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Hyperconnectedness and Resolvability of Soft Ideal Topological Spaces

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Abstract: This paper introduces and explores the concept of soft ideal dense sets, utilizing soft open sets and soft local functions, to examine their fundamental characteristics under some conditions for the following notions: soft ideal hyperconnectedness, soft ideal resolvability, soft ideal irresolvability, and soft ideal semi-irresolvability in soft ideal topological spaces. Moreover, it explores the relationship between these notions if $\tau \sqcap \bar{\mathcal{I}} = \phi_E$ is obtained in the soft set environment.

Keywords: soft open set; soft dense; soft ideal; soft ideal hyperconnected; soft ideal resolvable; soft ideal irresolvable; soft ideal semi-irresolvable

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1. Introduction

In 1999, Molodtsov [1] initially suggested the idea of soft sets as a broad mathematical tool for handling uncertain situations. Molodtsov effectively utilized soft theory in some areas, including probability, theory of measurement, smoothness of functions, Perron integration, operations research, Riemann integration, and so on, in [2].

Shabir and Naz [3] started researching soft topological spaces in 2011. They defined the topology on the collection τ of soft sets over X. Thus, they developed many features of soft regular spaces, soft normal spaces, soft separation axioms, soft open and soft closed sets, soft subspace, soft closure, and soft nbd of a point. They also defined the fundamental concepts of soft topological spaces.

Kandil and colleagues introduced the concept of the soft ideal for the first time [4]. Additionally, they presented the idea of soft local functions. These ideas are presented with the goal of identifying new soft topologies, termed soft topological spaces with soft ideal $(X_E, \tau, \overline{\mathcal{I}})$, from the original one. Numerous mathematical structures, such as soft group theory [5], soft ring theory [6], soft primals [7], soft algebras [8,9], soft category theory [10], ideal spaces [11], ideal resolvability [12], and so on, have been addressed by soft set theory. Similarly, the notion of soft topology through soft grills was introduced in [13]. Additionally, a large number of academics and researchers developed gentle versions of the traditional topological ideas, such as soft resolvable spaces [14], soft hyperconnected spaces [15], suitable soft spaces [7], soft ideal spaces [4,16,17], soft extremally disconnected spaces [18], soft Menger spaces [19], soft countable chain condition, and soft caliber [20]. From here on, we shall refer to a soft ideal topological space $(X_E, \tau, \overline{\mathcal{I}})$, a soft ideal space. The way this work is set out is as follows: Following the introduction, we discuss the definitions and findings that are necessary to understand the data in Section 2. Next, we recall the notion of soft local functions in Section 3. We study the fundamental operations on soft local functions. The definitions of soft hyperconnected and soft hyperconnected modulo ideal spaces, as well as a soft ideal topological space, are provided in Section 4.

We look at the basic characteristics and connections between soft hyperconnected and soft hyperconnected modulo ideals. A soft ideal resolvable space is defined in Section 5 and it is demonstrated that soft ideal resolvable topologies over soft ideal resolvable subspace are also soft ideal resolvable. The concept of soft ideal semi-irresolvable space and an overview of its properties are provided in Section 6. In Section 7, we finish off by providing an overview of the major contributions and some recommendations for the future.

2. Preliminary

Here, we provide the fundamental concepts and the outcomes of soft set theory that are required for the follow-up.

Definition 1 ([1])**.** *Let X be an initial universe and E be a set of parameters. Let $P(X)$ denote the power set of X and A be a non-null subset of parameters E. A pair (F,A) symbolized by F_A is a soft set over X_E, where F is a mapping given by $F : A \to P(X)$. Otherwise put, a soft set over X_E is a parameterized family of subsets of the universe X_E. For a particular $e \in E$, $F(e)$ might be regarded as the set of e-approximate elements of the soft set $(F,E) = F_E$ and, if $e \notin E$, then $F(e) = \phi$, i.e., $F_E = \{F(e) : e \in E, F : E \to P(X)\}$. The collection of all these soft sets is symbolized by $SS(X)_E$.*

Definition 2 ([21])**.** *Let $F_E, G_E \in SS(X)_E$. Then*

1. *F_E is called a soft subset of G_E, denoted by $F_E \sqsubseteq G_E$, if $F(e) \subseteq G(e)$, for all $e \in E$.*
2. *F_E is called absolute, symbolized by X_E, if $F(e) = X$ for all $e \in E$.*
3. *F_E is called null, symbolized by ϕ_E, if $F(e) = \phi$ for all $e \in E$.*

In this case F_E is said to be a soft subset of G_E and G_E is said to be a soft superset of F_E, $F_E \sqsubseteq G_E$.

Definition 3 ([22])**.** *1. A soft set $F_E \in SS(X)_E$ is called a soft point in X_E if there exist $x \in X$ and $e \in E$ such that $F(e) = \{x\}$ and $F(e^c) = \phi$ for each $e^c \in E - \{e\}$. This soft point F_E is denoted by x_e.*

2. *Let Δ be an arbitrary index set and $\Omega = \{(F_\alpha)_E : \alpha \in \Delta\}$ be a subfamily of $SS(X)_E$. Then:*
   (a) *The union of all $(F_\alpha)_E$ is the soft set H_E, where $H(e) = \cup_{\alpha\in\Delta}(F_\alpha)_E(e)$ for each $e \in E$. We write $\sqcup_{\alpha\in\Delta}(F_\alpha)_E = H_E$.*
   (b) *The intersection of all $(F_\alpha)_E$ is the soft set M_E, where $M(e) = \cap_{\alpha\in\Delta}(F_\alpha)_E(e)$ for each $e \in E$. We write $\sqcap_{\alpha\in\Delta}(F_\alpha)_E = M_E$.*
3. *A soft set G_E in a soft topological space (X_E, τ) is called a soft neighborhood of the soft point $x_e \in X_E$ if there exists a soft open set H_E such that $x_e \in H_E \sqsubseteq G_E$.*

Definition 4 ([3])**.** *Let (X_E, τ) be a soft topological space and $F_E \in SS(X)_E$.*

1. *The soft closure of F_E, symbolized by $cl(F_E)$, is the intersection of all soft closed supersets of F_E, i.e., $cl(F_E) = \sqcap\{H_E : H_E$ is soft closed and $F_E \sqsubseteq H_E\}$.*
2. *The soft interior of F_E is the set $Int(F_E) = \sqcup\{H_E : H_E$ is soft open and $H_E \sqsubseteq F_E\}$.*
3. *A difference of two soft sets F_E and G_E over the common universe X_E, symbolized by $F_E - G_E$, is the soft set H_E for all $e \in E$, $H(e) = F(e) - G(e)$.*
4. *A complement of a soft set F_E, symbolized by F_E^c, is defined as follows. $F^c : E \to P(X)$ is a mapping given by $F^c(e) = X_E(e) - F(e)$, for all $e \in E$, and F^c is called a soft complement function of F_E.*
5. *Let F_E be a soft set over X_E and $x_e \in X_E$. We say that $x_e \in F_E$ denotes that x_e belongs to the soft set F_E whenever $x_e(e) \in F(e)$, for all $e \in E$.*

For more details of soft set theory and its applications in a variety of mathematical structures, see [18,23–27].

