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Bifurcation-Type Results for the Fractional p-Laplacian with Parametric Nonlinear Reaction

Silvia Frassu and Antonio Iannizzotto *

Department of Mathematics and Computer Science, University of Cagliari, Via Ospedale 72, 09124 Cagliari, Italy
* Correspondence: antonio.iannizzotto@unica.it

Abstract: We consider a nonlinear, nonlocal elliptic equation driven by the degenerate fractional p-Laplacian with a Dirichlet boundary condition and involving a parameter $\lambda > 0$. The reaction is of general type, including concave–convex reactions as a special case. By means of variational methods and truncation techniques, we prove that there exists λ^* such that the problem has two positive solutions for $\lambda < \lambda^*$, one solution for $\lambda = \lambda^*$, and no solutions for $\lambda > \lambda^*$.

Keywords: fractional p-Laplacian; bifurcation; critical point theory

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1. Introduction and Main Result

In this paper, we deal with the following Dirichlet problem for a nonlinear, nonlocal equation:

$$(P_\lambda) \qquad \begin{cases} (-\Delta)_p^s\, u = f(x,u,\lambda) & \text{in } \Omega \\ u > 0 & \text{in } \Omega \\ u = 0 & \text{in } \Omega^c. \end{cases}$$

Here, $\Omega \subset \mathbb{R}^N$ ($N \geqslant 2$) is a bounded domain with $C^{1,1}$ boundary, $p \geqslant 2$, $s \in (0,1)$ s.t. $N > ps$, and the leading operator is the degenerate fractional p-Laplacian, which is defined for all $u : \mathbb{R}^N \to \mathbb{R}$ smooth enough and all $x \in \mathbb{R}^N$ by

$$(-\Delta)_p^s\, u(x) = 2 \lim_{\varepsilon \to 0^+} \int_{B_\varepsilon^c(x)} \frac{|u(x) - u(y)|^{p-2}(u(x) - u(y))}{|x-y|^{N+ps}}\, dy,$$

which embraces the (linear) fractional Laplacian, up to a dimensional constant, as a special case for $p = 2$. We consider a subcritical Carathéodory reaction $f : \Omega \times \mathbb{R} \times \mathbb{R}_0^+ \to \mathbb{R}$, also depending on a parameter $\lambda > 0$. Our hypotheses on the reaction include a $(p-1)$-sublinear behavior near the origin and a $(p-1)$-superlinear one at infinity along with a quasi-monotonicity condition and several conditions on the λ-dependance.

Under such assumptions, we prove a bifurcation-type result for problem (P_λ): namely, our problem admits at least two positive solutions for λ below a certain threshold $\lambda^* > 0$, at least one solution for $\lambda = \lambda^*$, and no solution for $\lambda > \lambda^*$. In addition, we study the behavior of solutions as $\lambda \to \lambda^*$.

Our reaction embraces the model case of the concave–convex reaction introduced in [1], i.e., the following pure power map with exponents $1 < q < p < r$:

$$t \mapsto \lambda t^{q-1} + t^{r-1} \ (t > 0).$$

Nonlocal elliptic equations driven by the fractional p-Laplacian with concave–convex reactions are investigated, for instance, in [2–6]. Other existence and bifurcation results for problems with several parametric reactions can be found in [7–10]. These are indeed only a few recent references out of a vast and increasing literature on fractional p-Laplacian equations, which is motivated by both intrinsic mathematical interest and applications in game theory and nonlinear Dirichlet-to-Neumann operators; see [11,12].

Here, we try to keep the λ-dependence as general as possible, assuming at the same time several conditions on the behavior of $f(\cdot,\cdot,\lambda)$. The main novelty of the present work, in the framework of nonlocal equations, is that we consider general parametric reactions rather than focusing on pure power type maps. In addition, with respect to previous results, we gain new monotonicity and convergence properties of the solutions with respect to λ.

We see (P_λ) as a variational problem, which can be treated by using critical point theory. Our approach mainly follows [13]. In particular, we shall often use two recent results on equivalence between Sobolev and Hölder minima of the energy functional from [14] and on strong maximum and comparison principles from [7]. This will allow us to establish a general sub-supersolution principle for problem (P_λ) and to slightly relax the assumptions on the mapping $\lambda \mapsto f(x,t,\lambda)$ with respect to [13]. In addition, in the proof of the nonexistence result, we will employ a recent anti-maximum principle proved in [15].

Our precise hypotheses on the reaction f are the following:

H $f : \Omega \times \mathbb{R} \times \mathbb{R}_0^+ \to \mathbb{R}$ is a Carathéodory map s.t. $f(x,0,\lambda) = 0$ for a.e. $x \in \Omega$ and all $\lambda > 0$, and for all $(x,t,\lambda) \in \Omega \times \mathbb{R} \times \mathbb{R}_0^+$, we set

$$F(x,t,\lambda) = \int_0^t f(x,\tau,\lambda)\, d\tau.$$

In addition, the following conditions hold:

(i) There exist $c_1 > 0$, $r \in (p, p_s^*)$, and for all $\lambda > 0$, a function $a_\lambda \in L^\infty(\Omega)_+$ s.t. $\lambda \mapsto \|a_\lambda\|_\infty$ is locally bounded, $\|a_\lambda\|_\infty \to 0$ as $\lambda \to 0$, and for a.e. $x \in \Omega$ and all $t \geqslant 0$, $\lambda > 0$
$$|f(x,t,\lambda)| \leqslant a_\lambda(x) + c_1 t^{r-1};$$

(ii) For all $\lambda > 0$, uniformly for a.e. $x \in \Omega$
$$\lim_{t\to\infty} \frac{f(x,t,\lambda)}{t^{p-1}} = \infty;$$

(iii) There exist $\rho \in (\frac{N}{ps}(r-p),\, p_s^*)$, and for all $\Lambda > 0$ a number $\theta > 0$ s.t. for all $\lambda \in (0,\Lambda]$, uniformly for a.e. $x \in \Omega$
$$\liminf_{t\to\infty} \frac{f(x,t,\lambda)t - pF(x,t,\lambda)}{t^\rho} > \theta;$$

(iv) For all $\Lambda > 0$ there exist $c_2, \delta > 0$, $q \in (1,p)$ s.t. for a.e. $x \in \Omega$ and all $t \in [0,\delta]$, $\lambda \geqslant \Lambda$
$$f(x,t,\lambda) \geqslant c_2 t^{q-1};$$

(v) For all $T, \Lambda > 0$ there exists $\sigma > 0$ s.t. for a.e. $x \in \Omega$ and for all $\lambda \in (0,\Lambda]$, the map $t \mapsto f(x,t,\lambda) + \sigma t^{p-1}$ is nondecreasing in $[0,T]$;

(vi) For a.e. $x \in \Omega$ and all $t > 0$, the map $\lambda \mapsto f(x,t,\lambda)$ is increasing in $\mathbb{R}_0^+$;

(vii) For all $0 < T_1 < T_2$, uniformly for a.e. $x \in \Omega$ and all $t \in [T_1, T_2]$
$$\lim_{\lambda\to\infty} f(x,t,\lambda) = \infty.$$

Hypothesis (i) is a subcritical growth condition. Hypotheses (ii) and (iii) govern the behavior of $f(x,\cdot,\lambda)$ at infinity, which is $(p-1)$-superlinear but tempered by an asymptotic condition of Ambrosetti–Rabinowitz type. By hypothesis (iv), $f(x,\cdot,\lambda)$ is $(p-1)$-sublinear near the origin, while hypothesis (v) is a quasi-monotonicity condition. Finally, hypotheses (vi) and (vii) are related to the λ-dependence of the reaction. For some examples of functions satisfying **H**, see the end of Section 3.

Under hypothesis **H**, we prove the following bifurcation-type result:

Theorem 1. *Let* **H** *hold. Then, there exists* $\lambda^* > 0$ *s.t.*

(i) *For all* $\lambda \in (0, \lambda^*)$ (P_λ) *has at least two solutions* $0 < u_\lambda < v_\lambda$, *s.t.* $u_\lambda < u_\mu$ *for all* $0 < \lambda < \mu < \lambda^*$;
(ii) (P_{λ^*}) *has at least one solution* $u^* > 0$ *s.t.* $u_\lambda \to u^*$ *uniformly in* Ω *as* $\lambda \to \lambda^*$;
(iii) *For all* $\lambda > \lambda^*$ (P_λ) *has no solutions.*

See Section 2 below for a proper definition of solution. Note that our result is new even in the semilinear case $p = 2$ (fractional Laplacian). In addition, note that we have no precise information on the behavior of the greater solution v_λ as $\lambda \to \lambda^*$ (this is why Theorem 1 is not literally a bifurcation result).

Notation: Throughout the paper, for any $A \subset \mathbb{R}^N$, we shall set $A^c = \mathbb{R}^N \setminus A$. For any two measurable functions $u, v : \Omega \to \mathbb{R}$, $u \leqslant v$ in Ω will mean that $u(x) \leqslant v(x)$ for a.e. $x \in \Omega$ (and similar expressions). The positive (resp., negative) part of u is denoted u^+ (resp., u^-). If X is an ordered Banach space, then X_+ will denote its non-negative order cone. For all $r \in [1, \infty]$, $\| \cdot \|_r$ denotes the standard norm of $L^r(\Omega)$ (or $L^r(\mathbb{R}^N)$), which will be clear from the context. Every function u defined in Ω will be identified with its 0 extension to $\mathbb{R}^N$. Moreover, C will denote a positive constant (whose value may change case by case).

