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Application of Wavelet Transform to Urysohn-Type Equations

V. Lukianenko *,†, M. Kozlova † and V. Belozub

Institute of Physics and Technology, V.I. Vernadsky Crimean Federal University, Simferopol 295007, Russia; kozlovamg@cfuv.ru (M.K.); belozubva@cfuv.ru (V.B.)
* Correspondence: lukyanenkova@cfuv.ru; Tel.: +7-978-7291655
† These authors contributed equally to this work.

Abstract: This paper deals with convolution-type Urysohn equations of the first kind. Finding a solution for such equations is an ill-posed problem. For it to be solved, regularization algorithms and the continuous wavelet transform are used. Similar to the Fourier transform, the continuous wavelet transform is applied to convolution-type equations (based on the Fourier and wavelet transforms) and to Urysohn equations with unknown shift. The wavelet transform is preferable for the cases with approximated right-hand sides and for type 1 equations. We demonstrated that the application of the wavelet transform to Urysohn-type equations with unknown shift translates into a solution of a nonlinear equation with an oscillating kernel. Depending on the availability of a priori information, a combination of regularization and iterative algorithms with the use of close equations are effective for solving convolution-type equations based on the continuous wavelet transform and Urysohn equation.

Keywords: convolution-type Urysohn equations of the first kind; ill-posed problem; regularization algorithms; the continuous wavelet transform

MSC: 45E10; 45G10; 65R20

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1. Introduction

Urysohn equations of convolution type and convolution-type equations based on the Fourier transform have already found wide application. The study of nonlinear Urysohn integral equations (NUIEs) begins with the classical work of P. Urysohn [1,2], and the basic results can be found in [3–5]. Historical development of the theory of nonlinear integral equations, and the contributions of Soviet mathematicians V. V. Nemitsky, M. A. Krasnoselsky, Ya. B. Rutitsky, and others, and the analysis of certain qualitative methods for solving Urysohn equations are covered in [6]. Examples of physical problems reduced to studying nonlinear integral equations with strong nonlinearities (combustion theory problems), as well as methods for studying equations with non-degenerate nonlinearities in Orlich spaces are demonstrated in [7].

Modern publications are mainly related to non-linear Urysohn equations of the second kind. For numerically solving such types of equations, different modifications of the Newton–Kantorovich quadrature method are mainly used [8–10]. For solving NUIEs of the second kind, decompositions by different polynomials are used, for example: Chebyshev's polynomials [11], Bernoulli's polynomials [12], and Bernstein's polynomials [13], which are combined with artificial neural networks.

In [14], to find a solution, the results of the fixed point in the complex space of the complex partial metric space are applied. The homotopy perturbation method (HPM) for solving NUIEs of the first kind is applied in [15]. The Adamian decomposition method is used in paper [16]. Haar wavelets are most commonly used in combination with approximation [17–19].

In [20,21], the authors analyze the problem of a solution reconstruction for Urysohn equations under the assumption that a priori information on the solution and additional information on the solution of close equations is known. Hence, two independent problems of processing and interpreting indirectly obtained data arise. Processing of experimental data is performed in order to extract the maximum valid information on real characteristics in reconstructed functions. However, only integral characteristics are practically observed, i.e., those that account for the total effect from all points of the observed object. Such characteristics are insensitive even to big changes in values that characterize the object, as long as these changes compensate each other. As a result, the task of interpretation involves solving the inverse problem $Az = u$. The direct problem involves measuring the object's integral characteristics, namely, the right-hand sides in Urysohn-type integral equations of the first kind, u, according to the given initial dependencies, z, which characterize the object. Information on monotonicity of the sought solution $z(s)$ makes it possible to reduce the problem of solving a first-kind nonlinear Urysohn equation to a linear convolution-type equation. Depending on the integral transform used (Fourier, Mellin, Laplace, or wavelet transform), the convolution of two functions is different, yet the solution procedure is mostly similar.

Boundary value problems reducible to the convolution-type equations, convolution-type integral equations (with two kernels, coupled, smooth transition, Wiener–Hopf, etc.), and their discrete analogues had been presented in monographs by Yu. I. Chersky, F. D. Gakhov [22,23], and other classical works [24,25]. However, scientific publications on the use of the continuous wavelet transform in solving convolution-type equations with respect to the Fourier and wavelet transforms are scarce. By analogy with the Fourier transform properties, here we provide the corresponding wavelet transform properties required to solve convolution-type equations. For convolution-type equations with continuous wavelet transformation, it is possible to summarize the results that are obtained for many classes of integral equations, such as convolution-type equations, singular integral equations, and boundary value problems by a Fourier transform, like in the works by F. D. Gakhov, Y. I. Chersky, M. G. Krein, G. S. Litvinchuk, N. I. Muskhelishvili, etc. The results for approximate solutions are based on the method, which was first proposed by J. I. Chersky and based on two proven theorems for abstract linear equations in abstract linear spaces [22]. Regarding the wavelet transform, we consider mainly the spaces $L_p(\mathbb{R})$ for $p = 1, 2$, although the results are still valid for $1 \leqslant p < \infty$ (given obvious corrections are made). We use the continuous wavelet transform (CWT) properties for the purpose of studying Urysohn equations and type of convolution.

The present study is a follow-up to a series of papers previously published by the authors [21,26].

Structure of article: Section 2 introduces the basic definition and statements for continuous wavelet transformation (CWT). The convolution CWT is presented in two modifications. The results on synthesis of mother wavelets and representing integral operators by convolution CWT are obtained, and also approximate solutions of the integral equation of the first kind with convolution CWT are found. A new perspective method for constructing approximate solutions, which is based on well-known solutions of close equations, is provided. In Section 3, the Urysohn equation with an unknown shift is transformed into the Urysohn equation with an oscillating kernel for which asymptotic methods are applicable. The case of the monotonic solution, which is transformed into a convolution-type equation with a difference kernel, has been selected. Using the proposed approach of searching for an approximate solution of NUIEs, iterative algorithms that can generalize the known solutions are constructed. An example of discretization NUIEs from gravimetry and iterative algorithms, which are based on Newton's method, are considered.

