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Article

Novel Formulas of Schröder Polynomials and Their Related Numbers

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Abstract: This paper explores the Schröder polynomials, a class of polynomials that produce the famous Schröder numbers when $x = 1$. The three-term recurrence relation and the inversion formula of these polynomials are a couple of the fundamental Schröder polynomial characteristics that are given. The derivatives of the moments of Schröder polynomials are given. From this formula, the moments of these polynomials and also their high-order derivatives are deduced as two significant special cases. The derivatives of Schröder polynomials are further expressed in new forms using other polynomials. Connection formulas between Schröder polynomials and a few other polynomials are provided as a direct result of these formulas. Furthermore, new expressions that link some celebrated numbers with Schröder numbers are also given. The formula for the repeated integrals of these polynomials is derived in terms of Schröder polynomials. Furthermore, some linearization formulas involving Schröder polynomials are established.

Keywords: Schröder numbers; Schröder polynomials; orthogonal polynomials; connection and linearization coefficients; generalized hypergeometric functions; symbolic computation

MSC: 11B37; 33C45

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1. Introduction

Special functions are crucial in numerous issues in various disciplines, including approximation theory and theoretical physics. For example, the authors in [1] presented some applications of special functions in mathematical physics, and some uses of special functions in numerical analysis can be found in [2]. Other examples of special function applications can be found in [3–5]. There is a large number of studies regarding the different celebrated sequences of polynomials and their related numbers. For example, there are several articles regarding the Fibonacci and Lucas sequences and their modifications and generalizations. In this direction, the authors in [6,7] introduced some identities, including the Fibonacci and Lucas polynomials. The authors of [8,9] developed some new formulas for certain types of generalized Fibonacci and generalized Lucas polynomials and their related numbers. Moreover, they developed some formulas for reducing some even and odd radicals based on making use of these sequences of polynomials. Recently, the authors of [10] established new formulas and integrals formulas involving Bernoulli polynomials. The Horadam sequence of polynomials was investigated in [11]. In [12], the authors established some determinant forms for general polynomial sequences. A sequence of polynomials, namely, poly-Genocchi polynomials, was investigated in [13]. A matrix approach was followed in [14] to treat certain Appel polynomials. Matrix calculus was employed in [15] to treat bivariate Appell polynomials. For some other articles that study different types of numbers and polynomials sequences, one can be referred to [16–22].

The Schröder number S_n is also called a large Schröder number. It arises in number theory (see [23]). The first few Schröder numbers are: 1, 2, 6, 22, 90, 394 1806 8558, . . . The name Schröder is due to the German mathematician Ernst Schröder. Several contributions were devoted to investigating these numbers and driving new identities for them. For example, the authors in [24] presented some interesting properties of these numbers, such as some determinant inequalities and product inequalities. Some recursive formulas of these numbers are also introduced in [25]. Furthermore, some arithmetic properties of these numbers were given in [26]. Some other contributions regarding these numbers can be found in [27,28]. The Schröder polynomials are defined in [23]. In spite of the existence of several contributions regarding Schröder numbers, the contributions regarding Schröder polynomials and their numerous formulas are not found in the literature. This gives us motivation for investigating these polynomials. In addition, investigating Schröder polynomials enables us to obtain new properties of Schröder numbers. This gives us another motivation for investigating Schröder numbers by following a different approach.

Various special function formulas are of theoretical and practical interest. Both approximation theory and numerical analysis rely heavily on the use of special functions. As an illustration, obtaining spectral solutions to various differential equations can be aided by expressing the derivatives of polynomials as certain combinations of the original ones. Specifically, in [29], the authors constructed formulas for the high-order derivatives of Chebyshev polynomials of the third kind and applied them to the treatment of certain differential equations. Furthermore, some other derivatives formulas of the polynomials that generalize Chebyshev polynomials of the third kind were established in [30]. Using the spectral Galerkin approach, these polynomials were used to find solutions to linear and non-linear even-order BVPs. In order to obtain an approximate solution to the non-linear one-dimensional Burgers equation, Abd-Elhameed in [31] constructed new derivative formulas of Chebyshev polynomials of the sixth kind.

Many areas would benefit greatly from knowing the formulas for linearization and connection between various special functions ([32,33]). Many works, both historical and contemporary, have investigated this issue for various polynomials. Rahman [34] and Gasper [35,36] have made significant contributions to this area in the past. The research in [37–41] is also useful. Regarding a few recent studies that discuss the linearization formulas of Jacobi polynomials and related classes, one may refer to the papers of Abd-Elhameed [42,43]. Some additional articles that study various other sequences of polynomials can be found in [44–50].

The study and use of hypergeometric functions is fundamental to mathematical analysis and its related fields. In fact, many hypergeometric functions serve as expressions for almost all the fundamental functions. In addition, the connection and linearization coefficients between various orthogonal polynomials, and special functions that are applicable in a wide variety of contexts, are typically expressed in terms of hypergeometric functions of varying arguments; see, for instance, [51–53].

Studying Schröder polynomials theoretically is the primary focus of this paper. In this paper, we develop several new formulas for Schröder polynomials. Moment derivative formulas for these polynomials are developed. Furthermore, their derivatives in terms of other polynomials are established. Connection formulas with different polynomials are also found. With the help of our connection formulas for Schröder polynomials with other celebrated polynomials, we can find new connections between Schröder numbers and some other well-known numbers. The formulas in this paper are, according to what we know, new and may be useful in some contexts.

We can categorize the information in the document as follows. Preliminary information and some fundamental formulas of Schröder polynomials are presented in the next section. In addition, we provide some elementary characteristics of a few well-known polynomials. The focus of Section 3 is on the detailed derivation of the formula that expresses the derivatives of the moments of Schröder polynomials. Derivatives of Schröder polynomials are expressed in terms of other polynomials in Section 4. Connection formulas between Schröder polynomials and other well-known polynomials are also shown in this section.

A new simplified linearization formula of Schröder polynomials is given in Section 5. In addition, the products of Schröder polynomials with some other celebrated polynomials are also given in this section. The repeated integrals formula for Schröder polynomials is given in Section 6. In the end, some concluding remarks and discussions are given in Section 7.

2. Preliminaries and Essential Formulas of Some Celebrated Polynomials

Some useful formulas for Schröder polynomials are discussed here. In addition, we present a brief overview of a selection of well-known polynomials that we connect to Schröder polynomials.

2.1. An Overview on Schröder Polynomials and Their Related Numbers

Schröder numbers appear in combinatorics and number theory. They can be defined as ([23])

$$S_i = \sum_{j=0}^{i} \frac{\binom{2j}{j}\binom{i+j}{i-j}}{j+1}.$$

These numbers satisfy the following recurrence relation: ([25])

$$S_{j+3} = 3\,S_{j+2} + \sum_{i=0}^{j} S_{i+1}\,S_{j-i+1}, \quad j \geq 0.$$

In [23], the author defined Schröder polynomials as

$$S_i(x) = \sum_{j=0}^{i} \frac{\binom{2j}{j}\binom{i+j}{i-j}}{j+1}\,x^j. \tag{1}$$

It is evident that $S_i = S_i(1)$.

For Schröder polynomials $S_i(x)$, we now state and demonstrate two fundamental lemmas. The first lemma illustrates the three-term recurrence relation that $S_i(x)$ satisfies, and the second lemma provides the inversion formula for these polynomials.

Lemma 1. *The following recurrence three-term recurrence relation is fulfilled by $S_i(x)$:*

$$xS_i(x) = \frac{i-1}{2(2i+1)}\,S_{i-1}(x) - \frac{1}{2}\,S_i(x) + \frac{i+2}{2(2i+1)}\,S_{i+1}(x), \quad i \geq 1. \tag{2}$$

Proof. By applying the power form representation of the polynomials $S_i(x)$ in (1), it is straightforward to demonstrate the correctness of the next relation:

$$\frac{i-1}{2(2i+1)}\,S_{i-1}(x) - \frac{1}{2}\,S_i(x) + \frac{i+2}{2(2i+1)}\,S_{i+1}(x) - xS_i(x) = 0.$$

Lamma 1 is now proved. □

Lemma 2. *The following inversion formula is valid for $S_i(x)$:*

$$x^i = i!(i+1)!\sum_{m=0}^{i} \frac{(-1)^m(2i-2m+1)}{(2i-m+1)!m!}\,S_{i-m}(x), \quad i \geq 0. \tag{3}$$

Proof. We will prove the inversion formula by induction. The introductory step is obvious for $i = 0$. Now, assume the validity of (3). In order to finish the proof, we need to show the correctness of the next identity:

$$x^{i+1} = \sum_{m=0}^{i+1} \frac{(-1)^m\,(2(i+1)-2m+1)\,(i+1)!\,(i+2)!}{(2(i+1)-m+1)!\,m!}\,S_{i-m+1}(x). \tag{4}$$

If we multiply both sides of Equation (3) by x and make use of the recurrence relation (2), then we get

$$x^{i+1}=\sum_{m=0}^{i+1}\frac{i!\,(i+1)!\,((-1)^m\,(2i-2m+1))}{(2i-m+1)!\,m!}\left(\frac{i-m-1}{2(2i-2m+1)}\,S_{i-m-1}(x)\right.$$
$$\left.-\frac{1}{2}\,S_{i-m}(x)+\frac{i-m+2}{2(2i-2m+1)}\,S_{i-m+1}(x)\right),$$

which after straightforward computations leads to Equation (4). $\square$

2.2. Several Characteristics of Jacobi Polynomials

Jacobi polynomials are well-known to be among the most significant categories of orthogonal polynomials. The Jacobi polynomials $P_s^{(\epsilon,\theta)}(x)$, $x\in[-1,1]$, $s\geq 0$ and $\epsilon>-1$, $\theta>-1$, (see [54]) can be expressed in the following form:

$$P_s^{(\epsilon,\theta)}(x)=\frac{(\epsilon+1)_s}{s!}\,{}_2F_1\left(\begin{matrix}-s,s+\epsilon+\theta+1\\ \epsilon+1\end{matrix}\,\middle|\,\frac{1-x}{2}\right),$$

where $(z)_\ell$ represents the well-known Pochhammer symbol.
The Jacobi polynomials $P_s^{(\epsilon,\theta)}$ can be normalized to define:

$$R_s^{(\epsilon,\theta)}(x)=\ {}_2F_1\left(\begin{matrix}-s,s+\epsilon+\theta+1\\ \epsilon+1\end{matrix}\,\middle|\,\frac{1-x}{2}\right).$$

We refer here that six important sub-classes can be obtained as special cases of the class of Jacobi polynomials. More precisely, we can write

$$T_s(x)=R_s^{(-\frac{1}{2},-\frac{1}{2})}(x),\qquad U_s(x)=(s+1)\,R_s^{(\frac{1}{2},\frac{1}{2})}(x),$$
$$V_s(x)=R_s^{(-\frac{1}{2},\frac{1}{2})}(x),\qquad W_s(x)=(2s+1)\,R_s^{(\frac{1}{2},-\frac{1}{2})}(x),$$
$$C_s^{(\alpha)}(x)=R_s^{(\alpha-\frac{1}{2},\alpha-\frac{1}{2})}(x),\qquad P_s(x)=R_s^{(0,0)}(x),$$

where $T_s(x)$, $U_s(x)$, $V_s(x)$, $W_s(x)$ stand, respectively, for the first, second, third, and fourth kinds of Chebyshev polynomials, while $C_s^{(\alpha)}(x)$ and $P_s(x)$ denote, respectively, the ultraspherical and Legendre polynomials.