2. Preliminaries

In this section, we recall some basic theory on the Dirichlet problem for a fractional p-Laplacian equation. We shall focus on such results which are most needed in our study and focus on simpler (if not most general) statements. We refer to [16] for a general introduction to variational methods for such a problem and to [17] for a detailed account on related regularity theory.

For all measurable $u : \Omega \to \mathbb{R}$, $s \in (0, 1)$, $p > 1$, we denote

$$[u]_{s,p,\Omega} = \Big[\iint_{\Omega \times \Omega} \frac{|u(x) - u(y)|^p}{|x - y|^{N+ps}} \, dx \, dy \Big]^{\frac{1}{p}}.$$

Accordingly, we define the fractional Sobolev space

$$W^{s,p}(\Omega) = \big\{ u \in L^p(\Omega) : \, [u]_{s,p,\Omega} < \infty \big\}.$$

If $\Omega \subset \mathbb{R}^N$ is a bounded domain with $C^{1,1}$-boundary, we also define

$$W^{s,p}_0(\Omega) = \big\{ u \in W^{s,p}(\mathbb{R}^N) : \, u = 0 \text{ in } \Omega^c \big\},$$

a uniformly convex, separable Banach space with norm $\|u\| = [u]_{s,p,\mathbb{R}^N}$ and dual space $W^{-s,p'}(\Omega)$. Assume now that $ps < N$ and set

$$p^*_s = \frac{Np}{N - ps},$$

then the embedding $W^{s,p}_0(\Omega) \hookrightarrow L^q(\Omega)$ is continuous for all $q \in [1, p^*_s]$ and compact for all $q \in [1, p^*_s)$ (see [18] for a quick introduction to fractional Sobolev spaces). We can now extend the definition of the fractional p-Laplacian (of order s) by setting for all $u, \varphi \in W^{s,p}_0(\Omega)$

$$\langle (-\Delta)^s_p \, u, \varphi \rangle = \iint_{\mathbb{R}^N \times \mathbb{R}^N} \frac{|u(x) - u(y)|^{p-2}(u(x) - u(y))(\varphi(x) - \varphi(y))}{|x - y|^{N+ps}} \, dx \, dy.$$

Such a definition is equivalent to the one given in Section 1, provided u is smooth enough, for instance if $u \in C^\infty_c(\Omega)$. More generally, we have defined $(-\Delta)^s_p : W^{s,p}_0(\Omega) \to W^{-s,p'}(\Omega)$ as a continuous, maximal monotone operator of $(S)_+$-type, i.e., whenever $u_n \rightharpoonup u$ in $W^{s,p}_0(\Omega)$ and

$$\limsup_n \langle (-\Delta)^s_p \, u_n, u_n - u \rangle \leqslant 0,$$

then we have $u_n \to u$ (strongly) in $W^{s,p}_0(\Omega)$. In addition, the map $(-\Delta)^s_p$ is strictly (T)-monotone, i.e., for all $u, v \in W^{s,p}_0(\Omega)$ s.t.

$$\langle(-\Delta)^s_p\, u - (-\Delta)^s_p\, v, (u-v)^+\rangle \leqslant 0,$$

we have $u \leqslant v$ in Ω. Finally, we recall that for all $u \in W^{s,p}_0(\Omega)$

$$\|u^\pm\|^p \leqslant \langle(-\Delta)^s_p\, u, \pm u^\pm\rangle.$$

All these properties are proved (in a slightly more general form) in [19], although some go back to previous works.

The general Dirichlet problem for the fractional p-Laplacian is stated as follows:

$$\begin{cases} (-\Delta)^s_p\, u = f_0(x,u) & \text{in } \Omega \\ u = 0 & \text{in } \Omega^c. \end{cases} \tag{1}$$

The reaction f_0 is subject to the following basic hypotheses:

$\mathbf{H}_0$ $f_0 : \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory map and there exist $c_0 > 0$, $r \in (1, p^*_s)$ s.t. for a.e. $x \in \Omega$ and all $t \in \mathbb{R}$

$$|f_0(x,t)| \leqslant c_0(1 + |t|^{r-1}).$$

By virtue of $\mathbf{H}_0$, the following definitions are well posed. We say that $u \in W^{s,p}_0(\Omega)$ is a (weak) supersolution of Equation (1) if for all $\varphi \in W^{s,p}_0(\Omega)_+$

$$\langle(-\Delta)^s_p\, u, \varphi\rangle \geqslant \int_\Omega f_0(x,u)\varphi\, dx.$$

The definition of a (weak) subsolution is analogous. Once again, we remark that these are not the most general definitions of super- and subsolution, as in general, one can require $u \geqslant 0$ or $u \leqslant 0$, respectively, in Ω^c (see [19]). Finally, we say that $u \in W^{s,p}_0(\Omega)$ is a (weak) solution of Equation (1) if it is both a super- and a subsolution, i.e., if for all $\varphi \in W^{s,p}_0(\Omega)$

$$\langle(-\Delta)^s_p\, u, \varphi\rangle = \int_\Omega f_0(x,u)\varphi\, dx.$$

For the solutions of Equation (1), we have the following a priori bound:

Proposition 1 ([20] (Theorem 3.3))**.** *Let $\mathbf{H}_0$ hold and $u \in W^{s,p}_0(\Omega)$ be a solution of Equation (1). Then, $u \in L^\infty(\Omega)$ with $\|u\|_\infty \leqslant C$, for some $C = C(\|u\|) > 0$.*

The regularity of solutions to nonlocal equations is a delicate issue, as such solutions fail in general to be smooth up to the boundary of the domain (no matter how regular it is). Such a problem can be overcome by comparing the solutions to a convenient power of the distance from the boundary, namely, set for all $x \in \mathbb{R}^N$

$$\mathrm{d}^s_\Omega(x) = \mathrm{dist}(x, \Omega^c)^s.$$

We define the following weighted Hölder spaces and the respective norms:

$$C^0_s(\overline{\Omega}) = \Big\{u \in C^0(\overline{\Omega}) : \frac{u}{\mathrm{d}^s_\Omega} \text{ has a continuous extension to } \overline{\Omega}\Big\},\ \|u\|_{0,s} = \Big\|\frac{u}{\mathrm{d}^s_\Omega}\Big\|_\infty,$$

and for all $\alpha \in (0,1)$

$$C^\alpha_s(\overline{\Omega}) = \Big\{u \in C^0(\overline{\Omega}) : \frac{u}{\mathrm{d}^s_\Omega} \text{ has a } \alpha\text{-Hölder continuous extension to } \overline{\Omega}\Big\},\ \|u\|_{\alpha,s} = \Big\|\frac{u}{\mathrm{d}^s_\Omega}\Big\|_{C^\alpha(\overline{\Omega})}.$$

The embedding $C^\alpha_s(\overline{\Omega}) \hookrightarrow C^0_s(\overline{\Omega})$ is compact for all $\alpha \in (0,1)$. In addition, the positive cone $C^0_s(\overline{\Omega})_+$ of $C^0_s(\overline{\Omega})$ has a nonempty interior given by

$$\mathrm{int}(C^0_s(\overline{\Omega})_+) = \Big\{ u \in C^0_s(\overline{\Omega}) : \inf_\Omega \frac{u}{\mathrm{d}^s_\Omega} > 0 \Big\}.$$

Combining Proposition 1 and [21] (Theorem 1.1), we have the following global regularity result for the degenerate case $p \geqslant 2$:

Proposition 2. *Let $p \geqslant 2$, $\mathbf{H}_0$ hold, and $u \in W^{s,p}_0(\Omega)$ be a solution of Equation (1). Then, $u \in C^\alpha_s(\overline{\Omega})$ for some $\alpha \in (0,s]$, with $\|u\|_{\alpha,s} \leqslant C(\|u\|)$.*

We recall now two recent strong maximum and comparison principles, which will be used in our study:

Proposition 3 ([7] (Theorem 2.6))**.** *Let $g \in C^0(\mathbb{R}) \cap BV_{\mathrm{loc}}(\mathbb{R})$, $u \in W^{s,p}_0(\Omega) \cap C^0(\overline{\Omega}) \setminus \{0\}$ s.t.*

$$\begin{cases} (-\Delta)^s_p\, u + g(u) \geqslant g(0) & \text{weakly in } \Omega \\ u \geqslant 0 & \text{in } \Omega. \end{cases}$$

Then,

$$\inf_\Omega \frac{u}{\mathrm{d}^s_\Omega} > 0.$$

In particular, if $u \in C^0_s(\overline{\Omega})$, then $u \in \mathrm{int}(C^0_s(\overline{\Omega})_+)$.

Proposition 4 ([7] (Theorem 2.7))**.** *Let $g \in C^0(\mathbb{R}) \cap BV_{\mathrm{loc}}(\mathbb{R})$, $u, v \in W^{s,p}_0(\Omega) \cap C^0(\overline{\Omega})$ s.t. $u \not\equiv v$, $C > 0$ satisfy*

$$\begin{cases} (-\Delta)^s_p\, v + g(v) \leqslant (-\Delta)^s_p\, u + g(u) \leqslant C & \text{weakly in } \Omega \\ 0 < v \leqslant u & \text{in } \Omega. \end{cases}$$

Then, $u > v$ in Ω. In particular, if $u, v \in \mathrm{int}(C^0_s(\overline{\Omega})_+)$, then $u - v \in \mathrm{int}(C^0_s(\overline{\Omega})_+)$.