The article uses the following terms and symbols:

$L_p(\mathbb{R})$, $0 \leq \rho < \infty$—is the space of measurable functions with norm $\|f\|^p = \int_{\mathbb{R}} |f(t)|^p dt < \infty$;
$W_2^1(\mathbb{R}) = \{f(t) : f(t), f'(t) \in L_2(\mathbb{R})\}$;
$\mathcal{F}$, $\mathcal{F}^{-1}$—are the Fourier transform and its inverse (FT), respectively;

W, W^{-1}—are the Wavelet transform and its inverse (CWT), respectively;
$(Wf)(a,b) = F(a,b)$, $f(t) = (W^{-1}F)(t)$;
W_φ—is the mother wavelet transform with $\varphi_{a,b}(t) = a^{-\frac{1}{2}}\,\varphi\left(\frac{t-b}{a}\right)$
$*$—the convolution by Fourier transform;
#—the convolution by Wavelet transform (CWT);
NUIE—nonlinear Urysohn integral equation.

2. Convolution Function Description

2.1. Review (Additional Information), Results and Symbols

In line with the works [26–28], here we provide the information needed further on.

Definition 1. *The continuous wavelet transform (CWT) of function $f(t) \in L_2(\mathbb{R})$ with wavelet function $\varphi(t) \in L_1(\mathbb{R}) \bigcap L_2(\mathbb{R})$ is defined by formula*

$$(W_\varphi f)(a,b) = F_\varphi(a,b) = \int_{\mathbb{R}} f(t)\overline{\varphi_{a,b}}(t)dt = (f, \varphi_{ab})_{L_2}, \tag{1}$$

where $\varphi_{a,b}(t) = a^{-\frac{1}{2}}\,\varphi\left(\frac{t-b}{a}\right)$, $a \in \mathbb{R}_+$, $a \neq 0$, $b \in \mathbb{R}$, and $\varphi(t)$ meets the following condition:

$$C_\varphi = \int_{\mathbb{R}} \frac{|\Phi(\xi)|^2}{|\xi|} d\xi < \infty, \tag{2}$$

where $\Phi(\xi)$ is the Fourier transform (FT) of function $\varphi(t)$:

$$\Phi(\xi) = (\mathcal{F}\varphi)(\xi) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \varphi(t)e^{it\xi}dt$$

Wavelet function $\varphi(t)$ is called the basis (mother) wavelet.

The inverse FT is defined by the formula

$$\varphi(t) = (\mathcal{F}^{-1}\Phi)(t) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \Phi(\xi)e^{-it\xi}d\xi.$$

It follows from (2), that $\Phi(\xi) = (\mathcal{F}\varphi)(\xi)$ is continuous in the vicinity of point $\xi = 0$ and $\Phi(0) = 0$ (wavelets in which this condition is not met are also taken into account):

$$\Phi(0) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \varphi(t)dt = 0.$$

Let us consider the properties of the basis wavelet functions. Let us introduce the convolution of functions $k(t) \in L_1(\mathbb{R})$ and $f(t) \in L_2(\mathbb{R})$ for the FT:

$$h(t) = (k * f)(t) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} k(t-s)f(s)ds. \tag{3}$$

Here are given some properties of the continuous wavelet transform. Note, that the FT of function $\varphi_{a,b}(t)$:

$$\mathcal{F}\{\varphi_{a,b}(t)\}(\xi) = \mathcal{F}\left\{\frac{1}{\sqrt{a}}\varphi\left(\frac{t-b}{a}\right)\right\} = |a|^{\frac{1}{2}}e^{ib\xi}\,\Phi(a\xi).$$

Parseval's (Plancherel's) identity in continuous wavelet transform is also a basis one, similar to the FT theory.

Theorem 1. *Let functions $f, g \in L_2(\mathbb{R})$. Then we have*

$$\big((W_\varphi f)(a,b), (W_\varphi g)(a,b)\big)_{L_2(\mathbb{R}\times\mathbb{R})} = 2\pi C_\varphi (f,g)_{L_2(\mathbb{R})}, \tag{4}$$

where constant C_φ is defined by Formula (2).

The proof follows from the representation of the CWT via the FT and Parseval's equality based on the FT. In (4) we use the CWT property

$$(W_\varphi f)(a,b) = \int_{\mathbb{R}} f(t)\, \frac{1}{|a|^{\frac{1}{2}}}\, \overline{\varphi\left(\frac{t-b}{a}\right)} dt = (f, \varphi_{a,b}) = (F, \Phi_{ab}) =$$

$$= \int_{\mathbb{R}} F(\xi) \overline{|a|^{\frac{1}{2}}\, e^{ib\xi}\, \Phi(a\xi)}\, d\xi = |a|^{\frac{1}{2}} \int_{\mathbb{R}} F(\xi) \overline{\Phi(a\xi)}\, e^{-ib\xi} d\xi.$$

Using Parseval's equality, we find the inverse wavelet formula (inversion formula).

Theorem 2. *Let $f(t) \in L_2(\mathbb{R})$, then*

$$f(t) = ((W_\varphi^{-1} F_\varphi(a,b))(t) = \frac{1}{2\pi C_\varphi} \int_{\mathbb{R}} \int_{\mathbb{R}} (W_\varphi f)(a,b)\ \varphi_{ab}(t) \frac{da\, db}{|a|^2}.$$

Proof. Let $f(t) \in L_2(\mathbb{R})$, then we have

$$2\pi C_\varphi (f,g)_{L_2} = \int_{\mathbb{R}} \int_{\mathbb{R}} (W_\varphi f)(a,b)\ \overline{(W_\varphi g)(a,b)}\ \frac{da\, db}{|a|^2} =$$

$$= \int_{\mathbb{R}} \int_{\mathbb{R}} (W_\varphi f)(a,b) \overline{\int_{\mathbb{R}} g(t) \overline{\varphi_{a,b}(t)}\ dt}\ \frac{da\, db}{|a|^2} =$$

$$= \int_{\mathbb{R}} \int_{\mathbb{R}} \int_{\mathbb{R}} (W_\varphi f)(a,b)\ \varphi_{a,b}(t)\ \overline{g(t)}\ \frac{da\, db}{|a|^2}\ dt = \left(\int_{\mathbb{R}} \int_{\mathbb{R}} (W_\varphi f)(a,b)\ \varphi_{a,b}(t)\ \frac{da\, db}{|a|^2},\ g(t) \right)_{L_2},$$

which leads to the sought result. □

Theorem 3. *The wavelet transform produces an isomorphism of space $L_2(\mathbb{R})$ on $L_2(\mathbb{R}^2)$.*

