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Quasi-Statistical Schouten–van Kampen Connections on the Tangent Bundle

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Abstract: We determine the general natural metrics G on the total space TM of the tangent bundle of a Riemannian manifold (M, g) such that the Schouten–van Kampen connection $\overline{\nabla}$ associated to the Levi-Civita connection of G is (quasi-)statistical. We prove that the base manifold must be a space form and in particular, when G is a natural diagonal metric, (M, g) must be locally flat. We prove that there exist one family of natural diagonal metrics and two families of proper general natural metrics such that $(TM, \overline{\nabla}, G)$ is a statistical manifold and one family of proper general natural metrics such that $(TM \setminus \{0\}, \overline{\nabla}, G)$ is a quasi-statistical manifold.

Keywords: (pseudo-)Riemannian manifold; Codazzi pair; statistical manifold; quasi-statistical manifold; Schouten–van Kampen connection; tangent bundle; general natural metric

MSC: 53B12; 53B05; 53B21

1. Introduction

Statistical manifolds, whose points correspond to probability distributions, provide a natural framework for information geometry, which uses differential geometry in the study of probability theory and statistics and which was initiated by C. R. Rao in [1], who was the first to treat a Fisher matrix as a Riemannian metric. The notion of statistical manifold, introduced in 1987 by S. L. Lauritzen in the paper [2] and studied, e.g., in [2–32] and the references therein, has various applications in information science, neural networks, and statistical physics.

According to T. Kurose [14], a statistical manifold is a differentiable manifold endowed with a symmetric linear connection ∇ and a (pseudo-)Riemannian metric h such that the covariant derivative ∇h is totally symmetric. A couple (∇, h) with this property is called a statistical structure or a Codazzi pair, while the metric h and the connection ∇ are said to be Codazzi-coupled (see [2,9,12,23]). Alternatively, the notion of statistical manifold was defined by H. Furuhata and I. Hasegawa in [11] as a (pseudo-)Riemannian manifold endowed with a pair of torsion-free conjugate connections. For the pairs of connections compatible with a g–structure, we go back in the literature to V. Cruceanu and R. Miron [33].

A classical example of statistical manifold is a (pseudo-)Riemannian manifold (M, h) endowed with the Levi-Civita connection of the metric h. The statistical manifolds generalize the (pseudo-)Riemannian manifolds by extending the parallelism of the metric h under the Levi-Civita connection to the Codazzi coupling of the metric with a torsion-free linear connection. Moreover, relaxing the Codazzi coupling to the case when the linear connection has nonzero torsion, T. Kurose introduced in [15] the notion of statistical manifold admitting torsion, also called quasi-statistical manifold (see [17]), which is the subject of quantum information geometry.

Codazzi couplings of an affine connection with a pseudo-Riemannian metric, a nondegenerate 2-form, and a tangent bundle isomorphism on smooth manifolds and in particular on an almost (para-)Hermitian manifold (M, g, L) endowed with the 2-form ω given as

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$\omega(X,Y) = g(LX,Y)$, were studied by T. Fei and J. Zhang in [9]. They proved that the Codazzi couplings of ∇ with both g and L lead to a (para-)Kähler structure, and subsequently, they defined Codazzi-(para-)Kähler manifolds as (para-)Kähler statistical manifolds. In [12], the study was extended to torsion couplings between an affine connection ∇ of nontrivial torsion and both g and L on an almost (para-)Hermitian manifold. The authors proved that the pair (∇, L) is torsion-coupled if and only if ∇ is (para-)holomorphic and the almost (para-)complex structure L is integrable. Statistical structures on almost anti-Hermitian (or Norden) manifolds were studied in [26,27] by A. Salimov and S. Turanli, who introduced the notion of anti-Kähler–Codazzi manifolds, then by L. Samereh, E. Peyghan, and I. Mihai in [28], and very recently by A. Gezer and H. Cakicioglu, who provided in [10] an alternative classification of anti-Kähler manifolds with respect to a torsion-free linear connection. Codazzi pairs on almost para-Norden manifolds were treated by S. Turanli and S. Uçan in [29]. F. Etayo et al. proved in [8] that Kähler–Codazzi type manifolds reduce to Kähler type manifolds in all the four types of (α, ε)-manifolds teated in an unified way in [34]. In [30], G. E. Vîlcu introduced the notion of para-Kähler-like statistical manifold and proved that if a manifold of this type has constant curvature in the Kurose's sense, then the statistical structure of the manifold is a Hessian structure.

Statistical structures on the tangent bundle of differentiable manifolds were treated in recent papers, such as [4,13,19,22,24].

The background of the present work is the total space TM of the tangent bundle of a Riemannian manifold (M,g), endowed with a metric G introduced by V. Oproiu in [35] as a general natural lift of the metric from the base manifold, by using Kowalski–Sekizawa's classification from [36] and the results in [37]. This metric, called a general natural metric, depends on six coefficients which are smooth real functions of the energy density t of a tangent vector y. We study the conditions under which the (pseudo-)Riemannian manifold (TM, G) endowed with the Schouten–van Kampen connection $\overline{\nabla}$ associated to the Levi-Civita connection of G is a statistical manifold admitting torsion (SMAT). A necessary condition for $(TM, \overline{\nabla}, G)$ to be a SMAT is that the base manifold is a space form. We prove that $(TM \setminus \{0\}, \overline{\nabla}, G)$ is a SMAT if and only if (M,g) has negative constant sectional curvature and the metric G depends on the energy density t, the constant sectional curvature of (M,g), an arbitrary nonzero real constant κ_2 and an arbitray smooth real function of t which is not $-\frac{\kappa_2}{2t^2}$. On the other hand, $(TM, \overline{\nabla}, G)$ is a statistical manifold (without torsion) if and only if the base manifold is locally flat and the metric G is of natural diagonal type (depending on two arbitrary nonzero smooth real functions of the energy density t and on an arbitrary nonzero real constant, satisfying the nondegeneracy conditions of the metric) or a proper general natural metric with two possible expressions. In one case, the expression of G depends on an arbitrary smooth real function c_3 of t different from $\frac{const}{\sqrt{t}}$ for every $const \in \mathbb{R}$, $t > 0$, such that $c_3(0) \neq 0$, and on two arbitrary nonzero real constants whose product is different from 1. In the other case, the metric G depends only on two arbitrary smooth real functions c_2, c_3 of the energy density, such that $c_2(0)c_3(0) \neq 0$, $c_3(t) \neq \frac{const}{\sqrt{t}}$ for every $const \in \mathbb{R}$, $t > 0$. If $c_2(t) \neq \kappa(c_3(t))^2$ for every $\kappa \in \mathbb{R}$, $t \geq 0$, then the Levi-Civita connection of G is different from its associated Schouten–van Kampen connection, and hence $(TM, \overline{\nabla}, G)$ is a nontrivial statistical manifold.

The results obtained in this work lead to new examples of (quasi-)statistical structures on the tangent bundle of a Riemann manifold. Unlike the majority of previous studies (see, e.g., [4,13,19,22,24]), which produce new examples of statistical structures on the tangent bundle by lifting a given statistical structure on the base space, the present article does not assume the a priori existence of a statistical structure on the base manifold. The new structures are, thus, uncorrelated with the ones from the base, therefore constituting a more convenient geometric setting to investigate the statistical behavior in depth. Thus, new opportunities are opened for applications in information theory, machine learning, neural networks, statistical mechanics and geometry of Ricci solitons, for which we cite [38–41] and the references therein.

We mention that in the present paper the manifolds, tensor fields, and other geometric objects are considered to be smooth and the Einstein summation convention is used, the range of the indices always being $\{1, \ldots, n\}$.

2. The Schouten–van Kampen Connection of a General Natural Metric on TM Revisited

In this section, we recall some results from our previous paper [42] concerning the Schouten–van Kampen connection associated to the Levi-Civita connection of a general natural metric on the total space TM of the tangent bundle of a Riemannian manifold. For the geometry of the tangent bundle we cite the monograph [43].

Let (M, g) be a Riemannian manifold of dimension n and let $(x^i)_{i=1}^n$ and $(x^i, y^j)_{i,j=1}^n$ be the local coordinates on an open subset U of M and on $\tau^{-1}(U) \subset TM$, respectively, where $\tau : TM \to M$ is the tangent bundle of M.

Denoting, by a slight abuse, the set of all vector fields tangent to TM by TTM, we have its direct sum decomposition, that is:

$$TTM = VTM \oplus HTM, \tag{1}$$

into the vertical distribution $VTM = \ker \tau_*$ and the horizontal distribution HTM, locally generated, respectively, by $\{\frac{\partial}{\partial y^i}\}_{i=1}^n$ and $\{\frac{\delta}{\delta x_j}\}_{j=1}^n$, the horizontal generators being $\frac{\delta}{\delta x^j} = \frac{\partial}{\partial x^j} - y^l \Gamma_{lj}^h \frac{\partial}{\partial y^h}$, where Γ_{lj}^h are the Christoffel symbols of the metric g. Then, the local frame field adapted to the direct sum decomposition (1) is $\{\frac{\partial}{\partial y^i}, \frac{\delta}{\delta x^j}\}_{i,j=1}^n$, denoted also by $\{\partial_i, \delta_j\}_{i,j=1}^n$. Its Lie brackets satisfy the identities:

$$[\partial_i, \partial_j] = 0, \ [\partial_i, \delta_j] = -\Gamma_{ij}^h \partial_h, \ [\delta_i, \delta_j] = -R_{lij}^h y^l \partial_h, \tag{2}$$

where R_{lij}^h are components of the curvature tensor field of (M, g) in a local chart $(U, x^i)_{i=1}^n$.