3. Soft Local Functions

Definition 5 ([4])**.** *The non-null collection of soft subsets $\overline{\mathcal{I}}$ of $SS(X)_E$ is called a soft ideal on X_E if*

(a) $F_E \in \overline{\mathcal{I}}$ *and* $G_E \sqsubseteq F_E$*, then* $G_E \in \overline{\mathcal{I}}$.
(b) $F_E \in \overline{\mathcal{I}}$ *and* $G_E \in \overline{\mathcal{I}}$*, then* $F_E \sqcup G_E \in \overline{\mathcal{I}}$.

Definition 6 ([4])**.** *Let* $(X_E, \tau, \overline{\mathcal{I}})$ *be a* $\mathcal{SITS}$*. Then,* $\overline{F_E}^*(\overline{\mathcal{I}}, \tau)$ *(or* $\overline{F}_E^*$*)* $= \sqcup\{x_e \in X_E : O_{x_e} \sqcap F_E \notin \overline{\mathcal{I}}$ *for every soft open set* $O_{x_e}\}$ *is called a soft local function of* F_E *with respect to* $\overline{\mathcal{I}}$ *and soft topology* τ*, where* O_{x_e} *is a soft open set containing* x_e*.*

A soft subset A_E of a soft ideal topological space "symbolized $\mathcal{SITS}$" $(X_E, \tau, \overline{\mathcal{I}})$ is said to be soft ideal dense if every soft point of X_E is in $\overline{A}_E^*$, i.e., if $\overline{A}_E^* = X_E$.

Remark 1. *For a* $\mathcal{SITS}$ $(X_E, \tau, \overline{\mathcal{I}})$*, if* $D_E \sqsubseteq X_E$ *is soft ideal dense, then* X_E *is also soft ideal dense, i.e.,* $\overline{X_E}^* = X_E$*.*

A soft set $S_E \in SS(X)_E$ is called soft co-dense [28] if $Int(S_E) = \phi_E$.

Theorem 1. *Let* $(X_E, \tau, \overline{\mathcal{I}})$ *be a* $\mathcal{SITS}$*. Then, the next characteristics are interchangeable:*

(a) $\tau \sqcap \overline{\mathcal{I}} = \phi_E$*, where* ϕ_E *is a null soft set;*
(b) *If* $S_E \in \overline{\mathcal{I}}$*, then* $Int(S_E) = \phi_E$*;*
(c) *For any soft open* F_E*, we have* $F_E \sqsubseteq \overline{F}_E^*$*;*
(d) $X_E = \overline{X}_E^*$*.*

Proof. (a) $\to$ (b): Assume that $\tau \sqcap \overline{\mathcal{I}} = \phi_E$ and $S_E \in \overline{\mathcal{I}}$. Suppose that $x_e \in Int(S_E)$. Then, there exists a soft open set U_E such that $x_e \in U_E \sqsubseteq S_E$. Since $S_E \in \overline{\mathcal{I}}$, $U_E \in \overline{\mathcal{I}}$. This is contrary to $\tau \sqcap \overline{\mathcal{I}} = \phi_E$. Therefore, $Int(S_E) = \phi_E$.
(b) $\to$ (c): Assume that $x_e \in F_E$. Let $x_e \notin \overline{F}_E^*$; then, there exists soft open set U_{x_e} containing x_e such that $F_E \sqcap U_{x_e} \in \overline{\mathcal{I}}$. Since F_E is a soft open set, by (b) $x_e \in F_E \sqcap U_{x_e} = Int[F_E \sqcap U_{x_e}] = \phi_E$. This is incoherent, and so $x_e \in \overline{F}_E^*$ and $F_E \sqsubseteq \overline{F}_E^*$.
(c) $\to$ (d): Since X_E is a soft open set, $X_E = \overline{X}_E^*$.
(d) $\to$ (a): $X_E = \overline{X}_E^* = \{x_e \in X_E : U_E \sqcap X_E = U_E \notin \overline{\mathcal{I}}$ for all soft open sets U_E and $x_e \in U_E\}$. Then, $\tau \sqcap \overline{\mathcal{I}} = \phi_E$. □

4. Soft Hyperconnected Spaces

Definition 7. *Let* $(X_E, \tau, \overline{\mathcal{I}})$ *be a* $\mathcal{SITS}$*. We say that this space is:*

1. *Soft hyperconnected "symbolized* $\mathcal{HC}$*" [17] if every pair of non-null soft open sets of* X_E *has non-null intersection.*
2. *Soft* $\mathcal{HC}$ *modulo* $\overline{\mathcal{I}}$ *if the intersection of every two non-null soft open sets is not in* $\overline{\mathcal{I}}$*.*
3. *Soft ideal* $\mathcal{HC}$ *if every non-null soft open set is soft ideal dense in* X_E*.*

Lemma 1. *A* $\mathcal{SITS}$ $(X_E, \tau, \overline{\mathcal{I}})$ *is soft* $\mathcal{HC}$ *modulo* $\overline{\mathcal{I}}$ *iff there are no proper soft closed sets* G_E *and* H_E *such that* $X_E - (G_E \sqcup H_E) \in \overline{\mathcal{I}}$*.*

Proof. If there are proper soft closed sets G_E and H_E such that $X_E - [G_E \sqcup H_E] \in \overline{\mathcal{I}}$. If $H_E = \phi_E$, then $X_E - G_E \in \overline{\mathcal{I}}$. $X_E - G_E$ and X_E are non-null soft open sets with $X_E \sqcap (X_E - G_E) = (X_E - G_E) \in \overline{\mathcal{I}}$. This is incoherent. Hence, $G_E \neq \phi_E$ and $H_E \neq \phi_E$ are both proper soft closed sets. Then, $X_E - G_E$ and $X_E - H_E$ are non-null soft open sets. So, $(X_E - G_E) \sqcap (X_E - H_E) = X_E - (G_E \sqcup H_E) \in \overline{\mathcal{I}}$, which contradicts.

Conversely, assume that $A_E \neq \phi_E$ and $B_E \neq \phi_E$ are soft open sets in X_E. So, $X_E - A_E$ and $X_E - B_E$ are proper soft closed sets in X_E and $X_E - [(X_E - A_E) \sqcup (X_E - B_E)] \notin \overline{\mathcal{I}}$. This implies that $X_E - [X_E - (A_E \sqcap B_E)] \notin \overline{\mathcal{I}}$. Thus, $(A_E \sqcap B_E) \notin \overline{\mathcal{I}}$. Hence, $(X_E, \tau, \overline{\mathcal{I}})$ is soft $\mathcal{HC}$ modulo $\overline{\mathcal{I}}$. □

Theorem 2. *Let* $(X_E, \tau, \overline{\mathcal{I}})$ *be a* $\mathcal{SITS}$ *and* $\tau \sqcap \overline{\mathcal{I}} = \phi_E$*. Then,* $(X_E, \tau, \overline{\mathcal{I}})$ *is soft* $\mathcal{HC}$ *modulo* $\overline{\mathcal{I}}$ *if and only if* (X_E, τ) *is soft* $\mathcal{HC}$*.*

Proof. Assume that $(X_E, \tau, \overline{\mathcal{I}})$ is a soft $\mathcal{HC}$ modulo $\overline{\mathcal{I}}$. So, since $\phi_E \in \overline{\mathcal{I}}$, (X_E, τ) is soft $\mathcal{HC}$.