Proof. The proof follows from Theorem 2 and formula

$$\|f\|^2_{L_2(\mathbb{R})} = \frac{1}{2\pi C_\varphi} \int_{\mathbb{R}} \int_{\mathbb{R}} |(W_\varphi f)(a,b)|^2 \frac{da\, db}{|a|^2},$$

which is based on Parseval's equality (4), if $\overline{f(t)} = g(t)$. □

2.2. Convolution of Functions Based on CWT

Let us consider the product of the wavelet transforms of functions $k(t)$ and $f(t)$ and find the inverse wavelet transform that results in the convolution of functions $k(t)$ and $f(t)$. When calculations include both types of convolution, we will indicate the convolution operation based on the CWT as "#" to differentiate it from symbol "$*$", widely employed when employing the FT. Let us consider several different representations for the convolution based on the CWT

$$(k\#f)(t) = W_\varphi^{-1}\{(W_\varphi k)(a,b) \cdot (W_\varphi f)(a,b)\} \equiv W_\varphi^{-1}\big(K_\varphi(a,b) F_\varphi(a,b)\big) =$$

$$= \frac{1}{2\pi C_\varphi} \int_{\mathbb{R}} \int_{\mathbb{R}} K_\varphi(a,b) F_\varphi(a,b) \varphi_{a,b}(t) \frac{da\, db}{|a|^2} =$$

$$= \frac{1}{2\pi C_\varphi} \int_{\mathbb{R}} \int_{\mathbb{R}} \left(\int_{\mathbb{R}} k(\tau) \overline{\varphi_{a,b}(\tau)} d\tau \right) \left(\int_{\mathbb{R}} f(s) \overline{\varphi_{a,b}(s)}\, ds \right) \varphi_{a,b}(t) \frac{da\, db}{|a|^2} =$$

$$= \frac{\sqrt{2\pi}}{2\pi C_\varphi} \int_{\mathbb{R}} \int_{\mathbb{R}} K(\xi) F(\zeta - \xi) \int_{\mathbb{R}} \overline{\Phi(a\xi)}\, \overline{\Phi(a(\zeta - \xi))}\; \Phi(a\zeta) \frac{da}{|a|^{\frac{1}{2}}}\, e^{-i\zeta t}\, d\xi\, d\zeta.$$

$$(k\#f)(t) = \int_{\mathbb{R}} \int_{\mathbb{R}} K(\xi) F(\zeta - \xi) Q(\xi, \zeta) e^{-i\zeta t} d\xi d\zeta, \tag{5}$$

where

$$Q(\xi, \zeta) = \frac{1}{\sqrt{2\pi} C_\varphi} \int_{\mathbb{R}} \overline{\Phi(a\xi)}\, \overline{\Phi(a(\zeta - \xi))}\, \Phi(a\zeta) \frac{da}{|a|^{\frac{1}{2}}}.$$

The result is the convolution expression via the FT $K(\xi), F(\xi)$ of functions $k(t), f(t)$. Next,

$$(k\#f)(t) = \int_{\mathbb{R}} \int_{\mathbb{R}} k(\tau) f(s) \left(\frac{1}{2\pi C_\varphi} \int_{\mathbb{R}} \int_{\mathbb{R}} \overline{\varphi_{a,b}(\tau)}\, \overline{\varphi_{a,b}(s)}\, \varphi_{a,b}(t) \frac{da\, db}{|a|^2} \right) d\tau\, ds =$$

$$= \int_{\mathbb{R}} \int_{\mathbb{R}} q(t, \tau, s) k(\tau) f(s) d\tau\, ds = \int_{\mathbb{R}} n(t, s)\, f(s)\, ds, \tag{6}$$

where

$$n(t, s) = \int_{\mathbb{R}} q(t, \tau, s)\, k(\tau)\, d\tau, \quad q(t, \tau, s) = \frac{1}{2\pi C_\varphi} \int_{\mathbb{R}} \int_{\mathbb{R}} \overline{\varphi_{a,b}(\tau)}\, \overline{\varphi_{a,b}(s)}\, \varphi_{a,b}(t) \frac{da\, db}{|a|^2}.$$

Let us consider the convolution of the wavelet transform. Let $F(a, b)$ be the wavelet transform of function $f(t)$ by means of wavelet $\varphi(t)$. Based on representation

$$(Wf)(a, b) = |a|^{\frac{1}{2}} \int_{\mathbb{R}} F(\xi) \overline{\Phi(a\xi)} e^{-ib\xi} d\xi = F(a, b),$$

$$(Wk)(a, b) = |a|^{\frac{1}{2}} \int_{\mathbb{R}} K(\xi) \overline{\Phi(a\xi)} e^{-ib\xi} d\xi = K(a, b),$$

the convolution formulas for the wavelet transforms may be obtained in the following forms

$$H(a, b) = (K * F)(a, b) = C \int_{\mathbb{R}} K(a, b - u) F(a, u) du =$$

$$= \frac{C}{|a|} \int_{\mathbb{R}} \left(\int_{\mathbb{R}} K(\xi) \overline{\Phi(a\xi)} e^{-i(b-u)\xi} d\xi \right) \left(\int_{\mathbb{R}} F(\eta) \overline{\Phi(a\eta)} e^{-iu\eta} d\eta \right) du =$$

$$= \frac{C}{|a|} \int_{\mathbb{R}} \int_{\mathbb{R}} K(\xi) F(\eta) \overline{\Phi(a\xi)}\, \overline{\Phi(a\eta)} \left(\int_{\mathbb{R}} e^{-i(b-u)\xi} e^{-iu\eta} du \right) d\xi d\eta =$$

$$= \frac{2\pi C}{|a|} \int_{\mathbb{R}} \int_{\mathbb{R}} K(\xi) F(\eta) \overline{\Phi(a\xi)}\, \overline{\Phi(a\eta)}\, e^{-ib\xi} \delta(\xi - \eta) d\eta d\xi =$$

$$= \frac{2\pi C}{|a|} \int_{\mathbb{R}} K(\xi) F(\xi) \overline{\Phi(a\xi)}\, \overline{\Phi(a\xi)}\, e^{-ib\xi} d\xi = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} k_\varphi(b - t) f_\varphi(t) dt = k_\varphi * f_\varphi, \tag{7}$$

where

$$k_\varphi(t) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} K(\xi)\, \overline{\Phi(a\xi)}\, e^{-it\xi} d\xi = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} k(t - s) \frac{1}{|a|}\, \overline{\varphi}\left(\frac{s}{a}\right) ds.$$

2.3. Synthesis of Mother Wavelets, Representing Integral Operators by Convolution CWT

Synthesis of mother wavelets of specified properties occurs in many applications. In [29] is proposed the modified algorithm of synthesis of CWT with guaranteed accuracy of approximation to the overlap match.