Note the the ultraspherical polynomials $C_s^{(\alpha)}(x)$ are the normalized Gegenbauer polynomials. That is,

$$C_s^{(\alpha)}(x)=\frac{s!}{(2\alpha)_s}\,G_s^{(\alpha)}(x),$$

where $G_k^{(\alpha)}(x)$ are the standard Gegenbauer polynomials.

Among the features of Chebyshev polynomials is that they may be written in trigonometric forms (see [55]):

$$T_s(x)=\cos(s\,\theta),\qquad U_s(x)=\frac{\sin((s+1)\,\theta)}{\sin\theta},$$
$$V_s(x)=\frac{\cos\left(\left(s+\frac{1}{2}\right)\theta\right)}{\cos\left(\frac{\theta}{2}\right)},\qquad W_s(x)=\frac{\sin\left(\left(s+\frac{1}{2}\right)\theta\right)}{\sin\left(\frac{\theta}{2}\right)},$$

where $\theta=\cos^{-1}(x)$.

The following unified recurrence relation can be used to generate all four families of Chebyshev polynomials:

$$\phi_k(x)=2x\,\phi_{k-1}(x)-\phi_{k-2}(x),\quad k\geq 2, \tag{5}$$

but with different initials.

Remark 1. *Since all four kinds of Chebyshev polynomials have a unified recurrence relation* (5), *then they have a unified moments formula. It can be easily derived from* (5) *to give*

$$x^m\,\phi_j(x) = \frac{1}{2^m}\sum_{s=0}^{m}\binom{m}{s}\phi_{j+m-2s}(x). \tag{6}$$

In addition, we comment here the ultraspherical polynomials are symmetric Jacobi polynomials, so the three classes of Legendre and Chebyshev polynomials of the first and second kinds are special ones of the ultraspherical polynomials. Furthermore, the ultraspherical polynomials have, respectively, the following power form and inversion formulas ([56]):

$$\begin{aligned} C_j^{(\lambda)}(x) =& \frac{j!\,\Gamma(2\lambda+1)}{2\,\Gamma(\lambda+1)\,\Gamma(j+2\lambda)}\sum_{r=0}^{\lfloor \frac{j}{2}\rfloor}\frac{(-1)^r\,2^{j-2r}\,\Gamma(j-r+\lambda)}{(j-2r)!\,r!}\,x^{j-2r}, \quad j\geq 0,\\ x^k =& \frac{\Gamma(\lambda+1)}{\Gamma(2\lambda+1)}\sum_{m=0}^{\lfloor \frac{k}{2}\rfloor}\frac{2^{-k+1}\,(k-2m+\lambda)\,k!\,\Gamma(k-2m+2\lambda)}{(k-2m)!\,m!\,\Gamma(k-m+\lambda+1)}\,C_{k-2m}^{(\lambda)}(x), \quad k\geq 0. \end{aligned} \tag{7}$$

2.3. An Account on Two Generalized Classes of Fibonacci and Lucas Polynomials

Recently, two forms of generalized Fibonacci and Lucas polynomials were studied in [9]. The following two recurrence relations can produce these two classes of polynomials:

$$F_k^{A,B}(x) = A\,x\,F_{k-1}^{A,B}(x) + B\,F_{k-2}^{A,B}(x), \quad F_0^{A,B}(x) = 1,\ F_1^{A,B}(x) = A\,x, \quad k\geq 2, \tag{8}$$

and

$$L_k^{R,S}(x) = R\,x\,L_{k-1}^{R,S}(x) + S\,L_{k-2}^{R,S}(x), \quad L_0^{R,S}(x) = 2,\ L_1^{R,S}(x) = R\,x, \quad k\geq 2. \tag{9}$$

It is to be noted that several celebrated classes of polynomials can be obtained as special cases of the two generalized classes of $F_k^{A,B}(x)$ and $L_k^{R,S}(x)$ (see [9]). For example, the Fibonacci polynomials $F_{k+1}(x)$ and Lucas polynomials $L_k(x)$ are particular ones of $F_k^{A,B}(x)$ and $L_k^{R,S}(x)$. In fact, we have:

$$F_{k+1}(x) = F_k^{1,1}(x), \quad L_k(x) = L_k^{1,1}(x).$$

The moment formulae for the two polynomials $F_k^{A,B}(x)$ and $L_k^{R,S}(x)$ are among their essential characteristics. If (8) and (9) are written as:

$$x\,F_k^{A,B}(x) = \frac{1}{A}\,F_{k+1}^{A,B}(x) - \frac{B}{A}\,F_{k-1}^{A,B}(x),$$

and

$$x\,L_k^{R,S}(x) = \frac{1}{R}\,L_{k+1}^{R,S}(x) - \frac{S}{R}\,L_{k-1}^{R,S}(x),$$

then it is not difficult to obtain the moment formulas for the two generalizing classes $F_k^{A,B}(x)$ and $L_k^{A,B}(x)$. This lemma gives these two formulas of the moments.

Lemma 3. *Choose any two non-negative numbers for r and k. The moment formulas listed below are valid.*

$$x^r F_k^{A,B}(x) = \sum_{m=0}^{r} \binom{r}{m} A^{-r} (-B)^m F_{k+r-2m}^{A,B}(x),$$
$$x^r L_k^{R,S}(x) = \sum_{m=0}^{r} \binom{r}{m} R^{-r} (-S)^m L_{k+r-2m}^{R,S}(x).$$

Proof. Applying the two recurrence relations, (8) and (9), makes the proof straightforward via induction. □

3. Derivatives of the Moments of Schröder Polynomials

In this section, we introduce a new formula for the derivatives of the moments of Schröder polynomials. Then, as special cases, the following two significant formulas can be derived from this one:

- The expression for the high-order derivatives of Schröder polynomials in terms of their original polynomials.
- The moment formula of Schröder polynomials.

We comment here that the above two mentioned formulas are of fundamental interest to deriving several formulas in this paper.

Theorem 1. *Let j, m and q be non-negative integers with $j+m \geq q$. The following formula holds:*

$$\begin{aligned} D^q\left(x^m S_j(x)\right) = &\frac{4^j (j+m)!\,(j+m-q+1)!\,\Gamma\left(j+\frac{1}{2}\right)}{\sqrt{\pi}\,(j+1)!} \times \\ &\sum_{p=0}^{j+m-q} \frac{(-1)^{p+1}(-2j-2m+2p+2q-1)}{p!\,(2j+2m-p-2q+1)!} \times \\ &{}_4F_3\left(\left.\begin{matrix} -p, -1-j, -j, -1-2j-2m+p+2q \\ -2j, -j-m, -1-j-m+q \end{matrix}\right| 1\right) S_{j+m-q-p}(x). \end{aligned} \tag{10}$$

Proof. The analytic form of $S_i(x)$ in (1) can be rewritten as

$$S_i(x) = \sum_{j=0}^{i} \frac{\binom{2(i-j)}{i-j}\binom{2i-j}{j}}{i-j+1}\, x^{i-j}. \tag{11}$$

Based on the last formula, one can write the following expression for $D^q\left(x^m S_j(x)\right)$:

$$D^q\left(x^m S_j(x)\right) = \sum_{r=0}^{j+m-q} \frac{(2j-r)!\,(j+m-q-r+1)_q}{r!\,(j-r+1)\,((j-r)!)^2}\, x^{j+m-r-q}.$$

Due to the inversion formula in (3), the previous expression turns into

$$\begin{aligned} D^q\left(x^m S_j(x)\right) = &\sum_{r=0}^{j+m-q} \frac{(2j-r)!\,(j+m-q-r+1)_q\,(j+m-q-r+1)!}{r!\,(j-r+1)\,((j-r)!)^2} \times \\ &\sum_{t=0}^{j+m-q-r} \frac{(-1)^t\,(2(j+m-q-r)-2t+1)\,(j+m-q-r)!}{t!\,\Gamma(2j+2m-2(q+r-1)-t)}\, S_{j+m-r-q-t}(x). \end{aligned} \tag{12}$$

Some algebraic manipulations lead to converting formula (12) into the following one:

$$D^q\left(x^m\,S_j(x)\right) = \sum_{p=0}^{j+m-q}(2j+2m-2p-2q+1)\times$$
$$\sum_{r=0}^{p}\frac{(-1)^{p-r+1}\,(2j-r)!\,(j+m-r)!\,(j+m-q-r+1)!}{(-j+r-1)((j-r)!)^2\,(p-r)!\,r!\,(2j+2m-p-2q-r+1)!}\,S_{j+m-q-p}(x),$$

which can be written in the following hypergeometric formula:

$$\begin{aligned}D^q\left(x^m\,S_j(x)\right) =&\frac{4^j\,(j+m)!\,(j+m-q+1)!\,\Gamma\left(j+\frac{1}{2}\right)}{\sqrt{\pi}\,(j+1)!}\times\\&\sum_{p=0}^{j+m-q}\frac{(-1)^{p+1}\,(-2j-2m+2p+2q-1)}{p!\,(2j+2m-p-2q+1)!}\times\\&{}_4F_3\left(\left.\begin{array}{c}-p,-j-1,-j,-2j-2m+p+2q-1\\-2j,-j-m,-1-j-m+q\end{array}\right|1\right)S_{j+m-q-p}(x).\end{aligned}$$

This finalizes the proof of Theorem 1. □

Remark 2. *As particular results of Theorem 1, two crucial formulas involving Schröder polynomials can be deduced. The first formula, obtained by putting $m=0$ in (10), provides the expressions of Schröder polynomial derivatives in terms of their original ones. If we set $q=0$ in (10), we can obtain the moment formula for Schröder polynomials. These vital formulae are displayed in the two corollaries that follow.*