Conversely, let (X_E, τ) be a soft $\mathcal{HC}$ and A_E, B_E be non-null soft open sets. Then, $A_E \sqcap B_E$ is a non-null soft open set in (X_E, τ). Since $\tau \sqcap \overline{\mathcal{I}} = \phi_E$, $A_E \sqcap B_E \notin \overline{\mathcal{I}}$. Thus, $(X_E, \tau, \overline{\mathcal{I}})$ is soft $\mathcal{HC}$ modulo $\overline{\mathcal{I}}$. □

The following example show that, if $\tau \sqcap \overline{\mathcal{I}} \neq \phi_E$, Theorem 2 is not true.

Example 1. *Let $(X_E, \tau, \overline{\mathcal{I}})$ be a $\mathcal{SITS}$, where $X = \{h_1, h_2\}$, $E = \{e_1, e_2\}$ $\tau = \{X_E, \phi_E, \{(e_1, \{h_1\}), (e_2, \{h_2\})\}, \{(e_1, X_E), (e_2, \{h_2\})\}, \{(e_1, \{h_1\}), (e_2, \{X_E\})\}\}$, and $\overline{\mathcal{I}} = \{\phi_E, \{(e_1, \{h_1\})\}, \{(e_2, \{h_2\})\}, \{(e_1, \{h_1\}), (e_2, \{h_2\})\}\}$. Then, $\tau \sqcap \overline{\mathcal{I}} \neq \phi_E$.*

Since every pair of non-null soft open sets of X_E has non-null soft intersection, $(X_E, \tau, \overline{\mathcal{I}})$ is soft $\mathcal{HC}$. But it is clear that it is not soft $\mathcal{HC}$ modulo $\overline{\mathcal{I}}$.

Theorem 3. *A soft topological space (X_E, τ) is soft $\mathcal{HC}$ iff the union of two not soft dense sets is a not soft dense set.*

Proof. Assume that (X_E, τ) is soft $\mathcal{HC}$ and G_E, F_E are two not soft dense sets in (X_E, τ). Then there exist two non-null soft open sets U_E and V_E such that $U_E \sqcap G_E = \phi_E$ and $V_E \sqcap F_E = \phi_E$. Since (X_E, τ) is soft $\mathcal{HC}$, $U_E \sqcap V_E \neq \phi_E$. But $(U_E \sqcap V_E) \sqcap (G_E \sqcup F_E) = \phi_E$ and, hence, $G_E \sqcup F_E$ is not soft dense in (X_E, τ).

Conversely, if the condition is true in (X_E, τ) but (X_E, τ) is not soft $\mathcal{HC}$, then there exist two non-null soft open sets U_E and V_E such that $U_E \sqcap V_E = \phi_E$. Hence, $U_E \sqsubseteq X_E - V_E$ and $V_E \sqsubseteq X_E - U_E$. Then, $X_E - U_E$ and $X_E - V_E$ are not soft dense in (X_E, τ). But $(X_E - U_E) \sqcup (X_E - V_E) = X_E$. This contradicts the assertion that a union of two non-soft dense sets is also not a soft dense set. The theorem is therefore now proven. □

Lemma 2. *Let $(X_E, \tau, \overline{\mathcal{I}})$ be a $\mathcal{SITS}$. Then, $(X_E, \tau, \overline{\mathcal{I}})$ is soft ideal $\mathcal{HC}$ if and only if (X_E, τ) is soft $\mathcal{HC}$ and $\tau \sqcap \overline{\mathcal{I}} = \phi_E$.*

Proof. Clearly, every soft ideal $\mathcal{HC}$ space is soft $\mathcal{HC}$. Let U_E be a non-null soft open set in the soft ideal. Then, $\overline{U}_E^* = X_E$. Conversely, yet, since $U_E \in \overline{\mathcal{I}}$, $\overline{U}_E^* = \phi_E$. Hence, $X_E = \phi_E$. There is inconsistency here. Consequently, $\tau \sqcap \overline{\mathcal{I}} = \phi_E$.

Conversely, let U_E be a non-null soft open set. Let $x_e \in X_E$. Due to the soft $\mathcal{HC}$ property of (X_E, τ), every soft open set V_E containing x_e meets U_E. Moreover, $U_E \sqcap V_E$ is a soft open set and $U_E \sqcap V_E \notin \overline{\mathcal{I}}$ because $\tau \sqcap \overline{\mathcal{I}} = \phi_E$. Thus, $x_e \in \overline{U}_E^*$. This shows that U_E is soft ideal dense. □

Theorem 4. *Let $(X_E, \tau, \overline{\mathcal{I}})$ be a $\mathcal{SITS}$, where $\tau \sqcap \overline{\mathcal{I}} = \phi_E$. Then, a set D_E is soft ideal dense if and only if $(U_E - A_E) \sqcap D_E \neq \phi_E$ whenever U_E is non-null soft open and $A_E \in \overline{\mathcal{I}}$.*

Proof. Let D_E be soft ideal dense. So, $U_E \sqcap D_E \notin \overline{\mathcal{I}}$ for all non-null soft open sets U_E. Hence, for all $A_E \in \overline{\mathcal{I}}$, $(U_E - A_E) \sqcap D_E \neq \phi_E$, for, otherwise, $(U_E - A_E) \sqcap D_E = \phi_E$ and, hence, $\phi_E = U_E \sqcap (X_E - A_E) \sqcap D_E = (U_E \sqcap D_E) \sqcap (X_E - A_E)$. Therefore, $U_E \sqcap D_E \sqsubseteq A_E$. Since $A_E \in \overline{\mathcal{I}}$, $U_E \sqcap D_E \in \overline{\mathcal{I}}$, which is contrary to $U_E \sqcap D_E \notin \overline{\mathcal{I}}$. Therefore, $(U_E - A_E) \sqcap D_E \neq \phi_E$.