Lemma 1. *If function $\varphi(t)$ is a mother wavelet and $\psi(t) \in L_1(\mathbb{R})$, then the convolution of functions $(\varphi * \psi)(t) = h(t)$ is a mother wavelet.*

The proof is given in [23].

Lemma 2. *Selecting mother wavelets. As a rule, mother wavelets are defined based on the problem stated. Well-chosen basis wavelets simplify finding a solution for a problem. According to Lemma 1, a transition from one wavelet to another is possible via employing a convolution operation.*

Let us consider the following problem. It is required to match a mother wavelet, $\varphi(t)$, with another mother wavelet, $h(t)$, using the convolution operator $\varphi * \psi = h$. This means that it is necessary to specify a function that is the kernel of the convolution integral operator, i.e., to solve the convolution integral equation of the first kind. Generally, such a solution can be obtained only in an approximate form. For this purpose, by means of solving a convolution-type equation in Fourier images, we obtain the following algebraic equation: $H(\xi) = \Psi(\xi)\Phi(\xi)$ and $|\Phi(\xi)| \to 0,\ |\xi| \to \infty$.

On using the first-order regularizer, we obtain a solution in Fourier images [30]:

$$\Psi_\alpha(\xi) = R_\alpha(\xi)H(\xi), \quad R_\alpha(\xi) = \frac{\overline{\Phi(\xi)}}{\alpha(1+\xi^2) + |\Phi(\xi)|^2},$$

$$\psi(t) = \psi_\alpha(t) = (\mathcal{F}^{-1}\Psi_\alpha)(t).$$

The regularization parameter $\alpha > 0$ depends on the error level of the convolution operator and the right-hand side, $h(t)$. On the other hand, by means of selecting different functions, $\psi(t)$, it is possible to obtain mother wavelets, $h(t)$, that possess a required set of properties, i.e., $h(t)$ can be found as a result of solving a direct problem.

Lemma 3. *Representing a convolution operator based on the CWT in the form (6)*

$$(k\#f)(t) = \int_{\mathbb{R}} n(t,s)f(s)ds$$

allows to represent certain integral operators

$$(Af)(t) = \int_{\mathbb{R}} n(t,s)f(s)ds$$

in the form of a convolution $k\#f$ based on the CWT. We can use such representation if it is possible for the kernel $n(t,s)$ to define functions

$$q(t,\tau,s) = \frac{1}{2\pi C_\varphi} \int_{\mathbb{R}} \int_{\mathbb{R}} \overline{\varphi_{a,b}(\tau)}\ \overline{\varphi_{a,b}(s)}\ \varphi_{a,b}(t) \frac{da\,db}{|a|^2} \tag{8}$$

and function $k(\tau)$ such that

$$(Qk)(t,s) = \int_{\mathbb{R}} q(t,\tau,s)\ k(\tau)\ d\tau = n(t,s), \tag{9}$$

i.e., to solve an integral equation of the first kind $Qk = n$ (9).

Lemmas 1–3 allow us to vary in a wide range mother wavelets and integral equations with convulation $k\#f$ depending on the nature of the application problems. The stage of transformation of mother wavelets and equations is associated with the solution of ill-posed problems for equations of the first kind. A special case, without having to solve an integral equation of the first kind, is given in Lemma 4.

In the case of defining a mother wavelet, $\varphi_{a,b}(t)$, function q is defined by Formula (8) and $k(\tau)$ is obtained by solving the integral Equation (9). To obtain $k(\tau)$ we can also use the representation (5) (via the FT). Approximate solutions of the integral equation of the first kind (9) obtained using the regularization method with a priori information are applicable to iterative algorithms and close solution theorems proposed earlier in the paper [21].

If instead of one parent function, $\varphi_{a,b}(t)$, we take different $\varphi_{a,b}$, $\psi_{a,b}$, $\theta_{a,b}$

$$F_\psi(a,b) = (W_\psi f)(a,b), \quad K_\theta(a,b) = (W_\theta k)(a,b),$$

$$H_\varphi(a,b) = (W_\varphi h)(a,b) = W_\varphi(k\#f)(a,b),$$

then similarly to Formula (6), we obtain

$$(k\#f)(t) = \frac{1}{2\pi C_\varphi}\int_{\mathbb{R}}\int_{\mathbb{R}} (W_\theta k)(a,b)(W_\varphi f)(a,b)\varphi_{a,b}(t)|a|^{-2}da\, db = \int_{\mathbb{R}} n(t,s)f(s)ds,$$

$$(Qk)(t,s) = \int_{\mathbb{R}} q(t,\tau,s)\, k(\tau)\, d\tau = n(t,s)$$

$$q(t,\tau,s) = \frac{1}{2\pi C_\varphi}\int_{\mathbb{R}}\int_{\mathbb{R}} \overline{\theta_{a,b}(\tau)}\ \overline{\psi_{a,b}(s)}\ \varphi_{a,b}(t)|a|^{-2}da\, db.$$

The wavelet transform of function $q(t,\tau,s)$ is found by the formula:

$$(W_\varphi q(t,\tau,s))(a,b) = \overline{\theta_{a,b}(\tau)}\ \overline{\psi_{a,b}(s)} \tag{10}$$

This representation has more arbitrariness in finding the function $k(\tau)$ with respect to $n(t,s)$ in Equation (9) $(Qk)(t,s) = n(t,s)$.

Lemma 4. *If function $n(t,s) = m(t)\ \overline{\varphi_{a,b}(s)}$, then $k(\tau) = m(\tau)$.*

Proof. Indeed, applying a CWT to equation $(Qk)(t,s) = n(t,s)$ with respect to variable t and taking into account (10) for $\theta_{a,b}(\tau) = \varphi_{a,b}(\tau)$, we obtain

$$(W_\varphi Qk)(a,b,s) = \int_{\mathbb{R}} \overline{\varphi_{a,b}(\tau)}\ \overline{\varphi_{a,b}(s)}\ k(\tau)\, d\tau = (W_\varphi n)(a,b,s)$$

or

$$\overline{\varphi_{a,b}(s)}(W_\varphi k)(a,b) = (W_\varphi n)(a,b,s),$$

$$(W_\varphi k)(a,b) = \int_{\mathbb{R}} n(t,s)\overline{\varphi_{a,b}(t)}\, dt\, \left[\overline{\varphi_{a,b}(s)}\right]^{-1} = \int_{\mathbb{R}} m(t)\overline{\varphi_{a,b}(t)}\, dt = (W_\varphi m)(a,b).$$