Corollary 1. *Let j and q be non-negative integers with $j\geq q$. The qth derivatives of Schröder polynomials are given in terms of their original ones as follows:*

$$\begin{aligned}D^qS_j(x) =&\frac{2^{2q}}{j\,(j+1)\,(q-1)!}\left(\sum_{p=0}^{\left\lfloor\frac{j-q}{2}\right\rfloor}\frac{(2j-4p-2q+1)\,\Gamma\left(j-p+\frac{1}{2}\right)(p+q-1)!}{2p!\,\Gamma\left(j-p-q+\frac{3}{2}\right)}\times\right.\\&\left(j^2-p+2p(p+q)-j(2p+q-1)\right)S_{j-q-2p}(x)\\&\left.+\sum_{p=0}^{\left\lfloor\frac{1}{2}(j-q-1)\right\rfloor}\frac{(-2j+4p+2q+1)\,(p+q)!\,\Gamma\left(j-p+\frac{1}{2}\right)}{p!\,\Gamma\left(j-p-q+\frac{1}{2}\right)}\,S_{j-q-2p-1}(x)\right).\end{aligned}$$

Proof. Setting $m=0$ in Formula (10) yields:

$$\begin{aligned}D^qS_j(x) =&\frac{4^j\,j!\,(j-q+1)!\,\Gamma\left(j+\frac{1}{2}\right)}{\sqrt{\pi}(j+1)!}\sum_{p=0}^{j-q}\frac{(-1)^{p+1}\,(-2j+2p+2q-1)}{p!\,(2j-p-2q+1)!}\times\\&{}_3F_2\left(\left.\begin{array}{c}-p,-1-j,-1-2j+p+2q\\-2j,-1-j+q\end{array}\right|1\right)S_{j-q-p}(x).\end{aligned}\tag{13}$$

Now, the terminating hypergeometric ${}_3F_2(1)$ that appears in (13) can be reduced as follows: Set

$$H_{p,j,q} = {}_3F_2\left(\left.\begin{array}{c}-p,-1-j,-1-2j+p+2q\\-2j,-1-j+q\end{array}\right|1\right),$$

and we utilize the celebrated algorithm of Zeilberger ([57]) to show that the following recurrence relation is fulfilled by $H_{p,j,q}$:

$$\begin{aligned}&(1-p)\,(j-p-q+1)^2\,(p+2q-2)\,H_{p-2,j,q}-(2j-2p-2q+3)\times\\&\left(-2-2j+3p+2jp-p^2+4q+2jq-2pq-2q^2\right)H_{p-1,j,q}\\&+(2j-p+1)(2j-p-2q+2)(j-p-q+2)^2\,H_{p,j,q}=0,\end{aligned}$$

accompanied with the following starting values:

$$H_{0,i,j}=1,\quad H_{1,i,j}=\frac{q}{j(j-q+1)}.$$

The exact solution to the preceding recurrence relation is

$$H_{p,j,q}=\frac{1}{j\,\sqrt{\pi}}\begin{cases}\dfrac{(2j^2-2j(p+q-1)+p(p+2q-1))\Gamma\left(\frac{p+1}{2}\right)(q)_{\frac{p}{2}}}{2\left(j-\frac{p}{2}+\frac{1}{2}\right)_{\frac{p}{2}}\left(j-\frac{p}{2}-q+1\right)_{\frac{p}{2}+1}}, & p \text{ even},\\[2ex] \dfrac{2\Gamma\left(j-\frac{p}{2}+1\right)\Gamma\left(\frac{p}{2}+1\right)(q)_{\frac{p+1}{2}}}{\Gamma\left(j+\frac{1}{2}\right)\left(j-\frac{p}{2}-q+\frac{3}{2}\right)_{\frac{p+1}{2}}}, & p \text{ odd}.\end{cases}$$

Some simplifications lead to the following formula:

$$\begin{aligned}D^q S_j(x)=&\frac{2^{2q}}{j(j+1)\,(q-1)!}\left(\sum_{p=0}^{\left\lfloor\frac{j-q}{2}\right\rfloor}\frac{(2j-4p-2q+1)\,\Gamma\left(j-p+\frac{1}{2}\right)(p+q-1)!}{2\,p!\,\Gamma\left(j-p-q+\frac{3}{2}\right)}\times\right.\\&\left(j^2-p+2p(p+q)-j(2p+q-1)\right)S_{j-q-2p}(x)\\&\left.+\sum_{p=0}^{\left\lfloor\frac{1}{2}(j-q-1)\right\rfloor}\frac{(-2j+4p+2q+1)\,(p+q)!\,\Gamma\left(j-p+\frac{1}{2}\right)}{p!\,\Gamma\left(j-p-q+\frac{1}{2}\right)}S_{j-q-2p-1}(x)\right).\end{aligned}$$

Corollary 1 is now proved. □

Corollary 2. *The following is the moment formula of Schröder polynomials:*

$$\begin{aligned}x^m\,S_j(x)=&\frac{4^j\,(j+m)!\,(j+m+1)!\,\Gamma\left(j+\frac{1}{2}\right)}{\sqrt{\pi}(j+1)!}\sum_{p=0}^{j+m}\frac{(-1)^{p+1}\,(-2j-2m+2p-1)}{(2j+2m-p+1)!\,p!}\times\\&{}_4F_3\left(\left.\begin{matrix}-p,-1-j,-j,-1-2j-2m+p\\-2j,-1-j-m,-j-m\end{matrix}\right|1\right)S_{j+m-p}(x).\end{aligned}\tag{14}$$

Proof. Setting $q=0$ in Formula (10) yields Formula (14) instantly. □

4. New Expressions for the Derivatives of Some Celebrated Polynomials

This section displays various derivative expressions of various well-known polynomials. More specifically, we provide new derivative expressions of Schröder polynomials in terms of ultraspherical, Hermite, generalized Laguerre, generalized Fibonacci, and generalized Lucas polynomials. The inversion formulas to the generated derivatives formulas are also presented in this section. In addition, we provide some new formulas for the relationships between Schröder polynomials and other types of polynomials.

4.1. Expressions for the Derivatives of Some Other Polynomials

Theorem 2. *If we have two non-negative integers j, q with $j \geq q$, then $D^q S_j(x)$ can be expressed in terms of the ultraspherical polynomials $C_i^{(\lambda)}(x)$ as:*

$$
\begin{aligned}
D^q S_j(x) =& \frac{2^{1-j+q}\,(2j)!\,\Gamma(\lambda+1)}{(j+1)!\,\Gamma(2\lambda+1)} \sum_{m=0}^{\left\lfloor \frac{j-q}{2} \right\rfloor} \frac{(j-2m-q+\lambda)\,\Gamma(j-2m-q+2\lambda)}{m!\,(j-2m-q)!\,\Gamma(j-m-q+\lambda+1)} \times \\
& {}_4F_3\left(\begin{matrix} -m, -\frac{1}{2}-\frac{j}{2}, -\frac{j}{2}, -j+m+q-\lambda \\ \frac{1}{2}, \frac{1}{2}-j, -j \end{matrix} \middle| 1 \right) C_{j-q-2m}^{(\lambda)}(x) \\
& + \frac{2^{2-j+q-2\lambda}\,\sqrt{\pi}\,(2j-1)!}{j!\,\Gamma\left(\lambda+\frac{1}{2}\right)} \sum_{m=0}^{\left\lfloor \frac{1}{2}(j-q-1) \right\rfloor} \frac{(j-2m-q+\lambda-1)\,\Gamma(j-2m-q+2\lambda-1)}{m!\,(j-2m-q-1)!\,\Gamma(j-m-q+\lambda)} \times \\
& {}_4F_3\left(\begin{matrix} -m, \frac{1}{2}-\frac{j}{2}, -\frac{j}{2}, -1-j+m+q-\lambda \\ \frac{3}{2}, \frac{1}{2}-j, 1-j \end{matrix} \middle| 1 \right) C_{j-q-2m-1}^{(\lambda)}(x).
\end{aligned}
\tag{15}
$$

Proof. Thanks to Schröder's polynomials power form, one can write

$$
D^q S_j(x) = \sum_{r=0}^{j-q} \frac{(2j-r)!}{r!\,(j-r+1)!\,(j-q-r)!}\, x^{j-r-q}.
$$

The previous formula is transformed into the following form utilizing Formula (7):

$$
\begin{aligned}
D^q S_j(x) =& \frac{\Gamma(\lambda+1)}{\Gamma(2\lambda+1)} \sum_{r=0}^{j-q} \frac{2^{1-j+q+r}\,(2j-r)!}{r!\,(j-r+1)!\,(j-q-r)!} \times \\
& \sum_{t=0}^{\left\lfloor \frac{1}{2}(j-r-q) \right\rfloor} \frac{(j-q-r-2t+\lambda)(j-q-r)!\,\Gamma(j-q-r-2t+2\lambda)}{(j-q-r-2t)!\,t!\,\Gamma(j-q-r-t+\lambda+1)}\, C_{j-r-q-2t}^{(\lambda)}(x).
\end{aligned}
$$

The last formula can be turned into a more convenient form after performing expanding and rearranging the terms

$$
D^q S_j(x) = \sum_{m=0}^{\left\lfloor \frac{j-q}{2} \right\rfloor} \sum_{r=0}^{m} \delta_{m,j,q}\, C_{j-q-2m}^{(\lambda)}(x) + \sum_{m=0}^{\left\lfloor \frac{1}{2}(j-q-1) \right\rfloor} \sum_{r=0}^{m} \bar{\delta}_{m,j,q}\, C_{j-q-2m-1}^{(\lambda)}(x), \tag{16}
$$

where $\delta_{m,j,q}$ and $\bar{\delta}_{m,j,q}$ are given as follows

$$
\begin{aligned}
\delta_{m,j,q} =& \frac{2^{1-j+q+2r}\,(j-2m-q+\lambda)(2j-2r)!\,\Gamma(\lambda+1)\,\Gamma(j-2m-q+2\lambda)}{(j-2m-q)!\,(j-2r+1)!\,(m-r)!\,(2r)!\,\Gamma(j-m-q-r+\lambda+1)\,\Gamma(2\lambda+1)}, \\
\bar{\delta}_{m,j,q} =& \frac{2^{2-j+q+2r}\,(j-2m-q+\lambda-1)\,(2j-2r-1)!\,\Gamma(\lambda+1)\,\Gamma(j-2m-q+2\lambda-1)}{(j-2m-q-1)!\,(j-2r)!\,(m-r)!\,(2r+1)!\,\Gamma(j-m-q-r+\lambda)\,\Gamma(2\lambda+1)}.
\end{aligned}
$$

