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The Sufficiency of Solutions for Non-Smooth Minimax Fractional Semi-Infinite Programming with $(B_K,\rho)-$Invexity

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Abstract: Minimax fractional semi-infinite programming is an important research direction for semi-infinite programming, and has a wide range of applications, such as military allocation problems, economic theory, cooperative games, and other fields. Convexity theory plays a key role in many aspects of mathematical programming and is the foundation of mathematical programming research. The relevant theories of semi-infinite programming based on different types of convex functions have their own applicable scope and limitations. It is of great value to study semi-infinite programming on the basis of more generalized convex functions and obtain more general results. In this paper, we defined a new type of generalized convex function, based on the concept of the $K-$directional derivative, that is, uniform $(B_K,\rho)-$invex, strictly uniform $(B_K,\rho)-$invex, uniform $(B_K,\rho)-$pseudoinvex, strictly uniform $(B_K,\rho)-$pseudoinvex, uniform $(B_K,\rho)-$quasiinvex and weakly uniform $(B_K,\rho)-$quasiinvex function. Then, we studied a class of non-smooth minimax fractional semi-infinite programming problems involving this generalized convexity and obtained sufficient optimality conditions.

Keywords: non-smooth programming; fractional semi-infinite programming; $K-$directional derivative; uniform $(B_K,\rho)-$invexity; optimality conditions

MSC: 90C26; 90C30; 90C32; 90C34; 90C46

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1. Introduction

1.1. Background

Semi-infinite programming is an optimization problem with finite decision variables and infinite constraints. Its mathematical model is

$$(SIP)\ \min f(x)$$

$$s.t. g(x,y) \le 0, y \in Y$$

where $f : R^n \to R, g : R^n \times R^m \to R$, Y is a non-empty bounded closed set in R^m.

The study of semi-infinite programming can be traced back to 1924, when Haar [1] first considered linear systems with infinite constraints in the study of Chebyshev approximation, that is, linear semi-infinite programming, which was then called "Haar" programming. Later, John [2] also mentioned optimization problems with infinite constraints when studying Fritz John conditions. In 1962, Charnes, Cooper, and Kortanek [3] put forward this type of problem and called it "semi-infinite programming", which marked semi-infinite programming becoming one of the independent research branches of mathematical programming. Semi-infinite programming arises in some engineering problems, such as robot trajectory planning [4], vibrating membranes [5], and air pollution control [6]. Once proposed, it attracted extensive attention from many scholars and became a research hotspot. Minimax fraction semi-infinite programming is an important research direction for semi-infinite programming, and is widely used in engineering design, information technology, optimal control, cooperative games, and other fields.

1.2. The Related Work

Convexity plays an important role in optimization theory because it is the basis for studying the optimization, duality, and other related theories of various programming problems. Based on convexity assumptions, mathematical programming problems can be solved efficiently. To weaken the convexity assumption, various generalized convex functions have been introduced. Hanson introduced the invex function [7]. Bector and Singh presented the $b-$convex function [8]. Hanson defined the $F-$convex function [9]. Vial introduced the $\rho-$convex function [10]. Preda introduced the $(F,\rho)-$convex function as an extension of the $F-$convex function and the $\rho-$convex function [11]. Liang, Huang, and Pardalos introduced $(F,\alpha,\rho,d)-$convexity [12]. Yuan, Liu, Chinchuluun, and Pardalos defined $(C,\alpha,\rho,d)-$convexity, which is a generalization of $(F,\alpha,\rho,d)-$convexity [13]. Antczak defined the $(\Phi,\rho)-$invex and $(\Phi,\rho)-V-$invex function [14,15]. Yang defined the $K-(F_b,\rho)-$convex function and established duality results for multi-objective semi-infinite programming involving the generalized convexity assumptions [16].

On the basis of various convexity assumptions, many scholars have studied the optimality and duality of semi-infinite programming with different convexity and obtained a series of research results. Ruckman and Shapiro established the first-order optimality conditions for generalized convex semi-infinite programming [17]. Kim and Lee obtained the optimality of non-smooth Lipschitz optimization problems [18]. Kuk, Lee, and Tanino studied the optimality and duality for non-smooth multi-objective fractional programming with generalized invexity [19]. Nader obtained the necessary conditions for the optimality of non-smooth semi-infinite programs [20]. Liu and Wu derived the sufficient optimality conditions for minmax fractional programming in the framework of the $(F,\rho)-$convex function and invex function [21,22]. Mishra discussed the duality of non-differentiable minimax fractional programming involving generalized $\alpha-$uniform convexity [23]. Vazquez, Ruckman, and Werner studied the saddle points of non-convex semi-infinite programs [24]. Mordukhovich, Boris, and Nghia discussed the application of non-smooth cone-constrained optimization [25]. Mishra and Jaiswal discussed the optimality conditions and duality for non-differentiable multi-objective semi-infinite programming problems with generalized $(C,\alpha,\rho,d)-$convexity [26]. Yang, Chen, and Zhou studied the optimality conditions for semi-infinite and generalized semi-infinite programs via lower-order exact penalty functions [27]. Liu and Goberna presented asymptotic optimality conditions for linear semi-infinite programming [28]. Mishra, Singh, and Verma discussed the saddle point criteria in non-smooth semi-infinite minimax fractional programming problems [29]. Fan and Qin studied the stability of generalized semi-infinite optimization problems under functional perturbation [30].

1.3. Our Contributions

Many scholars have studied the problem of semi-infinite programming in which both the objective function and the constraint function are differentiable. However, for some practical problems, such as flood discharge from a water dam, or multi-input-multi-output control system and seismic structure design, the situation that the objective function or constraint function is not differentiable will be involved. However, it is impossible to study this kind of semi-infinite programming problem only via a set of theories in the differentiable case. Therefore, it is necessary to do special research on non-smooth semi-infinite programming.

