1}^{\iota_2} t\upsilon'(t)\zeta(t)\,dt \\ &= -2\iota_1\int_{\iota_1}^{\iota_2} \upsilon'(s)\zeta(s)\,ds + 2\int_{\iota_1}^{\iota_2} s\upsilon'(s)\zeta(s)\,ds \\ &= 2\int_{\iota_1}^{\iota_2} \upsilon'(s)(s - \iota_1)\zeta(s)\,ds.\end{aligned} \tag{18}$$

Then, (17) and (18) yield

$$\xi'(\iota_2)\int_{\iota_1}^{\iota_2} \zeta(s)\,ds - \zeta(\iota_2)(\xi(\iota_2) - \xi(\iota_1)) \geq 2\int_{\iota_1}^{\iota_2} \upsilon'(s)(s - \iota_1)\zeta(s)\,ds$$

for all $\iota_1, \iota_2 \in \mathbb{J}$ with $\iota_1 < \iota_2$. Hence, for all $\iota_1 < t < \iota_2$, it holds that

$$\xi'(t) \int_{\iota_1}^{t} \zeta(s)\, ds - \zeta(t)(\xi(t) - \xi(\iota_1)) \geq 2 \int_{\iota_1}^{t} v'(s)(s - \iota_1)\zeta(s)\, ds.$$

Integrating w.r.t. $t \in]\iota_1, \iota_2[$, we obtain

$$\int_{\iota_1}^{\iota_2} \xi'(t) \int_{\iota_1}^{t} \zeta(s)\, ds\, dt - \int_{\iota_1}^{\iota_2} \zeta(t)\xi(t)\, dt + \xi(\iota_1) \int_{\iota_1}^{\iota_2} \zeta(t)\, dt \geq 2 \int_{\iota_1}^{\iota_2} \int_{\iota_1}^{t} v'(s)(s - \iota_1)\zeta(s)\, ds\, dt. \tag{19}$$

Integrating by parts, it holds that

$$\begin{aligned} \int_{\iota_1}^{\iota_2} \xi'(t) \int_{\iota_1}^{t} \zeta(s)\, ds\, dt &= \left[\xi(t) \int_{\iota_1}^{t} \zeta(s)\, ds\right]_{t=\iota_1}^{\iota_2} - \int_{\iota_1}^{\iota_2} \xi(t)\zeta(t)\, dt \\ &= \xi(\iota_2) \int_{\iota_1}^{\iota_2} \zeta(s)\, ds - \int_{\iota_1}^{\iota_2} \xi(t)\zeta(t)\, dt \end{aligned} \tag{20}$$

and

$$\begin{aligned} 2\int_{\iota_1}^{\iota_2} \int_{\iota_1}^{t} v'(s)(s - \iota_1)\zeta(s)\, ds\, dt &= 2\left[t \int_{\iota_1}^{t} v'(s)(s - \iota_1)\zeta(s)\, ds\right]_{t=\iota_1}^{\iota_2} - 2\int_{\iota_1}^{\iota_2} t v'(t)(t - \iota_1)\zeta(t)\, dt \\ &= 2\iota_2 \int_{\iota_1}^{\iota_2} v'(s)(s - \iota_1)\zeta(s)\, ds - 2\int_{\iota_1}^{\iota_2} t v'(t)(t - \iota_1)\zeta(t)\, dt \\ &= 2\int_{\iota_1}^{\iota_2} v'(s)(s - \iota_1)(\iota_2 - s)\zeta(s)\, ds. \end{aligned} \tag{21}$$