We can write the coefficients $\sum_{r=0}^{m} \delta_{m,j,q}$ and $\sum_{r=0}^{m} \bar{\delta}_{m,j,q}$ as:

$$\sum_{r=0}^{m} \delta_{m,j,q} = \frac{2^{1-j+q}\,(j-2m-q+\lambda)\,(2j)!\,\Gamma(\lambda+1)\,\Gamma(j-2m-q+2\lambda)}{(j+1)!\,m!\,(j-2m-q)!\,\Gamma(j-m-q+\lambda+1)\,\Gamma(2\lambda+1)} \times$$
$$ {}_4F_3\left(\left.\begin{array}{c} -m, \frac{1}{2}-\frac{j}{2}, -\frac{j}{2}, 1-j+m+q-\lambda \\ \frac{1}{2}, \frac{1}{2}-j, -j \end{array}\right| 1\right),$$
$$\sum_{r=0}^{m} \bar{\delta}_{m,j,q} = \frac{2^{2-j+q-2\lambda}\,\sqrt{\pi}\,(j-2m-q+\lambda-1)\,(2j-1)!\,\Gamma(j-2m-q+2\lambda-1)}{j!\,m!\,(j-2m-q-1)!\,\Gamma\left(\lambda+\frac{1}{2}\right)\Gamma(j-m-q+\lambda)} \times$$
$$ {}_4F_3\left(\left.\begin{array}{c} -m, \frac{1}{2}-\frac{j}{2}, -\frac{j}{2}, 1-j+m+q-\lambda \\ \frac{3}{2}, \frac{1}{2}-j, 1-j \end{array}\right| 1\right).$$

If we insert the last two identities into (16), then Formula (15) can be obtained. □

Remark 3. *Since the ultraspherical polynomials $C_j^{(\lambda)}(x)$, involve three important special classes of polynomials, namely, Legendre and Chebyshev polynomials of the first and second kinds, we can deduce three specific formulas of Formula (15). In the following, we write the expression of $D^q S_j(x)$ in terms of Chebyshev polynomials. The expressions in terms of Legendre and the second-kind Chebyshev polynomials can be also deduced.*

Corollary 3. *Given two non-negative integers j, q with $j \geq q$, $D^q S_j(x)$ can be represented as:*

$$D^q S_j(x) = \frac{2^{1-j+q}\,(2j)!}{(j+1)!} \sum_{m=0}^{\left\lfloor \frac{j-q}{2} \right\rfloor} \frac{c_{j-q-2m}}{m!\,(j-m-q)!}\, {}_4F_3\left(\left.\begin{array}{c} -m, -\frac{1}{2}-\frac{j}{2}, -\frac{j}{2}, -j+m+q \\ \frac{1}{2}, \frac{1}{2}-j, -j \end{array}\right| 1\right) T_{j-q-2m}(x)$$
$$+ \frac{2^{2-j+q}(2j-1)!}{j!} \sum_{m=0}^{\left\lfloor \frac{1}{2}(j-q-1) \right\rfloor} \frac{c_{j-q-2m-1}}{m!(j-m-q-1)!}\, {}_4F_3\left(\left.\begin{array}{c} -m, \frac{1}{2}-\frac{j}{2}, -\frac{j}{2}, 1-j+m+q \\ \frac{3}{2}, \frac{1}{2}-j, 1-j \end{array}\right| 1\right) \times$$
$$T_{j-q-2m-1}(x),$$

where the constants c_r are defined as

$$c_r = \begin{cases} \frac{1}{2}, & r = 0, \\ 1, & r \geq 1. \end{cases} \tag{17}$$

Remark 4. *Relying on the power form of Schröder polynomials and the inversion formula for ultraspherical polynomials, we were able to prove Theorem 2, so using similar procedures, one can derive other expressions for the derivatives of Schröder polynomials in terms of other polynomials. In the following, we give without proof other expressions for the derivatives of Schröder polynomials in terms of some celebrated polynomials.*

Theorem 3. *If we have two non-negative integers j, q with $j \geq q$, then $D^q S_j(x)$ can be expressed in terms of Hermite polynomials $H_i(x)$ as:*

$$
\begin{aligned}
D^q S_j(x) = & \frac{2^{-j+q}\,(2j)!}{(j+1)!} \sum_{m=0}^{\left\lfloor \frac{j-q}{2} \right\rfloor} \frac{1}{m!(j-2m-q)!} \times \\
& {}_3F_3\left(\left. \begin{array}{c} -\frac{1}{2}-\frac{j}{2}, -\frac{j}{2}, -m \\ \frac{1}{2}, \frac{1}{2}-j, -j \end{array} \right| -1 \right) H_{j-q-2m}(x) \\
& + \frac{2^{1-j+q}\,(2j-1)!}{j!} \sum_{m=0}^{\left\lfloor \frac{1}{2}(j-q-1) \right\rfloor} \frac{1}{m!(j-2m-q-1)!} \times \\
& {}_3F_3\left(\left. \begin{array}{c} -m, \frac{1}{2}-\frac{j}{2}, -\frac{j}{2} \\ \frac{3}{2}, \frac{1}{2}-j, 1-j \end{array} \right| -1 \right) H_{j-q-2m-1}(x).
\end{aligned}
$$

Theorem 4. *If we have two non-negative integers j, q with $j \geq q$, then $D^q S_j(x)$ can be expressed in terms of generalized Laguerre polynomials $L_i^{(\alpha)}(x)$ as:*

$$
\begin{aligned}
D^q S_j(x) = & \frac{(2j)!\,\Gamma(j-q+\alpha+1)}{(j+1)!} \sum_{m=0}^{j-q} \frac{(-1)^{j-m-q}}{m!\,\Gamma(j-m-q+\alpha+1)} \times \\
& {}_2F_2\left(\left. \begin{array}{c} -m, -1-j \\ -2j, -j+q-\alpha \end{array} \right| 1 \right) L_{j-q-m}^{(\alpha)}(x).
\end{aligned} \tag{18}
$$

Theorem 5. *If we have two non-negative integers j, q with $j \geq q$, $D^q S_j(x)$ can be expressed in terms of the generalized Fibonacci polynomials $F_i^{A,B}(x)$ as:*

$$
\begin{aligned}
D^q S_j(x) = & \frac{(2j)!\,A^{-j+q}}{(j+1)!} \sum_{m=0}^{\left\lfloor \frac{j-q}{2} \right\rfloor} \frac{(-1)^m\,B^m\,(j-2m-q+1)}{m!\,(j-m-q+1)!} \times \\
& {}_4F_3\left(\left. \begin{array}{c} -m, -\frac{1}{2}-\frac{j}{2}, -\frac{j}{2}, -1-j+m+q \\ \frac{1}{2}, \frac{1}{2}-j, -j \end{array} \right| -\frac{A^2}{4B} \right) F_{j-q-2m}^{A,B}(x) \\
& + \frac{(2j-1)!\,A^{1-j+q}}{j!} \sum_{m=0}^{\left\lfloor \frac{1}{2}(j-q-1) \right\rfloor} \frac{(-1)^{m+1}\,B^m\,(-j+2m+q)}{m!\,(j-m-q)!} \times \\
& {}_4F_3\left(\left. \begin{array}{c} -m, \frac{1}{2}-\frac{j}{2}, -\frac{j}{2}, -j+m+q \\ \frac{3}{2}, \frac{1}{2}-j, 1-j \end{array} \right| -\frac{A^2}{4B} \right) F_{j-q-2m-1}^{A,B}(x).
\end{aligned} \tag{19}
$$

Remark 5. *Since the class of the generalized Fibonacci polynomials, $F_i^{A,B}(x)$, includes some celebrated polynomials as special cases for specific choices of the two parameters A and B, we can easily deduce specific derivatives formulas from Formula (19). More precisely, we will give the derivatives of Schröder polynomials in terms of the well-known Fibonacci polynomials in the following corollary. Other formulas can be easily deduced from Formula (19).*

Corollary 4. *Let j and q be two non-negative integers with $j \geq q$. The derivatives of $S_j(x)$ can be expressed in terms of Fibonacci polynomials as:*

$$\begin{aligned} D^q S_j(x) =& \frac{(2j)!}{(j+1)!} \sum_{m=0}^{\left\lfloor \frac{j-q}{2} \right\rfloor} \frac{(-1)^m (j-2m-q+1)}{m!(j-m-q+1)!} \times \\ & {}_4F_3\left(\left. \begin{array}{c} -m, -\frac{1}{2}-\frac{j}{2}, -\frac{j}{2}, -1-j+m+q \\ \frac{1}{2}, \frac{1}{2}-j, -j \end{array} \right| -\frac{1}{4} \right) F_{j-q-2m}(x) \\ & + \frac{(2j-1)!}{j!} \sum_{m=0}^{\left\lfloor \frac{1}{2}(j-q-1) \right\rfloor} \frac{(-1)^{m+1}(-j+2m+q)}{m!\,(j-m-q)!} \times \\ & {}_4F_3\left(\left. \begin{array}{c} -m, \frac{1}{2}-\frac{j}{2}, -\frac{j}{2}, -j+m+q \\ \frac{3}{2}, \frac{1}{2}-j, 1-j \end{array} \right| -\frac{1}{4} \right) F_{j-q-2m-1}(x). \end{aligned} \tag{20}$$

Theorem 6. *If we have two non-negative integers j, q with $j \geq q$, then $D^q S_j(x)$ can be expressed in terms of the generalized Lucas polynomials $L_i^{R,S}(x)$ as:*

$$\begin{aligned} D^q S_j(x) =& \frac{R^{-j+q}\,(2j)!}{(j+1)!} \sum_{m=0}^{\left\lfloor \frac{j-q}{2} \right\rfloor} \frac{c_{j-q-2m}\,(-S)^m}{m!\,(j-m-q)!} \times \\ & {}_4F_3\left(\left. \begin{array}{c} -m, -\frac{1}{2}-\frac{j}{2}, -\frac{j}{2}, -j+m+q \\ \frac{1}{2}, \frac{1}{2}-j, -j \end{array} \right| -\frac{R^2}{4S} \right) L_{j-q-2m}^{R,S}(x) \\ & + \frac{R^{1-j+q}\,(2j-1)!}{j!} \sum_{m=0}^{\left\lfloor \frac{1}{2}(j-q-1) \right\rfloor} \frac{c_{j-q-2m-1}\,(-S)^m}{m!\,(j-m-q-1)!} \times \\ & {}_4F_3\left(\left. \begin{array}{c} -m, \frac{1}{2}-\frac{j}{2}, -\frac{j}{2}, 1-j+m+q \\ \frac{3}{2}, \frac{1}{2}-j, 1-j \end{array} \right| -\frac{R^2}{4S} \right) L_{j-q-2m-1}^{R,S}(x), \end{aligned} \tag{21}$$

where c_r are given by (17).