The horizontal and vertical lifts of a vector field $X = X^i \frac{\partial}{\partial x^i}$ from M to TM are denoted by X^H and X^V and with respect to the adapted local frame field, they have the expressions

$$X^H = X^i \frac{\delta}{\delta x^i}, \ X^V = X^i \frac{\partial}{\partial y^i}.$$

The kinetic energy or energy density of any tangent vector $y \in \tau^{-1}(U)$ with respect to the Riemannian metric g is given as:

$$t = \frac{1}{2}\|y\|^2 = \frac{1}{2} g_{\tau(y)}(y, y) = \frac{1}{2} g_{ik}(x) y^i y^k \geq 0. \tag{3}$$

An important tool in the geometry of the tangent bundle are the metrics constructed as natural lifts of the Riemannian metric from base manifold to the total space of the tangent bundle, classified by O. Kowalski and M. Sekizawa in [36]. By using this classification and the results in [37], V. Oproiu defined in [35] *a general natural metric* on TM, given locally as:

$$\begin{cases} G\left(\frac{\delta}{\delta x^i}, \frac{\delta}{\delta x^j}\right) = c_1 g_{ij} + d_1 g_{0i} g_{0j} = G_{ij}^{(1)} \\ G\left(\frac{\partial}{\partial y^i}, \frac{\partial}{\partial y^j}\right) = c_2 g_{ij} + d_2 g_{0i} g_{0j} = G_{ij}^{(2)} \\ G\left(\frac{\partial}{\partial y^i}, \frac{\delta}{\delta x^j}\right) = G\left(\frac{\delta}{\delta x^i}, \frac{\partial}{\partial y^j}\right) = c_3 g_{ij} + d_3 g_{0i} g_{0j} = G_{ij}^{(3)}, \end{cases} \tag{4}$$

where c_i, d_i $(i = 1, 2, 3)$ are smooth real functions of the energy density on TM and $g_{0i} = g_{li} y^l$.

The invariant expression of the metric G is:

$$\begin{cases} G(X_y^H, Y_y^H) = c_1(t) g_{\tau(y)}(X,Y) + d_1(t) g_{\tau(y)}(X,y) g_{\tau(y)}(Y,y), \\ G(X_y^V, Y_y^V) = c_2(t) g_{\tau(y)}(X,Y) + d_2(t) g_{\tau(y)}(X,y) g_{\tau(y)}(Y,y), \\ G(X_y^V, Y_y^H) = c_3(t) g_{\tau(y)}(X,Y) + d_3(t) g_{\tau(y)}(X,y) g_{\tau(y)}(Y,y), \end{cases} \tag{5}$$

for all $X, Y \in \mathcal{T}_0^1(M)$, $y \in TM$, where t is the energy density of y.

The nondegeneracy conditions for the metric G are as follows:

$$c_1 c_2 - c_3^2 \neq 0, \quad (c_1 + 2td_1)(c_2 + 2td_2) - (c_3 + 2td_3)^2 \neq 0. \tag{6}$$

The metric G is positive definite if:

$$c_1 + 2td_1 > 0, \quad c_2 + 2td_2 > 0, \ (c_1 + 2td_1)(c_2 + 2td_2) - (c_3 + 2td_3)^2 > 0. \tag{7}$$

When the horizontal and vertical distributions are orthogonal with respect to the metric G, we say that G is a *metric of natural diagonal lift type* or a *natural diagonal metric* on TM (see [44]). This type of metric has the expression (5), with $c_3 = d_3 = 0$. We say that a metric given by (5) is a *proper general natural metric* if it is not a natural diagonal metric.

The matrix of the metric G with respect to the adapted local frame field $\{\delta_i, \partial_j\}_{i,j=1}^n$ and the inverse matrix are, respectively:

$$\begin{pmatrix} \left(G_{ij}^{(1)}\right) & \left(G_{ij}^{(3)}\right) \\ \left(G_{ij}^{(3)}\right) & \left(G_{ij}^{(2)}\right) \end{pmatrix}_{i,j\in\{1,\ldots,n\}}, \quad \begin{pmatrix} \left(H_{(1)}^{jk}\right) & \left(H_{(3)}^{jk}\right) \\ \left(H_{(3)}^{jk}\right) & \left(H_{(2)}^{jk}\right) \end{pmatrix}_{j,k\in\{1,\ldots,n\}},$$

where:

$$H_{(1)}^{kl} = p_1 g^{kl} + q_1 y^k y^l, \ H_{(2)}^{kl} = p_2 g^{kl} + q_2 y^k y^l, \ H_{(3)}^{kl} = p_3 g^{kl} + q_3 y^k y^l, \tag{8}$$

with:

$$p_1 = \frac{c_2}{c_1 c_2 - c_3^2}, \ p_2 = \frac{c_1}{c_1 c_2 - c_3^2}, \ p_3 = -\frac{c_3}{c_1 c_2 - c_3^2} \tag{9}$$

$$\begin{aligned} q_1 &= -\frac{c_2 d_1 p_1 - c_3 d_3 p_1 - c_3 d_2 p_3 + c_2 d_3 p_3 + 2 d_1 d_2 p_1 t - 2 d_3^2 p_1 t}{(c_1 + 2d_1 t)(c_2 + 2d_2 t) - (c_3 + 2d_3 t)^2}, \\ q_2 &= \frac{(c_3 + 2d_3 t)[(d_3 p_1 + d_2 p_3)(c_1 + 2d_1 t) - (d_1 p_1 + d_3 p_3)(c_3 + 2d_3 t)]}{(c_2 + 2d_2 t)[(c_1 + 2d_1 t)(c_2 + 2d_2 t) - (c_3 + 2d_3 t)^2]} - \frac{d_2 p_2 + d_3 p_3}{c_2 + 2d_2 t}, \\ q_3 &= -\frac{(d_3 p_1 + d_2 p_3)(c_1 + 2d_1 t) - (d_1 p_1 + d_3 p_3)(c_3 + 2d_3 t)}{(c_1 + 2d_1 t)(c_2 + 2d_2 t) - (c_3 + 2d_3 t)^2}. \end{aligned} \tag{10}$$

Inspired by the Schouten–van Kampen connection associated to a linear connection on a smooth manifold with two globally complementary distributions (see [45] and [46]), we defined in [42] the Schouten–van Kampen connection $\overline{\nabla}$ associated to the Levi-Civita connection ∇ of a general natural metric G by the relation:

$$\overline{\nabla}_X Y = V \nabla_X V Y + H \nabla_X H Y, \tag{11}$$

for any vector fields X, Y on TM, where V and H are the projection tensor fields corresponding to VTM and HTM, respectively.

Proposition 1. (Proposition 3.1 [42]) *The Schouten–van Kampen connection* $\overline{\nabla}$ *associated to the Levi-Civita connection* ∇ *of a general natural metric* G *on* TM *has the following expression in the adapted local frame field* $\{\partial_i, \delta_j\}_{i=1}^n$*:*

$$\begin{cases} \overline{\nabla}_{\frac{\partial}{\partial y^i}} \frac{\partial}{\partial y^j} = Q_{ij}^h \frac{\partial}{\partial y^h}, \ \overline{\nabla}_{\frac{\delta}{\delta x^i}} \frac{\partial}{\partial y^j} = \left(\Gamma_{ij}^h + \overline{U}_{ji}^h\right) \frac{\partial}{\partial y^h} \\ \overline{\nabla}_{\frac{\partial}{\partial y^i}} \frac{\delta}{\delta x^j} = U_{ij}^h \frac{\delta}{\delta x^h}, \ \overline{\nabla}_{\frac{\delta}{\delta x^i}} \frac{\delta}{\delta x^j} = \left(\Gamma_{ij}^h + \overline{S}_{ij}^h\right) \frac{\delta}{\delta x^h}, \end{cases} \tag{12}$$

where Γ_{ij}^h *are the Christoffel symbols of the metric* g *of the base manifold* M*:*

$$\begin{cases} Q_{ij}^h = \frac{1}{2}(\partial_i G_{jk}^{(2)} + \partial_j G_{ik}^{(2)} - \partial_k G_{ij}^{(2)}) H_{(2)}^{kh} + \frac{1}{2}(\partial_i G_{jk}^{(3)} + \partial_j G_{ik}^{(3)}) H_{(3)}^{kh}, \\ U_{ij}^h = \frac{1}{2}(\partial_i G_{jk}^{(3)} - \partial_k G_{ij}^{(3)}) H_{(3)}^{kh} + \frac{1}{2}(\partial_i G_{jk}^{(1)} + R_{0jk}^l G_{li}^{(2)}) H_{(1)}^{kh}, \\ \overline{U}_{ij}^h = \frac{1}{2}(\partial_i G_{jk}^{(3)} - \partial_k G_{ij}^{(3)}) H_{(2)}^{kh} + \frac{1}{2}(\partial_i G_{jk}^{(1)} + R_{0jk}^l G_{li}^{(2)}) H_{(3)}^{kh}, \\ \overline{S}_{ij}^h = -\frac{1}{2}(\partial_k G_{ij}^{(1)} + R_{0ij}^l G_{lk}^{(2)}) H_{(3)}^{kh} + c_3 R_{i0jk} H_{(1)}^{kh}, \end{cases} \tag{13}$$

where R_{kij}^h *are the components of the curvature of the base manifold and:*

$$R_{0ij}^l = R_{hij}^l y^h, \ R_{i0jk} = R_{ihjk} y^h.$$

The torsion tensor field T of the connection $\overline{\nabla}$ is defined by the formula:

$$T(X,Y) = \overline{\nabla}_X Y - \overline{\nabla}_Y X - [X,Y], \quad \forall X, Y \in \mathcal{T}_0^1(TM). \tag{14}$$

Proposition 2. *The torsion tensor field of the Schouten–van Kampen connection* $\overline{\nabla}$ *given in Proposition 1 has the following components with respect to the adapted local frame field* $\{\partial_i, \delta_j\}_{i=1}^n$*:*

$$\begin{cases} T\left(\frac{\partial}{\partial y^i}, \frac{\partial}{\partial y^j}\right) = 0, \ T\left(\frac{\delta}{\delta x^i}, \frac{\delta}{\delta x^j}\right) = R_{0ij}^h \frac{\partial}{\partial y^h}, \\ T\left(\frac{\partial}{\partial y^i}, \frac{\delta}{\delta x^j}\right) = -T\left(\frac{\delta}{\delta x^j}, \frac{\partial}{\partial y^i}\right) = U_{ij}^h \frac{\delta}{\delta x^h} - \overline{U}_{ij}^h \frac{\partial}{\partial y^h}. \end{cases} \tag{15}$$

Proof. We showed in [42] Proposition 3.2 that the torsion tensor field of the Schouten–van Kampen connection $\overline{\nabla}$ has the components:

$$\begin{cases} T\left(\frac{\partial}{\partial y^i}, \frac{\partial}{\partial y^j}\right) = (Q_{ij}^h - Q_{ji}^h) \frac{\partial}{\partial y^h}, \\ T\left(\frac{\delta}{\delta x^i}, \frac{\delta}{\delta x^j}\right) = \left(\overline{S}_{ij}^h - \overline{S}_{ji}^h\right) \frac{\delta}{\delta x^h} + R_{0ij}^h \frac{\partial}{\partial y^h}, \\ T\left(\frac{\partial}{\partial y^i}, \frac{\delta}{\delta x^j}\right) = -T\left(\frac{\delta}{\delta x^j}, \frac{\partial}{\partial y^i}\right) = U_{ij}^h \frac{\delta}{\delta x^h} - \overline{U}_{ij}^h \frac{\partial}{\partial y^h}. \end{cases} \tag{16}$$