As mentioned above, many scholars have studied the optimality theory and duality results of non-smooth semi-infinite programming involving different convexity, and have achieved results. Unfortunately, these research results have their own limitations and scope of application and a lack of systematization. Therefore, it is necessary to define a new type of non-smooth generalized convex functions, which makes some existing convex functions its special cases. Based on the newly defined non-smooth generalized convex function, the results for the optimality, duality, and other related theories of semi-infinite programming involving such convexity are more general and valuable.

In the present paper, we define a new type of convexity based on the concept of the $K-$directional derivative and obtained sufficient optimality conditions for non-smooth minimax fractional semi-infinite programming involving uniform $(B_K, \rho)-$invexity. Compared with the results in [21,22], the optimality conditions of non-smooth minimax fraction semi-infinite programming in this paper are more general, because the new generalized convex functions defined in this paper are a generalization of many existing convex functions, such as the $b-$convex function, invex function, $(F, \rho)-$convex function, etc. The research is helpful for enriching the relevant theories of non-smooth fractional semi-infinite programming.

1.4. Organization of the Paper

The rest of this paper is organized as follows. In Section 2, we first introduced the notions of local cone approximation and the $K-$directional derivative. Then, we defined a new type of generalized convex function based on the concept of the $K-$directional derivative that is used in this paper, that is, the uniform $(B_K, \rho)-$invex, strictly uniform $(B_K, \rho)-$invex, uniform $(B_K, \rho)-$pseudoinvex, strictly uniform $(B_K, \rho)-$pseudoinvex, uniform $(B_K, \rho)-$quasiinvex, and weakly uniform $(B_K, \rho)-$quasiinvex functions. In Section 3, we study a class of non-smooth minimax fractional semi-infinite programming involving the generalized convexities defined in Section 2 and obtain sufficient optimality conditions. Finally, we conclude this paper in Section 4.

2. Definitions and Preliminaries

Let us, first, recall the definition of local cone approximation and the $K-$directional derivative, which will be needed subsequently.

Definition 1 ([31])**.** *Let $X = [X, \tau]$ be a local convex Hausdorff space. A mapping $K : 2^X \times X \to 2^X$ is called a local cone approximation if a cone $K[M, x]$ is associated with each set $M \in 2^X$ and each $x \in X$ such that*

(i) $K(M - x, 0) = K(M, x)$.
(ii) $K(M \cap U, x) = K(M, x), \forall U \in N(x)$, *where $N(x)$ is a system of neighborhoods of x.*
(iii) $K(M, x) = X, \forall x \in \mathrm{int}M$, *where $\mathrm{int}M$ is the interior of the set M.*
(iv) $K(M, x) = \varnothing, \forall x \notin \overline{M}$.
(v) $K(\varphi(M), \varphi(x)) = \varphi(K(M, x))$, *where $\varphi : X \to X$ is any linear homeomorphism.*
(vi) $0^+M \subset 0^+K(M, x)$, *where 0^+M is Rockafellar's recession cone of M.*

Definition 2 ([31])**.** *Let $K(\cdot, \cdot)$ be a local cone approximation, the function $f^K(x, \cdot) : X \to R$ with*

$$f^K(x; y) := \inf\{\xi \in R | (y, \xi) \in K(epif, (x, f(x)), y \in R^n)\}$$

is called $K-$directional derivative of f at x.

In order to study the optimality of minimax fraction semi-infinite programming in Section 3, we will use the following important concepts. Throughout this paper, we suppose that $C \subset R^n$ is a non-empty set, $f^K(x, \cdot) : C \to R$ is the $K-$directional derivative of f at $x_0 \in C$, $b : C \times C \times [0, 1] \to R_+$, $\phi : R \to R$, $\lim\limits_{\lambda \to 0^+} b(x, x_0; \lambda) = b(x, x_0)$, $\eta : C \times C \to R^n$, $\rho \in R$, $\theta : C \times C \to R^n$. Elster and Thierfelder defined the $K-$directional derivative and the $K-$subdifferential and pointed out that the $K-$subdifferential is the most generalized. Using the $K-$directional derivative introduced in [31], some new generalized convex functions are defined as follows:

Definition 3. *The function $f : C \to R$ is said to be uniform $(B_K, \rho)-$invex at $x_0 \in C$, if for any $x \in C$, there exist b, ϕ, η, θ and ρ, such that*

$$b(x, x_0)\phi[f(x) - f(x_0)] \geq f^K(x_0; \eta(x, x_0)) + \rho\|\theta(x, x_0)\|^2.$$

Definition 4. *The function $f : C \to R$ is said to be strictly uniform $(B_K, \rho)-$invex at $x_0 \in C$, if for any $x \in C$ and $x \neq x_0$, there exist b, ϕ, η, θ and ρ, such that*

$$b(x, x_0)\phi[f(x) - f(x_0)] > f^K(x_0; \eta(x, x_0)) + \rho\|\theta(x, x_0)\|^2.$$

Definition 5. *The function $f : C \to R$ is said to be uniform $(B_K, \rho)-$pseudoinvex at $x_0 \in C$, if for any $x \in C$, there exist b, ϕ, η, θ and ρ, such that*

$$b(x, x_0)\phi[f(x) - f(x_0)] < 0 \Rightarrow f^K(x_0; \eta(x, x_0)) + \rho\|\theta(x, x_0)\|^2 < 0.$$

Definition 6. *The function $f : C \to R$ is said to be strictly uniform $(B_K, \rho)-$ pseudoinvex at $x_0 \in C$, if for any $x \in C$ and $x \neq x_0$, there exist b, ϕ, η, θ and ρ, such that*

$$b(x, x_0)\phi[f(x) - f(x_0)] \leq 0 \Rightarrow f^K(x_0; \eta(x, x_0)) + \rho\|\theta(x, x_0)\|^2 < 0.$$