Next, we recall some properties related to the following nonlocal, nonlinear eigenvalue problem:

$$\begin{cases} (-\Delta)^s_p\, u = \hat{\lambda}|u|^{p-1}u & \text{in } \Omega \\ u = 0 & \text{in } \Omega^c. \end{cases} \tag{2}$$

Problem (2) admits an unbounded sequence of variational (Lusternik–Schnirelmann) eigenvalues $(\hat{\lambda}_n)$. In particular, we focus on the principal eigenvalue $\hat{\lambda}_1$:

Proposition 5 ([22] (Theorem 6) and [23] (Theorems 4.1 and 4.2))**.** *The smallest eigenvalue of Equation (2) is*

$$\hat{\lambda}_1 = \min_{u \in W^{s,p}_0(\Omega) \setminus \{0\}} \frac{\|u\|^p}{\|u\|^p_p} > 0.$$

In addition, $\hat{\lambda}_1$ is simple, isolated, and attained at a unique positive eigenfunction $e_1 \in \mathrm{int}(C^0_s(\overline{\Omega})_+)$ s.t. $\|e_1\|_p = 1$.

All the non-principal eigenfunctions of Equation (2) are nodal (i.e., sign-changing) in Ω. More generally, we have the following anti-maximum principle for the degenerate case:

Proposition 6 ([15] (Lemma 3.9))**.** *Let $p \geqslant 2$, $\lambda \geqslant \hat{\lambda}_1$, $\beta \in L^\infty(\Omega)_+ \setminus \{0\}$, and $u \in W^{s,p}_0(\Omega)$ be a solution of*

$$\begin{cases} (-\Delta)^s_p\, u = \lambda|u|^{p-2}u + \beta(x) & \text{in } \Omega \\ u = 0 & \text{in } \Omega^c. \end{cases}$$

Then, $u^- \neq 0$.

Finally, we introduce a variational framework for problem (1). For all $(x,t) \in \Omega \times \mathbb{R}$ set

$$F_0(x,t) = \int_0^t f_0(x,\tau)\,d\tau,$$

and for all $u \in W_0^{s,p}(\Omega)$ set

$$\Phi_0(u) = \frac{\|u\|^p}{p} - \int_\Omega F_0(x,u)\,dx.$$

Then, $\Phi_0 \in C^1(W_0^{s,p}(\Omega))$ and its critical points coincide with the solutions of (1). In addition, Φ_0 is sequentially weakly l.s.c. in $W_0^{s,p}(\Omega)$ and its local minimizers in the topologies of $W_0^{s,p}(\Omega)$ and $C_s^0(\overline{\Omega})$, respectively, coincide:

Proposition 7 ([14] (Theorem 1.1))**.** *Let $p \geqslant 2$, $\mathbf{H}_0$ hold, and $u \in W_0^{s,p}(\Omega)$. Then, the following are equivalent:*

(i) There exists $\rho > 0$ s.t. $\Phi_0(u+v) \geqslant \Phi_0(u)$ for all $v \in W_0^{s,p}(\Omega) \cap C_s^0(\overline{\Omega})$, $\|v\|_{0,s} \leqslant \rho$;
(ii) There exists $\sigma > 0$ s.t. $\Phi_0(u+v) \geqslant \Phi_0(u)$ for all $v \in W_0^{s,p}(\Omega)$, $\|v\| \leqslant \sigma$.

3. Bifurcation-Type Result

This section is devoted to the proof of Theorem 1, which we split into several lemmas. We recall that $p \geqslant 2$, $s \in (0,1)$ satisfies $ps < N$, that $\Omega \subset \mathbb{R}^N$ is a bounded domain with $C^{1,1}$-boundary, and that the reaction f in problem (P_λ) satisfies the standing hypotheses **H** (for simplicity we shall omit such assumptions in the results of this section). Since **H** only deals with $t \geqslant 0$, without loss of generality, we set for all $(x,t,\lambda) \in \Omega \times \mathbb{R}^- \times \mathbb{R}_0^+$

$$f(x,t,\lambda) = 0.$$

We note that by **H** (i), $f(\cdot,\cdot,\lambda) : \Omega \times \mathbb{R} \to \mathbb{R}$ satisfies $\mathbf{H}_0$ for all $\lambda > 0$. For all $\lambda > 0$, $u \in W_0^{s,p}(\Omega)$, we define the energy functional of (P_λ)

$$\Phi_\lambda(u) = \frac{\|u\|^p}{p} - \int_\Omega F(x,u,\lambda)\,dx.$$

We begin with a sub-supersolution principle:

Lemma 1. *Let $\lambda > 0$, $\bar{u} \in \mathrm{int}(C_s^0(\overline{\Omega})_+)$ be a supersolution of (P_λ). Then, there exists a solution $u \in \mathrm{int}(C_s^0(\overline{\Omega})_+)$ of (P_λ) s.t. $u \leqslant \bar{u}$ in Ω.*

Proof. We perform a truncation on the reaction, setting for all $(x,t) \in \Omega \times \mathbb{R}$

$$\bar{f}_\lambda(x,t) = \begin{cases} f(x,t,\lambda) & \text{if } t < \bar{u}(x) \\ f(x,\bar{u}(x),\lambda) & \text{if } t \geqslant \bar{u}(x) \end{cases}$$

and

$$\bar{F}_\lambda(x,t) = \int_0^t \bar{f}_\lambda(x,\tau)\,d\tau.$$

By **H** (i), $\bar{f}_\lambda : \Omega \times \mathbb{R} \to \mathbb{R}$ satisfies $\mathbf{H}_0$. Moreover, for a.e. $x \in \Omega$ and all $t > \bar{u}(x)$, we have

$$\begin{aligned}\bar{F}_\lambda(x,t) &= \int_0^{\bar{u}} f(x,\tau,\lambda)\,d\tau + \int_{\bar{u}}^t f(x,\bar{u},\lambda)\,dx \\ &\leqslant \int_0^{\bar{u}} \left(a_\lambda(x) + c_1\tau^{r-1}\right) d\tau + \int_{\bar{u}}^t \left(a_\lambda(x) + c_1\bar{u}^{r-1}\right) d\tau \leqslant C(1+t).\end{aligned}$$

More generally, for a.e. $x \in \Omega$ and all $t \in \mathbb{R}$

$$\bar{F}_\lambda(x,t) \leqslant C(1+|t|). \tag{3}$$

In addition, we set for all $u \in W_0^{s,p}(\Omega)$

$$\bar{\Phi}_\lambda(u) = \frac{\|u\|^p}{p} - \int_\Omega \bar{F}_\lambda(x,u)\,dx.$$

By **H** (i), $\bar{\Phi}_\lambda \in C^1(W_0^{s,p}(\Omega))$ is sequentially weakly l.s.c. In addition, by Equation (3) and the continuous embedding $W_0^{s,p}(\Omega) \hookrightarrow L^1(\Omega)$, we have for all $u \in W_0^{s,p}(\Omega)$

$$\begin{aligned}\bar{\Phi}_\lambda(x) &\geqslant \frac{\|u\|^p}{p} - \int_\Omega C(1+|u|)\,dx \\ &\geqslant \frac{\|u\|^p}{p} - C(1+\|u\|),\end{aligned}$$

and the latter tends to ∞ as $\|u\| \to \infty$. So, $\bar{\Phi}_\lambda$ is coercive in $W_0^{s,p}(\Omega)$. Thus, there exists $u \in W_0^{s,p}(\Omega)$ s.t.

$$\bar{\Phi}_\lambda(u) = \inf_{W_0^{s,p}(\Omega)} \bar{\Phi}_\lambda = \bar{m}_\lambda.$$

Now, let $e_1 \in \mathrm{int}(C_s^0(\overline{\Omega})_+)$ be defined as in Proposition 5, $\delta > 0$ be as in **H** (iv). Then, we can find $\tau > 0$ s.t. in Ω

$$0 < \tau e_1 < \min\{\bar{u}, \delta\}.$$

By **H** (iv) and the construction of $\bar{\Phi}_\lambda$, we have

$$\begin{aligned}\bar{\Phi}_\lambda(\tau e_1) &= \frac{\tau^p\|e_1\|^p}{p} - \int_\Omega F(x,\tau e_1,\lambda)\,dx \\ &\leqslant \frac{\tau^p\hat{\lambda}_1}{p} - \frac{\tau^q c_2\|e_1\|_q^q}{q},\end{aligned}$$

and the latter is negative for $\tau > 0$ and even smaller if necessary (by $q < p$). So $\bar{m}_\lambda < 0$, which in turn implies $u_\lambda \neq 0$. By minimization, we have weakly in Ω

$$(-\Delta)_p^s\, u = \bar{f}_\lambda(x,u). \tag{4}$$

By Proposition 2, we have $u \in C_s^\alpha(\overline{\Omega})$. Testing (4) with $-u^- \in W_0^{s,p}(\Omega)$, we have

$$\begin{aligned}\|u^-\|^p &\leqslant \langle(-\Delta)_p^s\, u, -u^-\rangle \\ &= \int_{\{u<0\}} f(x,u,\lambda)u\,dx = 0.\end{aligned}$$

