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A Finite-Dimensional Integrable System Related to the Kadometsev–Petviashvili Equation

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Abstract: In this paper, the Kadometsev–Petviashvili equation and the Bargmann system are obtained from a second-order operator spectral problem $L\varphi = (\partial^2 - v\partial - \lambda u)\varphi = \lambda\varphi_x$. By means of the Euler–Lagrange equations, a suitable Jacobi–Ostrogradsky coordinate system is established. Using Cao's method and the associated Bargmann constraint, the Lax pairs of the differential equations are nonlinearized. Then, a new kind of finite-dimensional Hamilton system is generated. Moreover, involutive representations of the solutions of the Kadometsev–Petviashvili equation are derived.

Keywords: nonlinearization of Lax pairs; Kadometsev–Petviashvili equation; involutive solution

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1. Introduction

Many nonlinear phenomena that occur in nature can be described by nonlinear integrable models. This is particularly important to solve nonlinear partial differential equations and study the properties of their solutions. However, solving nonlinear equations is much more complicated than solving linear equations. Up to now, there is no unified method to solve them. Thus, many scholars have been attracted to study methods to solve nonlinear partial differential equations. At the same time, many methods have emerged, such as the inverse scattering method [1,2], the Darboux transformation [3,4], the Hirota bilinear method [5], the Riemann–Hilbert approach [6,7], the algebro-geometric method [8–10], etc. By the Darboux transformation between Lax pairs, the exact solutions for a five-component generalized mKdV equation are obtained [11]. Using the Dbar dressing method, the N-soliton solutions of the derivative NLS equation are discussed [12]. The characteristic polynomial of the Lax matrix is used to construct the trigonal curve, which plays an important role in obtaining the quasi-periodic solutions of nonlinear equations [13].

As is well known, the nonlinearization of Lax pairs [14] plays an important role in solving nonlinear evolution equations. The key is establishing the connection between infinite dimensional nonlinear evolution equations and finite dimensional integrable systems. By the Bargmann or Neumann constraint of the potentials and eigenfunctions, we can obtain the involutive representations of the solutions to nonlinear evolution equations [15–17].

In our research, we found that the second-order spectral problem

$$\varphi_{xx} - v\varphi_x - \lambda u\varphi = \lambda\varphi_x, \tag{1}$$

is associated with the well-known Kadometsev–Petviashvili (KP) equation

$$w_t = \frac{1}{4}(w_{xx} + 6w^2)_x + \frac{3}{4}\partial^{-1}w_{yy}, \tag{2}$$

which was first proposed by Kadomtsev and Petviashvili in 1970 [18]. The outline of this paper is as follows. In the next section, we introduce a second-order spectral problem with two potentials and derive a hierarchy of nonlinear equations based on Lenard recursion sequences. In Section 3, resorting to the viewpoint of Hamiltonian mechanics [19], the Jacobi–Ostrogradsky coordinates are presented. Then, the Bargmann system for (1) is written as a Hamilton canonical system. In Section 4, the spectral problem is nonlinearized and a new kind of finite-dimensional Hamilton system is constructed by using Cao's method. The Liouville integrability of the resulting Hamilton systems is generated. Section 5 is devoted to deriving the (2+1)-dimensional KP equation and constructing its involutive solution. The conclusions are presented in the last section.

2. Nonlinear Evolution Equations

Throughout this paper, we suppose that $\Omega = (-\infty, +\infty)$ is the basic interval of (1). The functions $\{u, v\}$ and their derivatives on x decay at infinity. Suppose that the linear space is equipped with L_2 scalar product $(\cdot,\cdot)_{L_2(\Omega)}$:

$$(f, g)_{L_2(\Omega)} = \int_\Omega f g^* dx < \infty,$$

where the symbol $*$ is used to denote the complex conjugate.

Now, we consider the spectral problem

$$L\varphi = (\partial^2 - v\partial - \lambda u)\varphi = \lambda\varphi_x, \tag{3}$$

where $\partial = \frac{\partial}{\partial x}$, $u = u(x,t)$ and $v = v(x,t)$ are the potential functions, and the parameter λ is an eigenvalue of the spectral problem (3).