Thus, it follows from (19)–(21) that

$$\begin{aligned} &\xi(\iota_2) \int_{\iota_1}^{\iota_2} \zeta(s)\, ds - \int_{\iota_1}^{\iota_2} \xi(t)\zeta(t)\, dt - \int_{\iota_1}^{\iota_2} \zeta(t)\xi(t)\, dt + \xi(\iota_1) \int_{\iota_1}^{\iota_2} \zeta(t)\, dt \\ &\geq 2\int_{\iota_1}^{\iota_2} v'(s)(s - \iota_1)(\iota_2 - s)\zeta(s)\, ds, \end{aligned}$$

that is,

$$2\int_{\iota_1}^{\iota_2} \xi(t)\zeta(t)\, dt \leq (\xi(\iota_1) + \xi(\iota_2)) \int_{\iota_1}^{\iota_2} \zeta(s)\, ds - 2\int_{\iota_1}^{\iota_2} v'(s)(s - \iota_1)(\iota_2 - s)\zeta(s)\, ds,$$

which proves (16). □

2.3. The Set of Functions $Z_\zeta^\alpha(\mathbb{J})$

We define the set of functions $Z_\zeta^\alpha(\mathbb{J})$ as follows:

Definition 3. *Let $\zeta \in C(\mathbb{J})$ and $\alpha > 0$. Function $v \in Z_\zeta^\alpha(\mathbb{J})$ if the following conditions hold:*

(i) $v \in C^1(\mathbb{J})$;
(ii) *for all $\iota_1, \iota_2 \in \mathbb{J}$ with $\iota_1 < \iota_2$, we have*

$$\left|\zeta(\iota_2)(v(\iota_2) - v(\iota_1)) - v'(\iota_2) \int_{\iota_1}^{\iota_2} \zeta(s)\, ds\right| \leq \alpha\zeta(\iota_2) \int_{\iota_1}^{\iota_2} \zeta(s)\, ds. \tag{22}$$

Some examples of functions v and ζ such that $v \in Z_\zeta^\alpha(\mathbb{J})$ for some $\alpha > 0$ are given below.

Example 8. *Let $\zeta(s) = 1$ and $v \in C^1(\mathbb{J})$ be such that*

$$0 < M_v := \sup_{s\in\mathbb{J}} |v'(s)| < +\infty. \tag{23}$$

Then, by the mean value theorem, for all $\iota_1, \iota_2 \in \mathbb{J}$ with $\iota_1 < \iota_2$, we have

$$\begin{aligned} \left|\zeta(\iota_2)(v(\iota_2) - v(\iota_1)) - v'(\iota_2)\int_{\iota_1}^{\iota_2} \zeta(s)\,ds\right| &= \left|(v(\iota_2) - v(\iota_1)) - v'(\iota_2)(\iota_2 - \iota_1)\right| \\ &= |v'(c) - v'(\iota_2)|(\iota_2 - \iota_1) \end{aligned}$$

for some $\iota_1 < c < \iota_2$. Hence, by (23), it holds that

$$\left|\zeta(\iota_2)(v(\iota_2) - v(\iota_1)) - v'(\iota_2)\int_{\iota_1}^{\iota_2} \zeta(s)\,ds\right| \le 2M_v(\iota_2 - \iota_1). \tag{24}$$

Moreover, we have

$$\zeta(\iota_2)\int_{\iota_1}^{\iota_2} \zeta(s)\,ds = (\iota_2 - \iota_1). \tag{25}$$

Then, it follows from (24) and (25) that

$$\left|\zeta(\iota_2)(v(\iota_2) - v(\iota_1)) - v'(\iota_2)\int_{\iota_1}^{\iota_2} \zeta(s)\,ds\right| \le 2M_v\zeta(\iota_2)\int_{\iota_1}^{\iota_2} \zeta(s)\,ds$$

for all $\iota_1, \iota_2 \in \mathbb{J}$ with $\iota_1 < \iota_2$. Hence, v and ζ satisfy (22) with $\alpha = 2M_v$. Consequently, we have $v \in Z_1^{2M_v}(\mathbb{J})$.