So, $u \geqslant 0$ in Ω. On the other hand, testing (4) with $(u-\bar{u})^+ \in W_0^{s,p}(\Omega)$ and recalling that $\bar{u}$ is a supersolution of (P_λ), we have

$$\langle(-\Delta)_p^s\, u - (-\Delta)_p^s\, \bar{u}, (u-\bar{u})^+\rangle \leqslant \int_{\{u>\bar{u}\}} \big(\bar{f}_\lambda(x,u) - f(x,\bar{u},\lambda)\big)(u-\bar{u})\,dx = 0.$$

By strict (T)-monotonicity of $(-\Delta)_p^s$, we have $u \leqslant \bar{u}$ in Ω. By construction, then, we can rephrase (4) and have weakly in Ω

$$(-\Delta)_p^s\, u = f(x,u,\lambda).$$

By **H** (v) (with $T = \|\bar{u}\|_\infty$ and $\Lambda = \lambda$), there exists $\sigma > 0$ s.t. for a.e. $x \in \Omega$ the mapping

$$t \mapsto f(x,t,\lambda) + \sigma t^{p-1}$$

is nondecreasing in $[0, \|\bar{u}\|_\infty]$. So, weakly in Ω

$$(-\Delta)_p^s\, u + \sigma u^{p-1} = f(x, u, \lambda) + \sigma u^{p-1} \geqslant 0.$$

By Proposition 3 (with $g(t) = \sigma(t^+)^{p-1}$), we have $u \in \operatorname{int}(C_s^0(\overline{\Omega})_+)$; in particular, $u > 0$ in Ω, so u solves (P_λ). □

Set

$$\lambda^* = \sup\left\{\lambda > 0 : \,(P_\lambda) \text{ has a solution } u_\lambda \in \operatorname{int}(C_s^0(\overline{\Omega})_+)\right\} \tag{5}$$

(with the convention $\inf \emptyset = \infty$). We will now establish some properties of λ^*:

Lemma 2. *Let λ^* be defined by Equation (5). Then, we have*

(i) $0 < \lambda^* < \infty$;

(ii) *For all $\lambda \in (0, \lambda^*)$ (P_λ) has a solution $u_\lambda \in \operatorname{int}(C_s^0(\overline{\Omega})_+)$;*

(iii) *For all $\lambda, \mu \in (0, \lambda^*)$ s.t. $\lambda < \mu$, we have $u_\mu - u_\lambda \in \operatorname{int}(C_s^0(\overline{\Omega})_+)$.*

Proof. First, we consider the auxiliary problem (torsion equation)

$$\begin{cases} (-\Delta)_p^s\, w = 1 & \text{in } \Omega \\ w = 0 & \text{in } \Omega^c. \end{cases} \tag{6}$$

The corresponding energy functional $\Psi \in C^1(W_0^{s,p}(\Omega))$ is defined for all $w \in W_0^{s,p}(\Omega)$ by

$$\Psi(w) = \frac{\|w\|^p}{p} - \int_\Omega w\, dx.$$

As in Section 2, we see that Ψ is coercive and sequentially weakly l.s.c., so there exists $w \in W_0^{s,p}(\Omega)$ s.t.

$$\Psi(w) = \inf_{W_0^{s,p}(\Omega)} \Psi.$$

In particular, w is a critical point of Ψ and hence a solution of (6), so by Proposition 2, we have $w \in C_s^\alpha(\overline{\Omega})$. Testing (6) with $-w^- \in W_0^{s,p}(\Omega)$, we obtain

$$\begin{aligned} \|w^-\|^p &\leqslant \langle (-\Delta)_p^s\, w, -w^- \rangle \\ &= \int_{\{w<0\}} w\, dx \leqslant 0, \end{aligned}$$

so $w \geqslant 0$ in Ω. In addition, clearly, $w \neq 0$. By Proposition 3, then, we have $w \in \operatorname{int}(C_s^0(\overline{\Omega})_+)$.

Now, we prove (i). First, we claim that there exists $\check{\lambda} > 0$ with the following property: for all $\lambda \in (0, \check{\lambda})$ there is $\xi_\lambda > 0$ s.t.

$$\|a_\lambda\|_\infty + c_1(\xi_\lambda \|w\|_\infty)^{r-1} < \xi_\lambda^{p-1} \tag{7}$$

(with $a_\lambda \in L^\infty(\Omega)_+$, $c_1 > 0$ as in **H** (i)). Arguing by contradiction, let (λ_n) be a sequence s.t. $\lambda_n \to 0^+$ and for all $n \in \mathbb{N}$, $\xi > 0$

$$\|a_{\lambda_n}\|_\infty + c_1(\xi \|w\|_\infty)^{r-1} \geqslant \xi^{p-1}.$$

By **H** (i), we have $\|a_{\lambda_n}\|_\infty \to 0$, so passing to the limit as $n \to \infty$ we obtain for all $\xi > 0$

$$c_1 \|w\|_\infty^{r-1} > \xi^{p-r},$$

which yields a contradiction as $\xi \to 0^+$. We prove next that $\lambda^* \geqslant \check{\lambda}$. Indeed, for all $\lambda \in (0, \check{\lambda})$, let $\xi_\lambda > 0$ satisfy Equation (7), and set

$$\bar{u} = \xi_\lambda w \in \operatorname{int}(C_s^0(\overline{\Omega})_+).$$

By Equations (6) and (7), and **H** (i), we have weakly in Ω

$$(-\Delta)_p^s\,\bar{u} = \xi_\lambda^{p-1} \geqslant a_\lambda(x) + c_1\bar{u}^{r-1} \geqslant f(x,\bar{u},\lambda),$$

i.e., $\bar{u}$ is a (strict) supersolution of (P_λ). By Lemma 1, there exists a solution $u_\lambda \in \mathrm{int}(C_s^0(\overline{\Omega})_+)$ of (P_λ) s.t. $u_\lambda \leqslant \bar{u}$ in Ω. Hence, we have $\lambda^* \geqslant \lambda$. Taking the supremum over λ, we obtain as claimed

$$\lambda^* \geqslant \check{\lambda} > 0.$$

Looking on the opposite side, we claim that there exists $\hat{\lambda} > 0$ s.t. for all $\lambda \geqslant \hat{\lambda}$ we have for a.e. $x \in \Omega$ and all $t > 0$

$$f(x,t,\lambda) > \hat{\lambda}_1 t^{p-1} \tag{8}$$

(with $\hat{\lambda}_1 > 0$ as in Proposition 5). Indeed, by **H** (ii), given $\lambda = 1$, we can find $T_2 > 0$ s.t. for a.e. $x \in \Omega$ and all $t > T_2$

$$f(x,t,1) > \hat{\lambda}_1 t^{p-1}.$$

By **H** (vi), for all $\lambda \geqslant 1$, a.e. $x \in \Omega$, and all $t > T_2$, we have

$$f(x,t,\lambda) > \hat{\lambda}_1 t^{p-1}.$$

In addition, let $c_2, \delta > 0$, $q \in (1,p)$ be as in **H** (iv). Then, we can find $T_1 > 0$ s.t.

$$T_1 < \min\{T_2,\, \delta\},\ \frac{c_2}{T_1^{p-q}} > \hat{\lambda}_1.$$

Hence, for all $\lambda \geqslant 1$, a.e. $x \in \Omega$ and $t \in (0,T_1)$ we have by **H** (iv)

$$f(x,t,\lambda) \geqslant c_2 t^{q-1} > \frac{c_2}{T_1^{p-q}} t^{p-1} \geqslant \hat{\lambda}_1 t^{p-1}.$$

Finally, by **H** (vii), we have uniformly for a.e. $x \in \Omega$ and all $t \in [T_1, T_2]$

$$\lim_{\lambda\to\infty} f(x,t,\lambda) = \infty,$$

so we can find $\hat{\lambda} \geqslant 1$ s.t. for all $\lambda \geqslant \hat{\lambda}$, a.e. $x \in \Omega$, and all $t \in [T_1, T_2]$

$$f(x,t,\lambda) > \hat{\lambda}_1 T_2^{p-1} \geqslant \hat{\lambda}_1 t^{p-1}.$$

Putting the inequalities above in a row, we obtain Equation (8). We see that $\lambda^* \leqslant \hat{\lambda}$, arguing by contradiction. Let $\lambda > \hat{\lambda}$ be s.t. (P_λ) has a solution $u_\lambda \in \mathrm{int}(C_s^0(\overline{\Omega})_+)$. Then, by Equation (8), we have weakly in Ω

$$(-\Delta)_p^s\,u_\lambda = f(x,u_\lambda,\lambda) > \hat{\lambda}_1 u_\lambda^{p-1}.$$

Set for all $x \in \Omega$

$$\beta(x) = f(x,u_\lambda(x),\lambda) - \hat{\lambda}_1 u_\lambda(x)^{p-1},$$

then by **H** (i) and the inequality above, we have $\beta \in L^\infty(\Omega)_+$, $\beta \neq 0$. By Proposition 6, we have $u_\lambda^- \neq 0$, a contradiction. Thus, we have

$$\lambda^* \leqslant \hat{\lambda} < \infty.$$

Furthermore, we prove (ii). For all $\lambda \in (0,\lambda^*)$, we can find $\mu \in (\lambda,\lambda^*)$ s.t. (P_μ) has a solution $u_\mu \in \mathrm{int}(C_s^0(\overline{\Omega})_+)$. By **H** (vi), we have weakly in Ω

$$(-\Delta)_p^s\,u_\mu = f(x,u_\mu,\mu) > f(x,u_\mu,\lambda),$$

i.e., u_μ is a (strict) supersolution of (P_λ). By Lemma 1, there exists a solution $u_\lambda \in \mathrm{int}(C_s^0(\overline{\Omega})_+)$ of (P_λ) s.t. $u_\lambda \leqslant u_\mu$ in Ω.