From the first expression in (13) it follows that Q_{ij}^h is symmetric in i and j, and hence, from (16), we have:

$$T\left(\frac{\partial}{\partial y^i}, \frac{\partial}{\partial y^j}\right) = 0.$$

By substituting into the last relation (13) the components of the metric G from (4) and the entries of the inverse matrix form (8), then using in turn (10) and (9), we obtain that:

$$\overline{S}_{ij}^h - \overline{S}_{ji}^h = \frac{c_2 c_3}{c_1 c_2 - c_3^2} \left(R_{ij0}^h + R_{j0i}^h + R_{0ij}^h\right),$$

which vanishes due to the first Bianchi identity, and hence, the second relation in (16) reduces to:

$$T\left(\frac{\delta}{\delta x^i}, \frac{\delta}{\delta x^j}\right) = R_{0ij}^h \frac{\partial}{\partial y^h}.$$

Thus, the components of the torsion tensor field of the Schouten–van Kampen connection $\overline{\nabla}$ on (TM, G) are those given in the statement. □

Theorem 1. (Theorem 3.4 [42]) *The Schouten–van Kampen connection on (TM, G), given in Proposition 1, is torsion-free if and only if the base manifold (M, g) is locally flat and the metric G has the expression:*

$$\begin{cases} G(X_y^H, Y_y^H) = \kappa_1 g_{\tau(y)}(X, Y), \\ G(X_y^V, Y_y^V) = c_2(t) g_{\tau(y)}(X, Y) + d_2(t) g_{\tau(y)}(X, y) g_{\tau(y)}(Y, y), \\ G(X_y^V, Y_y^H) = c_3(t) g_{\tau(y)}(X, Y) + c_3'(t) g_{\tau(y)}(X, y) g_{\tau(y)}(Y, y), \end{cases} \tag{17}$$

where κ_1 is a real constant and c_2, d_2, c_3 are smooth functions depending on the energy density on TM, such that one of the following two sets of conditions is satisfied:

$$(i)\ \kappa_1 c_2 - c_3^2 \neq 0, \quad \kappa_1(c_2 + 2td_2) - (c_3 + 2tc_3')^2 \neq 0,$$
$$(ii)\ \kappa_1 > 0, \quad c_2 + 2td_2 > 0,\ \kappa_1(c_2 + 2td_2) - (c_3 + 2tc_3')^2 > 0.$$

In the first case, G is a pseudo-Riemannian metric and in the second one it is a Riemannian metric.

Proposition 3. (Proposition 3.5 [42]) *The torsion-free Schouten–van Kampen connection characterized in Theorem 1 coincides with the Levi-Civita connection of a pseudo-Riemannian general natural metric G given by (17) if and only if the coefficients of G fall in one of the instances:*

(i) $c_3(t) = 0$, $\forall t \geq 0$, c_2, d_2 *are some smooth functions of t such that:*

$$\kappa_1 c_2(t) \neq 0,\ c_2(t) + 2td_2(t) \neq 0,\ \forall t \geq 0;$$

(ii) $c_2(t) = d_2(t) = 0$, $\forall t \geq 0$ *and c_3 is an arbitrary nonzero smooth function of t, $c_3(t) \neq \frac{const}{\sqrt{t}}$, for all $const \in \mathbb{R}$ and all $t > 0$;*

(iii) $c_2(t) = \kappa_2 \in \mathbb{R}$, $c_3(t) = \kappa_3 \in \mathbb{R}$, *such that $\kappa_1\kappa_2 - \kappa_3^2 \neq 0$, $d_2(t) = 0$, for all $t \geq 0$;*

(iv) $c_2(t) = \kappa(c_3(t))^2$, $d_2(t) = 2\kappa c_3'(t)(c_3(t) + tc_3'(t))$, *where κ is a nonzero real constant, such that $\kappa_1\kappa \neq 1$ and c_3 is an arbitrary nonzero smooth function of t, $c_3(t) \neq \frac{const}{\sqrt{t}}$, for all $const \in \mathbb{R}$ and all $t > 0$.*

If the coefficients of the metric G from Proposition 3 have the expressions (*iv*) extended to the situation when κ is an arbitrary real constant such that $\kappa_1\kappa \neq 1$, then by taking $\kappa = 0$, we get the coefficients from (*ii*), and by taking $c_3(t) = \kappa_3 \in \mathbb{R} \setminus \{0\}$, we get the coefficients from (*iii*) for a proper general natural metric. Thus, we can state the following characterization of the proper general natural metrics on TM whose Levi-Civita connection coincides with the associated Schouten–van Kampen connection.

Proposition 4. *The proper general natural metrics G on TM for which the Levi-Civita connection coincides with its associated Schouten–van Kampen connection are given by (17), where c_3 is an arbitrary nonzero smooth function of t, $c_3(t) \neq \frac{const}{\sqrt{t}}$ for every $t > 0$, $const \in \mathbb{R}$, and the functions c_2 and d_2 have the particular expressions:*

$$c_2(t) = \kappa(c_3(t))^2,\ d_2(t) = 2\kappa c_3'(t)(c_3(t) + tc_3'(t)),$$

where κ is an arbitrary real constant such that $\kappa_1\kappa \neq 1$.

3. General Natural Metrics Torsion-Coupled with the Schouten–van Kampen Connection

Statistical manifolds, the main tool of classical information geometry, were defined in [14] as follows:

Definition 1. *Let (M,h) be a pseudo-Riemannian manifold, and let ∇ be a torsion-free affine connection on M. The triplet (M,∇,h) is called a statistical manifold if the tensor field ∇h is totally symmetric, that is:*

$$(\nabla_X h)(Y,Z) = (\nabla_Y h)(X,Z),\ \forall X,Y,Z \in \mathcal{T}_0^1(M). \tag{18}$$

A metric h and an affine connection ∇ satisfying (18) are called Codazzi-coupled. In this case, the couple (∇,h) is called a Codazzi pair or a statistical structure on M and ∇ is called a statistical connection on (M,h).

Extending the condition (18) to the case when the affine connection has nontrivial torsion, T. Kurose defined in [15] the statistical manifolds admitting torsion, also known as quasi-statistical manifolds (see [17]), which represent the subject of quantum information geometry.

Definition 2. *Let (M,h) be a pseudo-Riemannian manifold, and let ∇ be an affine connection of torsion T^∇ on M. If the metric h and the connection ∇ satisfy the relation:*

$$(\nabla_X h)(Y,Z) - (\nabla_Y h)(X,Z) = -h\Big(T^\nabla(X,Y),Z\Big),\ \forall X,Y,Z \in \mathcal{T}_0^1(M), \tag{19}$$

then the triplet (M,∇,h) is called a statistical manifold admitting torsion or a quasi-statistical manifold.

We say that a metric h and an affine connection ∇ with nonzero torsion T^∇ satisfying (19) are torsion-coupled. In this case the couple (∇,h) is called a statistical structure admitting torsion on M or a quasi-statistical structure on M and ∇ is called a quasi-statistical connection on (M,h).

In particular, if TM is the total space of the tangent bundle of a Riemannian manifold (M,g), endowed with a general natural metric G and with the corresponding Schouten–van Kampen connection $\overline{\nabla}$, we say that *the metric G and the connection $\overline{\nabla}$ are torsion-coupled*, $(\overline{\nabla},G)$ is *a statistical structure admitting torsion on TM* or *a quasi-statistical structure on TM*, $\overline{\nabla}$ is *a quasi-statistical connection* on (TM,G), and the triplet $(TM,\overline{\nabla},G)$ is *a statistical manifold admitting torsion* or *a quasi-statistical manifold* if:

$$(\overline{\nabla}_X G)(Y,Z) - (\overline{\nabla}_Y G)(X,Z) + G(T(X,Y),Z) = 0,\ \forall X,Y,Z \in \mathcal{T}_0^1(TM), \tag{20}$$

where T is the torsion tensor field of $\overline{\nabla}$.

If the connection $\overline{\nabla}$ is torsion-free, then the relation (20) reduces to:

$$(\overline{\nabla}_X G)(Y,Z) - (\overline{\nabla}_Y G)(X,Z) = 0,\ \forall X,Y,Z \in \mathcal{T}_0^1(TM). \tag{21}$$

If the metric G and the connection $\overline{\nabla}$ satisfy the relation (21), we say that G and $\overline{\nabla}$ are *Codazzi-coupled*, $(\overline{\nabla},G)$ is *a Codazzi pair* or *a statistical structure on TM*, $\overline{\nabla}$ is *a statistical connection* on (TM,G) and the triplet $(TM,\overline{\nabla},G)$ is *a statistical manifold*.

For simplicity of notations, we consider a $(0,3)$-tensor field $\mathcal{T}$ on TM:

$$\mathcal{T}(X,Y,Z) = (\overline{\nabla}_X G)(Y,Z) - (\overline{\nabla}_Y G)(X,Z) + G(T(X,Y),Z), \tag{22}$$

for every $X,Y,Z \in \mathcal{T}_0^1(TM)$. Thus, the relation (20) which characterizes the statistical manifold admitting torsion $(TM,\overline{\nabla},G)$ takes the simpler form:

$$\mathcal{T}(X,Y,Z) = 0,\ \forall X,Y,Z \in \mathcal{T}_0^1(M). \tag{23}$$

Taking into account the expressions (15) of the torsion of $\overline{\nabla}$ and the relation (22) which gives the tensor field $\mathcal{T}$, we obtain the components of $\mathcal{T}$ with respect to the adapted local frame field $\{\delta_i, \partial_j\}_{i,j=1}^n$:

$$\mathcal{T}(\partial_i, \partial_j, \partial_k) = (\overline{\nabla}_{\partial_i} G)(\partial_j, \partial_k) - (\overline{\nabla}_{\partial_j} G)(\partial_i, \partial_k); \tag{24}$$

$$\mathcal{T}(\partial_i, \partial_j, \delta_k) = (\overline{\nabla}_{\partial_i} G)(\partial_j, \delta_k) - (\overline{\nabla}_{\partial_j} G)(\partial_i, \delta_k); \tag{25}$$

$$\mathcal{T}(\partial_i, \delta_j, \partial_k) = (\overline{\nabla}_{\partial_i} G)(\delta_j, \partial_k) - (\overline{\nabla}_{\delta_j} G)(\partial_i, \partial_k) + U_{ij}^h G_{hk}^{(3)} - \overline{U}_{ij}^h G_{hk}^{(2)}; \tag{26}$$

$$\mathcal{T}(\partial_i, \delta_j, \delta_k) = (\overline{\nabla}_{\partial_i} G)(\delta_j, \delta_k) - (\overline{\nabla}_{\delta_j} G)(\partial_i, \delta_k) + U_{ij}^h G_{hk}^{(1)} - \overline{U}_{ij}^h G_{hk}^{(3)}; \tag{27}$$

$$\mathcal{T}(\delta_i, \delta_j, \partial_k) = (\overline{\nabla}_{\delta_i} G)(\delta_j, \partial_k) - (\overline{\nabla}_{\delta_j} G)(\delta_i, \partial_k) + R_{0ij}^h G_{hk}^{(2)}; \tag{28}$$

$$\mathcal{T}(\delta_i, \delta_j, \delta_k) = (\overline{\nabla}_{\delta_i} G)(\delta_j, \delta_k) - (\overline{\nabla}_{\delta_j} G)(\delta_i, \delta_k) + R_{0ij}^h G_{hk}^{(3)}. \tag{29}$$