Hence, the statement of the lemma. □

2.4. Equations of the First Kind Based on the Wavelet Transform

The problem of solving a convolution-type equation based on the CWT

$$Az \equiv k\#z = u \tag{11}$$

is an ill-posed one. In the simplest case, the regularization algorithm corresponds to the solution of the extremal problem for A. N. Tikhonov's functional

$$J(z(\cdot)) = \alpha\|z\|^2_{L_2} + \|k\#z - u\|^2_{L_2} \to \inf,$$

where α is the regularization parameter and is defined by the error level of the kernel and the right-hand side $\eta = (h,\delta)$. Here,

$$(k\#z)(t) = \int_{\mathbb{R}}\int_{\mathbb{R}} u(t,\tau,s)k(\tau)\, d\tau\, z(s)\, ds = \int_{\mathbb{R}} n(t,s)z(s)\, ds$$

Let us find the Fréchet derivative of the functional

$$J_1(z) = \|k\#z - u\|^2_{L_2} = (k\#z - u,\ k\#z - u),$$

$$\big(J_1'(z)\big)h = \frac{C_\varphi^{-1}}{2\pi}\big(K_\varphi(a,b)Z_\varphi(a,b) - U_\varphi(a,b), K_\varphi(a,b)H_\varphi(a,b)\big)_{L_2(\mathbb{R}^2)}.$$

The necessary conditions for the minimum of the initial A. N. Tikhonov's functional $J(f)$ leads us to the equation

$$\Big[\alpha + \big|K_\varphi(a,b)\big|^2\Big] Z_\varphi(a,b) = \overline{K_\varphi}(a,b)U_\varphi(a,b).$$

It follows that

$$Z_\varphi(a,b) = \overline{K_\varphi}(a,b)\Big[\alpha + \big|K_\varphi(a,b)\big|^2\Big]^{-1} U_\varphi(a,b) = R_\alpha(a,b)U_\varphi(a,b),$$

or

$$z_\alpha(t) = W_\varphi^{-1}\{R_\alpha(a,b)U_\varphi(a,b)\}(t) = (r_\alpha\#u)(t). \tag{12}$$

In contrast to the CWT solution of a convolution-type equation with respect to the FT, the solution of the convolution-type Equation (11) is expressed through the wavelet transform of the kernel and the right-hand side. From Parseval's equality (4) and the theory of regularization methods [30] and the given calculations follows:

Theorem 4. *Let D be a closed convex set of a priori constraints of the problem (e.g., $D = W_2' \subset L_2$), A is a one-to-one operator, $\overline{u} = A\overline{z}$, $\overline{z}$ is an exact solution (11), $\overline{z} \in D$. Approximate solution $z_\eta^{\alpha(\eta)}$ belongs to set D on a given set (A_h, u_δ, η), $\eta = (\delta, h)$, where δ is an error in the right-hand side of Equation (11), $\|A - A_h\| \leqslant h$, $h \geqslant 0$, $\|u_\delta - \overline{u}\| \leqslant \delta$, then $z_\eta^{\alpha(\eta)} \to \overline{z}$ with $\eta \to 0$ so that $(\eta + \delta)^2/\alpha(\eta) \to 0$. The solution can be selected according to the generalized discrepancy principle (e.g., by Formula (12) for $z \in L_2$).*

Regularization methods, which are based on minimization of regularizing functionals or on solutions of Euler equations corresponding to the necessary conditions of functional extremes, are used to solve the first kind of integral equations. If a problem with a good structure or a good solution algorithm is well known for one problem, then this problem can be used to find solutions to the studied problem. As closely related problems, we will take similar regularizing functionals or regularized equations. For proving, a modification of the results from [22] can be used.

Theorem 5. *Let the approximate solution of the equation $\mathbb{K}z = u$ be sought based on the solution of the equation $\tilde{\mathbb{K}}\tilde{z} = \tilde{u}$. Let us assume that*

1. *the equation $\tilde{\mathbb{K}}\tilde{z} = \tilde{u}$ has an unique solution $\tilde{z}$;*
2. *$u - \tilde{u} \in Y_0$, $Y_0 \subset Y$, Y_0 is a linear subset Y;*
3. *operator $\mathbb{K} - \tilde{\mathbb{K}}$ acts from X in X_0, X_0 — Banach space;*
4. *in Y_0 reverse operator is determined $\tilde{\mathbb{K}}^{-1}$, which is acting from Y_0 in X_0;*
5. *operator $\tilde{\mathbb{K}}^{-1}(\mathbb{K} - \tilde{\mathbb{K}})$ is limited in X_0 with norm*

$$\|\tilde{\mathbb{K}}^{-1}(\mathbb{K} - \tilde{\mathbb{K}})\| < 1, \tag{13}$$

then the equation $\mathbb{K}z = u$ has an unique solution such as

$$z = \tilde{z} + [I + \tilde{\mathbb{K}}^{-1}(\mathbb{K} - \tilde{\mathbb{K}})]^{-1}\tilde{\mathbb{K}}^{-1}(u - \mathbb{K}\tilde{z}). \tag{14}$$

It is clear, that

$$z - \tilde{z} \in X_0 \subset X \quad \textit{and} \quad \mathbb{K}\tilde{z} - u \in Y_0.$$

Error estimation is equal

$$\|z - \tilde{z}\|_{X_0} \leq \frac{\|\tilde{\mathbb{K}}^{-1}(u - \mathbb{K}\tilde{z}\|_{X_0}}{1 - \|\tilde{\mathbb{K}}^{-1}(\mathbb{K} - \tilde{\mathbb{K}})\|}. \tag{15}$$

The proof is based on simple transformation of elements and operators.

The scheme of Theorem (11) is applicable to regularized equations of convolution type CWT [11]. For $X_0 \in L_2$ or W_2^1, $X \in L_2$, $Y \in L_2$, $Y_0 \in L_2$

$$\mathbb{K}z = \alpha z + k\#z = u,$$

$$\tilde{\mathbb{K}}\tilde{z} = \tilde{\alpha}\tilde{z} + \tilde{k}\#\tilde{z} = \tilde{u}$$

conditions of the theorem are satisfied, and the solution can be found by (14) with error estimation (15). An estimate (13) follows from

$$\max |\frac{(\alpha - \tilde{\alpha}) + \mathbb{K}(a,b) - \tilde{\mathbb{K}}(a,b)}{\tilde{\alpha} + \tilde{\mathbb{K}}(a,b)}| < 1,$$

where $\mathbb{K}(a,b) = (Wk)(a,b)$, $\tilde{\mathbb{K}}(a,b) = (Wk)(a,b)$.