Finally, we prove (iii). For all $0 < \lambda < \mu < \lambda^*$, reasoning as above, we find $u_\lambda, u_\mu \in \mathrm{int}(C_s^0(\overline{\Omega})_+)$ solutions of (P_λ), (P_μ) respectively, s.t. $u_\lambda \leqslant u_\mu$ in Ω. Invoking **H** (v) (with $T = \|u_\mu\|_\infty$ and $\Lambda = \lambda^*$), we find $\sigma > 0$ s.t. the mapping

$$t \mapsto f(x,t,\lambda) + \sigma t^{p-1}$$

is nondecreasing in $[0, \|u_\mu\|_\infty]$. So, using also **H** (vi), weakly in Ω, we have

$$\begin{aligned}(-\Delta)_p^s\, u_\lambda + \sigma u_\lambda^{p-1} &= f(x,u_\lambda,\lambda) + \sigma u_\lambda^{p-1} \\ &\leqslant f(x,u_\mu,\lambda) + \sigma u_\mu^{p-1} \\ &< f(x,u_\mu,\mu) + \sigma u_\mu^{p-1} \\ &= (-\Delta)_p^s\, u_\mu + \sigma u_\mu^{p-1}.\end{aligned}$$

By Proposition 4 (with $g(t) = \sigma(t^+)^{p-1}$), we have $u_\mu - u_\lambda \in \mathrm{int}(C_s^0(\overline{\Omega})_+)$. $\square$

In the next result, we deal with the threshold case $\lambda = \lambda^*$:

Lemma 3. *Let $\lambda^* > 0$ be defined by Equation (5). Then, (P_{λ^*}) has at least one solution $u^* \in \mathrm{int}(C_s^0(\overline{\Omega})_+)$.*

Proof. Let (λ_n) be an increasing sequence in $\mathbb{R}_0^+$ s.t. $\lambda_n \to \lambda^*$. Recalling the proof of Lemma 2 (ii), we know that for all $n \in \mathbb{N}$, problem (P_{λ_n}) has a solution $u_n \in \mathrm{int}(C_s^0(\overline{\Omega})_+)$ with negative energy, i.e., weakly in Ω

$$(-\Delta)_p^s\, u_n = f(x,u_n,\lambda_n) \tag{9}$$

and

$$\|u_n\|^p - p\int_\Omega F(x,u_n,\lambda_n)\,dx < 0. \tag{10}$$

In addition, by Lemma 2 (iii), we have $u_n < u_m$ in Ω for all $n < m$. Testing (9) with $u_n \in W_0^{s,p}(\Omega)$, we obtain

$$\|u_n\|^p = \int_\Omega f(x,u_n,\lambda_n)u_n\,dx,$$

which along with Equation (10) gives

$$\int_\Omega \big(f(x,u_n,\lambda_n)u_n - pF(x,u_n,\lambda_n)\big)\,dx < 0. \tag{11}$$

By **H** (iii) (with $\Lambda = \lambda^*$), there exists $\theta, T > 0$ s.t. for all $n \in \mathbb{N}$, a.e. $x \in \Omega$, and all $t \geqslant T$ we have

$$f(x,t,\lambda_n)t - pF(x,t,\lambda_n) \geqslant \theta t^\rho.$$

In addition, by **H** (i) and since (λ_n) is bounded, we can find $C > 0$, independent of n, s.t. $\|a_{\lambda_n}\|_\infty \le C$, so for all $n \in \mathbb{N}$, a.e. $x \in \Omega$ and all $t \in [0,T]$ we have

$$|f(x,t,\lambda_n)| \le \|a_{\lambda_n}\|_\infty + c_1 T^{r-1} \le C$$

and

$$|F(x,t,\lambda_n)| \le CT,$$

so we obtain

$$\big|f(x,t,\lambda_n)t - pF(x,t,\lambda_n)\big| \leqslant C.$$

Summarizing, we have for all $n \in \mathbb{N}$, a.e. $x \in \Omega$, and all $t \geqslant 0$

$$f(x,t,\lambda_n)t - pF(x,t,\lambda_n) \geqslant \theta t^\rho - C,$$

with $\theta, C > 0$ independent of n. Plugging the estimate above into Equation (11), we obtain for all $n \in \mathbb{N}$

$$0 > \int_\Omega \big(\theta u_n^\rho - C\big)\, dx = \theta \|u_n\|_\rho^\rho - C.$$

So, (u_n) is a bounded sequence in $L^\rho(\Omega)$. Since $\rho \leqslant r < p_s^*$, we can find $\tau \in [0,1)$ s.t.

$$\frac{1}{r} = \frac{1-\tau}{\rho} + \frac{\tau}{p_s^*}.$$

By the interpolation and Sobolev's inequalities, we have for all $n \in \mathbb{N}$

$$\|u_n\|_r \leqslant \|u_n\|_\rho^{1-\tau} \|u_n\|_{p_s^*}^\tau \leqslant C\|u_n\|^\tau.$$

A straightforward calculation leads from the bounds on ρ in **H** (iii) to $\tau r < p$. Now, test (9) with $u_n \in W_0^{s,p}(\Omega)$ again and use **H** (i) to obtain

$$\begin{aligned}
\|u_n\|^p &= \int_\Omega f(x, u_n, \lambda_n) u_n\, dx \\
&\leqslant \int_\Omega \big(a_{\lambda_n}(x) + c_1 u_n^{r-1}\big) u_n\, dx \\
&\leqslant C\big(\|u_n\|_1 + \|u_n\|_r^r\big) \\
&\leqslant C\big(\|u_n\| + \|u_n\|^{\tau r}\big).
\end{aligned}$$

So, (u_n) is bounded in $W_0^{s,p}(\Omega)$. Passing to a subsequence if necessary, we find $u^* \in W_0^{s,p}(\Omega)$ s.t. $u_n \rightharpoonup u^*$ in $W_0^{s,p}(\Omega)$, $u_n \to u^*$ in $L^r(\Omega)$, and $u_n(x) \to u^*(x)$ for a.e. $x \in \Omega$. In particular, we have $u^* \geqslant 0$ in Ω. Test now (9) with $(u_n - u^*) \in W_0^{s,p}(\Omega)$, use **H** (i) and Hölder's inequality to obtain for all $n \in \mathbb{N}$

$$\begin{aligned}
\langle (-\Delta)_p^s\, u_n, u_n - u^* \rangle &= \int_\Omega f(x, u_n, \lambda_n)(u_n - u^*)\, dx \\
&\leqslant C \int_\Omega (1 + u_n^{r-1})(u_n - u^*)\, dx \\
&\leqslant C(1 + \|u_n\|_r^{r-1})\|u_n - u^*\|_r,
\end{aligned}$$

and the latter tends to 0 as $n \to \infty$. So, we have

$$\limsup_n \langle (-\Delta)_p^s\, u_n, u_n - u^* \rangle \leqslant 0,$$

which by the $(S)_+$-property of $(-\Delta)_p^s$ implies $u_n \to u^*$ in $W_0^{s,p}(\Omega)$. So, we can pass to the limit as $n \to \infty$ in Equation (9) and see that weakly in Ω

$$(-\Delta)_p^s\, u^* = f(x, u^*, \lambda^*).$$

By Proposition 2, we have $u^* \in C_s^\alpha(\overline{\Omega})$. Finally, since (u_n) is pointwise increasing, we have

$$\inf_\Omega \frac{u^*}{\mathrm{d}_\Omega^s} \geqslant \inf_\Omega \frac{u_n}{\mathrm{d}_\Omega^s} > 0.$$

So, $u^* \in \mathrm{int}(C_s^0(\overline{\Omega})_+)$ is a solution of (P_{λ^*}). $\square$

Finally, we prove that for any parameter below the threshold, there exists a second solution. This is in fact a fairly technical step in our study, involving some typical variational tricks. In particular, we recall the following notion:

Definition 1 ([24] (Definition 5.14))**.** *Let $(X, \|\cdot\|)$ be a Banach space, $\Phi \in C^1(X)$. Φ satisfies the Cerami* (C)*-condition if every sequence (u_n) in X, s.t. $(\Phi(u_n))$ is bounded and $(1 + \|u_n\|)\Phi'(u_n) \to 0$ in X^* has a (strongly) convergent subsequence.*

We can now prove our multiplicity result:

Lemma 4. *Let $\lambda^* > 0$ be defined by Equation (5). Then, for all $\lambda \in (0, \lambda^*)$, problem (P_λ) has a solution $v_\lambda \in \mathrm{int}(C_s^0(\overline{\Omega})_+)$ s.t. $v_\lambda - u_\lambda \in \mathrm{int}(C_s^0(\overline{\Omega})_+)$.*