Remark 6. *Since the class of the generalized Lucas polynomials, $L_j^{R,S}(x)$, includes some celebrated polynomials as special cases for specific choices of the two parameters R and S, we can write some expressions for the derivatives of Schröder polynomials in terms of some specific polynomials of $L_j^{R,S}(x)$. The following corollary exhibits the formula of the derivatives of Schröder polynomials in terms of Lucas polynomials. Other formulas can be deduced from Formula* (21).

Corollary 5. *Given two non-negative integers j, q with $j \geq q$. Consider the Lucas polynomials $L_j(x)$. The following derivatives formula is valid:*

$$\begin{aligned} D^q S_j(x) =& \frac{(2j)!}{(j+1)!} \sum_{m=0}^{\left\lfloor \frac{j-q}{2} \right\rfloor} \frac{c_{j-q-2m}}{m!(j-m-q)!} \times \\ & {}_4F_3\left(\left. \begin{array}{c} -m, -\frac{1}{2}-\frac{j}{2}, -\frac{j}{2}, -j+m+q \\ \frac{1}{2}, \frac{1}{2}-j, -j \end{array} \right| -\frac{1}{4} \right) L_{j-q-2m}(x) \\ & + \frac{(2j-1)!}{j!} \sum_{m=0}^{\left\lfloor \frac{1}{2}(j-q-1) \right\rfloor} \frac{c_{j-q-2m-1}}{m!(j-m-q-1)!} \times \\ & {}_4F_3\left(\left. \begin{array}{c} -m, \frac{1}{2}-\frac{j}{2}, -\frac{j}{2}, 1-j+m+q \\ \frac{3}{2}, \frac{1}{2}-j, 1-j \end{array} \right| -\frac{1}{4} \right) L_{j-q-2m-1}(x). \end{aligned}$$

4.2. Inversion Formulas for the Derivatives Formulas in Section 4.1

In this section, and following similar procedures to those given in the previous section, we introduce the inversion derivatives to those given in the previous section. Due to the similarity of the proofs to the proof of Theorem 2, the proofs are omitted.

Theorem 7. *Given two non-negative integers j, q with $j \geq q$, in terms of Schröder polynomials, the derivatives of ultraspherical polynomials have the following expression:*

$$
\begin{aligned}
D^q C_j^{(\lambda)}(x) =& \frac{2^{j+2\lambda-1}\, j!\,(j-q+1)!\,\Gamma\left(\lambda+\frac{1}{2}\right)\Gamma(j+\lambda)}{\sqrt{\pi}\,\Gamma(j+2\lambda)} \left(\sum_{p=0}^{\left\lfloor \frac{j-q}{2} \right\rfloor} \frac{2j-4p-2q+1}{(2p)!(2j-2p-2q+1)!} \times \right. \\
& {}_4F_3\left(\left. \begin{array}{c} -p, \frac{1}{2}-p, -\frac{1}{2}-j+p+q, -j+p+q \\ -\frac{1}{2}-\frac{j}{2}+\frac{q}{2}, -\frac{j}{2}+\frac{q}{2}, 1-j-\lambda \end{array} \right| 1 \right) S_{j-q-2p}(x) \\
& + \sum_{p=0}^{\left\lfloor \frac{1}{2}(j-q-1) \right\rfloor} \frac{1-2j+4p+2q}{(2p+1)!\,(2j-2p-2q)!} \times \\
& \left. {}_4F_3\left(\left. \begin{array}{c} -p, -\frac{1}{2}-p, -j+p+q, \frac{1}{2}-j+p+q \\ -\frac{1}{2}-\frac{j}{2}+\frac{q}{2}, -\frac{j}{2}+\frac{q}{2}, 1-j-\lambda \end{array} \right| 1 \right) S_{j-q-2p-1}(x) \right).
\end{aligned}
$$

Theorem 8. *Given two non-negative integers j, q with $j \geq q$, in terms of Schröder polynomials, the derivatives of Hermite polynomials have the following expression:*

$$
\begin{aligned}
D^q H_j(x) =& 2^j\, j!\,(j-q+1)! \times \\
& \left(\sum_{p=0}^{\left\lfloor \frac{j-q}{2} \right\rfloor} \frac{2j-4p-2q+1}{(2p)!(2j-2p-2q+1)!} \times \right. \\
& {}_4F_2\left(\left. \begin{array}{c} -p, \frac{1}{2}-p, -\frac{1}{2}-j+p+q, -j+p+q \\ -\frac{1}{2}-\frac{j}{2}+\frac{q}{2}, -\frac{j}{2}+\frac{q}{2} \end{array} \right| -1 \right) S_{j-q-2p}(x) \\
& + \sum_{p=0}^{\left\lfloor \frac{1}{2}(j-q-1) \right\rfloor} \frac{1-2j+4p+2q}{(2p+1)!(2j-2p-2q)!} \times \\
& \left. {}_4F_2\left(\left. \begin{array}{c} -p, -p-\frac{1}{2}, -j+p+q, \frac{1}{2}-j+p+q \\ -\frac{j}{2}+\frac{q}{2}, -\frac{1}{2}-\frac{j}{2}+\frac{q}{2} \end{array} \right| -1 \right) S_{j-q-2p-1}(x) \right).
\end{aligned}
$$

Theorem 9. *Given two non-negative integers j, q with $j \geq q$, in terms of Schröder polynomials, the derivatives of the generalized Laguerre polynomials have the following expression:*

$$
\begin{aligned}
D^q L_j^{(\alpha)}(x) =& (j-q+1)! \sum_{p=0}^{j-q} \frac{(-1)^{j+p}\,(2j-2p-2q+1)}{p!\,(2j-p-2q+1)!} \times \\
& {}_3F_1\left(\left. \begin{array}{c} -p, -\alpha-j, -1-2j+p+2q \\ -1-j+q \end{array} \right| 1 \right) S_{j-q-p}(x).
\end{aligned} \tag{22}
$$

Theorem 10. *Given two non-negative integers j, q with $j \geq q$, in terms of Schröder polynomials, the derivatives of the generalized Fibonacci polynomials have the following expression:*

$$\begin{aligned} D^q F_j^{A,B}(x) =& A^j\, j!\,(j-q+1)!\left(\sum_{p=0}^{\lfloor \frac{j-q}{2} \rfloor} \frac{2j-4p-2q+1}{(2p)!(2j-2p-2q+1)!} \times \right. \\ & {}_4F_3\left(\left. \begin{array}{c} -p, \frac{1}{2}-p, -\frac{1}{2}-j+p+q, -j+p+q \\ -j, -\frac{1}{2}-\frac{j}{2}+\frac{q}{2}, -\frac{j}{2}+\frac{q}{2} \end{array} \right| -\frac{4B}{A^2} \right) S_{j-q-2p}(x) \\ & + \sum_{p=0}^{\lfloor \frac{1}{2}(j-q-1) \rfloor} \frac{1-2j+4p+2q}{(2p+1)!(2j-2p-2q)!} \times \\ & \left. {}_4F_3\left(\left. \begin{array}{c} -p, -\frac{1}{2}-p, -j+p+q, \frac{1}{2}-j+p+q \\ -j, -\frac{1}{2}-\frac{j}{2}+\frac{q}{2}, -\frac{j}{2}+\frac{q}{2} \end{array} \right| -\frac{4B}{A^2} \right) S_{j-q-2p-1}(x) \right). \end{aligned} \tag{23}$$

Theorem 11. *Given two non-negative integers j, q with $j \geq q$, in terms of Schröder polynomials, the derivatives of the generalized Lucas polynomials have the following expression:*

$$\begin{aligned} D^q L_j^{R,S}(x) =& R^j\, j!\,(j-q+1)!\left(\sum_{p=0}^{\lfloor \frac{j-q}{2} \rfloor} \frac{2j-4p-2q+1}{(2p)!(2j-2p-2q+1)!} \times \right. \\ & {}_4F_3\left(\left. \begin{array}{c} -p, \frac{1}{2}-p, -\frac{1}{2}-j+p+q, -j+p+q \\ 1-j, -\frac{1}{2}-\frac{j}{2}+\frac{q}{2}, -\frac{j}{2}+\frac{q}{2} \end{array} \right| -\frac{4S}{R^2} \right) S_{j-q-2p}(x) \\ & + \sum_{p=0}^{\lfloor \frac{1}{2}(j-q-1) \rfloor} \frac{1-2j+4p+2q}{(2p+1)!(2j-2p-2q)!} \times \\ & \left. {}_4F_3\left(\left. \begin{array}{c} -p, -\frac{1}{2}-p, -j+p+q, \frac{1}{2}-j+p+q \\ 1-j, -\frac{1}{2}-\frac{j}{2}+\frac{q}{2}, -\frac{j}{2}+\frac{q}{2} \end{array} \right| -\frac{4S}{R^2} \right) S_{j-q-2p-1}(x) \right). \end{aligned} \tag{24}$$

4.3. Some Connection Formulas between Some Celebrated Polynomials

Some formulas relating Schröder polynomials to other polynomials are provided here. For the case where $q = 0$, all of the derivatives formulas presented in Sections 4.1 and 4.2 hold true. As a result, for every formula for a derivative, one can find a formula for a related connection. Some of these formulas are presented below.

Corollary 6. *For every non-negative integer j, the Schröder–Laguerre and the Laguerre–Schröder connection formulas are given by*

$$S_j(x) = \frac{(2j)!\,\Gamma(j+\alpha+1)}{(j+1)!} \sum_{m=0}^{j} \frac{(-1)^{j-m}}{m!\,\Gamma(j-m+\alpha+1)}\, {}_2F_2\left(\left. \begin{array}{c} -m, -1-j \\ -2j, -j-\alpha \end{array} \right| 1 \right) L_{j-m}^{(\alpha)}(x), \tag{25}$$

$$L_j^{(\alpha)}(x) = (j+1)! \sum_{m=0}^{j} \frac{(-1)^{j+m}(2j-2m+1)}{m!(2j-m+1)!}\, {}_3F_1\left(\left. \begin{array}{c} -m, -\alpha-j, -1-2j+m \\ -1-j \end{array} \right| 1 \right) S_{j-m}(x). \tag{26}$$

Proof. One can immediately acquire the two connection formulas in (25) and (26) by putting $q = 0$ in (18) and (22). □

Corollary 7. *For every $j \geq 1$, the Schröder-first kind Chebyshev connection formula is*

$$
\begin{aligned}
S_j(x) =& \frac{2^{1-j}\,(2j)!}{(j+1)!} \sum_{m=0}^{\lfloor \frac{j}{2} \rfloor} \frac{c_{j-2m}}{m!\,(j-m)!}\; {}_4F_3\left(\left. \begin{array}{c} -m, -\frac{1}{2} - \frac{j}{2}, -\frac{j}{2}, -j+m \\ \frac{1}{2}, \frac{1}{2} - j, -j \end{array} \right| 1 \right) T_{j-2m}(x) \\
&+ \frac{2^{2-j}(2j-1)!}{j!} \sum_{m=0}^{\lfloor \frac{1}{2}(j-1) \rfloor} \frac{c_{j-2m-1}}{m!(j-m-1)!}\; {}_4F_3\left(\left. \begin{array}{c} -m, \frac{1}{2} - \frac{j}{2}, -\frac{j}{2}, 1-j+m \\ \frac{3}{2}, \frac{1}{2} - j, 1-j \end{array} \right| 1 \right) \times \\
&T_{j-2m-1}(x),
\end{aligned} \tag{27}
$$

where the constants c_r are defined as in (17).