Let $\bar{L}$ represent the adjoint operator of L, so

$$\bar{L} = \partial^2 + \partial v - \lambda^* u. \tag{4}$$

Suppose φ, $\psi \in L_2(\Omega)$, and they satisfy

$$\begin{cases} L\varphi = \lambda\varphi_x, \\ \bar{L}\psi = -\lambda^*\psi_x, \end{cases} \tag{5}$$

then, we can easily obtain the following results.

(i) The eigenvalue λ is real, i.e., $\lambda = \lambda^* \in R$.

(ii) The functional gradient is as follows:

$$\mathrm{grad}\lambda = \begin{pmatrix} \frac{\delta\lambda}{\delta u} \\ \frac{\delta\lambda}{\delta v} \end{pmatrix} = \left(\int_\Omega (u\varphi\psi + \varphi_x\psi)dx\right)^{-1} \begin{pmatrix} \lambda\varphi\psi \\ \varphi_x\psi \end{pmatrix}. \tag{6}$$

We consider the stationary zero curvature equation

$$\lambda W_x + [W, L] = \lambda W_x + WL - LW = 0, \tag{7}$$

where

$$W = \sum_{j=0}^{\infty} (b_{j-1}\partial - \lambda a_{j-1})\lambda^{-j}. \tag{8}$$

Then, we introduce the Lenard recursive relation

$$Jg_j = Kg_{j-1}, \;\; g_j = (b_j, a_j)^T, \;\; j \geq 0, \tag{9}$$

with the initial values

$$a_{-1} = 0, b_{-1} = 1, \tag{10}$$

where the bi-Hamilton operators are defined as

$$J = \begin{pmatrix} 0 & -\partial \\ -\partial & 2\partial \end{pmatrix}, \qquad K = \begin{pmatrix} u\partial + \partial u & -\partial^2 + v\partial \\ \partial^2 + \partial v & 0 \end{pmatrix}. \tag{11}$$

Thus, we have the following result:

$$K\mathrm{grad}\lambda = \lambda J\mathrm{grad}\lambda. \tag{12}$$

Let φ satisfy spectral problem (3) and the auxiliary problem

$$\varphi_{t_m} = W_m\varphi, \tag{13}$$

with

$$W_m = \sum_{j=0}^{m}(b_{j-1}\partial - \lambda a_{j-1})\lambda^{m-j}.$$

Then, the compatible condition of (3) and (13) yields the equation $L_{t_m} = \lambda W_{mx} + [W_m, L]$, which is equivalent to a hierarchy of nonlinear equations

$$(u_{t_m}, v_{t_m})^T = Kg_m = Jg_{m+1}, \quad m \geq 0. \tag{14}$$

After a direct calculation, the first two nontrivial members in (14) are

$$\begin{cases} u_{t_1} = u_{xx} - 2u_xv - 2uv_x - 6uu_x, \\ v_{t_1} = -v_{xx} - 2u_{xx} - 2u_xv - 2uv_x - 2vv_x, \end{cases} \tag{15}$$

and

$$\begin{cases} u_{t_2} = u_{xxx} - 3u_{xx}(2u+v) + u_x(30u^2 + 3v^2 + 24uv - 3v_x - 6u_x) \\ \qquad + 12u^2v_x + 6uvv_x, \\ v_{t_2} = v_{xxx} + 3v_{xx}(2u+v) + v_x(12u_x + 3v_x + 12uv + 3v^2 + 6u^2) \\ \qquad + 6u_{xx}v + 12u_x^2 + 12uu_{xx} + 6u_xv^2 + 12uu_xv. \end{cases} \tag{16}$$