Proof. From Lemma 3, we know that (P_{λ^*}) has a solution $u^* \in \mathrm{int}(C_s^0(\overline{\Omega})_+)$. Now, fix $\lambda \in (0, \lambda^*)$. By **H** (vi), we have weakly in Ω

$$(-\Delta)_p^s\, u^* = f(x, u^*, \lambda^*) > f(x, u^*, \lambda),$$

so u^* is a strict supersolution of (P_λ). By Lemma 1, we see that (P_λ) has a solution $u_\lambda \in \mathrm{int}(C_s^0(\overline{\Omega})_+)$ s.t. $u_\lambda \leqslant u^*$ in Ω (without any loss of generality, we may assume that such u_λ is the same as in Lemma 2). By **H** (v) (with $T = \|u^*\|_\infty$ and $\Lambda = \lambda^*$), there exists $\sigma > 0$ s.t. for a.e. $x \in \Omega$ the mapping

$$t \mapsto f(x, t, \lambda) + \sigma t^{p-1}$$

is nondecreasing in $[0, \|u^*\|_\infty]$. By **H** (vi), we have weakly in Ω

$$\begin{aligned}
(-\Delta)_p^s\, u_\lambda + \sigma u_\lambda^{p-1} &= f(x, u_\lambda, \lambda) + \sigma u_\lambda^{p-1} \\
&\leqslant f(x, u^*, \lambda) + \sigma (u^*)^{p-1} \\
&< f(x, u^*, \lambda^*) + \sigma (u^*)^{p-1} \\
&= (-\Delta)_p^s\, u^* + \sigma (u^*)^{p-1}.
\end{aligned}$$

By Proposition 4 (with $g(t) = \sigma (t^+)^{p-1}$), we have

$$u^* - u_\lambda \in \mathrm{int}(C_s^0(\overline{\Omega})_+).$$

Set for all $(x, t) \in \Omega \times \mathbb{R}$

$$\hat{f}_\lambda(x, t) = \begin{cases} f(x, u_\lambda(x), \lambda) & \text{if } t \leqslant u_\lambda(x) \\ f(x, t, \lambda) & \text{if } t > u_\lambda(x) \end{cases}$$

and

$$\hat{F}_\lambda(x, t) = \int_0^t \hat{f}_\lambda(x, \tau)\, d\tau.$$

In addition, set for all $u \in W_0^{s,p}(\Omega)$

$$\hat{\Phi}_\lambda(u) = \frac{\|u\|^p}{p} - \int_\Omega \hat{F}_\lambda(x, u)\, dx.$$

Clearly, $\hat{f}_\lambda : \Omega \times \mathbb{R} \to \mathbb{R}$ satisfies $\mathbf{H}_0$, so $\hat{\Phi}_\lambda \in C^1(W_0^{s,p}(\Omega))$. The rest of the proof aims at showing the following claim:

$$\hat{\Phi}_\lambda \text{ has a critical point } v_\lambda \in \mathrm{int}(C_s^0(\overline{\Omega})_+) \text{ s.t. } u_\lambda \leqslant v_\lambda \text{ in } \Omega,\ u_\lambda \neq v_\lambda. \tag{12}$$

We proceed by dichotomy. First, we introduce a new truncation of the reaction, setting for all $(x, t) \in \Omega \times \mathbb{R}$

$$\tilde{f}_\lambda(x, t) = \begin{cases} f(x, u_\lambda(x), \lambda) & \text{if } t \leqslant u_\lambda(x) \\ f(x, t, \lambda) & \text{if } u_\lambda(x) < t < u^*(x) \\ f(x, u^*(x), \lambda) & \text{if } t \geqslant u^*(x) \end{cases}$$

and

$$\tilde{F}_\lambda(x, t) = \int_0^t \tilde{f}_\lambda(x, \tau)\, d\tau.$$

Since $u_\lambda, u^* \in \mathrm{int}(C_s^0(\overline{\Omega})_+)$, $\tilde{f}_\lambda : \Omega \times \mathbb{R} \to \mathbb{R}$ satisfies $\mathbf{H}_0$. In addition, reasoning as in the proof of Equation (3), we see that for a.e. $x \in \Omega$ and all $t \in \mathbb{R}$, we have

$$|\tilde{F}_\lambda(x,t)| \leqslant C(1+|t|).$$

Set for all $u \in W_0^{s,p}(\Omega)$

$$\tilde{\Phi}_\lambda(u) = \frac{\|u\|^p}{p} - \int_\Omega \tilde{F}_\lambda(x,u)\,dx.$$

Then, $\tilde{\Phi}_\lambda \in C^1(W_0^{s,p}(\Omega))$ is coercive and sequentially weakly l.s.c. So, there exists $v_\lambda \in W_0^{s,p}(\Omega)$ s.t.

$$\tilde{\Phi}_\lambda(v_\lambda) = \inf_{W_0^{s,p}(\Omega)} \tilde{\Phi}_\lambda. \tag{13}$$

In particular, we have weakly in Ω

$$(-\Delta)_p^s\, v_\lambda = \tilde{f}_\lambda(x, v_\lambda). \tag{14}$$

By Proposition 2, we have $v_\lambda \in C_s^\alpha(\overline{\Omega})$. Testing (P_λ) and (14) with $(u_\lambda - v_\lambda)^+ \in W_0^{s,p}(\Omega)$, we have

$$\langle (-\Delta)_p^s\, u_\lambda - (-\Delta)_p^s\, v_\lambda, (u_\lambda - v_\lambda)^+ \rangle = \int_{\{u_\lambda > v_\lambda\}} \big(f(x,u_\lambda,\lambda) - \tilde{f}_\lambda(x,v_\lambda,\lambda)\big)(u_\lambda - v_\lambda)\,dx = 0,$$

so by strict (T)-monotonicity $u_\lambda \leqslant v_\lambda$ in Ω. As a consequence, $v_\lambda \in \mathrm{int}(C_s^0(\overline{\Omega})_+)$. In addition, testing (14) and (P_{λ^*}) with $(v_\lambda - u^*)^+ \in W_0^{s,p}(\Omega)$, we obtain

$$\langle (-\Delta)_p^s\, v_\lambda - (-\Delta)_p^s\, u^*, (v_\lambda - u^*)^+ \rangle = \int_{\{v_\lambda > u^*\}} \big(\tilde{f}_\lambda(x,v_\lambda) - f(x,u^*,\lambda^*)\big)(v_\lambda - u^*)\,dx,$$

and the latter is non-positive by $\lambda < \lambda^*$ and $\mathbf{H}$ (vi). So, as above $v_\lambda \leqslant u^*$ in Ω. Thus, in Equation (14), we can replace $\tilde{f}_\lambda(x, v_\lambda)$ by $f(x, v_\lambda, \lambda)$ and see that $v_\lambda \geqslant u_\lambda$ is a critical point of $\hat{\Phi}_\lambda$.

Now, either $v_\lambda \neq u_\lambda$, and then Equation (12) is proved, or $v_\lambda = u_\lambda$, i.e., by Equation (13) we have

$$\tilde{\Phi}_\lambda(u_\lambda) = \inf_{W_0^{s,p}(\Omega)} \tilde{\Phi}_\lambda.$$

Set now

$$V = \{u^* - v : \ v \in \mathrm{int}(C_s^0(\overline{\Omega})_+)\},$$

an open set in the $C_s^0(\overline{\Omega})$-topology s.t. $u_\lambda \in V$. By construction, for all $u \in V$, we have

$$\hat{\Phi}_\lambda(u) = \tilde{\Phi}_\lambda(u) \geqslant \tilde{\Phi}_\lambda(u_\lambda) = \hat{\Phi}_\lambda(u_\lambda).$$

So, u_λ is a local minimizer of $\hat{\Phi}_\lambda$ in $C_s^0(\overline{\Omega})$. By Proposition 7, then, u_λ is a local minimizer of $\hat{\Phi}_\lambda$ in $W_0^{s,p}(\Omega)$ as well. Once again, an alternative shows: either there exists a critical point $v_\lambda \neq u_\lambda$ of $\hat{\Phi}_\lambda$, and as above, we deduce $v_\lambda \geqslant u_\lambda$; hence, Equation (12) is proved, or u_λ is a strict local minimizer of $\hat{\Phi}_\lambda$.

