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Exact Null Controllability of a One-Dimensional Wave Equation with a Mixed Boundary

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Abstract: In this paper, exact null controllability of one-dimensional wave equations in non-cylindrical domains was discussed. It is different from past papers, as we consider boundary conditions for more complex cases. The wave equations have a mixed Dirichlet–Neumann boundary condition. The control is put on the fixed endpoint with a Neumann boundary condition. By using the Hilbert Uniqueness Method, exact null controllability can be obtained.

Keywords: wave equation; non-cylindrical domain; exact null controllability

MSC: 35L05

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1. Introduction

Let $T > 0$. Define $\hat{Q}_T^k$ as a non-cylindrical domain on $\mathbb{R}^2$:

$$\hat{Q}_T^k = \Big\{(x,t) \in \mathbb{R}^2; 0 < x < \alpha_k(t) \ for\ all\ t \in (0,T)\Big\},$$

where

$$\alpha_k(t) = 1 + kt \quad k \in (0,1).$$

In this paper, we set

$$V(0,\alpha_k(t)) = \Big\{\varphi \in H^1(0,\alpha_k(t)); \varphi(\alpha_k(t)) = 0\Big\}, \ \ t \in [0,T].$$

We denote the conjugate space of $V(0,\alpha_k(t))$ with $[V(0,\alpha_k(t))]'$.

We study wave equation as follows:

$$\begin{cases} u_{tt} - u_{xx} = 0 & & in\ \hat{Q}_T^k, \\ u_x(0,t) = v, & u(\alpha_k(t),t) = 0 & on\ (0,T), \\ u(x,0) = u^0, & u_t(x,0) = u^1 & in\ (0,1), \end{cases} \tag{1}$$

where $v \in [H^1(0,T)]'$ is the control variable and u is the state variable. $(u^0,u^1) \in L^2(0,1) \times [V(0,1)]'$ is an any given initial value. The physical meaning of k is called the velocity of moving endpoint. By [1], we know that (1) has a unique wake solution u in the transposed sense.

Applications of control problems can be found everywhere in life; for example, in engineering practice and in science and technology. In modern mathematics, the distributed parameter energy control theory is an important branch. Control can be divided into exact control, null control and approximate control. In wave equations, we know that exact controllability is equivalent to null controllability.

In cylindrical domains, there are many studies on controllability of wave equations. However, not much work was performed on the wave equations defined in non-cylindrical domains ([1–14]). In [4], exact controllability was studied where the control is put on moving endpoints. In [5], exact controllability was discussed, and the system is as follows:

$$\begin{cases} u_{tt} - u_{yy} = 0 & & in\ \hat{Q}_T^k, \\ u(0,t) = v(t) & u(\alpha_k(t), t) = 0 & on\ (0,T), \\ u(y,0) = u^0(y) & u_t(y,0) = u^1(y) & in\ (0,1). \end{cases}$$

In [6,7], exact internal controllability was reviewed. We discuss one-dimensional wave equations with the Dirichlet–Neumann boundaries and the control is put on a fixed endpoint with the Neumann boundary condition. By performing the calculation directly in non-cylindrical domains, we obtain exact null controllability by using the Hilbert Uniqueness Method.

In Section 2, the definition of exact null controllability and some main theorems is provided. In Section 3, the dual system of system (1) by proving Theorem 2 can be obtained. In Section 4, by the nature of Hilbert's Uniqueness Method, we prove controllability of system (1) (Proof of Theorem 1).

2. Main Results and Preliminary Work

Definition 1. *Equation (1) is called null controllable at the time T, if for any given initial value*

$$(u^0, u^1) \in L^2(0,1) \times [V(0,1)]',$$

one can always find a control $v \in [H^1(0,T)]'$ such that solution u of (1) satisfies

$$u(T) = 0\,,\, u_t(T) = 0$$

in the transposed sense.

Remark 1. *If $\alpha_k(t)$ is a more general function that satisfies $0 < \alpha_k{}'(t) < 1$; then, it leads to the same conclusion as in this paper.*

We set controllability time as follows:

$$T_k^* = \frac{-1 + e^{\frac{2k(1+k)}{(1-k)^3}}}{k}.$$

The next theorem, Theorem 1, is the main proof of this paper (controllability).