Proposition 5. *Let (M, g) be a connected Riemannian manifold of dimension $n > 2$ and let TM be the total space of the tangent bundle, endowed with a general natural metric G given by (5). If the metric G and the corresponding Schouten–van Kampen connection are torsion-coupled, then the base manifold is a space form when $c_2(0)c_3(0) \neq 0$ and locally flat when $c_3(t) = 0$ for every $t \geq 0$.*

Proof. By using the relations (24), (4), (12), (13), and (8), we obtain:

$$\begin{aligned}\mathcal{T}(\partial_i, \partial_j, \partial_k) =& \frac{1}{2}[(2c_3' - 2d_3 - c_1'c_3p_1 + c_3d_1p_1 - 2c_3c_3'p_3 + 2c_3d_3p_3 \\ &+ 2d_1d_3p_1t - 2c_3'd_3p_3t + 2d_3^2p_3t + 2c_3d_1q_1t - 2c_3c_3'q_3t \\ &+ 2c_3d_3q_3t + 4d_1d_3q_1t^2 - 4c_3'd_3q_3t^2 + 4d_3^2q_3t^2)\cdot \\ &(g_{jk}g_{0i} - g_{ik}g_{0j}) - c_2c_3p_1(R_{lijk} - R_{ljik})y^l \\ &+ c_2(d_3p_1 + c_3q_1 + 2d_3q_1t)(R_{likm}g_{0j} - R_{ljkm}g_{0i})y^l y^m].\end{aligned} \tag{30}$$

The connection $\overline{\nabla}$ and the metric G are torsion-coupled if and only if the tensor field $\mathcal{T}$ vanishes, that is all its components with respect to the adapted local frame field $\{\delta_i, \partial_j\}_{i,j=1}^n$ vanish, and hence, a necessary condition for the torsion coupling between $\overline{\nabla}$ and G is $\mathcal{T}(\partial_i, \partial_j, \partial_k) = 0$. Differentiating the expression (30) with respect to the tangential coordinates y^l and taking the value of this derivative in $y = 0$, since the curvature of the base manifold does not depend on the tangent vector y, for $c_2(0)c_3(0) \neq 0$ we obtain that:

$$R_{lijk} - R_{ljik} = \left.\frac{2(1 - c_3p_3)(c_3' - d_3) - c_3p_1(c_1' - d_1)}{c_2c_3p_1}\right|_{t=0} (g_{li}g_{jk} - g_{lj}g_{ik}). \tag{31}$$

Due to the anti-symmetry of the Riemann-Christoffel tensor field in the last two arguments, the left-hand side of relation (31) becomes $R_{lijk} + R_{ljki}$, and from the first Bianchi identity it follows that:

$$R_{lkij} = c(g_{li}g_{jk} - g_{lj}g_{ik}),$$

where the function c depends on $x^1, \ldots, x^n$, only, having the expression:

$$c = -\left.\frac{2(1 - c_3p_3)(c_3' - d_3) - c_3p_1(c_1' - d_1)}{c_2c_3p_1}\right|_{t=0}.$$

Since the manifold M is connected and of dimension $n > 2$, from Schur's theorem we obtain that c is constant, i.e., M is a space form.

Now, we study the situation when $c_3(t) = 0$ for every $t \geq 0$. In this case, by using (10) and then (9), the expression (30) becomes simpler:

$$\mathcal{T}(\partial_i, \partial_j, \partial_k) = \frac{-2d_3^2 t(c_2 + c_2' t)}{c_1c_2 + 2t(c_2d_1 + c_1d_2) + 4t^2(d_1d_2 - d_3^2)}(g_{jk}g_{0i} - g_{ik}g_{0j}), \tag{32}$$

and its condition of vanishing does not involve the curvature of (M, g).

Analyzing the other components of the tensor field $\mathcal{T}$ in the same manner, we obtain:

$$\begin{aligned}\mathcal{T}(\delta_i, \delta_j, \partial_k) =& \frac{d_3^2 t}{c_1c_2 + 2t(c_2d_1 + c_1d_2) + 4t^2(d_1d_2 - d_3^2)} \cdot \\ & [c_1(g_{jk}g_{0i} - g_{ik}g_{0j}) + c_2(R_{hjkl}g_{0i} - R_{hikl}g_{0j})y^h y^l] \\ & - c_2 R_{hkij}y^h,\end{aligned} \tag{33}$$

whose derivative with respect to y^h computed in $y = 0$ is $c_2(0)R_{hkji}$.

Since $c_3(t) = 0$ for every $t \geq 0$, from the nondegeneracy condition (6) of the metric G it follows that $c_2(0) \neq 0$, and hence, $c_2(0)R_{hkji}$ vanishes if and only if $R_{hkji} = 0$, that is the base manifold is locally flat. □

One can easily prove the following lemma, which will be used to obtain the main results of the paper.

Lemma 1. *Let (M, g) be a Riemannian manifold of dimension $n > 2$ and α_1, α_2, α_3, α_4 be four smooth real functions of the energy density on TM. If these functions satisfy the following relation:*

$$\alpha_1(t)g_{jk}g_{0i} + \alpha_2(t)g_{ik}g_{0j} + \alpha_3(t)g_{ij}g_{0k} + \alpha_4(t)g_{0i}g_{0j}g_{0k} = 0, \ \forall t > 0,$$

where $g_{0i} = g_{hi}y^h$, then $\alpha_1(t) = \alpha_2(t) = \alpha_3(t) = \alpha_4(t) = 0$, for all $t \geq 0$.

Theorem 2. *Let (M, g) be a connected Riemannian manifold of dimension $n > 2$ and let the total space TM of the tangent bundle be endowed with a general natural metric G given by (5) such that $c_3(t) = 0$ for every $t \geq 0$. The following assertions are equivalent:*

(i) The metric G and the Schouten–van Kampen connection $\overline{\nabla}$ associated to the Levi-Civita connection ∇ of G are torsion-coupled;

(ii) The triplet $(TM, \overline{\nabla}, G)$ is a statistical manifold;

(iii) The base manifold is locally flat and the metric G is of natural diagonal lift type, given by:

$$\begin{cases} G(X_y^H, Y_y^H) = \kappa_1 g_{\tau(y)}(X, Y), \\ G(X_y^V, Y_y^V) = c_2(t)g_{\tau(y)}(X, Y) + d_2(t)g_{\tau(y)}(X, y)g_{\tau(y)}(Y, y), \\ G(X_y^V, Y_y^H) = 0, \end{cases} \tag{34}$$

for all $X, Y \in \mathcal{T}_0^1(M)$, $y \in TM$, $\kappa_1 \in \mathbb{R} \setminus \{0\}$, where c_2, d_2 are some arbitrary nonzero smooth real functions of energy density t of y such that $c_2(t) + 2td_2(t) \neq 0$, for every $t \geq 0$;

(iv) The Schouten–van Kampen connection $\overline{\nabla}$ coincides with the Levi-Civita connection ∇.

Proof. According to Proposition 5, if a metric G given by (5) such that $c_3(t) = 0$ for every $t \geq 0$ is torsion-coupled with the Schouten–van Kampen connection $\overline{\nabla}$, then the base manifold (M, g) is locally flat. Thus, the expression (33) of the component $\mathcal{T}(\delta_i, \delta_j, \partial_k)$ reduces to:

$$\mathcal{T}(\delta_i, \delta_j, \partial_k) = \frac{c_1 d_3^2 t}{c_1c_2 + 2t(c_2d_1 + c_1d_2) + 4t^2(d_1d_2 - d_3^2)}(g_{jk}g_{0i} - g_{ik}g_{0j}). \tag{35}$$

Applying Lemma 1, one has $\mathcal{T}(\delta_i, \delta_j, \partial_k) = 0$ for every $t \geq 0$ if and only if $c_1 d_3^2 = 0$. Since $c_3(t) = 0$ for every $t \geq 0$, from the nondegeneracy condition (6) of the metric G it follows that $c_1(t) \neq 0$ for every $t \geq 0$, and hence, the expressions (35) of $\mathcal{T}(\delta_i, \delta_j, \partial_k)$ and (32) of $\mathcal{T}(\partial_i, \partial_j, \partial_k)$ vanish simultaneously if and only if $d_3 = 0$, i.e., the metric is of natural diagonal lift type. We compute the other components of the tensor field $\mathcal{T}$ with resect to the adapted local frame field $\{\delta_i, \partial_j\}_{i,j=1}^n$ by imposing the conditions already obtained, that is $c_3 = d_3 = 0$ and the locally flatness of the base manifold, and we have that:

$$\mathcal{T}(\partial_i, \partial_j, \delta_k) = 0,\ \mathcal{T}(\partial_i, \delta_j, \partial_k) = 0,\ \mathcal{T}(\delta_i, \delta_j, \delta_k) = 0,$$
$$\mathcal{T}(\partial_i, \delta_j, \delta_k) = \frac{1}{2}(c_1' g_{jk} g_{0i} + d_1 g_{ik} g_{0j} + d_1 g_{ij} g_{0k} + d_1' g_{0i} g_{0j} g_{0k}).$$

By using Lemma 1, it follows that $\mathcal{T}(\partial_i, \delta_j, \delta_k) = 0$ if and only if $c_1(t) = \kappa_1 \in \mathbb{R}$ and $d_1(t) = 0$ for every $t \geq 0$. For the nondegeneracy of the metric G the real constant κ_1 and the functions c_2 and $c_2 + 2td_2$ must be nonzero. Thus, we prove that all the components of the tensor field $\mathcal{T}$ corresponding to the general natural metric G with $c_3 = 0$ vanish if and only if the base manifold (M, g) is locally flat and the metric G has the form (34). Hence, we proved the equivalence of the items (i) and (iii).