We prove now that u_λ is not a global minimizer of $\hat{\Phi}_\lambda$. Indeed, by $\mathbf{H}$ (ii) and de l'Hôpital's rule, we have uniformly for a.e. $x \in \Omega$

$$\lim_{t \to \infty} \frac{F(x,t,\lambda)}{t^p} = \infty.$$

Let $\hat{\lambda}_1 > 0$, $e_1 \in \mathrm{int}(C_s^0(\overline{\Omega})_+)$ be defined as in Proposition 5, and fix $\varepsilon > 0$. Then, we can find $T > 0$ s.t. for a.e. $x \in \Omega$ and all $t \geqslant T$

$$F(x,t,\lambda) \geqslant \frac{\hat{\lambda}_1 + \varepsilon}{p} t^p.$$

By **H** (i) and the construction of $\hat{f}_\lambda$, we can find $C > 0$ s.t. for a.e. $x \in \Omega$ and all $t \geqslant 0$

$$\hat{F}_\lambda(x,t) \geqslant \frac{\hat{\lambda}_1 + \varepsilon}{p} t^p - C.$$

So, for all $\tau > 0$, we have

$$\begin{aligned}\hat{\Phi}_\lambda(\tau e_1) &\leqslant \frac{\tau^p \|e_1\|^p}{p} - \int_\Omega \Big(\frac{\hat{\lambda}_1 + \varepsilon}{p} (\tau e_1)^p - C \Big)\, dx \\ &\leqslant \frac{\tau^p \hat{\lambda}_1}{p} - \frac{\tau^p (\hat{\lambda}_1 + \varepsilon)}{p} + C,\end{aligned}$$

and the latter tends to $-\infty$ as $\tau \to \infty$. So, there exists $\tau > 0$ s.t.

$$\hat{\Phi}_\lambda(\tau e_1) < \hat{\Phi}_\lambda(u_\lambda).$$

In order to complete the geometrical picture, we deduce from the previous estimates that there exists $R \in (0, \|\tau e_1 - u_\lambda\|)$ s.t.

$$\inf_{\|u - u_\lambda\| = R} \hat{\Phi}_\lambda(u) = \eta \geqslant \hat{\Phi}_\lambda(u_\lambda) > \hat{\Phi}_\lambda(\tau e_1).$$

The next step consists in proving that $\hat{\Phi}_\lambda$ satisfies (C) (see Definition 1 above). Let (v_n) be a sequence in $W^{s,p}_0(\Omega)$ s.t. $|\hat{\Phi}_\lambda(v_n)| \leqslant C$ for all $n \in \mathbb{N}$, and $(1 + \|v_n\|)\hat{\Phi}'_\lambda(v_n) \to 0$ in $W^{-s,p'}(\Omega)$ as $n \to \infty$. Then, we have for all $n \in \mathbb{N}$

$$\Big| \|v_n\|^p - p \int_\Omega \hat{F}_\lambda(x, v_n)\, dx \Big| \leqslant C,$$

and there exists a sequence (ε_n) s.t. $\varepsilon_n \to 0^+$ and for all $n \in \mathbb{N}$, $\varphi \in W^{s,p}_0(\Omega)$

$$\Big| \langle (-\Delta)^s_p\, v_n, \varphi \rangle - \int_\Omega \hat{f}_\lambda(x, v_n) \varphi\, dx \Big| \leqslant \frac{\varepsilon_n \|\varphi\|}{1 + \|v_n\|}. \tag{15}$$

Subtracting the inequalities above, we obtain for all $n \in \mathbb{N}$

$$\int_\Omega \big(\hat{f}_\lambda(x, v_n) v_n - p\hat{F}_\lambda(x, v_n) \big)\, dx \leqslant C. \tag{16}$$

By **H** (iii), we can find $\theta, T > 0$ s.t. for a.e. $x \in \Omega$ and all $t \geqslant T$

$$f(x,t,\lambda)t - pF(x,t,\lambda) \geqslant \theta t^\rho.$$

By **H** (i), and the construction of $\hat{f}_\lambda$, we can find $C > 0$ s.t. for a.e. $x \in \Omega$ and all $t \geqslant 0$

$$\hat{f}_\lambda(x,t)t - p\hat{F}_\lambda(x,t) \geqslant \theta t^\rho - C.$$

Plugging such estimate into Equation (16), we have for all $n \in \mathbb{N}$

$$\theta \|v_n\|_\rho^\rho \leqslant \int_\Omega \big(\hat{f}_\lambda(x, v_n) v_n - p\hat{F}_\lambda(x, v_n) \big)\, dx + C \leqslant C,$$

so (v_n) is bounded in $L^\rho(\Omega)$. By the interpolation and Sobolev's inequalities, for all $n \in \mathbb{N}$, we have

$$\|v_n\|_r \leqslant C \|v_n\|^\tau$$

for some $\tau \in [0,1)$ independent of $n \in \mathbb{N}$ s.t. $\tau r < p$ (see the proof of Lemma 3). By Equation (15) (with $\varphi = v_n$), **H** (i), and Hölder's inequality, we have for all $n \in \mathbb{N}$

$$\begin{aligned}\|v_n\|^p &\leqslant \int_\Omega \hat{f}_\lambda(x,v_n)v_n\,dx + \frac{\varepsilon_n\|v_n\|}{1+\|v_n\|}\\ &\leqslant C\int_\Omega (1+|v_n|^{r-1})|v_n|\,dx + \varepsilon_n\\ &\leqslant C\big(\|v_n\|_1 + \|v_n\|_r^r\big) + \varepsilon_n.\end{aligned}$$

So, by Sobolev's embedding and the inequality above, we obtain

$$\|v_n\|^p \le C\big(1 + \|v_n\| + \|v_n\|^{\tau r}\big).$$

Since $\tau r < p$, (v_n) is bounded in $W_0^{s,p}(\Omega)$. Passing to a subsequence, we have $v_n \rightharpoonup v$ in $W_0^{s,p}(\Omega)$ and $v_n \to v$ in both $L^1(\Omega)$ and $L^r(\Omega)$. Setting $\varphi = v_n - v$ in Equation (15) and using **H** (i) and Hölder's inequality again, we have for all $n \in \mathbb{N}$

$$\begin{aligned}\langle(-\Delta)_p^s\, v_n, v_n - v\rangle &\leqslant \int_\Omega \hat{f}_\lambda(x,v_n)(v_n - v)\,dx + \frac{\varepsilon_n\|v_n - v\|}{1+\|v_n\|}\\ &\leqslant C\int_\Omega (1+|v_n|^{r-1})|v_n - v|\,dx + C\varepsilon_n\\ &\leqslant C\big(\|v_n - v\|_1 + \|v_n\|_r^{r-1}\|v_n - v\|_r\big) + C\varepsilon_n,\end{aligned}$$

and the latter tends to 0 as $n \to \infty$. By the $(S)_+$-property of $(-\Delta)_p^s$, we have $v_n \to v$ in $W_0^{s,p}(\Omega)$, which proves (C).

We have now all the necessary ingredients to apply the mountain pass theorem (see for instance [24] (Theorem 5.40)). Set

$$\Gamma = \big\{\gamma \in C([0,1], W_0^{s,p}(\Omega)) : \gamma(0) = u_\lambda,\ \gamma(1) = \tau e_1\big\}$$

and

$$c = \inf_{\gamma\in\Gamma}\max_{t\in[0,1]} \hat{\Phi}_\lambda(\gamma(t)).$$

Then, $c \geqslant \eta$, and there exists a critical point $v_\lambda \in W_0^{s,p}(\Omega)$ of $\hat{\Phi}_\lambda$ s.t. $\hat{\Phi}_\lambda(v_\lambda) = c$. Moreover, if $c = \eta$, then $\|v_\lambda - u_\lambda\| = R$. So $v_\lambda \neq u_\lambda$ satisfies weakly in Ω

$$(-\Delta)_p^s\, v_\lambda = \hat{f}_\lambda(x, v_\lambda).$$

As above we see that $v_\lambda \geqslant u_\lambda$, thus proving (12) in all cases.

By construction of $\hat{f}_\lambda$, v_λ solves (P_λ), hence $v_\lambda \in C_s^\alpha(\overline{\Omega})$. In addition, from $v_\lambda \geqslant u_\lambda$ we deduce that $v_\lambda \in \mathrm{int}(C_s^0(\overline{\Omega})_+)$. By **H** (v) (with $T = \|v_\lambda\|_\infty$) there exists $\sigma > 0$ s.t. for a.e. $x \in \Omega$ the mapping

$$t \mapsto f(x,t,\lambda) + \sigma t^{p-1}$$

is nondecreasing in $[0, \|v_\lambda\|_\infty]$. So, we have weakly in Ω

$$\begin{aligned}(-\Delta)_p^s\, u_\lambda + \sigma u_\lambda^{p-1} &= f(x,u_\lambda,\lambda) + \sigma u_\lambda^{p-1}\\ &\leqslant f(x,v_\lambda,\lambda) + \sigma v_\lambda^{p-1}\\ &= (-\Delta)_p^s\, v_\lambda + \sigma v_\lambda^{p-1}.\end{aligned}$$

By Proposition 4 (with $g(t) = \sigma(t^+)^{p-1}$), we have $v_\lambda - u_\lambda \in \mathrm{int}(C_s^0(\overline{\Omega})_+)$. □

We can now complete the proof of Theorem 1:

Proof. Let $\lambda^* > 0$ be defined by (5). By Lemmas 2 and 4, for all $\lambda \in (0,\lambda^*)$, problem (P_λ) has at least two solutions $u_\lambda, v_\lambda \in \mathrm{int}(C_s^0(\overline{\Omega})_+)$ s.t. $v_\lambda - u_\lambda \in \mathrm{int}(C_s^0(\overline{\Omega})_+)$, in particular $u_\lambda < v_\lambda$ in Ω. In addition, Lemma 2 (iii) says that for all $0 < \lambda < \mu < \lambda^*$, we have $u_\mu - u_\lambda \in \mathrm{int}(C_s^0(\overline{\Omega})_+)$, in particular $u_\lambda < u_\mu$ in Ω. This proves (i).

By Lemma 3, as $\lambda \to \lambda^*$ we have $u_\lambda \to u^*$, with $u^* \in \mathrm{int}(C_s^0(\overline{\Omega})_+)$ solution of (P_{λ^*}). This proves (ii).