Theorem 1. *In the sense of Definition 1, (1) is called exactly controllable at time T for any given $T > T_k^*$.*

In order to prove controllability, we prove observability of its dual system. The dual system of system (1) is as follows:

$$\begin{cases} z_{tt} - z_{xx} = 0 & & in\ \hat{Q}_T^k, \\ z_x(0,t) = 0, & z(\alpha_k(t), t) = 0 & on\ (0,T), \\ z(x,0) = z^0, & z_t(x,0) = z^1 & in\ (0,1), \end{cases} \tag{2}$$

where $(z^0, z^1) \in L^2(0,1) \times V(0,1)$ is any given initial values. System (2) has a unique weak solution (for details refer to [1]):

$$z \in C([0,T], L^2(0, \alpha_k(t))) \cap C^1([0,T], V(0, \alpha_k(t))).$$

Remark 2. *C is a positive constant. Its value may vary from position to position.*

Next, we give two important inequalities (observability).

Theorem 2. *When $T > T_k^*$, for any $(z^0, z^1) \in L^2(0,1) \times V(0,1)$, there exists a constant $C > 0$ such that the solution of (2) satisfies*

$$C(|z^1|^2_{L^2(0,1)} + |z^0|^2_{V(0,1)}) \leq \int_0^T \alpha_k(t)|z_t(0,t)|^2 dt \leq C(|z^1|^2_{L^2(0,1)} + |z^0|^2_{V(0,1)}) \tag{3}$$

3. Observability: Proof of Theorem 2

For $t \geq 0$, we give the definition of the energy equation of (2) as follows:

$$E(t) = \frac{1}{2}\int_0^{\alpha_k(t)} [|z_t(x,t)|^2 + |z_x(x,t)|^2]dx. \tag{4}$$

Meanwhile, we define

$$E_0 \triangleq E(0) = \frac{1}{2}\int_0^1 [\left|z_x^0(x)\right|^2 + \left|z^1(x)\right|^2]dx. \tag{5}$$

Lemma 1. *When $t \in [0,T]$, for any $(z^0, z^1) \in L^2(0,1) \times V(0,1)$, the solution z of (2) satisfies*

$$E(0) - E(t) = \frac{k(1-k^2)}{2}\int_0^t |z_x(\alpha_k(s),s)|^2 ds. \tag{6}$$

Proof. For any $0 < t \leq T$, multiplying $z_{tt} - z_{xx} = 0$ by $z_s(x,s)$ and integrating on $(0,t) \times (0, \alpha_k(s))$, we obtain

$$\begin{aligned} 0 &= \textstyle\int_0^t \int_0^{\alpha_k(s)} z_s(x,s)[z_{ss}(x,s) - z_{xx}(x,s)]dxds \\ &= \textstyle\frac{1}{2}\int_0^t \int_0^{\alpha_k(s)} [|z_s(x,s)|^2 + |z_x(x,s)|^2]_s dxds \\ &\quad -\textstyle\int_0^t \int_0^{\alpha_k(s)} [z_x(x,s)z_s(x,s)]_x dxds. \end{aligned} \tag{7}$$

Since

$$\alpha_k(s) = 1 + ks. \tag{8}$$

it is easy to check

$$\alpha_{k,s}(s) = k. \tag{9}$$

It follows from (7) that

$$\begin{aligned} 0 &= \textstyle\frac{1}{2}\int_0^{\alpha_k(t)} [|z_x(x,t)|^2 + |z_t(x,t)|^2]dx \\ &\quad -\textstyle\frac{1}{2}\int_0^1 [|z_t(x,0)|^2 + |z_x(x,0)|^2]dx \\ &\quad -\textstyle\frac{k}{2}\int_0^t [||z_x(\alpha_k(s),s)|^2 + z_s(\alpha_k(s),s)|^2]ds \\ &\quad -\textstyle\int_0^t z_s(\alpha_k(s),s)z_x(\alpha_k(s),s)ds \\ &\quad +\textstyle\int_0^t z_s(0,s)z_x(0,s)ds. \end{aligned} \tag{10}$$