Conversely, let $(U_E - A_E) \sqcap D_E \neq \phi_E$ whenever U_E is a non-null soft open set and $A_E \in \overline{\mathcal{I}}$. Next, we assert that D_E is soft ideal dense. Let D_E be not soft ideal dense. Then, there exists some non-null soft open set U_E such that $U_E \sqcap D_E \in \overline{\mathcal{I}}$. Let $U_E \sqcap D_E = A_E$. So, since $\tau \sqcap \overline{\mathcal{I}} = \phi_E$, $U_E - A_E$ is non-null but $(U_E - A_E) \sqcap D_E = \phi_E$. This defies everything we had assumed. □

Theorem 5. *Let $(X_E, \tau, \overline{\mathcal{I}})$ be a $\mathcal{SITS}$, where $\tau \sqcap \overline{\mathcal{I}} = \phi_E$. Then, $(X_E, \tau, \overline{\mathcal{I}})$ is soft $\mathcal{HC}$ modulo $\overline{\mathcal{I}}$ if and only if $(U_E - A_E) \sqcap D_E \neq \phi_E$ whenever U_E and D_E are non-null soft open sets and $A_E \in \overline{\mathcal{I}}$.*

Proof. From Lemma 2 and Theorem 4, the proof follows. □

5. Soft Ideal Resolvable Spaces

A soft space (X_E, τ) is soft resolvable [14], symbolized ($\mathcal{RS}$), if X_E is the union of two soft dense subsets which are disjoint.

A $\mathcal{SITS}$ $(X_E, \tau, \overline{\mathcal{I}})$ is soft ideal $\mathcal{RS}$ if it has two disjoint soft ideal dense sets; alternatively, it is claimed to be soft ideal irresolvable, symbolized ($\mathcal{IRS}$).

Lemma 3. *Let $(X_E, \tau, \overline{\mathcal{I}})$ be a $\mathcal{SITS}$.*

(1) *$(X_E, \tau, \overline{\mathcal{I}})$ is soft ideal $\mathcal{RS}$ iff X_E is the union of two disjoint soft ideal dense sets.*
(2) *If $(X_E, \tau, \overline{\mathcal{I}})$ is soft ideal $\mathcal{RS}$, then $\tau \sqcap \overline{\mathcal{I}} = \phi_E$.*

Proof. (1) Let A_E and B_E be disjoint soft ideal dense sets. Then, $\overline{A}^*_E = X_E$ and $X_E = \overline{B}^*_E \sqsubseteq \overline{(X_E - A_E)}^*_E$, and, hence, $X_E = \overline{(X_E - A_E)}^*_E$. Therefore, X_E is the union of soft ideal dense sets A_E and $X_E - A_E$. The opposite is evident.

(2) Let A_E and B_E be disjoint soft ideal dense sets. So, by Theorem 3.2 of [4], we have $X_E = \overline{A}^*_E \sqsubseteq \overline{X}^*_E$. Therefore, X_E is soft ideal dense. Thus, by Theorem 1, $\tau \sqcap \overline{\mathcal{I}} = \phi_E$. □

Remark 2. *In citekandil it was obtained that $\overline{Cl}^*(A_E) = A_E \sqcup \overline{A}^*_E$ is a soft Kuratowski closure operator. We will denote by $(X_E, \tau^*, \overline{\mathcal{I}})$ the soft topology generated by $\overline{Cl}^*$, that is, $\tau^* = \{U_E \sqsubseteq X_E : \overline{Cl}^*(X_E - U_E) = X_E - U_E\}$.*

Theorem 6 ([29])**.** *Let $(X_E, \tau, \overline{\mathcal{I}})$ be a $\mathcal{SITS}$. Then $\beta(\tau^*, \overline{\mathcal{I}}) = \{V_E - I : V_E$ is soft open set of $(X_E, \tau), I \in \overline{\mathcal{I}}\}$ is a basis for (X_E, τ^*).*

Theorem 7. *A $\mathcal{SITS}$ $(X_E, \tau, \overline{\mathcal{I}})$ is soft ideal $\mathcal{RS}$ if and only if (X_E, τ^*) is soft $\mathcal{RS}$ and $\tau \sqcap \overline{\mathcal{I}} = \phi_E$.*

Proof. Let $(X_E, \tau, \overline{\mathcal{I}})$ be soft ideal $\mathcal{RS}$. So, by Lemma 3 (1), $X_E = A_E \sqcup B_E$, where A_E and B_E are disjoint soft ideal dense sets of X_E. Note that $\overline{Cl}^*(A_E) = A_E \sqcup \overline{A}^*_E = A_E \sqcup X_E = X_E$. Hence, A_E and B_E are soft dense in (X_E, τ^*). Thus, (X_E, τ^*) is soft $\mathcal{RS}$. By Lemma 3 (2), $\tau \sqcap \overline{\mathcal{I}} = \phi_E$.

Conversely, let (X_E, τ^*) be soft $\mathcal{RS}$ and $\tau \sqcap \overline{\mathcal{I}} = \phi_E$. Suppose that $X_E = A_E \sqcup B_E$, $A_E \sqcap B_E = \phi_E$, and both A_E and B_E are soft dense in (X_E, τ^*). Let $x_e \in X_E$ and $x_e \notin \overline{A}^*_E$; then, there exists a soft open set U_E containing x_e such that $V_E = U_E \sqcap A_E \in \overline{\mathcal{I}}$. Since B_E is soft dense in (X_E, τ^*) and $\tau \sqcap \overline{\mathcal{I}} = \phi_E$, V_E is non-null and also $U_E \not\sqsubseteq A_E$. Hence, by Theorem 6, $W_E = U_E - V_E \in (X_E, \tau^*)$ is a non-null set and $W_E \sqcap A_E = \phi_E$. This contradicts the fact that A_E is soft dense in (X_E, τ^*). Thus, $x_e \in \overline{A}^*_E$ and, hence, A_E is soft ideal dense. A related argument demonstrates that B_E is soft ideal dense. Thus, $(X_E, \tau, \overline{\mathcal{I}})$ is soft ideal $\mathcal{RS}$. □

Definition 8 ([3])**.** *Let $Y_E \neq \phi_E$ be a soft subset of (X_E, τ, E); then, $\tau_{Y_E} = \{G_E \sqcap Y_E : G_E \in \tau\}$ is called a relative soft topology over Y and (Y_E, τ_{Y_E}, E) is a soft subspace of (X_E, τ, E).*

Lemma 4. *Let $Y_E \sqsubseteq X_E$ and $\overline{\mathcal{I}}$ be soft ideal in X_E. Then, $\overline{\mathcal{I}}_{Y_E} = \{I \in \overline{\mathcal{I}} : I \subseteq Y_E\} = \{I \sqcap Y_E : I \in \overline{\mathcal{I}}\}$ is soft ideal in Y_E.*

Lemma 5. *Let $(X_E, \tau, \overline{\mathcal{I}})$ be a $\mathcal{SITS}$. The non-null soft τ^*-open subspace of a soft ideal $\mathcal{RS}$ space is a soft ideal $\mathcal{RS}$ space.*

Proof. First, we know that the intersection of a soft dense and a soft open set is soft dense, so the soft resolvability is a soft open hereditary. Also, for all $A_E \in \tau^*$ we have $\tau^*_{|A} = (\tau_{|A})^*$. Thus, by Theorem 7, if $(X_E, \tau, \overline{\mathcal{I}})$ is soft ideal $\mathcal{RS}$ and A is τ^*-open, then (X_E, τ^*) is soft $\mathcal{RS}$; hence, $(A_E, \tau^*_{|A}) = (A_E, (\tau_{|A})^*)$ is soft $\mathcal{RS}$ and, thus, $(A_E, \tau_{|A}, \overline{\mathcal{I}}_{A_E})$ is soft ideal $\mathcal{RS}$. □