A parameter of regularization $\tilde{\alpha}$ depends on the right part of $\tilde{u}$ and on error $\eta = (\delta, h)$ of the operator $\tilde{\mathbb{K}}$ and is optimal. Parameter α may not to be very small and no special procedure is needed to find it. The proposed approach provides a wide perspective for constructing approximate solutions. In the article by Lukianenko V. A. [31], this method was applied for approximate solutions for normally solvable smooth transition equations and corresponding boundary value Carleman's problem from the theory of analytical functions.

3. Urysohn-Type Equations

Reduction to an Equation with an Oscillating Kernel

Urysohn equations of the first kind emerge in various applied problems [21]. Depending on the availability of a priori information, the model's structure, and classes of functions, a variety of approaches are used to obtain approximate solutions based on asymptotic and regularization methods [20,21,26,32].

Let us consider a model of an Urysohn equation for the remote sensing problem in the following form

$$\int_{\mathbb{R}} f(s)\, n(t - z(\xi - s))ds = u(t, \xi), \quad t \in \mathbb{R}, \quad \xi \in \mathbb{R}. \tag{16}$$

The kernel of equation $n(t)$ in a number of models is a delta-like function; the unknown function $z(s)$ specifies the time it takes for the impulse to travel the double distance from the observation point to the target surface and back; the function $f(s)$ can be known or sought (in the case it describes the parameters of the impulse reflection from the surface). For the uniqueness of the solution, it is necessary to have a system of such equations or to register the reflected signal in the course of the movement of the observation point along a given trajectory.

In the case of a given function, $z(s)$, and the right-hand side, $u(t, \xi)$, the function $f(s)$ is obtained by solving a linear equation of the first kind. If $z(s)$ and $f(s)$ are defined, $u(t, \xi)$ is obtained as a result of solving the direct problem. The classification of nonlinearities of the function $z(s)$ with respect to the right-hand side, $u(t, \xi)$, can be indirectly determined as a result of solving the spectral problem for the linear with respect to the f operator.

Theorem 6. *Urysohn Equation (16) can be represented in the form of an integral equation with an oscillating kernel*

$$\int_{\mathbb{R}} f(\xi - s)\, e^{iwz(s)} ds = V(w, \xi).$$

Indeed, let us apply the FT with respect to the variable t to the Urysohn Equation (16). Based on the relationship

$$\mathcal{F}\{n(t-z(\xi-s))\} = \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} n(t-z(\xi-s))e^{iwt}dt = e^{iwz(\xi-s)}N(w),$$

we obtain the Urysohn equation with an oscillating kernel

$$\int_{\mathbb{R}} f(s)\, e^{iwz(\xi-s)}ds = N^{-1}(w)U(w,\xi) \equiv V(w,\xi).$$

Since $N(w)\to 0,\ |w|\to\infty$, we should employ a regularization method, i.e., represent $V(w,\xi)$ in the form

$$V(w,\xi) = \frac{\overline{N(\xi)}\,U(w,\xi)}{\alpha(1+\xi^2)+|N(\xi)|^2},$$

where the selection of regularization parameter α depends on the error level of the right-hand side and the kernel (see Lemma 2).

Depending on the domain of integration, we also consider integral operators (in a two-dimensional domain, on a finite interval) in the form

$$(A_w f)(\xi) = \int\limits_{\Omega} f(s)\, e^{iwg(\xi,s)}ds.$$

Of interest is calculating the spectrum $\sigma(A_w)$ for such operators. For example, for the one-dimensional case $|s|\leqslant 1$ and $w\gg 1$, the spectrum can be found as a result of solving the problem:

$$(A_w f)(t) = \int\limits_{-1}^{1} f(s)\, e^{iwz(t-s)}ds = \lambda f(t), \quad |t|\leqslant 1.$$

The spectrum depends on the unknown function $z(t)$. Yet, given (λ_j, f_j) for the known $z(t)$ and w, we have the right-hand side $u(t) = u_j \equiv \lambda_j f_j$. And, inversely, based on the set (λ_j, f_j) and the right-hand side we can assume the qualitative behavior of the function $z(t)$. This requires finding the most characteristic functions, $z = z_j(t)$, and their corresponding right-hand sides, $u = u_j(t)$. For $w\gg 1$, let us employ asymptotic analysis to the integral operators A_w [20,33].

Let us consider the case of the CWT application for solving the simplest version of the Urysohn equation

$$Az \equiv \int_{\mathbb{R}} f(s)\, n(t-z(s))ds = u(t), \quad t\in\mathbb{R}. \tag{17}$$

As the CWT can be represented via the FT, we obtain the result of Theorem 6. Indeed, let us use the wavelet transform

$$(W_\varphi A)(a,b) = \int_{\mathbb{R}} f(s)\, W_\varphi\{n(t-z(s))\}ds = U(a,b).$$

Since

$$W_\varphi\{n(t-z(s))\}(a,b) = |a|^{\frac{1}{2}}\int_{\mathbb{R}}\mathcal{F}\{n(t-z(s))\}(\xi)\bar{\Phi}(a\xi)e^{-ib\xi} = |a|^{\frac{1}{2}}\int_{\mathbb{R}} N(\xi)\, e^{i\xi z(s)}\bar{\Phi}(a\xi)e^{-ib\xi}ds,$$

we obtain

$$|a|^{\frac{1}{2}}\int_{\mathbb{R}}\left(\int_{\mathbb{R}} f(s)e^{i\xi z(s)ds}\right)N(\xi)\bar{\Phi}(a\xi)e^{-ib\xi} = |a|^{\frac{1}{2}}\int_{\mathbb{R}} U(\xi)\overline{\Phi(a\xi)}e^{-ib\xi}ds.$$

It follows that

$$\int_{\mathbb{R}} f(s)e^{i\xi z(s)}ds = N^{-1}(\xi)U(\xi), \quad N(\xi) \neq 0$$

or in regularized form with respect to the kernel $N(\xi)$

$$\int_{\mathbb{R}} f(s)e^{i\xi z(s)}ds = V(\xi) \equiv \frac{\overline{N(\xi)}U(\xi)}{\alpha + |N(\xi)|^2},$$

where α is the regularization parameter.