Example 9. *Let $\alpha > 0$, $\zeta \in C(\mathbb{J})$, $\zeta > 0$ and $v \in C^1(\mathbb{J})$ be such that*

$$\left|\frac{\zeta(\iota_1)v'(\iota_2) - \zeta(\iota_2)v'(\iota_1)}{\zeta(\iota_1)\zeta(\iota_2)}\right| \le \alpha \tag{26}$$

for all $\iota_1, \iota_2 \in \mathbb{J}$ with $\iota_1 < \iota_2$. Let us fix $\iota_1, \iota_2 \in \mathbb{J}$ with $\iota_1 < \iota_2$. By (26), for all $\iota_1 < t < \iota_2$, we have

$$-\alpha\zeta(t)\zeta(y) \le \zeta(t)v'(\iota_2) - \zeta(\iota_2)v'(t) \le \alpha\zeta(t)\zeta(\iota_2).$$

Integrating w.r.t. $t \in]\iota_1, \iota_2[$, we obtain

$$-\alpha\zeta(\iota_2)\int_{\iota_1}^{\iota_2} \zeta(t)\,dt \le v'(\iota_2)\int_{\iota_1}^{\iota_2} \zeta(t)\,dt - \zeta(\iota_2)\int_{\iota_1}^{\iota_2} v'(t)\,dt \le \alpha\zeta(\iota_2)\int_{\iota_1}^{\iota_2} \zeta(t)\,dt,$$

that is,

$$-\alpha\zeta(\iota_2)\int_{\iota_1}^{\iota_2} \zeta(t)\,dt \le v'(\iota_2)\int_{\iota_1}^{\iota_2} \zeta(t)\,dt - \zeta(\iota_2)(v(\iota_2) - v(\iota_1)) \le \alpha\zeta(\iota_2)\int_{\iota_1}^{\iota_2} \zeta(t)\,dt.$$

This shows that v and ζ satisfy (22). Then, $v \in Z_\zeta^\alpha(\mathbb{J})$.

Example 10. *Let $\zeta \in C(\mathbb{J})$. Assume that*

$$0 < \inf_{s\in\mathbb{J}} |\zeta(s)| := m_\zeta \le M_\zeta := \sup_{s\in\mathbb{J}} |\zeta(s)| < +\infty.$$

Let $v \in C^1(\mathbb{J})$ be such that

$$0 < M_v := \sup_{s\in\mathbb{J}} |v'(s)| < +\infty.$$

Then, for all $\iota_1, \iota_2 \in \mathbb{J}$, we have

$$\left|\frac{\zeta(\iota_1)v'(\iota_2) - \zeta(\iota_2)v'(x)}{\zeta(\iota_1)\zeta(\iota_2)}\right| \le \frac{1}{m_\zeta^2}(M_\zeta M_v + M_\zeta M_v) = \frac{2M_\zeta M_v}{m_\zeta^2}.$$

This shows that v and ζ satisfy (26) with $\alpha = \dfrac{2M_\zeta M_v}{m_\zeta^2}$. Hence, by Example 9, we have $v \in Z_\zeta^\alpha(\mathbb{J})$.

We have the following Fejér-type inequality for the class of functions $v \in Z_\zeta^\alpha(\mathbb{J})$.

Theorem 3. *Let $\zeta \in C(\mathbb{J})$ and $v \in Z_\zeta^\alpha(\mathbb{J})$ for some $\alpha > 0$. Then, it holds that*

$$\left|\int_{\iota_1}^{\iota_2} v(s)\zeta(s)\,ds - \frac{1}{2}\sum_{j=1}^{2} v(\iota_j)\int_{\iota_1}^{\iota_2}\zeta(s)\,ds\right| \le \frac{\alpha}{4}\left(\int_{\iota_1}^{\iota_2}\zeta(s)\,ds\right)^2, \quad (\iota_1,\iota_2) \in \mathbb{J}^2,\ \iota_1 < \iota_2. \tag{27}$$

Proof. Let $\iota_1, \iota_2 \in \mathbb{J}$ with $\iota_1 < \iota_2$. From (22), for all $\iota_1 < t < \iota_2$, we have

$$-\alpha\zeta(t)\int_{\iota_1}^{t}\zeta(s)\,ds \le \zeta(t)(v(t) - v(\iota_1)) - v'(t)\int_{\iota_1}^{t}\zeta(s)\,ds \le \alpha\zeta(t)\int_{\iota_1}^{t}\zeta(s)\,ds.$$