Theorem 8. *Let $(X_E, \tau, \overline{\mathcal{I}})$ be a $\mathcal{SITS}$. Simple expansion of soft ideal $\mathcal{RS}$ topologies over soft ideal $\mathcal{RS}$ subspace are soft ideal $\mathcal{RS}$.*

Proof. Let $(X_E, \tau, \overline{\mathcal{I}})$ be soft ideal $\mathcal{RS}$ and $S_E \sqsubseteq X_E$ be a soft ideal $\mathcal{RS}$ subspace. Let (D_E, D'_E) be the soft ideal resolution of $(S_E, \tau_{|S}, \overline{\mathcal{I}}_{S_E})$. We examine the next two instances:

Case (1): S_E is soft τ^*-dense in $(X_E, \tau, \overline{\mathcal{I}})$; that is, $X_E = S_E \sqcup S_E^*$. We first establish that D_E is soft ideal dense in $(X_E, \tau, \overline{\mathcal{I}})$. Let $x_e \in X_E$. Suppose that for some soft open set U_E with $x_e \in U_E$ we have $U_E \sqcap D_E \in \overline{\mathcal{I}}$. The two subcases that follow are ours.

Subcase (a): $x_e \in S_E$. Then, $V_E = U_E \sqcap S_E \in \tau_{|S}$ is a soft open set of x_e in $(S_E, \tau_{|S}, \overline{\mathcal{I}}_{S_E})$ such that $V_E \sqcap D_E = U_E \sqcap S_E \sqcap D_E \in \overline{\mathcal{I}}$ due to the heredity of $\overline{\mathcal{I}}$. This defies the assertion that D_E is soft ideal dense in $(S_E, \tau_{|S}, \overline{\mathcal{I}}_{S_E})$. So, D_E is soft ideal dense in $(X_E, \tau, \overline{\mathcal{I}})$.

Subcase (b): $x_e \notin S_E$. Since $X_E = S_E \sqcup S_E^*$, $x_e \in S_E$. To demonstrate that $x_e \in D_E^*$, we believe the opposite, i.e., there exists a soft open set U_E with $x_e \in U_E$ such that $U_E \sqcap D_R \in \overline{\mathcal{I}}$. Note that $U_E \sqcap S_E \neq \phi_E$; otherwise, $x_e \notin S_E^*$. Pick $y_e \in U_E \sqcap S_E \in \tau_{|S}$. Since $U_E \sqcap D_E \in \overline{\mathcal{I}}$, then, by heredity of $\overline{\mathcal{I}}$, $U_E \sqcap S_E \sqcap D_E \in \overline{\mathcal{I}}$. So, D_E is not soft ideal dense in $(S_E, \tau_{|S}, \overline{\mathcal{I}}_{S_E})$. By contradiction $x_e \in D_E^*$, i.e., D_E is soft ideal dense in $(X_E, \tau, \overline{\mathcal{I}})$. So, we have demonstrated that $D_E^* = X_E$. Using a comparable defense, $D_E^{'*} = X_E$. Let $x_e \in X_E$ and let $U_E \sqcup (V_E \sqcap S_E)$ be a soft open set of x_e in $(X_E, \tau(S_E), \overline{\mathcal{I}})$, where $\tau(S_E)$ is the simple expansion of τ over S_E. If $(U_E \sqcup (V_E \sqcap S_E)) \sqcap D_E \in \overline{\mathcal{I}}$, then, by heredity of $\overline{\mathcal{I}}$, $(V_E \sqcap S_E) \sqcap D_E$ is a member of $\overline{\mathcal{I}}$ so that V_E is a null set. Of course, $(V_E \sqcap S_E) \sqcap D_E$ cannot be a member of $\overline{\mathcal{I}}$ if V_E is non-null since then V_E must contain an element of S_E. So, x_e belongs to $U_E \sqcap D_E$, which is also not eligible to join with $\overline{\mathcal{I}}$ since $D_E^* = X_E$. This contradiction shows that D_E is soft $\tau(S_E)$-dense. Using a comparable defense of $D_E^{'}$, we determine that $(X_E, \tau(S_E), \overline{\mathcal{I}})$ is soft ideal $\mathcal{RS}$.

Case (2): S_E is not soft τ^*-dense in $(X_E, \tau, \overline{\mathcal{I}})$. Then, $S_E^{'} = X_E \setminus Cl^*(S_E)$, so it is τ^*-open and non-null. By Lemma 5, $S^{'}$ is soft ideal $\mathcal{RS}$ (more precisely said soft ideal $\mathcal{RS}$ with respect to S_E). Let (A_E, B_E) be the soft ideal resolution of $S^{'}$. By using reasoning akin to that of *Case (1)*, we can prove that $(D_E \sqcup A_E, D_E \sqcup B_E)$ is a soft ideal resolution of $(X_E, \tau, \overline{\mathcal{I}})$. Additionally, employing the same method as at the conclusion of *Case (1)*, we find that $(X_E \tau(S_E); \overline{\mathcal{I}})$ is soft ideal $\mathcal{RS}$.

□

Theorem 9. *A $\mathcal{SITS}$ $(X_E, \tau, \overline{\mathcal{I}})$ is soft ideal $\mathcal{RS}$ iff there exists a soft ideal dense set D_E such that, for all non-null soft open sets U_E and all $A_E \in \overline{\mathcal{I}}$, $U_E - A_E \neq \phi_E$ implies $(U_E - A_E) \not\sqsubseteq D_E$.*

Proof. Let $(X, \tau, \mathcal{I})$ be soft ideal $\mathcal{RS}$. So, by Remark 1 and Theorem 1, $\tau \sqcap \overline{\mathcal{I}} = \phi_E$. Now, there exist two disjoint soft ideal dense sets, say $D_E^{'}$ and $D_E^{''}$. We demonstrate that $(U_E - A_E) \not\sqsubseteq D_E^{'}$ whenever $U_E - A_E \neq \phi_E$ for all non-null soft open sets U_E and $A_E \in \overline{\mathcal{I}}$. If possible, let $(U_E - A_E) \sqsubseteq D_E^{'}$ for some non-null soft open set U_E and $A_E \in \overline{\mathcal{I}}$. Then, $(U_E - A_E) \sqcap D_E^{''} = \phi_E$. Now, since $\tau \sqcap \overline{\mathcal{I}} = \phi_E$, by Theorem 4 $D_E^{''}$ is not soft ideal dense. This is contrary to $D_E^{''}$ being soft ideal dense. Hence, $(U_E - A_E) \not\sqsubseteq D_E^{'}$ whenever $U_E - A_E \neq \phi_E$ for all non-null soft open sets U_E and $A_E \in \overline{\mathcal{I}}$.