3. The Hamilton Canonical Form

Suppose $\lambda_1 < \lambda_2 < \ldots < \lambda_N$ are N distinct eigenvalues of the spectral problems (5) and φ_j, ψ_j are the eigenfunctions for $\lambda_j (j = 1, 2, \ldots, N)$. Let

$$\Lambda = \mathrm{diag}(\lambda_1, \lambda_2, \ldots, \lambda_N), \quad \Phi = (\varphi_1, \varphi_2, \ldots, \varphi_N)^T, \quad \Psi = (\psi_1, \psi_2, \ldots, \psi_N)^T. \tag{17}$$

Take into consideration the following Bargmann constraint:

$$\begin{cases} u = -\langle \Phi_x, \Psi \rangle, \\ v = 2\langle \Phi_x, \Psi \rangle - \langle \Lambda\Phi, \Psi \rangle, \end{cases} \tag{18}$$

where the symbol $\langle \cdot, \cdot \rangle$ stands for the scalar product. Under the Bargmann constraint (18), we obtain that the Bargmann system of the eigenvalue problems (5) is equivalent to the following systems:

$$\begin{cases} \Phi_{xx} + \langle \Lambda\Phi, \Psi \rangle \Phi_x - 2\langle \Phi_x, \Psi \rangle \Phi_x - \Lambda\Phi_x + \langle \Phi_x, \Psi \rangle \Lambda\Phi = 0, \\ \Psi_{xx} + ((2\langle \Phi_x, \Psi \rangle - \langle \Lambda\Phi, \Psi \rangle)\Psi)_x + \Lambda\Phi_x + \langle \Phi_x, \Psi \rangle \Lambda\Psi = 0. \end{cases} \tag{19}$$

To derive the Hamilton canonical forms which correspond to the Bargmann system (19), we take the Lagrange function $\hat{I}$ as follows:

$$\hat{I} = \int_\Omega I dx, \tag{20}$$

where

$$I = \langle \Lambda\Phi, \Psi \rangle \langle \Phi_x, \Psi \rangle - \langle \Phi_x, \Psi \rangle^2 - \langle \Lambda\Phi_x, \Psi \rangle - \langle \Phi_x, \Psi_x \rangle.$$

From (20), we obtain

$$\begin{aligned} \frac{\delta \hat{I}}{\delta \Psi} &= \frac{\partial I}{\partial \Psi} - (\frac{\partial I}{\partial \Psi_x})_x \\ &= \langle \Lambda\Phi, \Psi \rangle \Phi_x + \langle \Phi_x, \Psi \rangle \Lambda\Psi - 2\langle \Phi_x, \Psi \rangle \Phi_x - \Lambda\Phi_x - \Phi_{xx} \\ &= 0. \end{aligned}$$

Similarly, $\frac{\delta \hat{I}}{\delta \Phi} = 0$, so we have the following results.

Proposition 1. *The Bargmann system (19) of the eigenvalue problems (5) is equivalent to the Euler–Lagrange equations:*

$$\begin{cases} \dfrac{\delta \hat{I}}{\delta \Phi} = 0, \\ \dfrac{\delta \hat{I}}{\delta \Psi} = 0. \end{cases} \tag{21}$$

Now, the Poisson bracket of the real-valued functions F and H in the symplectic space $\left(\omega = \sum\limits_{j=1}^{2} dq_j \wedge dp_j, R^{4N}\right)$ is defined as follows:

$$\{F, H\} = \sum_{j=1}^{2} \sum_{k=1}^{N} \left(\frac{\partial F}{\partial p_{jk}} \frac{\partial H}{\partial q_{jk}} - \frac{\partial F}{\partial q_{jk}} \frac{\partial H}{\partial p_{jk}} \right) = \sum_{j=1}^{2} (\langle F_{p_j}, H_{q_j} \rangle - \langle F_{q_j}, H_{p_j} \rangle). \tag{22}$$

Using the Euler–Lagrange equation (21), we will derive the Jacobi–Ostrogradsky coordinates to obtain the Hamilton canonical equations of the Bargmann system (19). Let

$$u_1 = \Phi, \quad u_2 = \Psi, \quad g = \sum_{j=1}^{2} \langle u_{jx}, v_j \rangle - I.$$