Taking $z_x(0,t) = 0\ for\ any\ t \in [0,T]$, it holds that

$$z_x(0,s) = 0\ for\ any\ s \in [0,t]. \tag{11}$$

Therefore, we can conclude that

$$\begin{aligned} 0 = &\tfrac{1}{2}\int_0^{\alpha_k(t)} [|z_t(x,t)|^2 + |z_x(x,t)|^2]dx \\ &-\tfrac{1}{2}\int_0^1 [|z_t(x,0)|^2 + |z_x(x,0)|^2]dx \\ &-\tfrac{k}{2}\int_0^t [|z_s(\alpha_k(s),s)|^2 + |z_x(\alpha_k(s),s)|^2]ds \\ &-\int_0^t z_x(\alpha_k(s),s)z_s(\alpha_k(s),s)ds. \end{aligned} \tag{12}$$

Due to (8) and $z(\alpha_k(s),s) = 0$, we have

$$kz_x(\alpha_k(s),s) = -z_s(\alpha_k(s),s). \tag{13}$$

Therefore, with (4), (5), (12) and (13), we obtain

$$E(0) - E(t) = \frac{k(1-k^2)}{2}\int_0^t |z_x(\alpha_k(s),s)|^2 ds.$$

□

Lemma 2. *When $t \in [0,T]$, for any $(z^0,z^1) \in L^2(0,1) \times V(0,1)$, the solution zof (2) satisfies*

$$\begin{aligned} &(1-k^2)\int_0^t \alpha_k(s)|z_x(\alpha_k(s),s)|^2 ds \\ &= 2\int_0^{\alpha_k(t)} xz_t(x,t)z_x(x,t)dx - 2\int_0^1 xz_t(x,0)z_x(x,0)dx + 2\int_0^t E(s)ds. \end{aligned} \tag{14}$$

Proof. For any $0 < t \le T$, multiplying $z_{tt} - z_{xx} = 0$ by $2xz_x(x,s)$ and integrating on $(0,t) \times (0,\alpha_k(s))$, we can deduce that

$$\begin{aligned} 0 = &2\int_0^t \int_0^{\alpha_k(s)} xz_x(x,s)[z_{ss}(x,s) - z_{xx}(x,s)]dxds \\ &= -\int_0^t \int_0^{\alpha_k(s)} [x|z_s(x,s)|^2 + x|z_x(x,s)|^2]_x dxds \\ &+ 2\int_0^t \int_0^{\alpha_k(s)} [xz_s(x,s)z_x(x,s)]_s dxds \\ &+ \int_0^t \int_0^{\alpha_k(s)} [|z_s(x,s)|^2 + |z_x(x,s)|^2]dxds. \end{aligned} \tag{15}$$

Considering (4) and (9), it follows from (15) that

$$\begin{aligned} 0 = &2\int_0^t \frac{\partial}{\partial s}\int_0^{\alpha_k(s)} xz_s(x,s)z_x(x,s)dxds \\ &-2k\int_0^t z_s(\alpha_k(s),s)\alpha_k(s)z_x(\alpha_k(s),s)ds \\ &-\int_0^t [x|z_s(x,s)|^2 + x|z_x(x,s)|^2]\Big|_0^{\alpha_k(s)} ds + 2\int_0^t E(s)ds. \end{aligned} \tag{16}$$

Further, we can derive

$$\begin{aligned} 0 = &2\int_0^{\alpha_k(t)} z_t(x,t)xz_x(x,t)dx - 2\int_0^1 z_t(x,0)xz_x(x,0)dx \\ &-2k\int_0^t z_s(\alpha_k(s),s)\alpha_k(s)z_x(\alpha_k(s),s)ds \\ &-\int_0^t \alpha_k(s)[|z_s(\alpha_k(s),s)|^2 + |z_x(\alpha_k(s),s)|^2]ds \\ &+2\int_0^t E(s)ds. \end{aligned} \tag{17}$$

Combining (13), we see

$$\begin{aligned} &(1-k^2)\int_0^t |z_x(\alpha_k(s),s)|^2\alpha_k(s)ds \\ &= 2\int_0^{\alpha_k(t)} z_t(x,t)xz_x(x,t)dx \\ &-2\int_0^1 z_t(x,0)xz_x(x,0)dx + 2\int_0^t E(s)ds. \end{aligned} \tag{18}$$