However, allow the condition to persist in $(X_E, \tau, \overline{\mathcal{I}})$. Then, there exists a soft ideal dense set D_E such that $(U_E - A_E) \not\subseteq D_E$ if $U_E - A_E \neq \phi_E$ for all non-null soft open sets U_E and $A_E \in \overline{\mathcal{I}}$. We show that $X_E - D_E$ is soft ideal dense. Let $X_E - D_E$ be not soft ideal dense. Then there exists a non-null soft open set V_E such that $V_E \sqcap (X_E - D_E) \in \overline{\mathcal{I}}$. Clearly, $V_E \sqcap (X_E - D_E) \neq \phi_E$, for otherwise $V_E \sqsubseteq D_E$, which is contrary to our assumption. Let $A_E = V_E \sqcap (X_E - D_E)$. Then, $V_E - A_E \neq \phi_E$. For if $V_E - A_E = \phi_E$ then $V_E \sqsubseteq A_E$ and, hence, $V_E \in \overline{\mathcal{I}}$, which suggests $V_E \sqcap D_E \in \overline{\mathcal{I}}$. Contrary to that, this D_E is soft ideal dense. Therefore, $V_E - A_E \sqsubseteq D_E$. It goes against our presumption once more. Thus, $X_E - D_E$ is soft ideal dense and so $(X_E, \tau, \overline{\mathcal{I}})$ is soft ideal $\mathcal{RS}$. □

Corollary 1. *A $\mathcal{SITS}$ $(X_E, \tau, \overline{\mathcal{I}})$ is soft ideal $\mathcal{IRS}$ iff, for each soft ideal dense set D_E, there exist a soft open set U_E and $A \in \overline{\mathcal{I}}$ such that $\phi_E \neq (U_E - A_E) \sqsubseteq D_E$.*

Theorem 10. *If $(X_E, \tau, \overline{\mathcal{I}})$ is a $\mathcal{SITS}$ such that $\tau \sqcap \overline{\mathcal{I}} = \phi_E$ and if D_E is soft ideal dense in $(X_E, \tau, \overline{\mathcal{I}})$, then, for all $Y_E = U_E - A_E$, where U_E is non-null soft open and $A_E \in \overline{\mathcal{I}}$, $Y_E \sqcap D_E$ is soft ideal dense in $(Y_E, \tau_{Y_E}, \overline{\mathcal{I}}_{Y_E})$.*

Proof. Clearly, we suppose that $\tau \sqcap \overline{\mathcal{I}} = \phi_E$. Then, by Proposition 11 of [3], a soft open set in Y_E is of the form $Y_E \sqcap O_E = (U_E - A_E) \sqcap O_E = (U_E \sqcap O_E) - A_E$, where O_E is a soft open set in (X_E, τ). Let $\phi_E \neq U_E \sqcap O_E - A_E$. Consider $\phi_E \neq ((U_E \sqcap O_E) - A_E) - B_E$, $B_E \in \overline{\mathcal{I}}_{Y_E}$. Then, since D_E is soft ideal dense and $U_E \sqcap O_E$ is a soft open set in (X_E, τ), by Theorem 4, $(U_E \sqcap O_E - (A_E \sqcup B_E)) \sqcap D_E \neq \phi_E$. Hence, $(((U_E \sqcap O_E) - A_E) - B_E) \sqcap D_E \neq \phi_E$. Therefore, again by Theorem 4, $Y_E \sqcap D_E$ is soft ideal dense in $(Y_E, \tau_{Y_E}, \overline{\mathcal{I}}_{Y_E})$. □

Theorem 11. *Let $(X_E, \tau, \overline{\mathcal{I}})$ be a $\mathcal{SITS}$ such that $\tau \sqcap \overline{\mathcal{I}} = \phi_E$ and $P_E \sqsubseteq Y_E = U_E - A_E$, where U_E is a non-null soft open set, $A_E \in \overline{\mathcal{I}}$. Then, P_E is soft ideal dense in $(Y_E, \tau_{Y_E}, \overline{\mathcal{I}}_{Y_E})$ if and only if $P_E = Y_E \sqcap D_E$, where D_E is soft ideal dense in $(X_E, \tau, \overline{\mathcal{I}})$.*

Proof. Assume that P_E is soft ideal dense in $(Y_E, \tau_{Y_E}, \overline{\mathcal{I}}_{Y_E})$. Consider the set $P_E \sqcup (X_E - Y_E)$. Then, $(P_E \sqcup (X_E - Y_E)) \sqcap O_E = (P_E \sqcap O_E) \sqcup ((X_E - Y_E) \sqcap O_E)$, where O_E is a non-null soft open set. Now, if $O_E \sqsubseteq X_E - Y_E$, then $P_E \sqsubseteq Y_E$ and $P_E \sqcap O_E = \phi_E$, and we have $(P_E \sqcup (X_E - Y_E)) \sqcap O_E = O_E$ which is not in $\overline{\mathcal{I}}$ because $\tau \sqcap \overline{\mathcal{I}} = \phi_E$. Moreover, if $O_E \sqcap Y_E \neq \phi_E$, then, since P_E is soft ideal dense in $(Y_E, \tau_{Y_E}, \overline{\mathcal{I}}_{Y_E})$, $P_E \sqcap (O_E \sqcap Y_E) \notin \overline{\mathcal{I}}_{Y_E}$ and so $P_E \sqcap O_E \notin \overline{\mathcal{I}}$. Therefore, $(P_E \sqcup (X_E - Y_E)) \sqcap O_E \notin \overline{\mathcal{I}}$. Thus, $(P_E \sqcup (X_E - Y_E)) = D_E$, say, is soft ideal dense in $(X_E, \tau, \overline{\mathcal{I}})$ and, hence, $P_E = Y_E \sqcap D_E$. Next, let $P_E = Y_E \sqcap D_E$, where D_E is soft ideal dense in $(X_E, \tau, \overline{\mathcal{I}})$. Hence, by Theorem 10, P_E is soft ideal dense in $(Y_E, \tau_{Y_E}, \overline{\mathcal{I}}_{Y_E})$. This completes the proof of the theorem. □

Note that, as per the condition in Theorem 11, for D_E soft ideal dense is necessary because if D_E is not soft ideal dense then $P_E = \phi_E$ for some non-null soft open set U_E, $A_E \in \overline{\mathcal{I}}$ and, hence, P_E is not soft ideal dense in $(Y_E, \tau_{Y_E}, \overline{\mathcal{I}}_{Y_E})$.

6. Soft Ideal Semi-Irresolvable Spaces

Next, we will define and go over the characteristics of a soft ideal semi-$\mathcal{IRS}$ space.