Remark 7. *The trigonometric identity corresponding to the connection formula (27) is:*

$$
\begin{aligned}
S_j(\cos\theta) =& \frac{2^{1-j}\,(2j)!}{(j+1)!} \sum_{m=0}^{\lfloor \frac{j}{2} \rfloor} \frac{c_{j-2m}}{m!\,(j-m)!}\; {}_4F_3\left(\left. \begin{array}{c} -m, -\frac{1}{2} - \frac{j}{2}, -\frac{j}{2}, -j+m \\ \frac{1}{2}, \frac{1}{2} - j, -j \end{array} \right| 1 \right) \cos((j-2m)\,\theta) \\
+& \frac{2^{2-j}\,(2j-1)!}{j!} \sum_{m=0}^{\lfloor \frac{1}{2}(j-1) \rfloor} \frac{c_{j-2m-1}}{m!\,(j-m-1)!}\; {}_4F_3\left(\left. \begin{array}{c} -m, \frac{1}{2} - \frac{j}{2}, -\frac{j}{2}, 1-j+m \\ \frac{3}{2}, \frac{1}{2} - j, 1-j \end{array} \right| 1 \right) \cos((j-2m-1)\,\theta).
\end{aligned}
$$

Remark 8. *Other trigonometric identities can be obtained using the connection formulas between Schröder polynomials and other kinds of Chebyshev polynomials.*

4.4. Relationships between Some Well-Known Numbers

Thanks to the connection formulas between some celebrated polynomials, some formulas linking some celebrated numbers can be obtained. In the following, we list some of these connections.

Corollary 8. *For every positive integer j, using the Fibonacci sequence, we can write the Schröder numbers as:*

$$
\begin{aligned}
S_j =& \frac{(2j)!}{(j+1)!} \sum_{m=0}^{\lfloor \frac{j}{2} \rfloor} \frac{(-1)^m (j-2m+1)}{m!(j-m+1)!}\; {}_4F_3\left(\left. \begin{array}{c} -\frac{1}{2} - \frac{j}{2}, -\frac{j}{2}, -m, -1-j+m \\ \frac{1}{2}, \frac{1}{2} - j, -j \end{array} \right| -\frac{1}{4} \right) F_{j-2m+1} \\
&+ \frac{(2j-1)!}{j!} \sum_{m=0}^{\lfloor \frac{j-1}{2} \rfloor} \frac{(-1)^{m+1}(-j+2m)}{m!(j-m)!}\; {}_4F_3\left(\left. \begin{array}{c} \frac{1}{2} - \frac{j}{2}, -\frac{j}{2}, -m, -j+m \\ \frac{3}{2}, \frac{1}{2} - j, 1-j \end{array} \right| -\frac{1}{4} \right) F_{j-2m}.
\end{aligned} \tag{28}
$$

Proof. Formulas (28) can be obtained by setting $q = 0$, $A = B = 1$, and $x = 1$ in Formula (19). □

Corollary 9. *For every positive integer j, using the Lucas numbers, we can write the Schröder numbers as:*

$$
\begin{aligned}
S_j =& \frac{(2j)!}{(j+1)!} \sum_{m=0}^{\lfloor \frac{j}{2} \rfloor} \frac{(-1)^m\, c_{j-2m}}{m!(j-m)!}\; {}_4F_3\left(\left. \begin{array}{c} -m, -\frac{1}{2} - \frac{j}{2}, -\frac{j}{2}, -j+m \\ \frac{1}{2}, \frac{1}{2} - j, -j \end{array} \right| -\frac{1}{4} \right) L_{j-2m} \\
&+ \frac{(2j-1)!}{j!} \sum_{m=0}^{\lfloor \frac{j-1}{2} \rfloor} \frac{(-1)^m\, c_{j-2m-1}}{m!(j-m-1)!}\; {}_4F_3\left(\left. \begin{array}{c} -m, \frac{1}{2} - \frac{j}{2}, -\frac{j}{2}, 1-j+m \\ \frac{3}{2}, \frac{1}{2} - j, 1-j \end{array} \right| -\frac{1}{4} \right) L_{j-2m-1}.
\end{aligned} \tag{29}
$$

Proof. Setting $q = 0$, $R = S = 1$, and $x = 1$ in Formula (21) yields Formula (29). □

Corollary 10. *In terms of Schröder numbers, the Fibonacci sequence F_{j+1} can be written for any positive integer j as:*

$$
\begin{aligned}
F_{j+1} =& j!\,(j+1)!\left(\sum_{p=0}^{\lfloor \frac{j}{2} \rfloor} \frac{2j-4p+1}{(2p)!\,(2j-2p+1)!}\;{}_4F_3\left(\begin{matrix} -p, \frac{1}{2}-\frac{1}{2}-j+p, -j+p \\ -\frac{1}{2}-\frac{j}{2}, -j, -\frac{j}{2} \end{matrix}\,\middle|\, -4\right) S_{j-2p}\right.\\
&\left. + \sum_{p=0}^{\lfloor \frac{j-1}{2} \rfloor} \frac{1-2j+4p}{(2p+1)!(2j-2p)!}\;{}_4F_3\left(\begin{matrix} -p, -\frac{1}{2}-p, \frac{1}{2}-j+p, -j+p \\ -\frac{1}{2}-\frac{j}{2}, -j, -\frac{j}{2} \end{matrix}\,\middle|\, -4\right) S_{j-2p-1}\right).
\end{aligned} \tag{30}
$$

Proof. Formulas (30) can be obtained by setting $q = 0$, $A = B = 1$ and $x = 1$ in Formula (23). □

Corollary 11. *In terms of Schröder numbers, the Lucas numbers L_j can be written for any positive integer j as:*

$$
\begin{aligned}
L_j =& j!\,(j+1)!\left(\sum_{p=0}^{\lfloor \frac{j}{2} \rfloor} \frac{2j-4p+1}{(2p)!\,(2j-2p+1)!}\;{}_4F_3\left(\begin{matrix} -p, \frac{1}{2}-p, -\frac{1}{2}-j+p, -j+p \\ -\frac{1}{2}-\frac{j}{2}, 1-j, -\frac{j}{2} \end{matrix}\,\middle|\, -4\right) S_{j-2p}\right.\\
&\left. + \sum_{p=0}^{\lfloor \frac{j-1}{2} \rfloor} \frac{1-2j+4p}{(2p+1)!(2j-2p)!}\;{}_4F_3\left(\begin{matrix} -p, -\frac{1}{2}-p, \frac{1}{2}-j+p, -j+p \\ -\frac{1}{2}-\frac{j}{2}, 1-j, -\frac{j}{2} \end{matrix}\,\middle|\, -4\right) S_{j-2p-1}\right).
\end{aligned} \tag{31}
$$

Proof. Formula (31) can be obtained by setting $q = 0$, $R = S = 1$, and $x = 1$ in Formula (24). □

5. Some Linearization Formulas Involving Schröder Polynomials

This section is interested in introducing some linearization formulas involving Schröder polynomials. To be more precise, we will state and prove three theorems concerned with Schröder polynomials.

- The first theorem gives the standard linearization formula of Schröder polynomials; that is, we solve the problem

$$S_m(x)S_n(x) = \sum_{r=0}^{m+n} G_{r,m,n}\, S_{m+n-r}(x),$$

  where $G_{r,m,n}$ are the linearization coefficients.
- The second theorem introduces an expression for the product of Schröder polynomials with any polynomial of the well-known four kinds of Chebyshev polynomials in terms of the same kind of polynomials.
- The third theorem introduces an expression for the product of Schröder polynomials with the generalized Fibonacci polynomials $F_m^{A,B}(x)$ in terms of $F_m^{A,B}(x)$.

Theorem 12. *Given two non-negative integers m and n, the next formula for linearization is valid:*

$$S_m(x)\,S_n(x) =$$
$$\frac{1}{2\,m\,n\,(m+1)\,(n+1)\pi}\left(\sum_{r=0}^{\lfloor\frac{m+n}{2}\rfloor}\frac{(2m+2n-4\,r+1)\,\Gamma\left(m-r+\frac{1}{2}\right)\,\Gamma\left(n-r+\frac{1}{2}\right)\,(m+n-r)!\,\Gamma\left(r+\frac{1}{2}\right)}{r!\,(m-r)!\,(n-r)!\,\Gamma\left(m+n-r+\frac{3}{2}\right)}\times\right.$$
$$\left(-(2n-2\,r+1)^2r^2+m^2\left(n+4nr-4r^2\right)+m(n-2r+1)\left(n+4nr-4r^2\right)\right)S_{m+n-2r}(x)$$
$$\left.+2\sum_{r=0}^{\lfloor\frac{1}{2}(m+n-1)\rfloor}\frac{(2m+2n-4r-1)\,\Gamma\left(m-r+\frac{1}{2}\right)\Gamma\left(n-r+\frac{1}{2}\right)(m+n-r)!\Gamma\left(r+\frac{3}{2}\right)}{r!(m-r-1)!(n-r-1)!\Gamma\left(m+n-r+\frac{1}{2}\right)}S_{m+n-2r-1}(x)\right).$$

Proof. As a starting point, we use the analytic form in (11) to get

$$S_m(x)\,S_n(x) = \sum_{\ell=0}^{m}\frac{\binom{2(m-\ell)}{m-\ell}\binom{2m-\ell}{\ell}}{m-\ell+1}\,x^{m-\ell}S_n(x).$$

The moment's formula of Schröder polynomials in (14) helps one get at the following equation:

$$\begin{aligned}S_m(x)\,S_n(x) =&\frac{4^n\,\Gamma\left(n+\frac{1}{2}\right)}{\sqrt{\pi}\,(n+1)!}\sum_{\ell=0}^{m}\frac{\binom{2(m-\ell)}{m-\ell}\binom{2m-\ell}{\ell}(m+n-\ell)!(m+n-\ell+1)!}{m-\ell+1}\times\\&\sum_{r=0}^{m-\ell+n}\frac{(-1)^{r+1}\,(-1-2m-2n+2\ell+2r)}{r!\,(2(m+n-\ell+1)-r-1)!}\times\\&{}_4F_3\left(\left.\begin{matrix}-r,-1-n,-n,-1-2m-2n+2\ell+r\\-2n,-1-m-n+\ell,-m-n+\ell\end{matrix}\right|1\right)S_{m+n-\ell-r}(x).\end{aligned}$$