□

Lemma 3. *When $t \in [0,T]$, for any $(z^0, z^1) \in L^2(0,1) \times V(0,1)$, the solution z of (2) satisfies*

$$k(1-k^2)\int_0^t s|z_x(\alpha_k(s),s)|^2 ds = -2tE(t) + 2\int_0^t E(s)ds.$$

Proof. For any $0 < t \le T$, multiplying $z_{tt} - z_{xx} = 0$ by $2sz_s(x,s)$ and integrating on $(0,t) \times (0, \alpha_k(s))$, we get

$$\begin{aligned} 0 &= 2\textstyle\int_0^t \int_0^{\alpha_k(s)} sz_s(x,s)[z_{tt}(x,s) - z_{xx}(x,s)]dxds \\ &= 2\textstyle\int_0^t \int_0^{\alpha_k(s)} [z_s(x,s)sz_x(x,s)]_x dxds \\ &\quad - \textstyle\int_0^t \int_0^{\alpha_k(s)} [s|z_s(x,s)|^2 + s|z_x(x,s)|^2]_s dxds \\ &\quad + \textstyle\int_0^t \int_0^{\alpha_k(s)} [|z_s(x,s)|^2 + |z_x(x,s)|^2]dxds. \end{aligned} \tag{19}$$

Considering (4) and (9), it is follows that

$$\begin{aligned} 0 = {} & 2\textstyle\int_0^t sz_x(\alpha_k(s),s)z_s(\alpha_k(s),s)ds - 2\int_0^t sz_x(0,s)z_s(0,s)ds \\ & -2tE(t) + k\textstyle\int_0^t s[|z_s(\alpha_k(s),s)|^2 + |z_x(\alpha_k(s),s)|^2]ds + 2\int_0^t E(s)ds. \end{aligned} \tag{20}$$

With (11) and (13), we have

$$k(1-k^2)\int_0^t s|z_x(\alpha_k(s),s)|^2 ds = -2tE(t) + 2\int_0^t E(s)ds. \tag{21}$$

□

Lemma 4. *When $t \in [0,T]$, for any $(z^0, z^1) \in L^2(0,1) \times V(0,1)$, the solution zof (2) satisfies*

$$\frac{1-k}{(1+k)(1+kt)}E(0) \le E(t) \le \frac{1+k}{(1-k)(1+kt)}E(0). \tag{22}$$

Proof. According to Lemmas 2 and 3, we can conclude that

$$\begin{aligned} &(1-k^2)\textstyle\int_0^t |z_x(\alpha_k(s),s)|^2 ds \\ &= 2\textstyle\int_0^{\alpha_k(t)} xz_x(x,t)z_t(x,t)dx \\ &-2\textstyle\int_0^1 xz_x(x,0)z_t(x,0)dx + 2tE(t). \end{aligned} \tag{23}$$

Combining Lemma 1, we have

$$\begin{aligned} &\tfrac{2}{k}E(0) + 2\textstyle\int_0^1 xz_x(x,0)z_t(x,0)dx \\ &= 2\textstyle\int_0^{\alpha_k(t)} z_t(x,t)xz_x(x,t)dx + \tfrac{2}{k}E(t) + 2tE(t). \end{aligned} \tag{24}$$

This follows from Cauchy's inequality:

$$\left| \int_0^{\alpha_k(t)} 2z_t(x,t)xz_x(x,t)dx \right| \le 2\alpha_k(t)E(t), \tag{25}$$

$$\left| \int_0^1 2xz_t(x,0)z_x(x,0)dx \right| \le 2E(0). \tag{26}$$

From (25) and (26), it follows from (24) that

$$-2E(0) + \frac{2}{k}E(0) \leq 2tE(t) + 2\alpha_k(t)E(t) + \frac{2}{k}E(t), \tag{27}$$

and

$$2E(0) + \frac{2}{k}E(0) \geq -2\alpha_k(t)E(t) + 2tE(t) + \frac{2}{k}E(t). \tag{28}$$

Therefore, we have

$$E(0) \leq \frac{1+k}{1-k}\alpha_k(t)E(t), \tag{29}$$

$$E(0) \geq \frac{1-k}{1+k}\alpha_k(t)E(t). \tag{30}$$

Hence, we see that (22) follows. □

Remark 3. *Lemma 4 implies that*

$$\frac{1-k}{1+k}(1+kT)E(0) \leq E(T) \leq \frac{1+k}{1-k}(1+kT)E(0). \tag{31}$$

We will give the proof of Theorem 2, which has three steps.