Definition 9. *A $\mathcal{SITS}$ $(X_E, \tau, \overline{\mathcal{I}})$ is a said to be soft ideal semi- $\mathcal{IRS}$ if for each soft ideal dense set D_E and each non-null soft open set U_E and $A_E \in \overline{\mathcal{I}}$ such that $U_E - A_E$ is non-null set, there exists a non-null soft open set V_E and $B_E \in \overline{\mathcal{I}}$ such that $\phi_E \neq (V_E - B_E) \sqsubseteq (U_E - A_E) \sqcap D_E$.*

Theorem 12. *A $\mathcal{SITS}$ $(X_E, \tau, \overline{\mathcal{I}})$ is a soft ideal semi- $\mathcal{IRS}$, iff the intersection of soft ideal dense sets is a soft ideal dense set, where $\tau \sqcap \overline{\mathcal{I}} = \phi_E$.*

Proof. Assume that $(X_E, \tau, \overline{\mathcal{I}})$ is a soft ideal semi- $\mathcal{IRS}$ and $\tau \sqcap \overline{\mathcal{I}} = \phi_E$. Let $D_E^{'}$ and $D_E^{''}$ be two soft ideal dense sets in $(X_E, \tau, \overline{\mathcal{I}})$. We demonstrate that $D_E^{'} \sqcap D_E^{''}$ is soft ideal dense. Consider $U_E - A_E$, where U is a non-null soft open set and $A_E \in \overline{\mathcal{I}}$. As we demonstrate, $(U_E - A_E) \sqcap D_E^{'} \sqcap D_E^{''} \neq \phi_E$. Since $D_E^{'}$ is soft ideal dense, by Theorem 4, $(U_E - A_E) \sqcap D_E^{'} \neq \phi_E$. Since $(X_E, \tau, \overline{\mathcal{I}})$ is soft ideal semi- $\mathcal{IRS}$, there exists a non-null soft open set $V_E^{'}$ and $B_E^{'} \in \overline{\mathcal{I}}$ such that $\phi_E \neq (V_E^{'} - B_E^{'}) \sqsubseteq (U_E - A_E) \sqcap D_E^{'}$. Again, since $D_E^{''}$ is soft ideal dense, there exists a non-null soft open set $V_E^{''}$ and $B_E^{''} \in \overline{\mathcal{I}}$ such that $\phi_E \neq (V''_E - B''_E) \sqsubseteq (V'_E - B'_E) \sqcap D''_E$. Hence, $\phi_E \neq V''_E - B''_E \sqsubseteq (U_E - A_E) \sqcap D'_E \sqcap D''_E$. Therefore, $(U_E - A_E) \sqcap (D'_E \sqcap D''_E) \neq \phi_E$ and, by Theorem 4, $D'_E \sqcap D''_E$ is soft ideal dense.

Conversely, assume that the intersection of soft ideal dense sets is soft ideal dense. Assume that $(X_E, \tau, \overline{\mathcal{I}})$ is not soft ideal semi- $\mathcal{IRS}$. Then, there exists a soft ideal dense set D'_E, and a non-null soft open set U_E and $A_E \in \overline{\mathcal{I}}$, where $\phi_E \neq U_E - A_E$, such that $(U_E - A_E) \sqcap D'_E$ does not contain $V_E - B_E$, for any non-null soft open set V_E and $B_E \in \overline{\mathcal{I}}$. Consider the set $D''_E = (X_E - (U_E - A_E)) \sqcup ((U_E - A_E) - (U_E - A_E) \sqcap D'_E)$. By Theorem 4, D''_E is soft ideal dense since $(V_E - B_E) \sqcap D''_E \neq \phi_E$. But $(U_E - A_E) \sqcap D'_E \sqcap D''_E = \phi_E$. This contradicts the reality that the intersection of two soft ideal dense sets is a soft ideal dense set. Hence, $(X_E, \tau, \overline{\mathcal{I}})$ must be soft ideal semi- $\mathcal{IRS}$. This concludes the theorem's proof. □

Example 2. *Let $(X_E, \tau, \overline{\mathcal{I}})$ be a $\mathcal{SITS}$, where $X = \{h_1, h_2, h_3\}$, $E = \{e\}$. Consider $\tau = \{X_E, \phi_E, \{(e, \{h_1, h_2\})\}$ and $\overline{\mathcal{I}} = \{\phi_E, \{(e, \{h_2\})\}, \{(e, \{h_3\})\}, \{(e, \{h_2, h_3\})\}\}$. Then, we have the following.*

1. $\tau \sqcap \overline{\mathcal{I}} = \phi_E$.
2. *The collection of all soft ideal dense sets are X_E, $\{(e, \{h_1\})\}$, $\{(e, \{h_1, h_2\})\}$ and $\{(e, \{h_1, h_3\})\}$.*
3. *The soft intersection of any soft ideal dense sets is soft ideal dense.*

Hence, by Theorem 12, $(X_E, \tau, \overline{\mathcal{I}})$ is soft ideal semi- $\mathcal{IRS}$.

Theorem 13. *Let $(X_E, \tau, \overline{\mathcal{I}})$ be a $\mathcal{SITS}$ and $\tau \sqcap \overline{\mathcal{I}} = \phi_E$. If $(X_E, \tau, \overline{\mathcal{I}})$ is soft ideal semi- $\mathcal{IRS}$, then $(Y_E, \tau_{Y_E}, \overline{\mathcal{I}}_{Y_E})$ is soft ideal semi- $\mathcal{IRS}$ whenever $Y_E = U_E - A_E$, for every non-null soft open set U_E and $A_E \in \overline{\mathcal{I}}$.*

Proof. Assume that D_E and G_E are soft ideal dense sets in $(Y_E, \tau_{Y_E}, \overline{\mathcal{I}}_{Y_E})$. Then, by Theorem 11, $D_E = (U_E - A_E) \sqcap D'_E$ and $G_E = (U_E - A_E) \sqcap D''_E$, where D'_E and D''_E are soft ideal dense sets in $(X_E, \tau, \overline{\mathcal{I}})$. Hence, $D_E \sqcap G_E = (U_E - A_E) \sqcap D'_E \sqcap D''_E$ and, since $D'_E \sqcap D''_E$ is a soft ideal dense set in $(X_E, \tau, \overline{\mathcal{I}})$, once more by Theorem 11, $D_E \sqcap G_E$ is soft ideal dense in $(Y_E, \tau_{Y_E}, \overline{\mathcal{I}}_{Y_E})$. So, by Theorem 12, $(Y_E, \tau_{Y_E}, \overline{\mathcal{I}}_{Y_E})$ is soft ideal semi-$\mathcal{IRS}$. □

Definition 10. *A $\mathcal{SITS}$ $(X_E, \tau, \overline{\mathcal{I}})$ is said to be soft ideal semi-$\mathcal{HC}$ if each $U_E - A_E \neq \phi_E$, where U_E is a soft open set and $A_E \in \overline{\mathcal{I}}$ is a soft ideal dense set.*

Theorem 14. *A $\mathcal{SITS}$ $(X_E, \tau, \overline{\mathcal{I}})$ is soft ideal semi-$\mathcal{HC}$ iff it is soft ideal $\mathcal{HC}$ and $\tau \sqcap \overline{\mathcal{I}} = \phi_E$.*

Proof. Let $(X_E, \tau, \overline{\mathcal{I}})$ be soft ideal semi-$\mathcal{HC}$. Clearly, $(X_E, \tau, \overline{\mathcal{I}})$ is soft ideal $\mathcal{HC}$. Let $U_E \neq \phi_E$ be a non-null soft open set and a member of the soft ideal $\overline{\mathcal{I}}$. Then, $\overline{U_E}^* = X_E$ since $(X_E, \tau, \overline{\mathcal{I}})$ is soft ideal $\mathcal{HC}$. Conversely, yet, since $U_E \in \overline{\mathcal{I}}$, $\overline{U_E}^* = \phi_E$, it is paradoxical. So $\tau \sqcap \overline{\mathcal{I}} = \phi_E$.