Integrating w.r.t. $t \in]\iota_1, \iota_2[$, it holds that

$$\left|\int_{\iota_1}^{\iota_2} v(t)\zeta(t)\,dt - v(\iota_1)\int_{\iota_1}^{\iota_2}\zeta(t)\,dt - \int_{\iota_1}^{\iota_2} v'(t)\int_{\iota_1}^{t}\zeta(s)\,ds\,dt\right| \le \alpha\int_{\iota_1}^{\iota_2}\zeta(t)\int_{\iota_1}^{t}\zeta(s)\,ds\,dt. \tag{28}$$

On the other hand, we have

$$\begin{aligned}\int_{\iota_1}^{\iota_2} v'(t)\int_{\iota_1}^{t}\zeta(s)\,ds\,dt &= \left[v(t)\int_{\iota_1}^{t}\zeta(s)\,ds\right]_{t=\iota_1}^{\iota_2} - \int_{\iota_1}^{\iota_2} v(t)\zeta(t)\,dt \\ &= v(\iota_2)\int_{\iota_1}^{\iota_2}\zeta(s)\,ds - \int_{\iota_1}^{\iota_2} v(t)\zeta(t)\,dt\end{aligned} \tag{29}$$

and

$$\begin{aligned}\int_{\iota_1}^{\iota_2}\zeta(t)\int_{\iota_1}^{t}\zeta(s)\,ds\,dt &= \left[\left(\int_{\iota_1}^{t}\zeta(s)\,ds\right)^2\right]_{t=\iota_1}^{\iota_2} - \int_{\iota_1}^{\iota_2}\zeta(t)\int_{\iota_1}^{t}\zeta(s)\,ds\,dt \\ &= \left(\int_{\iota_1}^{\iota_2}\zeta(s)\,ds\right)^2 - \int_{\iota_1}^{\iota_2}\zeta(t)\int_{\iota_1}^{t}\zeta(s)\,ds\,dt,\end{aligned}$$

that is,

$$\int_{\iota_1}^{\iota_2}\zeta(t)\int_{\iota_1}^{t}\zeta(s)\,ds\,dt = \frac{1}{2}\left(\int_{\iota_1}^{\iota_2}\zeta(s)\,ds\right)^2. \tag{30}$$

Then, it follows from (28)–(30) that

$$\left|2\int_{\iota_1}^{\iota_2} v(t)\zeta(t)\,dt - (v(\iota_1) + v(\iota_2))\int_{\iota_1}^{\iota_2}\zeta(t)\,dt\right| \le \frac{\alpha}{2}\left(\int_{\iota_1}^{\iota_2}\zeta(s)\,ds\right)^2,$$

which proves (27). □

Corollary 3. *Let $\alpha > 0$, $\zeta \in C(\mathbb{J})$, $\zeta > 0$ and $v \in C^1(\mathbb{J})$ be such that*

$$\left|\frac{\zeta(\iota_1)v'(\iota_2) - \zeta(\iota_2)v'(\iota_1)}{\zeta(\iota_1)\zeta(\iota_2)}\right| \le \alpha$$

for all $\iota_1, \iota_2 \in \mathbb{J}$ with $\iota_1 < \iota_2$. Then, (27) holds.

Proof. The result follows from Example 9 and Theorem 3. □

We now take $\zeta \equiv 1$ in Theorem 3. In this case, $v \in Z_\zeta^\alpha(\mathbb{J})$, $\alpha > 0$, means that

(i) $v \in C^1(\mathbb{J})$;
(ii) for all $\iota_1, \iota_2 \in \mathbb{J}$ with $\iota_1 < \iota_2$, we have

$$\left| \frac{v(\iota_2) - v(\iota_1)}{\iota_2 - \iota_1} - v'(\iota_2) \right| \leq \alpha.$$

By Theorem 3, we obtain the following result:

Corollary 4. *Let $v \in Z_1^\alpha(\mathbb{J})$ for some $\alpha > 0$. Then, it holds that*

$$\left| \frac{1}{\iota_2 - \iota_1} \int_{\iota_1}^{\iota_2} v(s)\, ds - \frac{1}{2} \sum_{j=1}^{2} v(\iota_j) \right| \leq \frac{\alpha}{4}(\iota_2 - \iota_1), \quad (\iota_1, \iota_2) \in \mathbb{J}^2,\ \iota_1 < \iota_2.$$

3. Fejér-Type Inequalities on a Disk

Let us denote by $\Delta^+(\Omega)$ the set of $C^2(\Omega)$-subharmonic functions v, that is,

(i) $v \in C^2(\Omega)$;
(ii) for all $z \in \Omega$, we have

$$\Delta v(z) \geq 0. \tag{31}$$

We have the following Fejér-type inequality for the class of functions $v \in \Delta^+(\Omega)$:

Theorem 4. *Let $\zeta : [0, +\infty[\to \mathbb{R}$ be a continuous and nonnegative function, and let $v \in \Delta^+(\Omega)$. Then, for all $R > 0$ with $\mathcal{D}_R \subset \Omega$, it holds that*

$$\int_{\mathcal{D}_R} v(x)\zeta(\|x\|)\, dx \leq \frac{1}{R} \int_0^R s\zeta(s)\, ds \int_{\partial \mathcal{D}_R} v(x)\, dS_x. \tag{32}$$

Proof. Let $R > 0$ with $\mathcal{D}_R \subset \Omega$. Let us introduce the function

$$\widetilde{\zeta}(s) = \int_s^R \tau\zeta(\tau)\, d\tau, \quad 0 \leq s \leq R.$$

Clearly, we have (since $\zeta \geq 0$)

$$\widetilde{\zeta}'(s) = -s\zeta(s) \leq 0 \tag{33}$$

and

$$\max_{0 \leq s \leq R} \widetilde{\zeta}(s) = \widetilde{\zeta}(0). \tag{34}$$

We also introduce the function

$$g(x) = -\int_{\|x\|}^R \frac{\widetilde{\zeta}(s) - \widetilde{\zeta}(0)}{s}\, ds, \quad x \in \mathcal{D}_R. \tag{35}$$

It follows from (34) that

$$g(x) \geq 0, \quad x \in \mathcal{D}_R. \tag{36}$$

Furthermore, since g is a radial function, that is,

$$g = g(r) := -\int_r^R \frac{\widetilde{\zeta}(s) - \widetilde{\zeta}(0)}{s}\, ds, \quad r = \|x\|,$$

for all $x \in \mathbb{R}^2$ with $0 < \|x\| < R$, we have

$$\begin{aligned} \Delta g(x) &= g''(r) + \frac{1}{r} g'(r) \\ &= \frac{r\widetilde{\zeta}'(r) - \widetilde{\zeta}(r) + \widetilde{\zeta}(0)}{r^2} + \frac{\widetilde{\zeta}(r) - \widetilde{\zeta}(0)}{r^2} \\ &= \frac{\widetilde{\zeta}'(r)}{r}, \end{aligned}$$

which implies by (33) that

$$\Delta g(x) = -\zeta(\|x\|), \quad 0 < \|x\| < R. \tag{37}$$

We also notice that by (35), we have

$$g(x) = 0, \quad x \in \partial \mathcal{D}_R. \tag{38}$$

On the other hand, making use of the Green's formula, we obtain

$$-\int_{\mathcal{D}_R} v(x) \Delta g(x)\, dx = \int_{\mathcal{D}_R} (\nabla v(x), \nabla g(x))\, dx - \int_{\partial \mathcal{D}_R} \left(\nabla g(x), \frac{x}{R} \right) v(x)\, dS_x. \tag{39}$$

Similarly, we have

$$-\int_{\mathcal{D}_R} g(x) \Delta v(x)\, dx = \int_{\mathcal{D}_R} (\nabla v(x), \nabla g(x))\, dx - \int_{\partial \mathcal{D}_R} \left(\nabla v(x), \frac{x}{R} \right) g(x)\, dS_x. \tag{40}$$