Proof of Theorem 2.

Step 1. Multiplying $z_{tt} - z_{xx} = 0$ by $(x - \alpha_k(t))z_x(x,t)$ and integrating on $\hat{Q}_T^k$, it follows that

$$\begin{aligned} 0 &= \int_0^T \int_0^{\alpha_k(t)} (x-\alpha_k(t))z_x(x,t)z_{tt}(x,t)dxdt \\ &\quad -\int_0^T \int_0^{\alpha_k(t)} (x-\alpha_k(t))z_x(x,t)z_{xx}(x,t)dxdt \\ &\triangleq J_1 - J_2. \end{aligned} \tag{32}$$

Next, we calculate $J_i (i = 1,2)$:

$$\begin{aligned} J_1 &= \int_0^T \int_0^{\alpha_k(t)} \tfrac{\partial}{\partial t}[z_t(x,t)(x-\alpha_k(t))z_x(x,t)]dxdt \\ &\quad +k\int_0^T \int_0^{\alpha_k(t)} z_x(x,t)z_t(x,t)dxdt \\ &\quad -\int_0^T \int_0^{\alpha_k(t)} (x-\alpha_k(t))z_{xt}(x,t)z_t(x,t)dxdt. \end{aligned} \tag{33}$$

Combining $\alpha_{k,t}(t) = k$, it follows that

$$\begin{aligned} J_1 &= [\int_0^{\alpha_k(t)} z_t(x,t)z_x(x,t)(x-\alpha_k(t))dx]\big|_0^T + k\int_0^T \int_0^{\alpha_k(t)} z_t(x,t)z_x(x,t)dxdt \\ &\quad -\tfrac{1}{2}\int_0^T \int_0^{\alpha_k(t)} (x-\alpha_k(t))\tfrac{\partial}{\partial x}(|z_t(x,t)|^2)dxdt \\ &= [\int_0^{\alpha_k(t)} z_t(x,t)(x-\alpha_k(t))z_x(x,t)dx]\big|_0^T + \int_0^T \int_0^{\alpha_k(t)} kz_x(x,t)z_t(x,t)dxdt \\ &\quad -\tfrac{1}{2}\int_0^T \alpha_k(t)|z_t(0,t)|^2dt + \tfrac{1}{2}\int_0^T \int_0^{\alpha_k(t)} |z_t(x,t)|^2dxdt. \end{aligned} \tag{34}$$

Calculating J_2, we get

$$\begin{aligned} J_2 &= \int_0^T \int_0^{\alpha_k(t)} (x-\alpha_k(t))\tfrac{\partial}{\partial x}(\tfrac{1}{2}|z_x(x,t)|^2)dxdt \\ &= \tfrac{1}{2}\int_0^T \alpha_k(t)|z_x(0,t)|^2dt - \tfrac{1}{2}\int_0^T \int_0^{\alpha_k(t)} |z_x(x,t)|^2dxdt. \end{aligned} \tag{35}$$

With $z_x(0,t) = 0$ *on* $(0,T)$, it is obvious that

$$J_2 = -\frac{1}{2}\int_0^T \int_0^{\alpha_k(t)} |z_x(x,t)|^2dxdt. \tag{36}$$

Therefore, with (34) and (36), we obtain

$$\begin{aligned}
&J_1 - J_2 \\
&= [\int_0^{\alpha_k(t)} (x - \alpha_k(t)) z_t(x,t) z_x(x,t) dx]\big|_0^T \\
&+ k\int_0^T \int_0^{\alpha_k(t)} z_x(x,t) z_t(x,t) dx dt \\
&- \tfrac{1}{2}\int_0^T \alpha_k(t) |z_t(0,t)|^2 dt + \tfrac{1}{2}\int_0^T \int_0^{\alpha_k(t)} |z_t(x,t)|^2 dx dt \\
&+ \tfrac{1}{2}\int_0^T \int_0^{\alpha_k(t)} |z_x(x,t)|^2 dx dt \\
&= 0.
\end{aligned} \tag{37}$$