Definition 7. *The function $f : C \to R$ is said to be uniform $(B_K, \rho)-$quasiinvex at $x_0 \in C$, if for any $x \in C$, there exist b, ϕ, η, θ and ρ, such that*

$$b(x, x_0)\phi[f(x) - f(x_0)] \leq 0 \Rightarrow f^K(x_0; \eta(x, x_0)) + \rho\|\theta(x, x_0)\|^2 \leq 0.$$

Definition 8. *The function $f : C \to R$ is said to be weakly uniform $(B_K, \rho)-$ quasiinvex at $x_0 \in C$, if for any $x \in C$, there exist b, ϕ, η, θ and ρ, such that*

$$b(x, x_0)\phi[f(x) - f(x_0)] < 0 \Rightarrow f^K(x_0; \eta(x, x_0)) + \rho\|\theta(x, x_0)\|^2 \leq 0.$$

Remark 1. *The uniform $(B_K, \rho)-$ invexity defined on the basis of $K-$directional derivatives is a kind of non-smooth generalized convex function, and many generalized convex functions are its special cases, such as $b-$convexity, $F-$convex function, $\rho-$convex function, $(F, \rho)-$convexity, etc.*

3. Sufficient Optimality Conditions

We consider the following minimax fractional semi-infinite programming problem:

$$(SIFP)\begin{cases} \min F(x) = \sup\limits_{y\in Y} \frac{f(x,y)}{h(x,y)}, \\ s.t. g(x, u) \leq 0, u \in U, x \in X. \end{cases}$$

where $X \neq \varnothing$ is an open subset of R^n, Y is a compact subset of R^m. $f(\cdot, \cdot) : X \times Y \to R$, $h(\cdot, \cdot) : X \times Y \to R$, $f(x, \cdot), h(x, \cdot)$ are continuous on Y for every $x \in X, g : X \times Y \to R$, $U \subset R$ is an infinite index set. $f(x, y) \geq 0$ and $h(x, y) > 0$ for each $(x, y) \in X \times Y$.

The feasible set of (SIFP) is denoted by X^0, i.e.,

$$X^0 = \{x | g(x, u) \leq 0, u \in U, x \in X\}.$$

We let

$$\Delta = \left\{ i \middle| g(x, u^i) \leq 0, x \in X, u^i \in U \right\},$$

$$I(x_0) = \left\{ i \middle| g(x_0, u^i) = 0, x_0 \in X, u^i \in U \right\},$$

$$U^* = \left\{ u^i \in U \middle| g(x, u^i) \leq 0, x \in X, i \in \Delta \right\} \text{ is a countable subset of } U,$$

$$\Lambda = \{\mu_j | \mu_j \geq 0, j \in \Delta, \text{ there is only finite } \mu_j \text{ such that } \mu_j > 0\},$$

$$\overline{Y}(x) = \left\{ y \in Y \middle| \frac{f(x,y)}{h(x,y)} = \sup_{z \in Y} \frac{f(x,z)}{h(x,z)} \right\},$$

$$Q = \{(s, \lambda, \overline{y}) \in N \times R^s_+ \times R^{ms} | 1 \leq s \leq n+1, \lambda = (\lambda_1, \lambda_2, \cdots, \lambda_s) \in R^s_+\}$$

with $\sum_{i=1}^{s} \lambda_i = 1$, and $\overline{y} = (\overline{y}_1, \overline{y}_2, \cdots, \overline{y}_s)$ with $\overline{y}_i \in \overline{Y}(x), i = 1, 2, \cdots, s\}$.

As Y is compact and $f(x,\cdot)$ and $h(x,\cdot)$ are continuous on Y, it is obviously that $\overline{Y}(x)$ is a non-empty compact subset of Y for each $x \in X$. For any $\overline{y}_i \in \overline{Y}(x_0)$, we let $q^* = \frac{f(x_0, \overline{y}_i)}{h(x_0, \overline{y}_i)}$, which is always a constant.

Definition 9. *A point $x^* \subset X^0$ is called an optimal solution for $(SIFP)$, for any $x \subset X^0$such that*

$$\sup_{y \in Y} \frac{f(x^*,y)}{h(x^*,y)} \leq \sup_{y \in Y} \frac{f(x,y)}{h(x,y)}.$$

Theorem 1. *Assume that $x^* \in X^0$, if for any $x \in X^0$, there exist $(s^*, \lambda^*, \overline{y}) \in Q$, $q^* \in R_+$, $\mu^*_j \in \Lambda$, $j \in \Delta$ and $b_1, \phi_1, b_2, \phi_2, \eta, \theta, \rho^* \in R^{s^*}$, $\tau^* \in R^{(\Delta)}$, such that*

(i) *For any $\overline{y}_i \in \overline{Y}(x^*)$, $(f - q^*h)(\cdot, \overline{y}_i)$ is uniform $(B_K, \rho^*_i)-$ invex at x^* with respect to b_1 and ϕ_1, $i = 1, 2, \cdots, s^*$;*

(ii) *For any $u^j \in U^*$, $g(\cdot, u^j)$ is uniform $(B_K, \tau^*_j)-$invex at x^* with respect to b_2 and ϕ_2, $j \in I(x^*)$;*

(iii) $\sum_{i=1}^{s^*} \lambda^*_i (f - q^*h)^K_x(x^*, \overline{y}_i; \eta(x, x^*)) + \sum_{j \in \Delta} \mu^*_j g^K_x(x^*, u_j; \eta(x, x^*)) \geq 0, \forall u^j \in U^*, j \in \Delta;$

(iv) $f(x^*, \overline{y}_i) - q^*h(x^*, \overline{y}_i) = 0, i = 1, 2, \cdots, s^*;$

(v) $\mu^*_j g(x^*, u^j) = 0, \forall u^j \in U^*, j \in \Delta;$

(vi) $\alpha < 0 \Rightarrow \phi_1(\alpha) < 0, \alpha \leq 0 \Rightarrow \phi_2(\alpha) \leq 0, b_1(x, x^*) > 0, b_2(x, x^*) \geq 0;$

(vii) $\sum_{i=1}^{s^*} \lambda^*_i \rho^*_i + \sum_{j \in \Delta} \mu^*_j \tau^*_j \geq 0.$

Then x^* is an optimal solution of $(SIFP)$.