Our goal is to find the coordinates $\{v_1, v_2\}$ and g that satisfy the following Hamilton canonical equations:

$$\begin{cases} u_{jx} = \{u_j, g\} = \dfrac{\partial g}{\partial v_j}, \\ v_{jx} = \{v_j, g\} = -\dfrac{\partial g}{\partial u_j}, \end{cases} \quad j = 1, 2.$$

In fact, by using the expression $g = \sum\limits_{j=1}^{2} \langle u_{jx}, v_j \rangle - I$, one obtains

$$\mathrm{d}g = \sum_{j=1}^{2} (\langle v_j, \mathrm{d}u_{jx} \rangle + \langle u_{jx}, \mathrm{d}v_j \rangle) - \mathrm{d}I.$$

Moreover, since $g = g(u_j, v_j | j = 1, 2)$, we obtain

$$\mathrm{d}g = \sum_{j=1}^{2} (\langle \frac{\partial h}{\partial u_j}, \mathrm{d}u_j \rangle + \langle \frac{\partial h}{\partial v_j}, \mathrm{d}v_j \rangle) = \sum_{j=1}^{2} (-\langle v_{jx}, \mathrm{d}u_j \rangle + \langle u_{jx}, \mathrm{d}v_j \rangle),$$

and

$$\begin{aligned} \mathrm{d}I &= \langle v_1, \mathrm{d}u_{1x} \rangle + \langle v_2, \mathrm{d}u_{2x} \rangle + \langle v_{1x}, \mathrm{d}u_1 \rangle + \langle v_{2x}, \mathrm{d}u_2 \rangle \\ &= \langle v_1, \mathrm{d}\Phi_x \rangle + \langle v_2, \mathrm{d}\Psi_x \rangle + \langle v_{1x}, \mathrm{d}\Phi \rangle + \langle v_{2x}, \mathrm{d}\Psi \rangle. \end{aligned}$$

By directly computing this, we obtain

$$v_1 = -\Psi_x - (\Lambda + 2\langle \Phi_x, \Psi \rangle - \langle \Lambda\Phi, \Psi \rangle)\Psi, \quad v_2 = -\Phi_x.$$

Given the above preparations, we take the Jacobi–Ostrogradsky coordinates as follows:

$$\begin{cases} p_1 = \Phi, \\ p_2 = \Phi_x, \\ q_1 = -\Psi_x - (\Lambda + 2\langle \Phi_x, \Psi \rangle - \langle \Lambda\Phi, \Psi \rangle)\Psi, \\ q_2 = \Psi, \end{cases} \tag{23}$$

and the following result holds.

Theorem 1. *The Bargmann system (19) for the eigenvalue problems (5) is equivalent to the Hamilton canonical system*

$$\begin{cases} p_{jx} = \dfrac{\partial H}{\partial q_j}, \\ q_{jx} = -\dfrac{\partial H}{\partial p_j}, \end{cases} \quad j = 1, 2, \tag{24}$$

where

$$H = \langle p_2, q_1 \rangle + \langle \Lambda p_2, q_2 \rangle - \langle p_2, q_2 \rangle \langle \Lambda p_1, q_2 \rangle + \langle p_2, q_2 \rangle^2. \tag{25}$$

4. The Classical Liouville Completely Integrable Systems

Based on the Jacobi–Ostrogradsky coordinates (23), the nonlinearized Lax pairs are written as a Hamilton equation system. Then, completely integrable systems in the Liouville sense are obtained.