Following some algebraic manipulation, the last formula can be rewritten as the following equation:

$$S_m(x)\,S_n(x) = \sum_{r=0}^{m+n}\sum_{\ell=0}^{r}F_{\ell,r,m,n}\,S_{m+n-r}(x), \tag{32}$$

where the coefficients $F_{\ell,r,m,n}$ are given by the following formula:

$$\begin{aligned}F_{\ell,r,m,n} =&\frac{(-1)^{r-\ell}\,4^n\,(2m+2n-2r+1)\binom{2\,(m-\ell)}{m-\ell}\binom{2m-\ell}{\ell}(m+n-\ell)!\,(m+n-\ell+1)!\Gamma\left(n+\frac{1}{2}\right)}{(-1-m+\ell)\,\sqrt{\pi}\,(n+1)!\,(2m+2n-\ell-r+1)!\,(r-\ell)!}\times\\&{}_4F_3\left(\left.\begin{matrix}\ell-r,-1-n,-n,-1-2m-2n+\ell+r\\-2n,-1-m-n+\ell,-m-n+\ell\end{matrix}\right|1\right).\end{aligned}$$

We make use of a suitable symbolic algorithm, such as Zeilberger's algorithm (see [57]), to reduce the linearization coefficients in (32) given by: $G_{r,m,n} = \sum_{\ell=0}^{r}F_{\ell,r,m,n}$ in the form

$$G_{r,m,n} =\frac{1}{m\,n\,(m+1)\,(n+1)\pi}\times$$
$$\begin{cases}\dfrac{(2m+2n-2r+1)\,\Gamma\left(m-\frac{r}{2}+\frac{1}{2}\right)\,\Gamma\left(n-\frac{r}{2}+\frac{1}{2}\right)\,\Gamma\left(m+n-\frac{r}{2}\right)!\,\Gamma\left(\frac{r+1}{2}\right)}{8\,\left(m-\frac{r}{2}\right)!\left(n-\frac{r}{2}\right)!\,\Gamma\left(m+n-\frac{r}{2}+\frac{3}{2}\right)\,\left(\frac{r}{2}\right)!}\times\\ \left(-r^2(-1-2n+r)^2+4m^2\left(n+2nr-r^2\right)+4m(n-r+1)\left(n+2nr-r^2\right)\right), & r\text{ even},\\ \dfrac{(2m+2n-2r+1)\,r\,\Gamma\left(m-\frac{r}{2}+1\right)\,\Gamma\left(n-\frac{r}{2}+1\right)\,\Gamma\left(m+n-\frac{r}{2}+\frac{3}{2}\right)\,\Gamma\left(\frac{r}{2}\right)}{\left(m-\frac{r+1}{2}\right)!\left(n-\frac{r+1}{2}\right)!\,\Gamma\left(m+n-\frac{r}{2}+1\right)\left(\frac{r-1}{2}\right)!}, & r\text{ odd}.\end{cases} \tag{33}$$

Now, using the reduction formula to the linearization coefficients $G_{p,i,n}$ that is given in (33), we have

$$S_m(x)\,S_n(x) = \sum_{r=0}^{\lfloor\frac{m+n}{2}\rfloor} G_{2r,m,n}\,S_{m+n-2r}(x) + \sum_{r=0}^{\lfloor\frac{1}{2}(m+n-1)\rfloor} G_{2r+1,m,n}\,S_{m+n-2r-1}(x),$$

which turns into the following relation:

$$\begin{aligned}
&S_m(x)\,S_n(x) = \\
&\frac{1}{2\,m\,n\,(m+1)\,(n+1)\pi}\left(\sum_{r=0}^{\lfloor\frac{m+n}{2}\rfloor}\frac{(2m+2n-4\,r+1)\,\Gamma\left(m-r+\frac{1}{2}\right)\Gamma\left(n-r+\frac{1}{2}\right)(m+n-r)!\,\Gamma\left(r+\frac{1}{2}\right)}{r!\,(m-r)!\,(n-r)!\,\Gamma\left(m+n-r+\frac{3}{2}\right)}\times\right.\\
&\left(-(2n-2\,r+1)^2r^2+m^2\left(n+4nr-4r^2\right)+m(n-2r+1)\left(n+4nr-4r^2\right)\right)S_{m+n-2r}(x)\\
&\left.+2\sum_{r=0}^{\lfloor\frac{1}{2}(m+n-1)\rfloor}\frac{(2m+2n-4r-1)\,\Gamma\left(m-r+\frac{1}{2}\right)\Gamma\left(n-r+\frac{1}{2}\right)(m+n-r)!\Gamma\left(r+\frac{3}{2}\right)}{r!(m-r-1)!(n-r-1)!\Gamma\left(m+n-r+\frac{1}{2}\right)}S_{m+n-2r-1}(x)\right).
\end{aligned}$$

This finishes the proof of Theorem 12. □

Theorem 13. *Assume $\phi_n(x)$ is one of the four well-known Chebyshev polynomials. The next linearization formula holds*

$$\begin{aligned}
S_m(x)\,\phi_n(x) =& \frac{\binom{2m}{m}}{2^m\,(m+1)}\sum_{r=0}^{m}\binom{m}{r}\,{}_4F_3\left(\left.\begin{matrix}-r,-\frac{1}{2}-\frac{m}{2},-\frac{m}{2},-m+r\\ \frac{1}{2},\frac{1}{2}-m,-m\end{matrix}\right|1\right)\phi_{m+n-2r}(x)\\
&+\frac{2^{1-m}(2m-1)!}{m!}\sum_{r=0}^{m-1}\frac{1}{r!(m-r-1)!}\,{}_4F_3\left(\left.\begin{matrix}-r,\frac{1}{2}-\frac{m}{2},-\frac{m}{2},1-m+r\\ \frac{3}{2},\frac{1}{2}-m,1-m\end{matrix}\right|1\right)\phi_{m+n-2r-1}(x).
\end{aligned}\tag{34}$$

Proof. Combining the unified moment's formula for the four types of Chebyshev polynomials (11) with the analytic formula (6), one obtains

$$S_m(x)\,\phi_n(x) = \sum_{\ell=0}^{m}\frac{2^{-m+\ell}\,(2m-\ell)!}{(m-\ell+1)((m-\ell)!)^2\,\ell!}\sum_{s=0}^{m-\ell}\binom{m-\ell}{s}\phi_{m+n-\ell-2s}(x),$$

which turns into the form:

$$\begin{aligned}
S_m(x)\,\phi_n(x) =& \sum_{r=0}^{\lfloor\frac{m+n}{2}\rfloor}\sum_{\ell=0}^{r}\frac{2^{-m+2\ell}\binom{m-2\ell}{r-\ell}\binom{2(m-2\ell)}{m-2\ell}\binom{2m-2\ell}{2\ell}}{m-2\ell+1}\phi_{m+n-2r}(x)\\
&+\sum_{r=0}^{\lfloor\frac{1}{2}(m+n-1)\rfloor}\sum_{\ell=0}^{r}\frac{2^{1-m+2\ell}\,(2m-2\ell-1)!}{(m-2\ell)!\,(2\ell+1)!\,(m-\ell-r-1)!\,(r-\ell)!}\phi_{m+n-2r-1}(x).
\end{aligned}\tag{35}$$

Now, we make use of the two identities:

$$\begin{aligned}
&\sum_{\ell=0}^{r}\frac{2^{-m+2\ell}\binom{m-2\ell}{r-\ell}\binom{2(m-2\ell)}{m-2\ell}\binom{2m-2\ell}{2\ell}}{m-2\ell+1}\\
&=\frac{2^{-m}\binom{i}{r}\binom{2m}{m}}{m+1}\,{}_4F_3\left(\left.\begin{matrix}-r,-\frac{1}{2}-\frac{m}{2},-\frac{m}{2},-m+r\\ \frac{1}{2},\frac{1}{2}-m,-m\end{matrix}\right|1\right),\\
&\sum_{\ell=0}^{r}\frac{2^{1-m+2\ell}(2m-2\ell-1)!}{(m-2\ell)!\,(2\ell+1)!\,(m-\ell-r-1)!\,(r-\ell)!}\\
&=\frac{2^{1-m}(2m-1)!}{m!\,r!\,(m-r-1)!}\,{}_4F_3\left(\left.\begin{matrix}-r,\frac{1}{2}-\frac{m}{2},-\frac{m}{2},1-m+r\\ \frac{3}{2},\frac{1}{2}-m,1-m\end{matrix}\right|1\right),
\end{aligned}$$

to convert Formula (35) into the following one:

$$S_m(x)\,\phi_n(x) = \frac{\binom{2m}{m}}{2^m\,(m+1)} \sum_{r=0}^{m} \binom{m}{r}\, {}_4F_3\left(\begin{array}{c} -r, -\frac{1}{2}-\frac{m}{2}, -\frac{m}{2}, -m+r \\ \frac{1}{2}, \frac{1}{2}-m, -m \end{array} \middle| 1 \right) \phi_{m+n-2r}(x)$$
$$+ \frac{2^{1-m}(2m-1)!}{m!} \sum_{r=0}^{m-1} \frac{1}{r!(m-r-1)!}\, {}_4F_3\left(\begin{array}{c} -r, \frac{1}{2}-\frac{m}{2}, -\frac{m}{2}, 1-m+r \\ \frac{3}{2}, \frac{1}{2}-m, 1-m \end{array} \middle| 1 \right) \phi_{m+n-2r-1}(x).$$

This proves Formula (34). □

Theorem 14. *Let $F_n^{A,B}(x)$ be the generalized Fibonacci polynomials that can be constructed by (8). The next formula for linearization is valid:*

$$S_m(x)\,F_n^{A,B}(x) = \frac{A^{-m}\binom{2m}{m}}{m+1} \sum_{r=0}^{m} (-B)^r \binom{m}{r}\, {}_4F_3\left(\begin{array}{c} -r, -\frac{1}{2}-\frac{m}{2}, -\frac{m}{2}, -m+r \\ \frac{1}{2}, \frac{1}{2}-m, -m \end{array} \middle| -\frac{A^2}{4B} \right) F_{m+n-2r}^{A,B}(x)$$
$$+ \frac{(2m-1)!}{m!} \sum_{r=0}^{m-1} \frac{A^{1-m}(-B)^r}{r!(m-r-1)!}\, {}_4F_3\left(\begin{array}{c} -r, \frac{1}{2}-\frac{m}{2}, -\frac{m}{2}, 1-m+r \\ \frac{3}{2}, \frac{1}{2}-m, 1-m \end{array} \middle| -\frac{A^2}{4B} \right) F_{m+n-2r-1}^{A,B}(x).$$