Proof. If we suppose that x^* is not an optimal solution of $(SIFP)$, then there exists $\overline{x} \in X^0$, such that

$$\sup_{y \in Y} \frac{f(\overline{x},y)}{h(\overline{x},y)} < \sup_{y \in Y} \frac{f(x^*,y)}{h(x^*,y)}.$$

We observe that

$$\sup_{y \in Y} \frac{f(x^*,y)}{h(x^*,y)} = \frac{f(x^*,\overline{y}_i)}{h(x^*,\overline{y}_i)} = q^*, \forall \overline{y}_i \in \overline{Y}(x^*), i = 1, 2, \cdots, s^*.$$

Because

$$\frac{f(\overline{x},\overline{y}_i)}{h(\overline{x},\overline{y}_i)} \leq \sup_{y \in Y} \frac{f(\overline{x},y)}{h(\overline{x},y)},$$

we thus have

$$\frac{f(\overline{x},\overline{y}_i)}{h(\overline{x},\overline{y}_i)} < q^*, i = 1, 2, \cdots, s^*.$$

That is,

$$f(\overline{x},\overline{y}_i)-q^*h(\overline{x},\overline{y}_i)<0, i=1,2,\cdots,s^*.$$

By (iv), we have

$$f(\overline{x},\overline{y}_i)-q^*h(\overline{x},\overline{y}_i)<0=f(x^*,\overline{y}_i)-q^*h(x^*,\overline{y}_i), i=1,2,\cdots,s^*.$$

By (vi), we get

$$b_1(\overline{x},x^*)\phi_1[(f(\overline{x},\overline{y}_i)-q^*h(\overline{x},\overline{y}_i))-(f(x^*,\overline{y}_i)-q^*h(x^*,\overline{y}_i))]<0.$$

By (i), we have

$$(f-q^*h)_x^K(x^*,\overline{y}_i;\eta(\overline{x},x^*))+\rho_i^*\|\theta(\overline{x},x^*)\|^2<0.$$

As $\lambda_i^*\geq 0$ and $\sum_{i=1}^{s^*}\lambda_i^*=1$, we have

$$\sum_{i=1}^{s^*}\lambda_i^*(f-q^*h)_x^K(x^*,\overline{y}_i;\eta(\overline{x},x^*))+\sum_{i=1}^{s^*}\lambda_i^*\rho_i^*\|\theta(\overline{x},x^*)\|^2<0. \tag{1}$$

Observing that $g(\overline{x},u^j)\leq 0=g(x^*,u^j), \forall u^j\in U^*, j\in I(x^*)$, we have

$$g(\overline{x},u^j)-g(x^*,u^j)\leq 0, \forall u^j\in U^*, j\in I(x^*).$$

By (vi), we can obtain

$$b_2(\overline{x},x^*)\phi_2[g(\overline{x},u^j)-g(x^*,u^j)]\leq 0, \forall u^j\in U^*, j\in I(x^*).$$

By (ii), we get

$$g_x^K(x^*,u^j;\eta(\overline{x},x^*))+\tau_j^*\|\theta(\overline{x},x^*)\|^2\leq 0, \forall u^j\in U^*, j\in I(x^*).$$

As $\mu_j^*\in\Lambda, j\in I(x^*)$, we have

$$\sum_{j\in I(x^*)}\mu_j^*g_x^K(x^*,u^j;\eta(\overline{x},x^*))+\sum_{j\in I(x^*)}\mu_j^*\tau_j^*\|\theta(\overline{x},x^*)\|^2\leq 0, \forall u^j\in U^*, j\in I(x^*).$$

By hypothesis (v), we known that as $j\in\Delta\backslash I(x^*)$, $\mu_j^*=0$ always holds for any $u^j\in U^*$. This implies that

$$\sum_{j\in\Delta}\mu_j^*g_x^K(x^*,u^j;\eta(\overline{x},x^*))+\sum_{j\in\Delta}\mu_j^*\tau_j^*\|\theta(\overline{x},x^*)\|^2\leq 0, \forall u^j\in U^*, j\in\Delta. \tag{2}$$

Adding (1) and (2), we can obtain

$$\sum_{i=1}^{s^*}\lambda_i^*(f-q^*h)_x^K(x^*,\overline{y}_i;\eta(\overline{x},x^*))+\sum_{j\in\Delta}\mu_j^*g_x^K(x^*,u^j;\eta(\overline{x},x^*))$$

$$+(\sum_{i=1}^{s^*}\lambda_i^*\rho_i^*+\sum_{j\in\Delta}\mu_j^*\tau_j^*)\|\theta(\overline{x},x^*)\|^2<0, \forall u^j\in U^*, j\in\Delta.$$