If assertion (iii) holds, i.e., the base manifold is locally flat and the metric G is given by (34), we obtain by using Theorem 1 that the Schouten–van Kampen connection $\overline{\nabla}$ associated to the Levi-Civita connection ∇ of G is torsion-free. On the other hand, we showed that (iii) is equivalent to (i), and since $\overline{\nabla}$ is torsion-free, items (i), (ii), and (iii) are equivalent. Moreover, since the metric G given by (34) is the metric from Proposition 3 (i), it follows that $\overline{\nabla}$ coincides with ∇, i.e., the items (iii) and (iv) in the statement are equivalent. □

Remark 1. *Let (M, g) be a locally flat connected Riemannian manifold of dimension $n > 2$. A natural diagonal metric whose corresponding Schouten–van Kampen connection is a statistical connection on TM depends on an arbitrary nonzero real constant and on two arbitrary nonzero smooth real functions c_2 and d_2 of the energy density t, such that $c_2(t) + 2td_2(t) \neq 0$ for all $t \geq 0$. For every metric in this family the Levi-Civita connection and its associated Schouten–van Kampen connection are identical, and hence, there is no natural diagonal metric G on TM such that (TM, G) endowed the corresponding Schouten–van Kampen connection is a statistical manifold admitting torsion.*

Theorem 3. *Let (M, g) be a connected Riemannian manifold of dimension $n > 2$ and let TM be the total space of the tangent bundle, endowed with a proper general natural metric G given by (5) such that $c_2(0)c_3(0) \neq 0$ and with the Schouten–van Kampen connection $\overline{\nabla}$ associated to the Levi-Civita connection ∇ of G. The following assertions hold:*

(a) *$(TM, \overline{\nabla}, G)$ is a statistical manifold if and only if the base manifold (M, g) is locally flat and the metric G has one of the following expressions:*

$$(i)\begin{cases} G(X_y^H, Y_y^H) = \kappa_1 g_{\tau(y)}(X, Y), \\ G(X_y^V, Y_y^V) = \kappa_2 (c_3(t))^2 g_{\tau(y)}(X, Y) \\ \qquad\qquad + 2\kappa_2 c_3'(t)(c_3(t) + c_3'(t)t) g_{\tau(y)}(X, y) g_{\tau(y)}(Y, y), \\ G(X_y^V, Y_y^H) = c_3(t) g_{\tau(y)}(X, Y) + c_3'(t) g_{\tau(y)}(X, y) g_{\tau(y)}(Y, y), \end{cases}$$

for every $X, Y \in \mathcal{T}_0^1(TM)$, $y \in TM$, *where* κ_1, κ_2 *are some arbitrary nonzero real constants such that* $\kappa_1\kappa_2 \neq 1$ *and* c_3 *is an arbitrary smooth function of the energy density* t *of* y, *such that* $c_3(0) \neq 0$, $c_3(t) \neq \frac{const}{\sqrt{t}}$ *for all* $const \in \mathbb{R}$ *and all* $t > 0$;

$$ii)\begin{cases} G(X_y^H, Y_y^H) = 0, \\ G(X_y^V, Y_y^V) = c_2(t) g_{\tau(y)}(X,Y) \\ \qquad\qquad + \frac{c_2'(t)c_3(t)^2 + 2c_2'(t)c_3(t)c_3'(t)t - 2c_2(t)c_3'^2(t)t}{c_3^2(t)} g_{\tau(y)}(X,y) g_{\tau(y)}(Y,y), \\ G(X_y^V, Y_y^H) = c_3(t) g_{\tau(y)}(X,Y) + c_3'(t) g_{\tau(y)}(X,y) g_{\tau(y)}(Y,y), \end{cases}$$

for every $X, Y \in \mathcal{T}_0^1(TM)$, $y \in TM$, *where* c_2, c_3 *are some arbitrary smooth functions of the energy density* t *of* $y \in TM$ *such that* $c_2(0)c_3(0) \neq 0$, $c_3(t) \neq \frac{const}{\sqrt{t}}$ *for all* $const \in \mathbb{R}$ *and all* $t > 0$.

The Levi-Civita connection ∇ *of* G *and its associated Schouten–van Kampen connection* $\overline{\nabla}$ *coincide for every metric* G *given by* $i)$.

The connections ∇ *and* $\overline{\nabla}$ *are different, i.e.,* $(TM, \overline{\nabla}, G)$ *is a nontrivial statistical manifold if the metric* G *has the expression* $ii)$ *with* $c_2(t) \neq \kappa_2 c_3^2(t)$ *for every* $t \geq 0$ *and every* $\kappa_2 \in \mathbb{R}$.

(b) $(TM \setminus \{0\}, \overline{\nabla}, G)$ *is a quasi-statistical manifold if and only if the base manifold* (M, g) *has constant sectional curvature* $c < 0$ *and the metric* G *has the following expression:*

$$\begin{cases} G(X_y^H, Y_y^H) = -\frac{c\kappa_2}{t} g_{\tau(y)}(X,y) g_{\tau(y)}(Y,y), \\ G(X_y^V, Y_y^V) = \frac{\kappa_2}{t} g_{\tau(y)}(X,Y) + d_2(t) g_{\tau(y)}(X,y) g_{\tau(y)}(Y,y), \\ G(X_y^V, Y_y^H) = \pm\Big[\frac{\kappa_2\sqrt{-2c}}{\sqrt{t}} g_{\tau(y)}(X,Y) - \frac{\kappa_2\sqrt{-2c}}{2t\sqrt{t}}(t) g_{\tau(y)}(X,y) g_{\tau(y)}(Y,y)\Big], \end{cases}$$

for every $X, Y \in \mathcal{T}_0^1(TM)$, $y \in TM$, *where* κ_2 *is an arbitrary nonzero real constant and* d_2 *is an arbitrary smooth real function of the energy density* t *of* y *such that* $d_2(t) \neq -\frac{\kappa_2}{2t^2}$, *for every* $t > 0$.

Proof. Our purpose is to determine the proper general natural metrics G such that the manifold $(TM, \overline{\nabla}, G)$ is a statistical manifold admitting torsion. To this aim, we study the conditions of vanishing for all the components of the tensor field $\mathcal{T}$ given by (22) with respect to the adapted local frame field $\{\delta_i, \partial_j\}_{i,j=1}^n$.

From Proposition 5, a necessary condition for $(TM, \overline{\nabla}, G)$ to be a statistical manifold admitting torsion is that the base manifold (M, g) has constant sectional curvature c, and hence, we take from the beginning:

$$R_{kij}^h = c(\delta_i^h g_{kj} - \delta_j^h g_{ki}),$$

where δ_i^h is the Kronecker delta.

By using the expressions (24)–(29), in which we substitute the components of the metric from (4), the components of the torsion T from (15), the expressions (12) of the Schouten–van Kampen connection, its coefficients from (13), the entries of the inverse matrix H from (8) and their coefficients from (10) and (9), we obtain that the components of the tensor field $\mathcal{T}$ have the forms:

$$\begin{aligned} &\mathcal{T}(\partial_i, \partial_j, \partial_k) = A_1(t)(g_{jk}g_{0i} - g_{ik}g_{0j});\ \mathcal{T}(\partial_i, \partial_j, \delta_k) = A_2(t)(g_{jk}g_{0i} - g_{ik}g_{0j}); \\ &\mathcal{T}(\partial_i, \delta_j, \partial_k) = A_3(t)g_{jk}g_{0i} + \widetilde{A}_3 g_{ik}g_{0j} + B_3(t)g_{ij}g_{0k} + C_3(t)g_{0i}g_{0j}g_{0k}; \\ &\mathcal{T}(\partial_i, \delta_j, \delta_k) = A_4(t)g_{jk}g_{0i} + \widetilde{A}_4 g_{ik}g_{0j} + B_4(t)g_{ij}g_{0k} + C_4(t)g_{0i}g_{0j}g_{0k}; \\ &\mathcal{T}(\delta_i, \delta_j, \partial_k) = A_5(t)(g_{jk}g_{0i} - g_{ik}g_{0j});\ \mathcal{T}(\delta_i, \delta_j, \delta_k) = A_6(t)(g_{jk}g_{0i} - g_{ik}g_{0j}), \end{aligned}$$

where A_i, $i = 1, \ldots, 6$ and $\widetilde{A}_j, B_j$, $j = 3, 4$, are some rational functions depending on the coefficients of the metric G, their derivatives, the constant sectional curvature c of (M, g),

and the energy density t. Since the expressions of A_i and B_i are quite long, we present here the shorter ones:

$$\begin{aligned} B_3(t) &= \frac{c_1'c_2c_3 + c_1(c_2'c_3 - c_2c_3') - c_3^2c_3' + (c_3^2 - c_1c_2)d_3}{2(c_1c_2 - c_3^2)}, \\ B_4(t) &= \tfrac{1}{2}(cc_2 + d_1). \end{aligned} \tag{36}$$

From Lemma 1, we have that all the components of the tensor field $\mathcal{T}$ from above vanish if and only if $A_i(t) = 0$, $i = 1, \ldots, 6$, $\widetilde{A}_j(t) = 0$, $B_j(t) = 0$, $j = 3, 4$.