In this way, we obtain a nonlinear Urysohn-type equation with an oscillating kernel. Although there is no particular gain compared with the FT, the use of the wavelet transform in the case of numerical calculations on finite intervals makes it possible to eliminate high-frequency interferences (errors).

Lemma 5. *If function $z(s)$ monotonically increases on the segment $[a, b]$ and*

$$\alpha = \min z(s) \leqslant z(s) \leqslant \max z(s) = \beta,$$

then Equation (17) on making replacements $\tau = z(s)$, $s = \varphi(\tau)$ takes the form

$$\int_{\alpha}^{\beta} f(\varphi(t))n(t-\tau)\varphi'(\tau)d\tau = u(t), \quad \alpha \leqslant t \leqslant \beta$$

For $n(t) = \delta(t)$ (the delta function), we obtain an ordinary differential equation

$$f(\varphi(\tau))\varphi'(t) = u(t)$$

or

$$f(s) = u(z)z'(s), \quad \int_{\alpha}^{z} u(z)dz = \int_{a}^{s} f(s)ds + C.$$

In areas of monotony, the solution can be used as an initial approximation for regularized iterative processes. In the case when the function $z(s)$ has several extremum points (minimum and maximum), the problem of finding extremum points and other characteristic (stationary, «brilliant») points is considered.

The contribution of monotony sections to the right side of the equation is summed up, which complicates the search for a solution. To restore the solution, several equations are needed that allow you to localize the solution. In applied problems (restoration of surface areas using antenna devices), the antenna radiation pattern corresponds to the core of the equation. The radiation pattern localizes the surface area from which the pulse probing signal is reflected. The presence of several channels (or the procedure of movement over the surface of the antenna device) allows you to restore the surface more accurately.

Let us consider an Urysohn equation analogue with respect to the CWT:

$$(Az)(t) \equiv \int_{\mathbb{R}} n(t - z(s), s)\, f(s)\, ds = u(t), \quad t \in \mathbb{R}. \tag{18}$$

Remark 1. *In some cases, the integral operator A can be presented as*

$$(k\#f) = \int_{\mathbb{R}} n(t,s)f(s)\, ds$$

is a convolution operation with respect to the CWT and

$$n(t,s) = \int_{\mathbb{R}} q(t,\tau,s)k(\tau)\, d\tau.$$

Let us apply the CWT with respect to variable t to Equation (18)

$$(W_\varphi A)(a,b) = (W_\varphi u)(a,b).$$

$$W_\varphi\{n(t-z(s),s)\}(a,b) = |a|^{\frac{1}{c}} \int_{\mathbb{R}} N(\xi,s) e^{i\xi z(s)} \overline{\Phi(a,\xi)}\, e^{-ib\xi}\, d\xi,$$

$$|a|^{\frac{1}{c}} \int_{\mathbb{R}} \left(\int_{\mathbb{R}} N(\xi,s) e^{i\xi z(s)}\, ds \; - U(\xi) \right) \overline{\Phi(a,\xi)}\, e^{-ib\xi}\, d\xi = 0,$$

where

$$N(\xi,s) = (\mathcal{F}n)(\xi,s) = \int_{\mathbb{R}} \left(\frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} q(t,\tau,s) e^{it\xi}\, dt \right) k(\tau)\, d\tau = \int_{\mathbb{R}} Q(\xi,\tau,s)\, k(\tau)\, d\tau.$$

As a result, we have a Urysohn equation with an oscillating kernel

$$\int_{\mathbb{R}} N(\xi,s) e^{i\xi z(s)}\, ds \; = U(\xi),\;\; \xi \in \mathbb{R}. \tag{19}$$

For non-monotonic operators in [34], substantiation of the regularization algorithm in a discrete case is obtained.

The results previously obtained by authors [20,21,32] on iterative algorithms based on the use of the close equation $\tilde{A}\tilde{z} = \tilde{u}$ can be extended to the Urysohn-type equation $Az = u$ (18). Depending on the availability of a priori and other information about the solution and the indirect measurement model, one can use the whole range of known regularization algorithms. The most close results can be found in the works by V. V. Vasin and his co-authors [35–39]. For example, for Lavrentiev's regularization method $F(z) = Az - u + \alpha(z - z_m)$, the iterative procedure of constructing z_{n+1} for approximation $z_{n+1} = z_n + h$ leads to the algorithm of finding h from the linear equation $[\alpha I + (A'z_n)]h - F(z_n) = 0$ or

$$z_{n+1} = z_n - \gamma[\alpha I + (A'z_n)]^{-1}[Az_n - u + \alpha(z_n - z_m)].$$

The following equations should be considered as close ones

$$\begin{aligned} \mathbb{K}h &\equiv [\alpha I + (A'z_n)]h = Az_n - u + \alpha(z_n - z_m) \equiv g, \\ \tilde{\mathbb{K}}\tilde{h} &\equiv [\tilde{\alpha} I + (\tilde{A}'\tilde{z}_n)]\tilde{h} = \tilde{A}\tilde{z}_n - \tilde{u} + \tilde{\alpha}(\tilde{z}_n - \tilde{z}_m) \equiv \tilde{g}, \end{aligned} \tag{20}$$

where, having provided $\|\tilde{\mathbb{K}}^{-1}(\mathbb{K} - \tilde{\mathbb{K}})\| < 1$, we can either obtain a solution for the equation $\mathbb{K}h = g$ by means of solving close equation $\tilde{\mathbb{K}}\tilde{h} = \tilde{g}$, simpler in its structure, or use a solution that has already been found for different levels of error (precedent information). For the modified version of the Levenberg–Marquardt method, the solution of nonlinear Urysohn-type equations, in special cases, has the form

$$z_{n+1} = z_n - \gamma[\tilde{A}'(z_n)^*\tilde{A}'(z_n) + \alpha I]^{-1}[\tilde{A}'(z_n)^*(Az_n - u_\delta) + \alpha(z_n - \tilde{z}_m)]. \tag{21}$$

Here, $\tilde{z}_m$ is the initial approximation for the sought solution,

$$(A'z)h \equiv \int_{\mathbb{R}} n'(t - z(s),s)\, f(s)\, h(s)\, ds \tag{22}$$

and valid is Theorem 2 from [21] on error estimation.

In Equation (20), for $\mathbb{K}$ and $\tilde{\mathbb{K}}$ in terms of $\alpha(z_n - z_m)$ and $\tilde{\alpha}(\tilde{z}_n - \tilde{z}_m)$, instead of the parameters α and $\tilde{\alpha}$, you can use the new parameters β and $\tilde{\beta}$, which are responsible for the proximity of the solution z_n to the specified approximation z_m ($\tilde{z}_n$ and $\tilde{z}_m$).