From the Jacobi–Ostrogradsky coordinates (23) and Theorem 1, the eigenvalue problem (19) can be rewritten as follows:

$$\begin{pmatrix} p_1 \\ p_2 \end{pmatrix}_x = M \begin{pmatrix} p_1 \\ p_2 \end{pmatrix}, \begin{pmatrix} q_1 \\ q_2 \end{pmatrix}_x = -M^T \begin{pmatrix} q_1 \\ q_2 \end{pmatrix}. \tag{26}$$

$$M = \begin{pmatrix} 0 & E \\ \Lambda u & \Lambda + vE \end{pmatrix}, \quad E = E_{N\times N} = \mathrm{diag}(1, 1, \ldots, 1).$$

Proposition 2. *The Lax pairs (3) and (13) for the evolution Equation (14) are equivalent to the following systems:*

$$\begin{cases} \begin{pmatrix} p_1 \\ p_2 \end{pmatrix}_x = M \begin{pmatrix} p_1 \\ p_2 \end{pmatrix}, & \begin{pmatrix} q_1 \\ q_2 \end{pmatrix}_x = -M^T \begin{pmatrix} q_1 \\ q_2 \end{pmatrix}, \\ \begin{pmatrix} p_1 \\ p_2 \end{pmatrix}_{t_m} = \begin{pmatrix} A_m & B_m \\ C_m & D_m \end{pmatrix} \begin{pmatrix} p_1 \\ p_2 \end{pmatrix}, & \begin{pmatrix} q_1 \\ q_2 \end{pmatrix}_{t_m} = -\begin{pmatrix} A_m & B_m \\ C_m & D_m \end{pmatrix}^T \begin{pmatrix} q_1 \\ q_2 \end{pmatrix}, \quad m \geq 0, \end{cases} \tag{27}$$

where

$$
\begin{aligned}
A_m &= -\sum_{j=0}^{m} a_{j-1}\Lambda^{m-j+1},\\
B_m &= \sum_{j=0}^{m} b_{j-1}\Lambda^{m-j},\\
C_m &= \sum_{j=0}^{m} (-a_{j-1,x}+ub_{j-1})\Lambda^{m-j+1},\\
D_m &= \sum_{j=0}^{m} (b_{j-1,x}+vb_{j-1}-\Lambda a_{j-1}+\Lambda b_{j-1})\Lambda^{m-j}.
\end{aligned} \tag{28}
$$

By (18) and (23), we have the Bargmann constraint

$$
\begin{cases} u = -\langle p_2, q_2\rangle, \\ v = 2\langle p_2, q_2\rangle - \langle \Lambda p_1, q_2\rangle. \end{cases} \tag{29}
$$

Furthermore, using (9) and (12), a straightforward calculation shows that

$$
\begin{cases} a_j = \langle \Lambda^j p_2, q_2\rangle, \\ b_j = \langle \Lambda^{j+1} p_1, q_2\rangle, \quad j = 0,1,2,\ldots. \end{cases} \tag{30}
$$

Substituting (29)–(30) into (27), we obtain

$$
\begin{pmatrix} p_1 \\ p_2 \end{pmatrix}_x = \bar{M}\begin{pmatrix} p_1 \\ p_2 \end{pmatrix}, \quad \begin{pmatrix} q_1 \\ q_2 \end{pmatrix}_x = -\bar{M}^T\begin{pmatrix} q_1 \\ q_2 \end{pmatrix}, \tag{31}
$$

$$
\begin{pmatrix} p_1 \\ p_2 \end{pmatrix}_{t_m} = \bar{W}\begin{pmatrix} p_1 \\ p_2 \end{pmatrix}, \quad \begin{pmatrix} q_1 \\ q_2 \end{pmatrix}_{t_m} = -\bar{W}^T\begin{pmatrix} q_1 \\ q_2 \end{pmatrix}, \quad m = 0,1,2,\ldots, \tag{32}
$$

where

$$
\bar{M} = \begin{pmatrix} 0 & E \\ -\langle p_2, q_2\rangle\Lambda & \Delta \end{pmatrix}, \quad \bar{W} = \begin{pmatrix} \bar{A}_m & \bar{B}_m \\ \bar{C}_m & \bar{D}_m \end{pmatrix},
$$