Hence, it follows from (39) and (40) that

$$\begin{aligned} -\int_{\mathcal{D}_R} v(x) \Delta g(x)\, dx = & \int_{\partial \mathcal{D}_R} \left(\nabla v(x), \frac{x}{R} \right) g(x)\, dS_x - \int_{\partial \mathcal{D}_R} \left(\nabla g(x), \frac{x}{R} \right) v(x)\, dS_x \\ & - \int_{\mathcal{D}_R} g(x) \Delta v(x)\, dx. \end{aligned} \tag{41}$$

Moreover, due to (38), we have

$$\int_{\partial \mathcal{D}_R} \left(\nabla v(x), \frac{x}{R} \right) g(x)\, dS_x = 0. \tag{42}$$

Since g is a radial function, for all $x \in \partial \mathcal{D}_R$, we have

$$\begin{aligned} \left(\nabla g(x), \frac{x}{R} \right) &= g'(r)|_{r=R} \\ &= \frac{\widetilde{\zeta}(r) - \widetilde{\zeta}(0)}{r}\Big|_{r=R} \\ &= \frac{\widetilde{\zeta}(R) - \widetilde{\zeta}(0)}{R} \\ &= -\frac{\widetilde{\zeta}(0)}{R}. \end{aligned} \tag{43}$$

Thus, from (37), (41)–(43), we deduce that

$$\int_{\mathcal{D}_R} v(x) \zeta(\|x\|)\, dx = \frac{\widetilde{\zeta}(0)}{R} \int_{\partial \mathcal{D}_R} v(x)\, dS_x - \int_{\mathcal{D}_R} g(x) \Delta v(x)\, dx.$$

Finally, due to (31) and since $g \geq 0$ by (36), the above inequality yields

$$\int_{\mathcal{D}_R} v(x) \zeta(\|x\|)\, dx \leq \frac{\widetilde{\zeta}(0)}{R} \int_{\partial \mathcal{D}_R} v(x)\, dS_x,$$

which is equivalent to (32). □

As a special case of Theorem 4, let us consider the weight function

$$\zeta(s) = s^k, \quad s \geq 0,$$

where $k \geq 0$. In this case, we have

$$\int_0^R s\zeta(s)\,ds = \int_0^R s^{k+1}\,ds = \frac{R^{k+2}}{k+2}.$$

Thus, from Theorem 4, we deduce the following result:

Corollary 5. *Let $v \in \Delta^+(\Omega)$. Then, for all $k \geq 0$ and $R > 0$ with $\mathcal{D}_R \subset \Omega$, it holds that*

$$\int_{\mathcal{D}_R} v(x)\|x\|^k\,dx \leq \frac{R^{k+1}}{k+2}\int_{\partial\mathcal{D}_R} v(x)\,dS_x. \tag{44}$$

Remark 2. *If $k = 0$, then* (44) *reduces to* (6).

4. Conclusions

New Féjer-type inequalities are established in one- and two-dimensional cases. In the one-dimensional case, three classes of functions are introduced, namely $X_\zeta(\mathbb{J})$, $Y_\zeta(\mathbb{J})$ and $Z_p^\alpha(\mathbb{J})$, where $\zeta \in C(\mathbb{J})$ and $\alpha > 0$. If $v \in X_\zeta(\mathbb{J})$, it is proven that the Fejér inequality (1) holds for all $\iota_1, \iota_2 \in \mathbb{J}$ with $\iota_1 < \iota_2$. If $v \in Y_\zeta(\mathbb{J})$, it is proven that the Abramovich–Persson inequality (4) holds for all $\iota_1, \iota_2 \in \mathbb{J}$ with $\iota_1 < \iota_2$. Next, a weighted version of Dragomir et al. inequality (3) is established for class of functions $v \in Z_\zeta^\alpha(\mathbb{J})$. In all the obtained results, no symmetry condition is imposed on weight function ζ. In the two-dimensional case, a weighted version of Dragomir inequality (6) is derived for the class of subharmonic functions.

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