In inverse operators $(\alpha I + A'(z_n))^{-1}$ and $(\alpha I + \tilde{A}'(z_n)^*\tilde{A}'(z_n))^{-1}$, instead of z_n, a fixed solution of the close equation $\tilde{z}_n$ or $\tilde{z}_0$ is used, which greatly simplifies calculations.

In applied problems, the variable s in (22) belongs to the interval $(a, b) \in \mathbb{R}$. The extension to $\mathbb{R}$ allows you to apply the FT and the wavelet transform to $\mathbb{R}$. For finite intervals, the equations are discretized using fast discrete FT and discrete wavelet transforms.

The proposed approach is promising. Many regularized iterative algorithms can be built on its basis. To perform this, already known solutions are used as close ones. For example, irregular nonlinear operator equations with a normally solvable derivative and iteratively regularized algorithms of the Gauss–Newton type can be used as close equations. M. Kokurin [40] provides the necessary a priori and a posteriori methods for stopping iterations, and error estimates for approximate solutions are established. As applied to Equations (16)–(19), it is sufficient to assume that the operator $A : H_1 \to H_2$ is differentiable by Frechet, H_1, H_2 are Hilbert spaces, and the derivative by Frechet A' in the ball $\Omega_R(z^*) = \|z - z^*\|_{H_1} \leqslant R$ satisfies the Lipschitz condition

$$\|A'z - A'x\|_{L(H_1,H_2)} \leqslant L\|z - x\|_{H_1}, \quad z, x \in \Omega_R(z^*).$$

Here, z^* is the desired solution of the Urysohn equation.

Estimates of the convergence rate and the error of stable iterative processes of the Gauss–Newton type are found when the representability conditions are met

$$\xi - z^* = [A'^*(z^*)A'(z^*)]^s v, \quad v \in H_1,\ s \geqslant \frac{1}{2}$$

and ξ—fixed approximation of the solution z^*. In a particular case, an iteratively regularized Gauss–Newton type algorithm has the form

$$z_{n+1} = \xi - [A'^*(z_n)A'(z_n) + \alpha_n I]^{-1}A'^*(z_n)(A(z_n) - u - A'(z_n)(z_n - \xi)).$$

When using this algorithm as a close one, an iterative process can be written as

$$z_{n+1} = \xi - [\tilde{A}'^*(\tilde{z}_n)\tilde{A}'(\tilde{z}_n) + \tilde{\alpha}_n I]^{-1}\tilde{A}'^*(\tilde{z}_n)(A(z_n) - u - \tilde{A}'(\tilde{z}_n)(z_n - \xi)).$$

To summarize, the availability of effectively solvable close equations allows constructing algorithms for the original equations.

Theoretical results for convolution-type equations with CWT and for Urysohn equations on the whole real axis are obtained. For finite areas, the solution algorithm is retained, but the appropriate discretization is required. For Urysohn equations with an unknown shift or oscillating kernel due to the nonuniqueness of the solutions, a system of such equations is considered. In practice, they occur naturally, as they correspond to a moving antenna device with a delta type and pulse function $n(t)$, and the directional diagram defines the area of the surface that can be restored. Consider as an example the Urysohn-type equation of gravimetry.

The basic equation of the gravity problem

$$(Az)(t) = \frac{\rho}{4\pi} \int_{-l}^{l} ln \frac{(t-s)^2 + H^2}{(t-s)^2 + (H - z(s))^2} ds = u(t), \tag{23}$$

where $u(t)$—is the anomaly of the gravity pull force, and is determined experimentally; sources of a gravitational field with a density of ρ, which are disturbed the gravitational field of the Earth, are dispersed in the area $D = \{-l \leq t \leq l,\ -H \leq z \leq -H + z(t)\}$.

The operator equation $Az = u$ is usually considered on a pair of Hilbert spaces $Z = W_2^1[-1, 1]$, $U = L_2[-1, 1]$, where the operator A in (23) is not monotone in L_2. It is easy to check by denoting $z_1(t) = 0$, $z_2(t) = 3H$, then we obtain

$$(Az - 2 - Az_1, z_2 - z_1) = \frac{3H\rho}{4\pi} \int_{-l}^{l} ln \frac{(t-s)^2 + H^2}{(t-s)^2 + (H - z(s))^2} dsdt < 0.$$

The problem of determining the surface of a partition between two media with different densities $\rho_1 > \rho_2$ by gravity anomaly is reduced to solving the nonlinear integral equation of Uryson's type (23), where $\rho = \rho_1 - \rho_2$. Due to the incorrect solution of thenonlinear equation

$$Az = \int_a^b k(t, s, z(s))ds = u(t),\ a \leq t \leq b \tag{24}$$

regularization is needed, for example in the following form

$$\alpha z + Az = u,$$

whose discretization leads to nonlinear algebraic equations.

For non-monotone operators in [34], the justification of the regularization algorithm in a discrete case is obtained.

The discrete analogue of the Urysohn equation of the gravimetry problem, an iterative process based on the Newton method, is applied:

$$z_n^{k+1} = z_n^k - [F' z_n^k]^{-1} F z_n^k,\ F z_n^k = A_n z_n + \alpha z_n - u_n,\ z_n = z(t_n),$$

$$(A_n z_n)i = \sum_{j=0}^{n} \beta_j k(t_i, s_j, z(s_j)) = u(t_i),\ i = 1, 2, \ldots, n,$$

where β_j is the coefficient of quadrature formula.

Computational algorithms and results of numerical experiments are proposed to be presented in the next article.

4. Conclusions

The considered approaches to solving linear and nonlinear convolution-type equations with respect to the Fourier transform or the wavelet transform, depending on a priori or other additional information, allow the construction of regularization algorithms for the first-order equations. Convolution-type Urysohn equations can be reduced to equations with an oscillating kernel. Applicable to such equations are Prony's methods, which reduce discrete versions of equations to solving linear difference equations and finding the roots of a polynomial. In the paper, the equations are considered on the entire axis, but the methodology is applicable to the equations on finite space.

For equations on a finite interval, it is reasonable to use fast discrete Fourier transforms and wavelet transforms in combination with regularization algorithms. Analogues of the convolution-type Urysohn equations with respect to the wavelet transform can be considered. Here one can use replacement of one wavelet by another, apply close equations, and build iterative algorithms.

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