$$
\begin{aligned}
\Delta &= \Lambda + (2\langle p_2, q_2\rangle - \langle \Lambda p_1, z_q\rangle)E,\\
\bar{A}_m &= -\sum_{j=0}^{m}\langle \Lambda^j p_2, q_2\rangle\Lambda^{m-j} + \langle \Lambda^m p_2, q_2\rangle E,\\
\bar{B}_m &= \sum_{j=0}^{m}\langle \Lambda^j p_1, q_2\rangle\Lambda^{m-j} + \Lambda^m - \langle p_1, q_2\rangle\Lambda^m,\\
\bar{C}_m &= \sum_{j=0}^{m}\langle \Lambda^j p_2, q_1\rangle\Lambda^{m-j} - \langle p_2, q_2\rangle\Lambda^{m+1} - \langle \Lambda^m p_2, q_1\rangle E,\\
\bar{D}_m &= -\sum_{j=0}^{m}\langle \Lambda^j p_1, q_1\rangle\Lambda^{m-j} + \langle \Lambda^m p_2, q_2\rangle E + \langle p_2, q_2\rangle\Lambda^m + \langle p_1, q_1\rangle\Lambda^m\\
&\quad + \Lambda^{m+1} - \langle \Lambda^{m+1} p_1, q_2\rangle E.
\end{aligned}
$$

Denote

$$
P = (p_1, p_2, q_1, q_2)^T, \quad I = \begin{pmatrix} 0 & E_{2N} \\ -E_{2N} & 0 \end{pmatrix},
$$

then we have the following results.

Theorem 2. *Using the Bargmann constraint (29), the nonlinearized Lax pairs (31) and (32) for evolution Equation (14) can be written as follows:*

$$P_x = \left(\frac{\partial H}{\partial q_1}, \frac{\partial H}{\partial q_2}, -\frac{\partial H}{\partial p_1}, -\frac{\partial H}{\partial p_2}\right)^T = I\nabla H, \tag{33}$$

$$P_{t_m} = \left(\frac{\partial H_m}{\partial q_1}, \frac{\partial H_m}{\partial q_2}, -\frac{\partial H_m}{\partial p_1}, -\frac{\partial H_m}{\partial p_2}\right)^T = I\nabla H_m, \quad m = 0, 1, 2, \dots, \tag{34}$$

where H is defined by (25), and

$$\begin{aligned} H_m = {} & \langle \Lambda^m p_2, q_1\rangle - \langle \Lambda^m p_2, q_1\rangle\langle p_1, q_2\rangle + \langle \Lambda^{m+1} p_2, q_2\rangle - \langle \Lambda^{m+1} p_1, q_2\rangle\langle p_2, q_2\rangle \\ & + \langle \Lambda^m p_2, q_2\rangle(\langle p_2, q_2\rangle + \langle p_1, q_1\rangle) - \sum_{j=0}^{m} \begin{vmatrix} \langle \Lambda^j p_1, q_1\rangle & \langle \Lambda^{m-j} p_1, q_2\rangle \\ \langle \Lambda^j p_2, q_1\rangle & \langle \Lambda^{m-j} p_2, q_2\rangle \end{vmatrix}. \end{aligned} \tag{35}$$

In what follows, we shall discuss the completely integrability of the Bargmann systems (33) and (34). We introduce the generators as follows:

$$\begin{aligned} E_k^{(1)} = {} & \tfrac{1}{\lambda_k} p_{2k}q_{1k} + \tfrac{1}{\lambda_k} p_{2k}q_{2k}\langle p_2, q_2\rangle + \tfrac{1}{\lambda_k} p_{2k}q_{2k}\langle p_1, q_2\rangle - \tfrac{1}{\lambda_k} p_{2k}q_{1k}\langle p_1, q_2\rangle \\ & + p_{2k}q_{2k} - p_{1k}q_{2k}\langle p_2, q_2\rangle - \Gamma_k^{(1,2)}, \\ E_k^{(2)} = {} & \Gamma_k, \end{aligned} \tag{36}$$

where

$$\begin{aligned} \Gamma_k^{(1,2)} &= \sum_{l=1, l\neq k}^{N} \frac{1}{\lambda_k - \lambda_l} \begin{vmatrix} p_{1k} & p_{1l} \\ p_{2k} & p_{2l} \end{vmatrix} \begin{vmatrix} q_{1k} & q_{1l} \\ q_{2k} & q_{2l} \end{vmatrix}, \\ \Gamma_k &= \sum_{l=1, l\neq k}^{N} \frac{1}{\lambda_k - \lambda_l} (p_{1l}q_{1l} + p_{2l}q_{2l})(p_{1k}q_{1k} + p_{2k}q_{2k}). \end{aligned}$$