Considering (4), it follows from (37) that

$$\begin{aligned}
&\tfrac{1}{2}\int_0^T \alpha_k(t) |z_t(0,t)|^2 dt \\
&= [\int_0^{\alpha_k(t)} z_x(x,t)(x - \alpha_k(t)) z_t(x,t) dx]\big|_0^T \\
&+ k\int_0^T \int_0^{\alpha_k(t)} z_x(x,t) z_t(x,t) dx dt + \int_0^T E(t) dt.
\end{aligned} \tag{38}$$

We have

$$\begin{aligned}
&\left| \int_0^{\alpha_k(t)} z_x(x,t)(x - \alpha_k(t)) z_t(x,t) dx \right| \\
&\le \tfrac{1}{2}\int_0^{\alpha_k(t)} [|z_t(x,t)|^2 + |z_x(x,t)|^2](\alpha_k(t) - x) dx \\
&\le \alpha_k(t) E(t).
\end{aligned} \tag{39}$$

This inequality implies that

$$\left| [\int_0^{\alpha_k(t)} z_x(x,t)(x - \alpha_k(t)) z_t(x,t) dx]\Big|_0^T \right| \le \alpha_k(T) E(T) + E(0), \tag{40}$$

$$\left| k\int_0^T \int_0^{\alpha_k(t)} z_t(x,t) z_x(x,t) dx dt \right| \le k\int_0^T E(t) dt. \tag{41}$$

Step 2. From (22), (31), (40) and (41), it follows from (38) that

$$\begin{aligned}
&\tfrac{1}{2}\int_0^T |z_t(0,t)|^2 \alpha_k(t) dt \\
&\ge \int_0^T E(t) dt - E(0) - k\int_0^T E(t) dt - \alpha_k(T) E(T) \\
&\ge [-\tfrac{1+k}{1-k} - 1 + \tfrac{(1-k)^2}{(1+k)k} \ln(1+kT)] E(0).
\end{aligned} \tag{42}$$

If $T > T_k^* = \frac{-1+e^{\frac{2k(1+k)}{(1-k)^3}}}{k}$, we have

$$-\frac{1+k}{1-k} - 1 + \frac{(1-k)^2}{(1+k)k} \ln(1+kT) > 0.$$

This implies that one can find a positive constant C to satisfy

$$\begin{aligned}
&\int_0^T |z_t(0,t)|^2 \alpha_k(t) dt \\
&\ge C[\ln(1+kT) \tfrac{(1-k)^2}{k(1+k)} - 1 - \tfrac{1+k}{1-k}](|z^0|^2_{V(0,\alpha_k(t))} + |z^1|^2_{L^2(0,\alpha_k(t))}).
\end{aligned}$$

Step 3. From (22), (31), (40) and (41), one concludes from (38) that

$$\begin{aligned}&\tfrac{1}{2}\int_0^T \alpha_k(t)|z_t(0,t)|^2dt\\ &\leq \alpha_k(T)E(T)+E(0)+k\int_0^T E(t)dt+\int_0^T E(t)dt\\ &\leq \tfrac{1+k}{1-k}E(0)+E(0)+(1+k)E(0)\int_0^T \tfrac{1+k}{(1-k)(1+kt)}dt\\ &\leq C[1+\tfrac{k+1}{-k+1}+\tfrac{(k+1)^2}{k(-k+1)}\ln(1+kT)](|z^0|^2_{V(0,1)}+|z^1|^2_{L^2(0,1)}).\end{aligned} \tag{43}$$

With (42) and (43), we get the desired result in Theorem 2. □

Remark 4.

$$T_0 \triangleq \lim_{k\to 0} T_k^* = \lim_{k\to 0}\frac{-1+e^{\frac{2k(k+1)}{(-k+1)^3}}}{k} = \lim_{k\to 0}\frac{\frac{2k(k+1)}{(-k+1)^3}}{k} = 2.$$

In the non-cylindrical domain $\hat{Q}_T^k$, for any time $T > T_0$, it is well known that (1) is controllable. However, T_k^* is not sharp.

4. Controllability: Proof of Theorem 1

We use Hilbert's Uniqueness Method to prove controllability. The specific proof is divided into three steps.