By (vii), we have

$$\sum_{i=1}^{s^*}\lambda_i^*\rho_i^*+\sum_{j\in\Delta}\mu_j^*\tau_j^*\geq 0.$$

So,

$$\sum_{i=1}^{s^*} \lambda_i^*(f - q^*h)_x^K(x^*, \overline{y}_i; \eta(\overline{x}, x^*)) + \sum_{j\in\Delta} \mu_j^* g_x^K(x^*, u^j; \eta(\overline{x}, x^*)) < 0, \forall u^j \in U^*, j \in \Delta. \quad (3)$$

which contradicts (iii). Therefore, x^* is an optimal solution for $(SIFP)$. □

Theorem 2. *Assume that $x^* \in X^0$, if for any $x \in X^0$, there exist $(s^*, \lambda^*, \overline{y}) \in Q$, $q^* \in R_+$, $\mu_j^* \in \Lambda$, $j \in \Delta$ and $b_1, \phi_1, b_2, \phi_2, \eta, \theta, \rho^* \in R^{s^*}$, $\tau^* \in R^{(\Delta)}$, such that*

(i) *For any $\overline{y}_i \in \overline{Y}(x^*)$, $(f - q^*h)(\cdot, \overline{y}_i)$ is uniform $(B_K, \rho_i^*)-$pseudoinvex at x^* with respect to b_1 and ϕ_1, $i = 1, 2, \cdots, s^*$;*
(ii) *For any $u^j \in U^*, g(\cdot, u^j)$ is uniform $(B_K, \tau_j^*)-$quasiinvex at x^* with respect to b_2 and ϕ_2, $j \in I(x^*)$;*
(iii) $\sum_{i=1}^{s^*}\lambda_i^*(f - q^*h)_x^K(x^*, \overline{y}_i; \eta(x, x^*)) + \sum_{j\in\Delta}\mu_j^* g_x^K(x^*, u_j; \eta(x, x^*)) \geq 0, \forall u^j \in U^*, j \in \Delta;$
(iv) $f(x^*, \overline{y}_i) - q^*h(x^*, \overline{y}_i) = 0, i = 1, 2, \cdots, s^*;$
(v) $\mu_j^* g(x^*, u^j) = 0, \forall u^j \in U^*, j \in \Delta;$
(vi) $\alpha < 0 \Rightarrow \phi_1(\alpha) < 0, \alpha \leq 0 \Rightarrow \phi_2(\alpha) \leq 0, b_1(x, x^*) > 0, b_2(x, x^*) \geq 0;$
(vii) $\sum_{i=1}^{s^*}\lambda_i^*\rho_i^* + \sum_{j\in\Delta}\mu_j^*\tau_j^* \geq 0.$

Then x^* is an optimal solution of $(SIFP)$.

Proof. If we suppose that x^* is not an optimal solution of $(SIFP)$, then there exists $\overline{x} \in X^0$, such that

$$\sup_{y\in Y}\frac{f(\overline{x}, y)}{h(\overline{x}, y)} < \sup_{y\in Y}\frac{f(x^*, y)}{h(x^*, y)}.$$

We observe that

$$\sup_{y\in Y}\frac{f(x^*, y)}{h(x^*, y)} = \frac{f(x^*, \overline{y}_i)}{h(x^*, \overline{y}_i)} = q^*, \forall \overline{y}_i \in \overline{Y}(x^*), i = 1, 2, \cdots, s^*.$$

Because

$$\frac{f(\overline{x}, \overline{y}_i)}{h(\overline{x}, \overline{y}_i)} \leq \sup_{y\in Y}\frac{f(\overline{x}, y)}{h(\overline{x}, y)},$$

we thus have

$$\frac{f(\overline{x}, \overline{y}_i)}{h(\overline{x}, \overline{y}_i)} < q^*, i = 1, 2, \cdots, s^*.$$

That is,

$$f(\overline{x}, \overline{y}_i) - q^*h(\overline{x}, \overline{y}_i) < 0, i = 1, 2, \cdots, s^*.$$

By (iv), we have

$$f(\overline{x}, \overline{y}_i) - q^*h(\overline{x}, \overline{y}_i) < 0 = f(x^*, \overline{y}_i) - q^*h(x^*, \overline{y}_i), i = 1, 2, \cdots, s^*.$$

By (vi), we get

$$b_1(\overline{x}, x^*)\phi_1[(f(\overline{x}, \overline{y}_i) - q^*h(\overline{x}, \overline{y}_i)) - (f(x^*, \overline{y}_i) - q^*h(x^*, \overline{y}_i))] < 0.$$

By (i), we have

$$(f - q^*h)_x^K(x^*, \overline{y}_i; \eta(\overline{x}, x^*)) + \rho_i^*\|\theta(\overline{x}, x^*)\|^2 < 0.$$

As $\lambda_i^* \geq 0$ and $\sum_{i=1}^{s^*} \lambda_i^* = 1$, we have

$$\sum_{i=1}^{s^*} \lambda_i^* (f - q^* h)_x^K (x^*, \overline{y}_i; \eta(\overline{x}, x^*)) + \sum_{i=1}^{s^*} \lambda_i^* \rho_i^* \|\theta(\overline{x}, x^*)\|^2 < 0. \tag{4}$$

We observe that $g(\overline{x}, u^j) \leq 0 = g(x^*, u^j), \forall u^j \in U^*, j \in I(x^*)$, we have

$$g(\overline{x}, u^j) - g(x^*, u^j) \leq 0, \forall u^j \in U^*, j \in I(x^*).$$

By (vi), we can obtain

$$b_2(\overline{x}, x^*)\phi_2[g(\overline{x}, u^j) - g(x^*, u^j)] \leq 0, \forall u^j \in U^*, j \in I(x^*).$$

By (ii), we get

$$g_x^K(x^*, u^j; \eta(\overline{x}, x^*)) + \tau_j^* \|\theta(\overline{x}, x^*)\|^2 \leq 0, \forall u^j \in U^*, j \in I(x^*).$$