From the conditions of vanishing of $B_3(t)$ and $B_4(t)$ given in (36), we obtain two necessary conditions for $(TM, \overline{\nabla}, G)$ to be a quasi-statistical manifold:

$$d_3 = \frac{c_1'c_2c_3 + c_1(c_2'c_3 - c_2c_3') - c_3^2c_3'}{c_1c_2 - c_3^2}, \tag{37}$$

$$d_1 = -cc_2. \tag{38}$$

After substituting the value obtained for d_1 into the expression of $\widetilde{A}_3(t)$ this turns into:

$$\begin{aligned} \widetilde{A}_3(t) =& (2cc_1c_2^3c_3 + c_1^2c_2c_2''c_3 - 2cc_2^2c_3^3 - c_1c_2'c_3^3 - 2c_1^2c_2c_3d_2 \\ &+ 2c_1c_3^3d_2 + 2c_1^2c_2^2d_3 - 2c_1c_2c_3^2d_3 - 4c^2c_2^4c_3t - 2cc_1c_2^2c_2'c_3t \\ &+ 2cc_2c_2'c_3^3t + 8cc_1c_2^2c_3d_2t - 4cc_2c_3^3d_2t - 4cc_1c_2^3d_3t + 2c_1^2c_2c_2'd_3t \\ &- 4cc_2^2c_3^2d_3t - 2c_1c_2'c_3^2d_3t - 8c^2c_2^3c_3d_2t^2 - 4cc_1c_2^2c_2'd_3t^2 \\ &+ 4cc_2c_2'c_3^2d_3t^2 - 8cc_2^2c_3d_3^2t^2)/[2(c_1c_2 - c_3^2)(c_1c_2 - c_3^2 \\ &- 2cc_2^2t + 2c_1d_2t - 4c_3d_3t - 4cc_2d_2t^2 - 4d_3^2t^2)], \end{aligned} \tag{39}$$

To obtain the necessary and sufficient conditions for $\widetilde{A}_3(t) = 0$, we have to treat the following cases:

(Case I) $c_1 - 2cc_2t \neq 0$ and $c_1c_2 - c_3^2 \neq 2cc_2^2t$;
(Case II) $c_1 - 2cc_2t = 0$;
(Case III) $c_1c_2 - c_3^2 = 2cc_2^2t$.

Next, we study each case separately.

(Case I) When $c_1 - 2cc_2t \neq 0$ (i.e., $c_1 + 2td_1 \neq 0$) and $c_1c_2 - c_3^2 \neq 2cc_2^2t$, from (39) we obtain that $\widetilde{A}_3(t) = 0$ if and only if:

$$\begin{aligned} d_2 =& \frac{1}{2c_3(c_1 - 2cc_2t)(c_3^2 - c_1c_2 + 2cc_2^2t)}\{4c^2c_2^4c_3t - c_1(c_1c_2 - c_3^2)[2c_2d_3 \\ &+ c_2'(c_3 + 2d_3t)] + 2cc_2\{c_1c_2[c_2't(c_3 + 2d_3t) - c_2(c_3 - 2d_3t)] \\ &+ c_3[c_2(c_3^2 + 2c_3d_3t + 4d_3^2t^2) - c_2'c_3t(c_3 + 2d_3t)]\}\}. \end{aligned} \tag{40}$$

By using (40) and then (37) and (38), we obtain that the numerators of $\widetilde{A}_4(t)$ and $A_6(t)$ become, respectively:

$$\begin{aligned} N_{\widetilde{A}_4}(t) =& 2cc_1^3c_2^4 - c_1^2c_1'c_2^2c_3^2 - 4cc_1^2c_2^3c_3^2 - c_1^3c_2c_2'c_3^2 + c_1c_3^4(c_1c_2)' \\ &+ 2cc_1c_2^2c_3^4 + 2c_1^3c_2^2c_3c_3' - 2c_1^2c_2c_3^3c_3' - 4c^2c_1^2c_2^5t + 2cc_1^3c_2^3c_2't \\ &+ 4c^2c_1c_2^4c_3^2t - 2cc_1^2c_2^2c_2'c_3^2t - 2cc_2c_3^4t(c_1c_2)' - 4cc_1^2c_2^3c_3c_3't \\ &+ 8cc_1c_2^2c_3^3c_3't - 4c^2c_1^2c_2^4c_2't^2 - 4c^2c_1'c_2^4c_3^2t^2 + 8c^2c_1c_2^4c_3c_3't^2, \end{aligned} \tag{41}$$

$$\begin{aligned} N_{A_6}(t) =& c_1^3c_1'c_2^2 + 2cc_1^3c_2^3 - 2c_1^2c_1'c_2c_3^2 - 2cc_1^2c_2^2c_3^2 + c_1c_1'c_3^4 - 2cc_1c_2c_3^4 \\ &+ 2cc_3^6 - 4cc_1^2c_1'c_2^3t - 4c^2c_1^2c_2^4t + 2cc_1^3c_2^2c_2't + 6cc_1c_1'c_2^2c_3^2t \\ &- 2cc_1^2c_2c_2'c_3^2t - 6cc_1'c_2c_3^4t + 4c^2c_2^2c_3^4t - 4cc_1c_2'c_3^4t + 4cc_1c_2c_3^3c_3't \\ &+ 4cc_3^5c_3't + 4c^2c_1c_1'c_2^4t^2 - 4c^2c_1^2c_2^3c_2't^2 - 12c^2c_1'c_2^3c_3^2t^2 \\ &- 4c^2c_1c_2^2c_2'c_3^2t^2 + 8c^2c_1c_2^3c_3c_3't^2 + 8c^2c_2^2c_3^3c_3't^2. \end{aligned} \tag{42}$$

Studying the simultaneous vanishing of $\widetilde{A}_4$ and A_6 we distinguish the following subcases of Case I:

(I.1) $c_3^2 \neq 2cc_2^2t$ and $c_1 \neq 0$;
(I.2) $c_1 \neq 0$, $c > 0$ and $c_3 = \pm\sqrt{2ct}c_2$;
(I.3) $c_1 = 0$.

We treat each subcase separately.

(I.1) When $c_3^2 \neq 2cc_2^2t$ and $c_1 \neq 0$, solving the system of equations given by $N_{\widetilde{A}_4}(t) = 0$ and $N_{A_6}(t) = 0$, we obtain that the derivatives of the functions c_1 and c_2 have the expressions:

$$c_1' = \frac{2c(c_1c_2c_3^2 + c_3^4 + 2c_1c_2c_3c_3't + 2c_3^3c_3't)}{(c_1 - 2cc_2t)(2cc_2^2t - c_3^2)}, \tag{43}$$

$$c_2' = \frac{2c_2[cc_1^2c_2^2 + cc_3^4 + c_1^2c_3c_3' + 2c(c_3^2 - c_1c_2)(cc_2^2 + c_3c_3')t + 4c^2c_2^2c_3c_3't^2]}{-c_1(c_1 - 2cc_2t)(2cc_2^2t - c_3^2)}. \tag{44}$$

Substituting (43) and (44) into the expression of $\mathcal{T}(\partial_i, \delta_j, \delta_k)$, this reduces to:

$$\begin{aligned} \mathcal{T}(\partial_i, \delta_j, \delta_k) =& \frac{cc_3(2c_1c_2 + c_3^2 - 2cc_2^2t)(c_3 + 2c_3't)}{(c_1 - 2cc_2t)(2cc_2^2t - c_3^2)} g_{jk}g_{0i} \\ &+ \frac{c^2c_2c_3(c_3^2 + 2cc_2^2t)(c_3 + 2c_3't)}{c_1(c_1 - 2cc_2t)(2cc_2^2t - c_3^2)} g_{0i}g_{0j}g_{0k}, \end{aligned} \tag{45}$$

and it vanishes, according to Lemma 1, if and only if the involved coefficients vanish simultaneously. Since the metric G is proper general natural, i.e., $c_3 \neq 0, d_3 \neq 0$, the coefficient of $g_{0i}g_{0j}g_{0k}$ vanishes if and only if one of the following instances happens:

(I.1.*i*) $c = 0$, which together with (37), (38), (40), (43), (44) leads to:

$$c_1 = \kappa_1 \in \mathbb{R} \setminus \{0\},\ d_1 = 0,\ c_2' = \frac{2c_2c_3'}{c_3},\ i.e.,\ c_2 = \kappa_2c_3^2,\ \ d_2 = 2\kappa_2c_3'(c_3 + c_3't),\ d_3 = c_3',$$

where κ_2 is an arbitrary nonzero real constant and c_3 is an arbitrary smooth nonzero real function of t such that $c_3(0) \neq 0$ and the nondegeneracy conditions (6) of the metric G are satisfied, i.e., $\kappa_1\kappa_2 \neq 1$ and $(\kappa_1\kappa_2 - 1)(c_3 + 2tc_3')^2 \neq 0$, and hence, $c_3(t) \neq \frac{const}{\sqrt{t}}$ for all $const \in \mathbb{R}$ and all $t > 0$. By substituting the values of the coefficients of the metric G obtained in Case I.1.*i*) and $c = 0$ into each component of the tensor field $\mathcal{T}$ with respect to the adapted local frame filed $\{\delta_i, \partial_j\}_{i,j=1}^n$, we obtain, by using Mathematica, that $\mathcal{T} = 0$. On the other hand, the obtained metric satisfies Proposition 4, and hence, the Schouten–van Kampen connection $\overline{\nabla}$ coincides with the Levi-Civita connection of the metric G, i.e., in Case (I.1.*i*), $(TM, \overline{\nabla}, G)$ is obviously a statistical manifold.

(I.1.*ii*) $c_3 + 2c_3't = 0$, i.e., $c_3(t) = \frac{\kappa_3}{\sqrt{t}}$, for every $\kappa_3 \in \mathbb{R} \setminus \{0\}$, $t > 0$, but together with (38), (40), (37), and (43), which would imply $d_2 = -\frac{c_2}{2t}$, i.e., $c_2 + 2td_2 = 0$ and $d_3 = c_3'$, i.e., $c_3 + 2td_3 = 0$, and hence, the second nondegeneracy condition (6) of the metric G would not be satisfied.

(I.1.*iii*) $c_2 = 0$ does not satisfy the condition $c_2(0)c_3(0) = 0$ from the hypothesis.

(I.1.*iv*) $c < 0$ and $c_3^2 = -2cc_2^2t$, which substituted into (45), turns the factor $2c_1c_2 + c_3^2 - 2cc_2^2t$ from the coefficient of $g_{jk}g_{0i}$ into $2c_2(c_1 - 2cc_2t)$ and this vanishes if and

only if $c_2 = 0$ (see I.1.*iii*) or $c_1 = 2cc_2t$, which together with $d_1 = -cc_2$ yields $c_1 + 2td_1 = 0$, which does not hold in Case I.

We conclude that the only favorable subcase of Case I.1 is (I.1.*i*), rended in the statement at *a* (*i*). We already showed that in Case (I.1.*i*) the Schouten–van Kampen connection $\overline{\nabla}$ coincides with the Levi-Civita connection of G.