By directly computing from the definition of the Poisson bracket (22), we obtain the following results:

(i) $\{E_j^{(i)}, i = 1, 2; j = 1, 2, \dots, N\}$ are involution systems, i.e.,

$$\{E_j^{(i)}, E_k^{(l)}\} = 0, \ \forall\, i, l = 1, 2; j, k = 1, 2, \dots, N. \tag{37}$$

(ii) $\{\mathrm{d}E_j^{(i)}, j = 1, 2, \dots, N; i = 1, 2\}$ are linearly independent.

Based on the above preparations, we can obtain the following theorem.

Theorem 3. *The Bargmann systems (33) and (34) are completely integrable systems in the Liouville sense, i.e.,*

$$\{H, E_j^{(i)}\} = 0, \qquad i = 1, 2; j = 1, 2, \dots, N. \tag{38}$$

$$\{H_m, E_j^{(i)}\} = 0, \qquad i = 1, 2; j = 1, 2, \dots, N. \tag{39}$$

$$\{H_m, H_n\} = 0, \qquad m, n = 0, 1, 2, \dots. \tag{40}$$

$$\{H, H_m\} = 0, \qquad m = 0, 1, 2, \dots. \tag{41}$$

Proof. A direct calculation shows that

$$H_{m-1} = \sum_{j=1}^{N} \lambda_j^m E_j^{(1)}, \quad m = 1, 2, \dots. \tag{42}$$

Combining (36), (37) and (42), we have

$$\{H_m, H_n\} = 0, \quad m, n = 0, 1, 2, \ldots.$$

On the other hand, we notice that $H = H_0$, so

$$\{H, E_j^{(i)}\} = 0, \quad i = 1, 2; \ j = 1, 2, \ldots, N.$$

$$\{H, H_m\} = 0, \quad m = 0, 1, 2, \ldots.$$

Using the Arnold theorem [19], the Bargmann systems (33) and (34) are completely integrable systems in the Liouville sense. □

We consider the canonical equation of the H_m flow (34) and the solution of the initial value problem:

$$P(t_m) = \begin{pmatrix} p_1(t_m) \\ p_2(t_m) \\ q_1(t_m) \\ q_2(t_m) \end{pmatrix} = h_m^{t_m} \begin{pmatrix} p_1(0) \\ p_2(0) \\ q_1(0) \\ q_2(0) \end{pmatrix} = h_m^{t_m} P(0). \tag{43}$$

Specifically, $t_0 = x$. According to Theorem 3, $h_k^{t_k}$ and $h_j^{t_j}$ are commutable.

Remark 1. *(1) When $m = 1$, we denote $t_1 = y$. Let $(p_1(x, y), p_2(x, y), q_1(x, y), q_2(x, y))$ be a compatible solution of*

$$\begin{aligned} P_x &= I\nabla H, \\ P_y &= I\nabla H_1, \end{aligned} \tag{44}$$

then $u(x, y) = -\langle p_2, q_2\rangle$, $v(x, y) = 2\langle p_2, q_2\rangle - \langle \Lambda p_1, q_2\rangle$ satisfies the coupled Equation (15).

(2) When $m = 2$, we denote $t_2 = t$. Let $(p_1(x, t), p_2(x, t), q_1(x, t), q_2(x, t))$ be a compatible solution of

$$\begin{aligned} P_x &= I\nabla H, \\ P_t &= I\nabla H_2, \end{aligned} \tag{45}$$

then $u(x, t) = -\langle p_2, q_2\rangle$, $v(x, t) = 2\langle p_2, q_2\rangle - \langle \Lambda p_1, q_2\rangle$ satisfies the coupled Equation (16).