As $\mu_j^* \in \Lambda, j \in I(x^*)$, we have

$$\sum_{j \in I(x^*)} \mu_j^* g_x^K(x^*, u^j; \eta(\overline{x}, x^*)) + \sum_{j \in I(x^*)} \mu_j^* \tau_j^* \|\theta(\overline{x}, x^*)\|^2 \leq 0, \forall u^j \in U^*, j \in I(x^*).$$

By hypothesis (v), we known that as $j \in \Delta \backslash I(x^*), \mu_j^* = 0$ always holds for any $u^j \in U^*$. This implies that

$$\sum_{j \in \Delta} \mu_j^* g_x^K(x^*, u^j; \eta(\overline{x}, x^*)) + \sum_{j \in \Delta} \mu_j^* \tau_j^* \|\theta(\overline{x}, x^*)\|^2 \leq 0, \forall u^j \in U^*, j \in \Delta. \tag{5}$$

Adding (4) and (5), we can obtain

$$\sum_{i=1}^{s^*} \lambda_i^* (f - q^* h)_x^K (x^*, \overline{y}_i; \eta(\overline{x}, x^*)) + \sum_{j \in \Delta} \mu_j^* g_x^K(x^*, u^j; \eta(\overline{x}, x^*))$$

$$+ (\sum_{i=1}^{s^*} \lambda_i^* \rho_i^* + \sum_{j \in \Delta} \mu_j^* \tau_j^*) \|\theta(\overline{x}, x^*)\|^2 < 0, \forall u^j \in U^*, j \in \Delta.$$

By (vii), we have

$$\sum_{i=1}^{s^*} \lambda_i^* \rho_i^* + \sum_{j \in \Delta} \mu_j^* \tau_j^* \geq 0.$$

So,

$$\sum_{i=1}^{s^*} \lambda_i^* (f - q^* h)_x^K (x^*, \overline{y}_i; \eta(\overline{x}, x^*)) + \sum_{j \in \Delta} \mu_j^* g_x^K(x^*, u^j; \eta(\overline{x}, x^*)) < 0, \forall u^j \in U^*, j \in \Delta. \tag{6}$$

which contradicts (iii). Therefore, x^* is an optimal solution for $(SIFP)$. □

Theorem 3. *Assume that $x^* \in X^0$, if for any $x \in X^0$, there exist $(s^*, \lambda^*, \overline{y}) \in Q$, $q^* \in R_+$, $\mu_j^* \in \Lambda$, $j \in \Delta$ and $b_1, \phi_1, b_2, \phi_2, \eta, \theta, \rho^* \in R^{s^*}$, $\tau^* \in R^{(\Delta)}$, such that*

(i) *For any $\overline{y}_i \in \overline{Y}(x^*)$, $(f - q^* h)(\cdot, \overline{y}_i)$ is uniform $(B_K, \rho_i^*)-$ in vex at x^* with respect to b_1 and ϕ_1, $i = 1, 2, \cdots, s^*$;*
(ii) *For any $u^j \in U^*$, $g(\cdot, u^j)$ isuniform $(B_K, \tau_j^*)-$ quasiinvex at x^* with respect to b_2 and ϕ_2, $j \in I(x^*)$;*

(iii) $\sum_{i=1}^{s^*}\lambda_i^*(f-q^*h)_x^K(x^*,\overline{y}_i;\eta(x,x^*))+\sum_{j\in\Delta}\mu_j^*g_x^K(x^*,u_j;\eta(x,x^*))\geq 0,\forall u^j\in U^*,j\in\Delta;$
(iv) $f(x^*,\overline{y}_i)-q^*h(x^*,\overline{y}_i)=0,i=1,2,\cdots,s^*;$
(v) $\mu_j^*g(x^*,u^j)=0,\forall u^j\in U^*,j\in\Delta;$
(vi) $\alpha<0\Rightarrow\phi_1(\alpha)<0,\alpha\leq 0\Rightarrow\phi_2(\alpha)\leq 0,b_1(x,x^*)>0,b_2(x,x^*)\geq 0;$
(vii) $\sum_{i=1}^{s^*}\lambda_i^*\rho_i^*+\sum_{j\in\Delta}\mu_j^*\tau_j^*\geq 0.$

Then, x^* is an optimal solution of $(SIFP)$.

Theorem 4. *Assume that $x^*\in X^0$, if for any $x\in X^0$, there exist $(s^*,\lambda^*,\overline{y})\in Q$, $q^*\in R_+$, $\mu_j^*\in\Lambda$, $j\in\Delta$ and $b_1,\phi_1,b_2,\phi_2,\eta,\theta,\rho^*\in R^{s^*},\tau^*\in R^{(\Delta)}$, such that*

(i) *For any $\overline{y}_i\in\overline{Y}(x^*)$, $(f-q^*h)(\cdot,\overline{y}_i)$ is uniform $(B_K,\rho_i^*)-$quasiinvex at x^* with respect to b_1 and ϕ_1,$i=1,2,\cdots,s^*$;*
(ii) *For any $u^j\in U^*$, $g(\cdot,u^j)$ is strictly uniform $(B_K,\tau_j^*)-$invex at x^* with respect to b_2 and ϕ_2,$j\in I(x^*)$;*
(iii) $\sum_{i=1}^{s^*}\lambda_i^*(f-q^*h)_x^K(x^*,\overline{y}_i;\eta(x,x^*))+\sum_{j\in\Delta}\mu_j^*g_x^K(x^*,u_j;\eta(x,x^*))\geq 0,\forall u^j\in U^*,j\in\Delta;$
(iv) $f(x^*,\overline{y}_i)-q^*h(x^*,\overline{y}_i)=0,i=1,2,\cdots,s^*;$
(v) $\mu_j^*g(x^*,u^j)=0,\forall u^j\in U^*,j\in\Delta;$
(vi) $\alpha<0\Rightarrow\phi_1(\alpha)<0,\alpha\leq 0\Rightarrow\phi_2(\alpha)\leq 0,b_1(x,x^*)>0,b_2(x,x^*)\geq 0;$
(vii) $\sum_{i=1}^{s^*}\lambda_i^*\rho_i^*+\sum_{j\in\Delta}\mu_j^*\tau_j^*\geq 0.$

Then, x^* is an optimal solution for $(SIFP)$.