(I.2) $c_1 \neq 0$, $c > 0$ and $c_3 = \pm\sqrt{2ct}c_2$, i.e., $c_2 = \pm\frac{c_3}{\sqrt{2ct}}$ for every $t > 0$. Substituting $c_2 = \epsilon\frac{c_3}{\sqrt{2ct}}$ (where $\epsilon = 1$ or $\epsilon = -1$) into $N_{\widetilde{A}_4}$ we obtain:

$$\begin{aligned} N_{\widetilde{A}_4}(t) =& \sqrt{c}c_3[c_1(2c_1^2 - 5\epsilon c_1c_3\sqrt{2ct} + 10cc_3^2t)(c_3 + 2c_3't) \\ & - 2c_1'c_3t(c_1^2 - \epsilon c_1c_3t\sqrt{2ct} + 4cc_3^2t)], \end{aligned}$$

which vanishes if and only if one of the following instances happens:

$$\text{(I.2.}i\text{)} \begin{cases} c_1^2 - \epsilon c_1c_3t\sqrt{2ct} + 4cc_3^2t = 0, \\ (2c_1^2 - 5\epsilon c_1c_3\sqrt{2ct} + 10cc_3^2t)(c_3 + 2c_3't) = 0. \end{cases}$$

If $c_3 + 2c_3't = 0$, i.e., $c_3(t) = \frac{\kappa_3}{\sqrt{t}}$ for every $t > 0$, $\kappa_3 \in \mathbb{R} \setminus \{0\}$, then the first relation in (I.2.*i*) turns into:

$$c_1^2 - \epsilon\kappa_3\sqrt{2ct}c_1 + 4\kappa_3^2ct = 0,$$

which is not satisfied by any real function c_1 of t.

If $2c_1^2 - 5\epsilon c_1c_3\sqrt{2ct} + 10cc_3^2t = 0$, since the first relation in (I.2.*i*) holds, it follows that $3c_1^2 + 10cc_3^2t = 0$, where $c > 0$, and hence $c_1 = c_3 = 0$, which do not satisfy neither the nondegeneracy condition (6) for the metric G nor the conditions (I.2).

$$\text{(I.2.}ii\text{)} \begin{cases} c_1 = \kappa_1 \in \mathbb{R} \setminus \{0\}, \\ (2c_1^2 - 5\epsilon c_1c_3\sqrt{2ct} + 10cc_3^2t)(c_3 + 2c_3't) = 0. \end{cases}$$

In this case, the first factor in the second relation of I.2.*ii*) becomes:

$$10ctc_3^2 - 5\epsilon\kappa_1\sqrt{2ct}c_3 + 2\kappa_1^2 \neq 0$$

for every real function c_3 of t. On the other hand, if the first relation in (I.2.*ii*) is satisfied and $c_3 + 2c_3't = 0$, i.e., $c_3 = \frac{\kappa_3}{\sqrt{t}}$, for every $t > 0$, $\kappa_3 \in \mathbb{R} \setminus \{0\}$, by taking into account the expressions (38), (40), (37) of the coefficients d_1, d_2, d_3 and the expression of c_2 in Case I.2, it follows that:

$$(c_1 + 2td_1)(c_2 + 2td_2) - (c_3 + 2td_3)^2 = 0,$$

hence, the second nedegeneracy condition (6) is not satisfied.

$$\text{(I.2.}iii\text{)} \begin{cases} c_1^2 - \epsilon c_1c_3t\sqrt{2ct} + 4cc_3^2t \neq 0 \\ c_1' = \dfrac{2c_1^3c_3 - 5\epsilon c_1^2c_3^2\sqrt{2ct} + 10cc_1c_3^3t + 4c_1^3c_3't - 10\epsilon c_1^2c_3c_3't\sqrt{2ct} + 20cc_1c_3^2c_3't^2}{2c_3t(c_1^2 - \epsilon c_1c_3t\sqrt{2ct} + 4cc_3^2t)}, \end{cases}$$

for every $t > 0$. By using the expressions (38), (40), (37) of d_1, d_2, d_3, the expression of c_1' in Case (I.2.*iii*) and that of c_2 in Case I.2, we obtain that the component $\mathcal{T}(\delta_i, \delta_j, \delta_k)$ reduces to:

$$\mathcal{T}(\delta_i, \delta_j, \delta_k) = \frac{2\epsilon\sqrt{c}(c_1^2 - \epsilon\sqrt{2ct}c_1c_3 + 4cc_3^2t)}{\sqrt{t}(\sqrt{2}c_1 - 6\epsilon\sqrt{ct}c_3)} \neq 0,$$

for every c_1, c_3 nonzero smooth real functions of t. Subsequently, a general natural metric G whose coefficients satisfy Case (I.2.*iii*) is not torsion-coupled with the corresponding Schouten–van Kampen connection.

(I.3) $c_1 = 0$, which substituted into the expression of $\mathcal{T}(\partial_i, \delta_j, \delta_k)$ together with (38), (40), (37) turns the coefficient $A_4(t)$ into:

$$A_4(t) = \frac{ct(2c_2c_3' - c_2'c_3)(c_3 + 2c_3't)}{c_3(c_3 + 2tc_3') + 2cc_2t(c_2 + 2tc_2')}$$

and this vanishes if and only if one of the following situations happens:

(I.3.*i*) $c = 0$, which turns relation (38) into $d_1 = 0$. In this subcase, it follows that the first condition of Case I, $c_1 + 2td_1 \neq 0$ is not satisfied, and hence, the subcase (I.3.*i*) is not possible.

(I.3.*ii*) $c_3 + 2tc_3' = 0$, i.e., $c_3 = \frac{\kappa_3}{\sqrt{t}}$, for all $\kappa_3 \in \mathbb{R} \setminus \{0\}$, $t > 0$, which together with $c_1 = 0$ and the expressions (38), (40), and (37) leads to $c_2 + 2td_2 = c_3 + 2td_3 = 0$, i.e., the second nondegeneracy condition in (6) is not satisfied, and hence, there is no metric G whose coefficients satisfy Case I.3.*ii*).

(I.3.*iii*) $2c_2c_3' - c_2'c_3 = 0$, i.e., $c_2 = \kappa_2 c_3^2$, for every $\kappa_2 \in \mathbb{R} \setminus \{0\}$. Together with (38), (40), and (37), the expression obtained for c_2 yields:

$$\mathcal{T}(\partial_i, \delta_j, \delta_k) = -\frac{c\kappa_2 c_3(c_3 + 2tc_3')}{2t} g_{0i}g_{0j}g_{0k}. \tag{46}$$

The expression (46) vanishes if and only if $c = 0$ or $c_3 + 2tc_3' = 0$, relations which are not possible in Case I.3 (see the discussion from I.3.*i* and I.3.*ii*).

(Case II) When $c_1 - 2cc_2t = 0$, i.e., $c_1 + 2td_1 = 0$, by using (37), we obtain that the coefficient $B_4(t)$ from the expression of $\mathcal{T}(\partial_i, \delta_j, \delta_k)$ reduces to:

$$B_4(t) = \frac{2c^2c_2^3}{c_3^2 - 2cc_2^2t},$$

and hence, it vanishes if and only if one of the following subcases holds:

(II.1) $c = 0$, which due to relations (38) and (37) yields $c_1 = d_1 = 0$ and $d_3 = c_3'$, then the expression of the component $\mathcal{T}(\partial_i, \partial_j, \partial_k)$ reduces to:

$$\mathcal{T}(\partial_i, \partial_j, \partial_k) = \frac{c_2'c_3^2 - c_3^2d_2 + 2c_2'c_3c_3't - 2c_2c_3'^2t}{c_3(c_3 + 2tc_3')}(g_{jk}g_{0i} - g_{ik}g_{0j}),$$

and according to Lemma 1 it is zero if and only if:

$$d_2 = \frac{c_2'c_3^2 + 2c_2'c_3c_3't - 2c_2c_3'^2t}{c_3^2}.$$

If the coefficients of the metric G have the expressions obtained in Case II.1 and the base manifold is locally flat, we verify by using Mathematica that all the components of the tensor field $\mathcal{T}$ with respect to the adapted local frame field $\{\delta_i, \partial_j\}_{i,j=1}^n$ vanish. The metric whose coefficients are those in Case II.1 is the metric from item *a ii*) in the statement. From Theorem 1, it follows that the Schouten–van Kampen connection associated to the Levi-Civita connection of the metric G given at *a ii*) is torsion-free, and since we proved that $\mathcal{T} = 0$, the triplet $(TM, \overline{\nabla}, G)$ is a statistical manifold. If in the expression *a* (*ii*) we take $c_2(t) = \kappa(c_3(t))^2$, where κ is an arbitrary nonzero real constant, it follows that $d_2(t) = 2\kappa c_3'(t)(c_3(t) + tc_3'(t))$, and hence, the metric G satisfies Proposition 4. It follows that the Levi-Civita connection of the metric G given at *a ii*) coincides with the associated Schouten–van Kampen connection only when $c_2(t) = \kappa(c_3(t))^2$ for every $t \geq 0$, $\kappa \in \mathbb{R} \setminus \{0\}$.

If the metric G has the expression from *a ii*) with $c_2(t) \neq \kappa(c_3(t))^2$ for every $t \geq 0$, $\kappa \in \mathbb{R}$, then the Levi-Civita connection of G and its associated Schouten–van Kampen connection do not coincide, and hence, the statistical manifold $(TM, \overline{\nabla}, G)$ is nontrivial.

(II.2) $c_2 = 0$ doest not verify the condition $c_2(0)c_3(0) = 0$ from the hypothesis.