Conversely let $(X_E, \tau, \overline{\mathcal{I}})$ be a soft ideal $\mathcal{HC}$ and $\tau \sqcap \overline{\mathcal{I}} = \phi_E$. Let $U_E - A_E$, where U_E is a non-null soft open set and $A_E \in \overline{\mathcal{I}}$. Then $U_E - A_E \neq \phi_E$ because $\tau \sqcap \overline{\mathcal{I}} = \phi_E$. We show that $U_E - A_E$ is soft ideal dense. Let $x_e \in X_E$ and V_E be a soft open set containing x_e. By Lemma 2, (X_E, τ) is soft $\mathcal{HC}$ and $V_E \sqcap (U_E - A_E) \neq \phi_E$ because $V_E \sqcap (U_E - A_E) = V_E \sqcap U_E - A_E \neq \phi_E$ and $\tau \sqcap \overline{\mathcal{I}} = \phi_E$. Thus, $(X_E, \tau, \overline{\mathcal{I}})$ is soft ideal semi-$\mathcal{HC}$. □

Example 3. *Let $(X_E, \tau, \overline{\mathcal{I}})$ be a $\mathcal{SITS}$, where $X = \{h_1, h_2, h_3\}$, $E = \{e\}$. Consider $\tau = \{X_E, \phi_E, \{(e, \{h_1, h_2\})\}$ and $\overline{\mathcal{I}} = \{\phi_E, \{(e, \{h_2\})\}, \{(e, \{h_3\})\}, \{(e, \{h_2, h_3\})\}\}$. Then*

1. $\tau \sqcap \overline{\mathcal{I}} = \phi_E$.
2. *Every non-null soft open set is soft ideal dense. So, $(X_E, \tau, \overline{\mathcal{I}})$ is soft ideal $\mathcal{HC}$.*

Hence, by Theorem 14, $(X_E, \tau, \overline{\mathcal{I}})$ is soft ideal semi- $\mathcal{IRS}$.

Theorem 15. *If a $\mathcal{SITS}$ $(X_E, \tau, \overline{\mathcal{I}})$ is soft ideal semi-$\mathcal{HC}$ and soft ideal $\mathcal{IRS}$, then it is soft ideal semi-$\mathcal{IRS}$.*

Proof. By Theorem 14, $\tau \sqcap \overline{\mathcal{I}} = \phi_E$. Let D'_E and D''_E be two soft ideal dense sets in $(X_E, \tau, \overline{\mathcal{I}})$. We demonstrate that $D'_E \cap D''_E$ is soft ideal dense. By Theorem 4, it suffices to demonstrate that $(D'_E \cap D''_E) \sqcap (U_E - A_E) \neq \phi_E$ for all non-null soft open sets U_E and $A_E \in \overline{\mathcal{I}}$. So, since $(X_E, \tau, \overline{\mathcal{I}})$ is soft ideal $\mathcal{IRS}$, by Corollary 1, there exists a non-null soft open set V_E and $B_E \in \overline{\mathcal{I}}$ such that $\phi_E \neq V_E - B_E \sqsubseteq D'_E$. Similarly, there exists a non-null soft open set W_E and $C_E \in \overline{\mathcal{I}}$ such that $\phi_E \neq W_E - C_E \sqsubseteq D''_E$. Now, (X_E, τ) is soft $\mathcal{HC}$ by Lemma 2 and Theorem 14; we have $V_E \sqcap W_E \neq \phi_E$. Since $\tau \sqcap \overline{\mathcal{I}} = \phi_E$, $(V_E - B_E) \sqcap (W_E - C_E) = (V_E \sqcap W_E) - (B_E \sqcup C_E) \neq \phi_E$ and, hence, $(V_E \sqcap W_E) - (B_E \sqcup C_E) \sqsubseteq D'_E \sqcap D''_E$. Therefore, by the soft ideal semi-$\mathcal{HC}$ property of $(X_E, \tau, \overline{\mathcal{I}})$, $(V_E \sqcap W_E) - (B_E \sqcup C_E)$ is soft ideal dense and, by Theorem 4, we have $\phi_E \neq (U_E - A_E) \sqcap [(V_E \sqcap W_E) - (B_E \sqcup C_E)]$ and, hence, $(U_E - A_E) \sqcap (D'_E \sqcap D''_E) \neq \phi_E$. Therefore, $D'_E \sqcap D''_E$ is soft ideal dense. So, by Theorem 12, $(X_E, \tau, \overline{\mathcal{I}})$ is soft ideal semi-$\mathcal{IRS}$. □

Remark 3. *For a $\mathcal{SITS}$ $(X_E, \tau, \overline{\mathcal{I}})$, if $\tau \sqcap \overline{\mathcal{I}} \neq \phi_E$. Then, no soft ideal dense set exists, because, if $\tau \sqcap \overline{\mathcal{I}} \neq \phi_E$ and there exists D_E, any soft ideal dense, then $\overline{D}^*_E = X_E$, so by Remark 1 we have $\overline{X}^*_E = X_E$. Hence, by Theorem 1, $\tau \sqcap \overline{\mathcal{I}} = \phi_E$, which is a contradiction. Therefore, if $\tau \sqcap \overline{\mathcal{I}} \neq \phi_E$ then no soft ideal dense set exists.*

Question: Is there any example of soft ideal topological space such that $\tau \sqcap \overline{\mathcal{I}} \neq \phi_E$, and Theorems 10–14 are true?

7. Conclusions and Future Work

As an extension of the classical (crisp) topology, the idea of a soft topology on a universal set was independently proven by Shabir and Naz [3], and Çağman et al. [30]. The study of this topological generalization has becoming more fascinating. Numerous techniques for building soft topologies have been documented in the literature. We have added to the body of knowledge in soft topology by delving into the ideas of soft hyperconnected modulo ideal, soft ideal resolvable, and soft ideal semi-irresolvable spaces. This research is based on the hyperconnectedness and resolvability of soft ideal spaces. We spoken about several fundamental operations on soft ideal spaces. A concept of a soft ideal semi-irresolvable space and an overview of its properties are provided. Furthermore, we have determined the basic characteristics of soft ideal resolvable spaces and connections between the other concepts. The findings presented in this work are preliminary and further research will examine additional aspects of the soft ideal resolvable space. By integrating these two approaches, our work creates opportunities for potential contributions to this trend using hyperconnectedness and resolvability structures with generalized rough approximation spaces, as well as the resolvability of primal soft topologies and the resolvability of fuzzy soft topologies in classical and soft settings.

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Abbreviations

The following abbreviations are utilized in this document:

$\mathcal{SITS}$	soft ideal topological space
$\mathcal{HC}$	hyperconnected
$\mathcal{IRS}$	irresolvable
$\mathcal{RS}$	resolvable

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