Theorem 5. *Assume that $x^*\in X^0$,if for any $x\in X^0$, there exist $(s^*,\lambda^*,\overline{y})\in Q$, $q^*\in R_+$, $\mu_j^*\in\Lambda$, $j\in\Delta$ and $b_1,\phi_1,b_2,\phi_2,\eta,\theta,\rho^*\in R^{s^*},\tau^*\in R^{(\Delta)}$, such that*

(i) *For any $\overline{y}_i\in\overline{Y}(x^*)$, $(f-q^*h)(\cdot,\overline{y}_i)$ is uniform $(B_K,\rho_i^*)-$quasiinvex at x^* with respect to b_1 and ϕ_1, $i=1,2,\cdots,s^*$;*
(ii) *For any $u^j\in U^*$, $g(\cdot,u^j)$ is strictly uniform $(B_K,\tau_j^*)-$pseudoinvex at x^* with respect to b_2 and ϕ_2, $j\in I(x^*)$;*
(iii) $\sum_{i=1}^{s^*}\lambda_i^*(f-q^*h)_x^K(x^*,\overline{y}_i;\eta(x,x^*))+\sum_{j\in\Delta}\mu_j^*g_x^K(x^*,u_j;\eta(x,x^*))\geq 0,\forall u^j\in U^*,j\in\Delta;$
(iv) $f(x^*,\overline{y}_i)-q^*h(x^*,\overline{y}_i)=0,i=1,2,\cdots,s^*;$
(v) $\mu_j^*g(x^*,u^j)=0,\forall u^j\in U^*,j\in\Delta;$
(vi) $\alpha<0\Rightarrow\phi_1(\alpha)<0,\alpha\leq 0\Rightarrow\phi_2(\alpha)\leq 0,b_1(x,x^*)>0,b_2(x,x^*)\geq 0;$
(vii) $\sum_{i=1}^{s^*}\lambda_i^*\rho_i^*+\sum_{j\in\Delta}\mu_j^*\tau_j^*\geq 0.$

Then, x^* is an optimal solution for $(SIFP)$.

The proofs of Theorems 3–5 are similar to Theorem 2.

Theorem 6. *Assume that $x^*\in X^0$, if for any $x\in X^0$, there exist $(s^*,\lambda^*,\overline{y})\in Q$, $q^*\in R_+$, $\mu_j^*\in\Lambda$, $j\in\Delta$ and $b_1,\phi_1,b_2,\phi_2,\eta,\theta,\rho^*\in R^{s^*},\tau^*\in R^{(\Delta)}$, such that*

(i) *For any $\overline{y}_i\in\overline{Y}(x^*)$, $(f-q^*h)(\cdot,\overline{y}_i)$ is weakly uniform $(B_K,\rho_i^*)-$ quasi in vex at x^* with respect to b_1 and ϕ_1, $i=1,2,\cdots,s^*$;*
(ii) *For any $u^j\in U^*$, $g(\cdot,u^j)$ is strictly uniform $(B_K,\tau_j^*)-$pseudoinvex at x^* with respect to b_2 and ϕ_2, $j\in I(x^*)$;*

(iii) $\sum_{i=1}^{s^*}\lambda_i^*(f-q^*h)_x^K(x^*,\overline{y}_i;\eta(x,x^*))+\sum_{j\in\Delta}\mu_j^* g_x^K(x^*,u_j;\eta(x,x^*))\geq 0, \forall u^j\in U^*, j\in\Delta;$

(iv) $f(x^*,\overline{y}_i)-q^*h(x^*,\overline{y}_i)=0, i=1,2,\cdots,s^*;$

(v) $\mu_j^* g(x^*,u^j)=0, \forall u^j\in U^*, j\in\Delta;$

(vi) $\alpha<0\Rightarrow\phi_1(\alpha)<0, \alpha\leq 0\Rightarrow\phi_2(\alpha)\leq 0, b_1(x,x^*)>0, b_2(x,x^*)\geq 0;$

(vii) $\sum_{i=1}^{s^*}\lambda_i^*\rho_i^*+\sum_{j\in\Delta}\mu_j^*\tau_j^*\geq 0.$

Then, x^* is an optimal solution for $(SIFP)$.

Proof. If we suppose that x^* is not an optimal solution of $(SIFP)$, then there exists $\overline{x}\in X^0$, such that

$$\sup_{y\in Y}\frac{f(\overline{x},y)}{h(\overline{x},y)}<\sup_{y\in Y}\frac{f(x^*,y)}{h(x^*,y)}.$$

Observe that

$$\sup_{y\in Y}\frac{f(x^*,y)}{h(x^*,y)}=\frac{f(x^*,\overline{y}_i)}{h(x^*,\overline{y}_i)}=q^*, \forall\overline{y}_i\in\overline{Y}(x^*), i=1,2,\cdots,s^*.$$

As

$$\frac{f(\overline{x},\overline{y}_i)}{h(\overline{x},\overline{y}_i)}\leq\sup_{y\in Y}\frac{f(\overline{x},y)}{h(\overline{x},y)},$$

we thus have

$$\frac{f(\overline{x},\overline{y}_i)}{h(\overline{x},\overline{y}_i)}<q^*, i=1,2,\cdots,s^*.$$