(Case III) When $c_1c_2 - c_3^2 = 2cc_2^2t$, it follows from the nondegeneracy condition (6) of the metric G that the base manifold (M, g) is not locally flat, $c_2(t) \neq 0$ and $t \neq 0$, and hence, in Case III, the metric G is defined on $TM \setminus \{0\}$, the total space of the bundle of nonzero vector fields tangent to the space form (M, g). In this case, one has:

$$c_1 = \frac{c_3^2 + 2cc_2^2t}{c_2} \tag{47}$$

and then the expression (37) of d_3 reduces to:

$$d_3 = \frac{c_2c_3 + 2c_2'c_3t - c_2c_3't}{c_2t}. \tag{48}$$

Substituting the expressions (38), (47), and (48) into the expression of $\mathcal{T}(\delta_i, \delta_j, \delta_k)$, we obtain that the numerator of its coefficient is:

$$\begin{aligned} N_{A_6}(t) =& c_3^2[8c_2^2c_3^3 - 8cc_2^4c_3t + 27c_2c_2'c_3^3t - 18c_2^2c_3^2c_3't - 14cc_2^3c_2'c_3t^2 \\ &+ 20c_2'^2c_3^3t^2 + 4cc_2^4c_3't^2 - 26c_2c_2'c_3^2c_3't^2 + 8c_2^2c_3c_3'^2t^2 \\ &- 8cc_2^2c_2'^2c_3t^3 + 4cc_2^3c_2'c_3't^3 - 2c_2c_3d_2t(c_3^2 + 2cc_2^2t)]. \end{aligned}$$

To obtain necessary and sufficient conditions for $N_{A_6}(t) = 0$, we have to study two subcases of Case III:

(III.1) If $c_3^2 + 2cc_2^2t \neq 0$, then $N_{A_6}(t) = 0$ if and only if:

$$\begin{aligned} d_2 =& (8c_2^2c_3^3 - 8cc_2^4c_3t + 27c_2c_2'c_3^3t - 18c_2^2c_3^2c_3't - 14cc_2^3c_2'c_3t^2 \\ &+ 20c_2'^2c_3^3t^2 + 4cc_2^4c_3't^2 - 26c_2c_2'c_3^2c_3't^2 + 8c_2^2c_3c_3'^2t^2 \\ &- 8cc_2^2c_2'^2c_3t^3 + 4cc_2^3c_2'c_3't^3)/[2c_2c_3t(c_3^2 + 2cc_2^2t)]. \end{aligned} \tag{49}$$

Taking into account the expressions (38), (47), (48), and (49), we obtain that the numerator of the coefficient of $g_{ik}g_{0j}$ involved in the expression of $\mathcal{T}(\partial_i, \delta_j, \partial_k)$ is of the form:

$$N_{\tilde{A}_3}(t) = c_3(c_3^2 + 2cc_2^2t)(2c_2c_3 + 3c_2'c_3t - 2c_2c_3't).$$

Since $c_3^2 + 2cc_2^2t \neq 0$ and G is a proper general natural metric, $N_{\tilde{A}_3}(t) = 0$ if and only if:

$$c_3' = \frac{2c_2c_3 + 3c_2'c_3t}{2c_2t},$$

which yields a simpler form of the coefficient of $g_{jk}g_{0i}$ in the same component of $\mathcal{T}$, namely:

$$A_3(t) = \frac{c_3(c_3^2 + 3cc_2^2t)(c_2 + c_2't)}{4cc_2^3t^2}, \tag{50}$$

while the coefficient involved in the expression of $\mathcal{T}(\partial_i, \partial_j, \delta_k)$ becomes:

$$A_2(t) = \frac{c_3(cc_2^2t - c_3^2)(c_2 + c_2't)}{4cc_2^3t^2}. \tag{51}$$

The expressions (50) and (51) vanish simultaneously if and only if $c_2 + c_2't = 0$ or $c_3^2 + 3cc_2^2t = cc_2^2t - c_3^2 = 0$.

If $c_2 + c_2't = 0$, i.e., $c_2 = \frac{\kappa_2}{t}$ for every $t > 0$, where κ_2 is an arbitrary nonzero real constant, then by taking into account (38), (47), (48) and (49) it follows that the second nondegeneracy condition (6) for the metric G is not satisfied.

If $c_3^2 + 3cc_2^2t = cc_2^2t - c_3^2 = 0$, i.e., $c_3^2 = cc_2^2t = 0$, then under the condition of Case III, it follows that $c_1c_2 - c_3^2 = 0$, i.e., the metric G is degenerate. We conclude that in

Case III.1, there is no proper general natural metric G torsion-coupled with the corresponding Schouten–van Kampen connection $\overline{\nabla}$.

(III.2) The subcase $c_3^2 + 2cc_2^2 t = 0$ holds for $c < 0$ and $t > 0$, and due to (47), it reduces to the condition $c_1 = 0$. Then, the relation (37) turns into $d_3 = c_3'$, and together with (38), it yields:

$$\mathcal{T}(\partial_i, \partial_j, \delta_k) = 0,\ \mathcal{T}(\delta_i, \delta_j, \partial_k) = 0,$$
$$\mathcal{T}(\delta_i, \delta_j, \delta_k) = \frac{cc_3(c_3 + 2c_3' t)^2}{c_3^2 + 4c_3 c_3' t + 2cc_2 t + 4c_3'^2 t^2 + 4cc_2 d_2 t^2}(g_{ik}g_{0j} - g_{jk}g_{0i}).$$

Then, $\mathcal{T}(\delta_i, \delta_j, \delta_k) = 0$ if and only if $c_3(t) = \frac{\kappa_3}{\sqrt{t}}$, for every $t > 0$, $\kappa_3 \in \mathbb{R} \setminus \{0\}$.

By using (38) and the coefficients obtained in Case III.2:

$$c_1 = 0,\ c_3 = \frac{\kappa_3}{\sqrt{t}},\ d_3 = -\frac{\kappa_3}{2t\sqrt{t}}, \tag{52}$$

we have:

$$\mathcal{T}(\partial_i, \partial_j, \partial_k) = \frac{c_2 + c_2' t}{2t}(g_{jk}g_{0i} - g_{ik}g_{0j}),$$
$$\mathcal{T}(\partial_i, \delta_j, \delta_k) = -\frac{c(c_2 + c_2' t)}{2t} g_{0i}g_{0j}g_{0k},$$
$$\mathcal{T}(\partial_i, \delta_j, \partial_k) = \frac{\kappa_3^2 + 2cc_2^2 t^2}{4\kappa_3 t^2\sqrt{t}}(g_{0i}g_{0j}g_{0k} - 2t g_{ik}g_{0j}),$$

which vanish simultaneously if and only if:

$$c_2 = \frac{\kappa_2}{t},\quad \kappa_3 = \pm\kappa_2\sqrt{-2c},\quad \forall t > 0,\ \kappa_2 \in \mathbb{R} \setminus \{0\}. \tag{53}$$

Subsequently, in Case III.2, all the components of the tensor field $\mathcal{T}$ with respect to the adapted local frame field $\{\delta_i, \partial_j\}_{i,j=1}^n$ vanish simultaneously if and only if the coefficients of the metric G satisfy the relations (38), (52), and (53) and d_2 is an arbitrary smooth real function of t, such that $d_2(t) \neq -\frac{\kappa_2}{2t^2}$, because if $d_2(t) = -\frac{\kappa_2}{2t^2}$, then the nondegeneracy condition (6) for the metric G would not be satisfied. Thus, we proved that the triplet $(TM \setminus \{0\}, \overline{\nabla}, G)$ is a statistical manifold admitting torsion if and only if the metric G has the expression given in the statement at item (b). □

Remark 2. *Let (M, g) be a locally flat connected Riemannian manifold of dimension $n > 2$. There are two families of proper general natural metrics on TM such that the Schouten–van Kampen connection associated to the Levi-Civita connection of a metric is a statistical connection on TM. One family of metrics depends on an arbitrary smooth function c_3 of the energy density t, different from $\frac{const}{\sqrt{t}}$ with $const \in \mathbb{R}$, $c_3(0) \neq 0$, and on two nonzero arbitrary real constants, provided that their product is not* 1*. The other family of metrics depends on two nonzero arbitrary smooth real functions c_2, c_3 of t, provided that $c_2(0)c_3(0) \neq 0$, $c_3(t) \neq \frac{const}{\sqrt{t}}$ for every $t > 0$, $const \in \mathbb{R}$. If, moreover, $c_2(t) \neq \kappa_2 c_3^2(t)$ for every $t \geq 0$, $\kappa_2 \in \mathbb{R}$, then the statistical structure on TM is nontrivial.*

Remark 3. *Let (M, g) be a connected $n > 2$–dimensional Riemannian manifold of constant sectional curvature $c < 0$. The family of proper general natural metrics on $TM \setminus \{0\}$ such that the Schouten–van Kampen connection associated to the Levi-Civita connection of a metric is a quasi-statistical connection on $TM \setminus \{0\}$ depends on the constant sectional curvature c of (M, g), the energy density t, an arbitrary nonzero real constant κ_2 and an arbitrary smooth function of t, different from $-\frac{\kappa_2}{2t^2}$.*

4. Conclusions

Investigating the quasi-statistical Schouten–van Kampen connection $\overline{\nabla}$ associated to the Levi-Civita connection of a general natural metric G given by (5) on the total space TM of the tangent bundle of a Riemannian manifold (M, g), we conclude the following:

(1) The base manifold must be a space form when $c_2(0)c_3(0) \neq 0$ and locally flat when $c_3(t) = 0$. Implicitly, when the metric G is of natural diagonal lift type, (M, g) must be locally flat.
(2) There exists one family of natural diagonal metrics such that $(TM, \overline{\nabla}, G)$ is a statistical manifold. The metrics in this family depend on two arbitrary nonzero smooth real functions of the energy density t and on an arbitrary nonzero real constant such that the nondegeneracy conditions of the metric are satisfied.
(3) When G is a proper general natural metric G on TM, $\overline{\nabla}$ is a statistical connection if and only if (M, g) is locally flat and the metric G has two possible expressions. Hence, there are two families of proper general natural metrics such that $(TM, \overline{\nabla}, G)$ is a statistical manifold. The metrics in the first family depend on two arbitrary nonzero real constants, κ_1, κ_2, and on an arbitrary smooth nonzero real function c_3 of the energy density t such that $c_3(0) \neq 0$, while the metrics in the second family depend only on two arbitrary smooth nonzero real functions of t, c_2 and c_3, for which $c_2(0)c_3(0) \neq 0$, such that the nondegeneracy conditions of the metric are satisfied in each case.
(4) If $c_2(t) \neq \kappa_2 c_3^2(t)$, then the statistical manifold $(TM, \overline{\nabla}, G)$ is nontrivial, i.e., the Levi-Civita connection is different from its associated Schouten–van Kampen connection.
(5) The manifold $(TM \setminus \{0\}, \overline{\nabla}, G)$ is quasi-statistical if and only if (M, g) has constant sectional curvature $c < 0$ and the metric G depends on c, t, on an arbitrary nonzero real constant κ_2 and on an arbitrary smooth real function of t, different from $-\frac{\kappa_2}{2t^2}$.

In a forthcoming paper we will determine the conditions under which the general natural α-structures characterized in [47] are torsion coupled (in particular Codazzi coupled) with the (quasi-)statistical Schouten–van Kampen connection $\overline{\nabla}$ associated to the Levi-Civita connection ∇ of a general natural metric G on TM. Another goal will be to characterize the para-Kähler-like statistical manifolds $(TM, \overline{\nabla}, P, G)$, where the almost product structure P and the metric G are of general natural lift type on TM.

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