Step 1. Define linear operator $\Gamma : V(0,1)\times L^2(0,1) \to [V(0,1)]'\times L^2(0,1)$. We consider

$$\begin{cases}\xi_{tt}-\xi_{xx}=0 & in\ \hat{Q}_T^k,\\ \xi_x(0,t)=G_{z_t(0,t)},\ \xi(\alpha_k(t),t)=0 & on\ (0,T),\\ \xi(T)=\xi_t(T)=0 & in\ (0,1).\end{cases} \tag{44}$$

For any $\phi \in H^1(0,T)$, $G_{z_t(0,t)}$ is defined as:

$$\left\langle G_{z_t(0,t)},\phi\right\rangle_{((H^1(0,T))',H^1(0,T))} = \int_0^T z_t(0,t)\phi_t(t)dt. \tag{45}$$

We set

$$(\xi^0,\xi^1) \triangleq (\xi(x,0),\ \xi_t(x,0)) \in L^2(0,1)\times[V(0,1)]'.$$

We can conclude that

$$(z^0,z^1)\to(-\xi^0,\xi^1).$$

Therefore,

$$\left\langle \Gamma(z^0,z^1),(z^0,z^1)\right\rangle = \int_0^1 (\xi_t^1 z^0-\xi^0 z^1)dx.$$

Step 2. Multiplying $\xi_{tt}-\xi_{xx}=0$ by $z(x,t)$ and integrating on $\hat{Q}_T^k$, we can derive

$$\begin{aligned}0&=\int_0^T\int_0^{\alpha_k(t)} z(x,t)[-\xi_{xx}(x,t)+\xi_{tt}(x,t)]dxdt\\ &=-\int_0^T\int_0^{\alpha_k(t)}[z(x,t)\xi_x(x,t)-z_x(x,t)\xi(x,t)]_x dxdt\\ &\quad+\int_0^T\int_0^{\alpha_k(t)}[z(x,t)\xi_t(x,t)-z_t(x,t)\xi(x,t)]_t dxdt.\end{aligned}$$

From $\alpha_{k,t}(t) = k$, we get

$$\begin{aligned}
0 = &-\int_0^T [z(\alpha_k(t),t)\xi_x(\alpha_k(t),t) - z_x(\alpha_k(t),t)\xi(\alpha_k(t),t)]dt \\
&-k\int_0^T [z(\alpha_k(t),t)\xi_t(\alpha_k(t),t) - z_t(\alpha_k(t),t)\xi(\alpha_k(t),t)]dt \\
&+\int_0^T [z(0,t)\xi_x(0,t) - z_x(0,t)\xi(0,t)]dt \\
&-\int_0^1 z(x,0)\xi_t(x,0) - z_t(x,0)\xi(x,0)dx \\
&+\int_0^{\alpha_k(T)} z(x,T)\xi_t(x,T) - z_t(x,T)\xi(x,T)dx.
\end{aligned} \tag{46}$$

Based on the conditions:

$$\xi_t(T) = z_x(0,t) = \xi(T) = z(\alpha_k(t),t) = \xi(\alpha_k(t),t) = 0.$$

Part (46) can conclude that

$$\int_0^T G_{z_t(0,t)} z(0,t)dt = \int_0^1 z(x,0)\xi_t(x,0) - z_t(x,0)\xi(x,0)dt. \tag{47}$$

Combining (45), we derive

$$\int_0^T |z_t(0,t)|^2 dt = \int_0^1 [z(x,0)\xi_t(x,0) - z_t(x,0)\xi(x,0)]dt. \tag{48}$$

With Theorem 2, Γ is proved to be coercive and bounded. Further, combining with the definition of the Lax–Milgram Theorem, we are able to obtain that Γ is an isomorphic mapping.

Step 3. For any given initial value

$$(u^0, u^1) \in L^2(0,1) \times [V(0,1)]',$$

we can define

$$v(\cdot) = G_{z_t(0,\cdot)} \in (H^1(0,T))',$$

where z is the solution of (2). There exists z^0, z^1 satisfying

$$(z^0, z^1) = \Gamma^{-1}(-u^0, u^1).$$

By combining the definitions of Γ we get

$$\Gamma(z^0, z^1) = (-\xi^0, \xi^1),$$

where ξ is the solution of (44).

Therefore, the following equation holds:

$$(-\xi^0, \xi^1) = (-u^0, u^1).$$

Due to the uniqueness of (44) we can obtain

$$(u(x,T), u_t(x,T)) = (0,0).$$

Therefore, we complete the proof of exact null controllability of (1).

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