That is,

$$f(\overline{x},\overline{y}_i)-q^*h(\overline{x},\overline{y}_i)<0, i=1,2,\cdots,s^*.$$

By (iv), we have

$$f(\overline{x},\overline{y}_i)-q^*h(\overline{x},\overline{y}_i)<0=f(x^*,\overline{y}_i)-q^*h(x^*,\overline{y}_i), i=1,2,\cdots,s^*.$$

By (vi), we get

$$b_1(\overline{x},x^*)\phi_1[(f(\overline{x},\overline{y}_i)-q^*h(\overline{x},\overline{y}_i))-(f(x^*,\overline{y}_i)-q^*h(x^*,\overline{y}_i))]<0.$$

By (i), we have

$$(f-q^*h)_x^K(x^*,\overline{y}_i;\eta(\overline{x},x^*))+\rho_i^*\|\theta(\overline{x},x^*)\|^2<0.$$

As $\lambda_i^*\geq 0$ and $\sum_{i=1}^{s^*}\lambda_i^*=1$, we have

$$\sum_{i=1}^{s^*}\lambda_i^*(f-q^*h)_x^K(x^*,\overline{y}_i;\eta(\overline{x},x^*))+\sum_{i=1}^{s^*}\lambda_i^*\rho_i^*\|\theta(\overline{x},x^*)\|^2<0. \tag{7}$$

Observing that $g(\overline{x},u^j)\leq 0=g(x^*,u^j), \forall u^j\in U^*, j\in I(x^*)$, we have

$$g(\overline{x},u^j)-g(x^*,u^j)\leq 0, \forall u^j\in U^*, j\in I(x^*).$$

By (vi), we can obtain

$$b_2(\overline{x},x^*)\phi_2[g(\overline{x},u^j)-g(x^*,u^j)]\leq 0, \forall u^j\in U^*, j\in I(x^*).$$

By (ii), we get

$$g_x^K(x^*, u^j; \eta(\overline{x}, x^*)) + \tau_j^* \|\theta(\overline{x}, x^*)\|^2 \leq 0, \forall u^j \in U^*, j \in I(x^*).$$

As $\mu_j^* \in \Lambda, j \in I(x^*)$, we have

$$\sum_{j \in I(x^*)} \mu_j^* g_x^K(x^*, u^j; \eta(\overline{x}, x^*)) + \sum_{j \in I(x^*)} \mu_j^* \tau_j^* \|\theta(\overline{x}, x^*)\|^2 \leq 0, \forall u^j \in U^*, j \in I(x^*).$$

By hypothesis (v), we known that as $j \in \Delta \backslash I(x^*)$, $\mu_j^* = 0$ always holds for any $u^j \in U^*$. This implies that

$$\sum_{j \in \Delta} \mu_j^* g_x^K(x^*, u^j; \eta(\overline{x}, x^*)) + \sum_{j \in \Delta} \mu_j^* \tau_j^* \|\theta(\overline{x}, x^*)\|^2 \leq 0, \forall u^j \in U^*, j \in \Delta. \tag{8}$$

Adding (7) and (8), we can obtain

$$\sum_{i=1}^{s^*} \lambda_i^* (f - q^* h)_x^K(x^*, \overline{y}_i; \eta(\overline{x}, x^*)) + \sum_{j \in \Delta} \mu_j^* g_x^K(x^*, u^j; \eta(\overline{x}, x^*))$$

$$+(\sum_{i=1}^{s^*} \lambda_i^* \rho_i^* + \sum_{j \in \Delta} \mu_j^* \tau_j^*) \|\theta(\overline{x}, x^*)\|^2 < 0, \forall u^j \in U^*, j \in \Delta.$$

By (vii), we have

$$\sum_{i=1}^{s^*} \lambda_i^* \rho_i^* + \sum_{j \in \Delta} \mu_j^* \tau_j^* \geq 0.$$

So,

$$\sum_{i=1}^{s^*} \lambda_i^* (f - q^* h)_x^K(x^*, \overline{y}_i; \eta(\overline{x}, x^*)) + \sum_{j \in \Delta} \mu_j^* g_x^K(x^*, u^j; \eta(\overline{x}, x^*)) < 0, \forall u^j \in U^*, j \in \Delta. \tag{9}$$

which contradicts (iii). Therefore, x^* is an optimal solution for $(SIFP)$. □

4. Discussion

Because there are many kinds of generalized convex functions, it is not easy to define a new class of more generalized convex functions. To this end, we consulted a number of relevant materials for inspiration. Finally, we chose to start with local cone approximation and the $K-$directional derivative and defined a new class of more generalized convex functions. On the basis of the newly defined generalized convex functions, the optimality of minimax fraction semi-infinite programming is studied, and more general results are obtained.

5. Conclusions

In this paper, we first defined a new type of generalized convex function based on the concept of the $K-$directional derivative, that is, the uniform $(B_K, \rho)-$invex, strictly uniform $(B_K, \rho)-$invex, uniform $(B_K, \rho)-$pseudoinvex, strictly uniform $(B_K, \rho)-$pseudoinvex, uniform $(B_K, \rho)-$quasiinvex, and weakly uniform $(B_K, \rho)-$quasiinvex functions. Then, we studied a class of non-smooth minimax fractional semi-infinite programming problems involving this generalized convexity and obtained sufficient optimality conditions. Compared with existing results, the optimality conditions of the non-smooth minimax fraction semi-infinite programming in this paper are more general. The research is helpful for enriching the relevant theories of non-smooth fractional semi-infinite programming.

Minimax fraction semi-infinite programming is an important research direction for semi-infinite programming. There are still some related problems that require further study. Subsequently, we will continue to study the duality and saddle point theory of minimax fraction semi-infinite programming involving this generalized convexity.

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