That is to say, the Lax pairs of the coupled Equation (15) are nonlinearized into the confocal flows H and H_1, while the Lax pairs of the coupled Equation (16) are nonlinearized into the confocal flows H and H_2.

5. Involutive Solutions of the KP Equation

In this section, the special solution of the KP equation is separated into three confocal flows: H, H_1 and H_2. The involutive solution to the KP equation is generated.

Proposition 3. *Let $u(x, y, t), v(x, y, t)$ be a compatible solution of the coupled Equations (15) and (16), then*

$$w(x, y, t) = u^2(x, y, t) + u(x, y, t)v(x, y, t), \tag{46}$$

solves the KP equation:

$$w_t = \frac{1}{4}(w_{xx} + 6w^2)_x + \frac{3}{4}\partial^{-1} w_{yy}. \tag{47}$$

Proof. By a complex calculation, one obtains:

$$
\begin{aligned}
w_{xt} &= w_{xxxx} + 15(w^2)_{xx} - u_{xxx}(3v^2 + 6uv + 6u_x + 3v_x) + v_{xxx}(6u^2 + 3uv + 3u_x) \\
&\quad + u_{xx}(3v^2 + 6uv^2 - 18u_x v - 9vv_x) + v_{xx}(6u^2 v + 9uv^2 + 18uu_x + 9uv_x) \\
&\quad + u_x v_x(24uv + 18v^2) + v_x^2(6u^2 + 18uv) + 6u_x^2 v^2. \\
\frac{3}{4}w_{yy} &= \frac{3}{4}w_{xxxx} + \frac{27}{2}(w^2)_{xx} - u_{xxx}(3v^2 + 6uv + 6u_x + 3v_x) + v_{xxx}(6u^2 + 3uv + 3u_x) \\
&\quad + u_{xx}(3v^2 + 6uv^2 - 18u_x v - 9vv_x) + v_{xx}(6u^2 v + 9uv^2 + 18uu_x + 9uv_x) \\
&\quad + u_x v_x(24uv + 18v^2) + v_x^2(6u^2 + 18uv) + 6u_x^2 v^2.
\end{aligned} \tag{48}
$$

Thus, Equation (47) holds. □

Theorem 4. *Let $(p(x, y, t), q(x, y, t))$ be a compatible solution of the following equations*

$$P_x = I\nabla H, \quad P_y = I\nabla H_1, \quad P_t = I\nabla H_2, \tag{49}$$

then

$$
\begin{aligned}
w(x, y, t) &=< p_2, q_2 > (< \Lambda p_1, q_2 > - < p_2, q_2 >) \\
&=< p_2, q_1 > + < \Lambda p_2, q_2 > -H,
\end{aligned} \tag{50}
$$

solves the KP Equation (47).

Proof. Since the flow operators h_0^x, h_1^y, h_2^t are commutable, the compatible solution can be written in two ways:

$$P(x, y, t) = h_0^x h_1^y \{h_2^t P(0, 0, 0)\} = h_0^x h_2^t \{h_1^y P(0, 0, 0)\}, \tag{51}$$

where the element of brace $\{\cdot\}$ can be regarded as an initial value. According to the Bargmann constraint (29) and Proposition 3, we infer that

$$w = u^2 + uv =< p_2, q_2 > (< \Lambda p_1, q_2 > - < p_2, q_2 >),$$

is a involutive solution of the KP Equation (47). □

6. Conclusions

Starting from a second-order operator spectral problem, we obtain a new hierarchy for a nonlinear evolution equation (14). Moreover, the (2+1)-dimensional KP equation (47) is decomposed into the first two (1+1)-dimensional nontrivial equations (15) and (16). By constructing the Bargmann constraint of the potential functions and eigenfunctions, and based on the nonlinearization of Lax pairs, we establish the relations between the infinite-dimensional nonlinear equations of soliton systems and finite-dimensional integrable systems. Furthermore, we obtain the involutive solution of the KP equation.

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