Proof. The analytic formula of Schröder polynomials enables one to get

$$S_m(x)\,F_n^{A,B}(x) = \sum_{\ell=0}^{m} \frac{\binom{2(m-\ell)}{m-\ell}\binom{2m-\ell}{\ell}}{m-\ell+1} x^{m-\ell} F_n^{A,B}(x).$$

Based on the moment formula of $F_n^{A,B}(x)$, the last formula turns into the following formula:

$$S_m(x)\,F_n^{A,B}(x) = \sum_{\ell=0}^{m} \frac{\binom{2(m-\ell)}{m-\ell}\binom{2m-\ell}{\ell}}{m-\ell+1} \sum_{s=0}^{m-\ell} A^{-m+\ell}\,(-B)^s \binom{m-\ell}{s} F_{m+n-\ell-2s}^{A,B},$$

which can be also converted into the following form:

$$S_m(x)\,F_n^{A,B}(x) = \sum_{r=0}^{m}\sum_{\ell=0}^{r} \frac{A^{-m+2\ell}(-B)^{r-\ell}\binom{m-2\ell}{r-\ell}\binom{2(m-2\ell)}{m-2\ell}\binom{2m-2\ell}{2\ell}}{m-2\ell+1} F_{m+n-2r}^{A,B}(x)$$
$$+ \sum_{r=0}^{m-1}\sum_{\ell=0}^{r} \frac{A^{1-m+2\ell}\,(-B)^{r-\ell}\,(2m-2\ell-1)!}{(m-2\ell)!\,(2\ell+1)!\,(m-\ell-r-1)!(r-\ell)!} F_{m+n-2r-1}^{A,B}(x).$$

Based on the following two identities:

$$\sum_{\ell=0}^{r} \frac{A^{-m+2\ell}\,(-B)^{r-\ell}\binom{m-2\ell}{r-\ell}\binom{2(m-2\ell)}{m-2\ell}\binom{2m-2\ell}{2\ell}}{m-2\ell+1}$$
$$= \frac{A^{-m}\,(-B)^r\binom{m}{r}\binom{2m}{m}}{m+1}\, {}_4F_3\left(\begin{array}{c} -r, -\frac{1}{2}-\frac{m}{2}, -\frac{m}{2}, -m+r \\ \frac{1}{2}, \frac{1}{2}-m, -m \end{array} \middle| -\frac{A^2}{4B} \right),$$
$$\sum_{\ell=0}^{r} \frac{A^{1-m+2\ell}(-B)^{r-\ell}(2m-2\ell-1)!}{(m-2\ell)!(2\ell+1)!(m-\ell-r-1)!(r-\ell)!}$$
$$= \frac{A^{1-m}(-B)^r(2m-1)!}{m!r!(m-r-1)!}\, {}_4F_3\left(\begin{array}{c} -r, \frac{1}{2}-\frac{m}{2}, -\frac{m}{2}, 1-m+r \\ \frac{3}{2}, \frac{1}{2}-m, 1-m \end{array} \middle| -\frac{A^2}{4B} \right),$$

the following formula can be obtained:

$$S_m(x)\, F_n^{A,B}(x) = \frac{A^{-m}\binom{2m}{m}}{m+1} \sum_{r=0}^{m} (-B)^r \binom{m}{r}\, {}_4F_3\left(\begin{matrix} -r, -\frac{1}{2}-\frac{m}{2}, -\frac{m}{2}, -m+r \\ \frac{1}{2}, \frac{1}{2}-m, -m \end{matrix} \,\middle|\, -\frac{A^2}{4B}\right) F_{m+n-2r}^{A,B}(x)$$
$$+ \frac{(2m-1)!}{m!} \sum_{r=0}^{m-1} \frac{A^{1-m}(-B)^r}{r!(m-r-1)!}\, {}_4F_3\left(\begin{matrix} -r, \frac{1}{2}-\frac{m}{2}, -\frac{m}{2}, 1-m+r \\ \frac{3}{2}, \frac{1}{2}-m, 1-m \end{matrix} \,\middle|\, -\frac{A^2}{4B}\right) F_{m+n-2r-1}^{A,B}(x).$$

This finishes the proof of Theorem 14. □

6. Repeated Integrals of Schröder Polynomials

In this section, we give a formula that expresses the repeated integrals of Schröder polynomials in terms of their original ones.

Theorem 15. *Let the q-times repeated integration of $S_r(x)$ be written as*

$$I_j^{(q)}(x) = \int^{(q)} S_j(x)\,(dx)^q = \overbrace{\int\int\ldots\int}^{q \text{ times}} S_j(x)\, \overbrace{dx\,dx\,\ldots\,dx}^{q \text{ times}};$$

then,

$$\begin{aligned} I_j^{(q)}(x) =& \frac{2^{-2q-1}}{j\,(j+1)} \sum_{\ell=0}^{q} \frac{(-1)^{\ell}\,(2j-4\ell+2q+1)\,\Gamma\left(j-\ell+\frac{1}{2}\right)(q-\ell+1)_{\ell}}{\ell!\,\Gamma\left(j-\ell+q+\frac{3}{2}\right)} \times \\ & \left(j^2+\ell(2\ell-2q-1)+j(1-2\ell+q)\right) S_{j+q-2\ell}(x) \\ & + \frac{4^{-q}}{j\,(j+1)} \sum_{\ell=0}^{q} \frac{(-1)^{\ell}(2j-4\ell+2q-1)\,\Gamma\left(j-\ell+\frac{1}{2}\right)(q-\ell)_{\ell+1}}{\ell!\,\Gamma\left(j-\ell+q+\frac{1}{2}\right)} S_{j+q-2\ell-1}(x) \\ & + \Omega_{q-1}(x), \end{aligned}$$

and $\Omega_{q-1}(x)$ is a polynomial whose degree does not exceed $(q-1)$.

Proof. If we integrate relation (11) q-times and make use of the identity

$$\int^{(q)} x^i\,(dx)^q = \frac{x^{i+q}}{(i+1)_q} + \Omega_{q-1}(x),$$

where $\Omega_{q-1}(x)$ is a polynomials of degree does not exceed $(q-1)$, then we get

$$I_j^{(q)}(x) = \sum_{r=0}^{j} \frac{(2j-r)!}{(j-r+1)!(j+q-r)!r!} x^{j-r+q} + \Omega_{q-1}(x).$$

Making use of relation (3) enables one to write

$$\begin{aligned} I_j^{(q)}(x) = & \sum_{r=0}^{j} \frac{(2j-r)!}{(j-r+1)!r!} \sum_{t=0}^{j+q-r} \frac{(-1)^t\,(2(j+q-r)-2t+1)\,(j+q-r+1)!}{t!\,(2(j+q-r)-t+1)!} S_{j-r+q-t}(x) \\ & + \Omega_{q-1}(x). \end{aligned}$$

The final formula can also take the following form by rearranging the terms:

$$\begin{aligned} I_j^{(q)}(x) = & \sum_{\ell=0}^{j} (2j-2\ell+2q+1) \sum_{r=0}^{\ell} \frac{(-1)^{\ell-r}\,(2j-r)!\,(j+q-r+1)!}{(j-r+1)!\,(\ell-r)!\,(2j-\ell+2q-r+1)!\,r!} S_{j+q-\ell}(x) \\ & + \Omega_{q-1}(x). \end{aligned}$$

Thanks to Zeilberger's algorithm [57], it can be demonstrated that the following identity holds:

$$\sum_{r=0}^{\ell}\frac{(-1)^{\ell-r}\,(2j-r)!\,(j+q-r+1)!}{(j-r+1)!\,(\ell-r)!\,(2j-\ell+2q-r+1)!\,r!}=\frac{q!}{j(j+1)}\times$$
$$\begin{cases}\dfrac{(-1)^{\ell/2}2^{-2(q+1)}\left(2j^2+\ell(\ell-2q-1)+2j(1-\ell+q)\right)\Gamma\left(j-\frac{\ell}{2}+\frac{1}{2}\right)}{\left(\frac{\ell}{2}\right)!\left(q-\frac{\ell}{2}\right)!\,\Gamma\left(j-\frac{\ell}{2}+q+\frac{3}{2}\right)}, & \ell \text{ even},\\[2ex] \dfrac{(-1)^{\frac{\ell-1}{2}}\,4^{-q}\,\Gamma\left(j-\frac{\ell}{2}+1\right)}{\left(\frac{\ell-1}{2}\right)!\left(q-\left(\frac{\ell-3}{2}\right)\right)!\Gamma\left(j-\frac{\ell}{2}+q+1\right)}, & \ell \text{ odd},\end{cases}$$

and accordingly, the following formula can be obtained:

$$\begin{aligned}I_j^{(q)}(x)=&\frac{2^{-1-2q}}{j\,(j+1)}\sum_{\ell=0}^{q}\frac{(-1)^{\ell}\,(2j-4\ell+2q+1)\,\Gamma\left(j-\ell+\frac{1}{2}\right)(q-\ell+1)_{\ell}}{\ell!\,\Gamma\left(j-\ell+q+\frac{3}{2}\right)}\times\\&\left(j^2+\ell(2\ell-2q-1)+j(1-2\ell+q)\right)S_{j+q-2\ell}(x)\\&+\frac{4^{-q}}{j\,(j+1)}\sum_{\ell=0}^{q}\frac{(-1)^{\ell}(2j-4\ell+2q-1)\,\Gamma\left(j-\ell+\frac{1}{2}\right)(q-\ell)_{\ell+1}}{\ell!\,\Gamma\left(j-\ell+q+\frac{1}{2}\right)}S_{j+q-2\ell-1}(x)\\&+\Omega_{q-1}(x).\end{aligned}$$

With that, we have completed our proof of Theorem 15. □

7. Concluding Remarks

This article was devoted to presenting some formulas concerned with Schröder polynomials. Several new formulas were established. New high-order derivatives and moment formulas of Schröder polynomials were given in terms of different polynomials; hence, new connection formulas between these polynomials with different polynomials could be obtained. The connection coefficients are often expressed in terms of the well-known generalized hypergeometric functions of different arguments. New connections between Schröder numbers and some other celebrated numbers were obtained. Furthermore, some new linearization formulas involving the Schröder polynomials were given. We do believe that the derived formulas in this article will be useful in some applications. In a forthcoming paper, we aim to employ polynomials of this kind from a practical point of view.

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