Finally, by Lemma 2, (i) we have $\lambda^* < \infty$, and by Equation (5), for all $\lambda > \lambda^*$, there is no positive solution to (P_λ). This proves (iii). $\square$

Example 1. *We collect here some functions $f : \Omega \times \mathbb{R}^+ \times \mathbb{R}^+_0 \to \mathbb{R}$ satisfying hypotheses* **H** *(as usual, we assume $f(x,t,\lambda) = 0$ for all $x \in \Omega$, $t < 0$, and $\lambda > 0$:*

(a) *(Non-autonomous concave–convex reaction) let $1 < q < p < r < p_s^*$, $a, b \in L^\infty(\Omega)$ s.t. $a \geqslant a_0$, $b \geqslant b_0$ in Ω for some $a_0, b_0 > 0$, and set for all $(x,t,\lambda) \in \Omega \times \mathbb{R}^+ \times \mathbb{R}^+_0$*

$$f(x,t,\lambda) = \lambda a(x) t^{q-1} + b(x) t^{r-1};$$

(b) *(Autonomous reaction) let $1 < q < p < r < (Np + p^2 s)/N$, and set for all $(t,\lambda) \in \mathbb{R}^+ \times \mathbb{R}^+_0$*

$$f(t,\lambda) = \begin{cases} \lambda t^{q-1} & \text{if } t \in [0,1] \\ \lambda t^{p-1}(\ln(t)+1) & \text{if } t > 1, \end{cases}$$

noting that f does not satisfy the classical Ambrosetti–Rabinowitz condition.

Notably, our approach also works in the case when $f(x,\cdot,\lambda)$ is asymptotically $(p-1)$-linear at the origin:

Remark 1. *Assume that* **H** *holds, just replacing hypothesis* **H** *(iv) with the following:*

(iv) For all $\Lambda > 0$, there exist $\hat{\lambda}_1 < c_2 \leqslant c_3$ s.t. uniformly for a.e. $x \in \Omega$ and all $\lambda \in (0,\Lambda]$

$$c_2 \leqslant \liminf_{t\to 0^+} \frac{f(x,t,\lambda)}{t^{p-1}} \leqslant \limsup_{t\to 0^+} \frac{f(x,t,\lambda)}{t^{p-1}} \leqslant c_3.$$

Then, all the conclusions of Theorem 1 hold. Indeed, there are only two main steps at which the arguments for the present case differ from those seen above. The first is in the proof of Lemma 1, in proving that

$$\inf_{W_0^{s,p}(\Omega)} \tilde{\Phi}_\lambda < 0.$$

Indeed, fix $\varepsilon > 0$ s.t.

$$\varepsilon < c_2 - \hat{\lambda}_1.$$

We can find $\delta > 0$ s.t. for a.e. $x \in \Omega$ and all $t \in [0,\delta]$

$$f(x,t,\lambda) > (c_2 - \varepsilon) t^{p-1}.$$

Furthermore, since $e_1 \in \mathrm{int}(C_s^0(\overline{\Omega})_+)$, find $\tau > 0$ s.t. in Ω

$$0 < \tau e_1 \leqslant \min\{\delta, \bar{u}\}.$$

By de l'Hôpital's rule, we have

$$\begin{aligned}\tilde{\Phi}_\lambda(\tau e_1) &\leqslant \frac{\tau^p \|e_1\|^p}{p} - \int_\Omega (c_2 - \varepsilon) \frac{(\tau e_1)^p}{p}\, dx \\ &\leqslant \frac{\tau^p}{p}\big(\hat{\lambda}_1 - c_2 + \varepsilon\big) < 0.\end{aligned}$$

A second difference appears in the proof of Lemma 2, precisely in proving that $\lambda^* < \infty$. Using (iv) in the place of **H** *(iv), we easily obtain Equation (8), and the rest follows as above.*

An example of an (autonomous) reaction satisfying the modified hypotheses is the following: let $1 < q < p < r < p^*$, $\eta > \hat{\lambda}_1$, and set for all $(t,\lambda) \in \mathbb{R}^+ \times \mathbb{R}^+_0$

$$f(x,t,\lambda) = \begin{cases} \lambda t^{r-1} + \eta t^{p-1} & \text{if } t \in [0,1] \\ \lambda t^{q-1} + \eta t^{r-1} & \text{if } t > 1. \end{cases}$$

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References

1. Ambrosetti, A.; Brezis, H.; Cerami, G. Combined effects of concave and convex nonlinearities in some elliptic problems. *J. Funct. Anal.* **1994**, *122*, 519–543. [CrossRef]
2. Bhakta, M.; Mukherjee, D. Sign-changing solutions of p-fractional equations with concave–convex nonlinearities. *Topol. Methods Nonlinear Anal.* **2018**, *51*, 511–544. [CrossRef]
3. Carboni, G.; Mugnai, D. On some fractional equations with convex-concave and logistic-type nonlinearities. *J. Differ. Equ.* **2017**, *262*, 2393–2413. [CrossRef]
4. Daoues, A.; Hammami, A.; Saoudi, K. Multiplicity results of a nonlocal problem involving concave-convex nonlinearities. *Math. Notes* **2021**, *109*, 192–207. [CrossRef]
5. Kim, Y.H. Existence and multiplicity of solutions to a class of fractional p-Laplacian equations of Schrödinger type with concave-convex nonlinearities in $\mathbb{R}^N$. *Mathematics* **2020**, *8*, 1792. [CrossRef]
6. Lou, Q.; Luo, H. Multiplicity and concentration of positive solutions for fractional p-Laplacian problem involving concave-convex nonlinearity. *Nonlinear Anal. Real World Appl.* **2018**, *42*, 387–408. [CrossRef]
7. Iannizzotto, A.; Mosconi, S.; Papageorgiou, N.S. On the logistic equation for the fractional p-Laplacian. *arXiv* **2021**, arXiv:2101.05535.
8. Perera, K.; Squassina, M.; Yang, Y. Bifurcation and multiplicity results for critical fractional p-Laplacian problems. *Math. Nachr.* **2016**, *289*, 332–342. [CrossRef]
9. Xiang, M.; Zhang, B.; Rădulescu, V.D. Existence of solutions for perturbed fractional p-Laplacian equations. *J. Differ. Equ.* **2016**, *260*, 1392–1413. [CrossRef]
10. Zhi, Z.; Yan, L.; Yang, Z. Existence and multiplicity of solutions for a fractional p-Laplacian equation with perturbation. *J. Inequalities Appl.* **2020**, *2020*, 97. [CrossRef]
11. Bjorland, C.; Caffarelli, L.; Figalli, A. Nonlocal tug-of-war and the infinity fractional Laplacian. *Commun. Pure Appl. Math.* **2012**, *65*, 337–380. [CrossRef]
12. Warma, M. On a fractional (s, p)-Dirichlet-to-Neumann operator on bounded Lipschitz domains. *J. Elliptic Parabol. Equ.* **2018**, *4*, 223–269. [CrossRef]
13. Iannizzotto, A.; Papageorgiou, N.S. Existence, nonexistence and multiplicity of positive solutions for parametric nonlinear elliptic equations. *Osaka J. Math.* **2014**, *51*, 179–202.
14. Iannizzotto, A.; Mosconi, S.; Squassina, M. Sobolev versus Hölder minimizers for the degenerate fractional p-Laplacian. *Nonlinear Anal.* **2020**, *191*, 111635. [CrossRef]
15. Frassu, S.; Iannizzotto, A. Multiple solutions for the fractional p-Laplacian with jumping reactions. *J. Fixed Point Theory Appl.* **2023**, *25*, 25. [CrossRef]
16. Iannizzotto, A.; Liu, S.; Perera, K.; Squassina, M. Existence results for fractional p-Laplacian problems via Morse theory. *Adv. Calc. Var.* **2016**, *9*, 101–125. [CrossRef]
17. Palatucci, G. The Dirichlet problem for the p-fractional Laplace equation. *Nonlinear Anal.* **2018**, *177*, 699–732. [CrossRef]
18. Nezza, E.D.; Palatucci, G.; Valdinoci, E. Hitchhiker's guide to the fractional Sobolev spaces. *Bull. Sci. Math.* **2012**, *136*, 521–573. [CrossRef]
19. Frassu, S.; Iannizzotto, A. Extremal constant sign solutions and nodal solutions for the fractional p-Laplacian. *J. Math. Anal. Appl.* **2021**, *501*, 124205. [CrossRef]
20. Chen, W.; Mosconi, S.; Squassina, M. Nonlocal problems with critical Hardy nonlinearity. *J. Funct. Anal.* **2018**, *275*, 3065–3114. [CrossRef]
21. Iannizzotto, A.; Mosconi, S.; Squassina, M. Fine boundary regularity for the degenerate fractional p-Laplacian. *J. Funct. Anal.* **2020**, *279*, 108659. [CrossRef]

22. Lindgren, E.; Lindqvist, P. Fractional eigenvalues. *Calc. Var. Partial. Differ. Equ.* **2014**, *49*, 795–826. [CrossRef]
23. Franzina, G.; Palatucci, G. Fractional p-eigenvalues. *Riv. Math. Univ. Parma* **2014**, *5*, 373–386.
24. Motreanu, D.; Motreanu, V.V.; Papageorgiou, N.S. *Topological and Variational Methods with Applications to Nonlinear Boundary Value Problems*; Springer: New York, NY, USA, 2014.