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Lightlike Hypersurfaces of Meta-Golden Semi-Riemannian Manifolds

Feyza Esra Erdoğan [1,*,†], Selcen Yüksel Perktaş [2,†], Şerife Nur Bozdağ [1,†] and Bilal Eftal Acet [2,†]

1 Department of Mathematics, Faculty of Science, Ege University, İzmir 35100, Türkiye; serife.nur.yalcin@ege.edu.tr
2 Department of Mathematics, Faculty of Science and Arts, Adıyaman University, Adıyaman 02040, Türkiye; sperktas@adiyaman.edu.tr (S.Y.P.); eacet@adiyaman.edu.tr (B.E.A.)
* Correspondence: feyza.esra.erdogan@ege.edu.tr
† These authors contributed equally to this work.

Abstract: In this research, we embark on the examination of lightlike hypersurfaces within an almost meta-Golden semi-Riemannian manifold. We investigate the properties of the induced structure on a lightlike hypersurface by meta-Golden semi-Riemannian structure. Then, we introduce invariant lightlike hypersurfaces, anti-invariant lightlike hypersurfaces and screen semi-invariant lightlike hypersurfaces of almost meta-Golden semi-Riemannian manifolds and give examples.

Keywords: Chi ratio; golden structure; meta-Golden structure; lightlike hypersurface

MSC: 53C15; 57R15

1. Introduction

It has been shown that there is a close connection between the transition from Newtonian physics to relativity mechanics and the Golden ratio. Moreover, the Golden ratio was also used to derive the special theory of relativity, Lorentz contraction of lengths and expansion of time intervals. This case reveals the research on numberless objects that satisfy the Golden ratio necessity through the world. One of the results was the view that a logarithmic spiral provides the Golden ratio. Recently, however, Barlett [1] has shown that this assertion is untrue. It was also proved that an important class of logarithmic spirals delivers the meta-Golden Chi ratio wonderfully. In [1], the same fulfillment was built around the meta-Golden Chi ratio given by $\chi = \frac{1+\sqrt{4\dot{\varphi}+5}}{2\dot{\varphi}}$, where $\dot{\varphi} = \frac{1+\sqrt{5}}{2}$.

In Riemannian (also semi-Riemannian) manifolds, different geometric structures allow important consequences to occur while investigating the geometric and differential properties of submanifolds. Manifolds with such differential geometric structures have been studied by several authors (see [2–7]).

A major shortcoming in manifold theory is the limited study of isometries between manifolds with non-positive metrics. This is a significant gap, particularly in the context of applications in physics and engineering. In fact, Riemannian submersions and isometric immersions are extensively studied topics, but degenerate cases have received scant attention due to the challenges posed by metric complexities. Nevertheless, transitioning from the non-degenerate case to the degenerate one, both in terms of applications and mathematics, holds the potential to yield more general and robust results. The degeneracy version of isometric immersions has been examined by a large group of geometers under the name of lightlike submanifolds which were firstly defined by Duggal and Bejancu [8], (see also [9–11]).

Recently, Şahin [12] introduced a new type of manifold and named it the meta-Golden Riemannian manifold. This manifold was constructed by means of the meta-Golden Chi ratio and the Golden manifolds.

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In this research, we embark on the study of lightlike geometry in meta-Golden semi-Riemannian manifolds.

2. Preliminaries

A structure similar to the Golden ratio is presented as follows (see Hylebrouck [13]): From Figure 1 in [12], we obtain $\dot{\chi} = \frac{1}{\dot{\varphi}} + \frac{1}{\dot{\chi}}$, which suggests that $\dot{\chi}^2 - \frac{1}{\dot{\varphi}}\dot{\chi} - 1 = 0$. Thus, the roots are found as $\frac{\frac{1}{\dot{\varphi}} \mp \sqrt{4+\frac{1}{\dot{\varphi}^2}}}{2}$. The correlation between the meta-Golden Chi ratio $\dot{\chi}$ and continued fractions was found in [13]. By denoting the positive and negative roots by $\dot{\chi} = \frac{\frac{1}{\dot{\varphi}} + \sqrt{4+\frac{1}{\dot{\varphi}^2}}}{2}$ and $\ddot{\chi} = \frac{\frac{1}{\dot{\varphi}} - \sqrt{4+\frac{1}{\dot{\varphi}^2}}}{2}$, respectively, we have [13]

$$\ddot{\chi} = \frac{1}{\dot{\varphi}} - \dot{\chi}, \tag{1}$$

$$\dot{\varphi}\dot{\chi}^2 = \dot{\varphi} + \dot{\chi}, \tag{2}$$

and

$$\dot{\varphi}\ddot{\chi}^2 = \dot{\varphi} + \ddot{\chi}. \tag{3}$$

In [3], it was stated that an endomorphism $\check{\beta}$ on a manifold $\check{\mathfrak{M}}^*$ is an almost Golden structure , if

$$\check{\beta}^2\mathbb{X}_1 = \check{\beta}\mathbb{X}_1 + \mathbb{X}_1, \tag{4}$$

for $\mathbb{X}_1 \in \Gamma(T\check{\mathfrak{M}}^*)$. Hence, let $\check{g}$ be the semi-Riemannian metric on $\check{\mathfrak{M}}^*$; then, $(\check{g}, \check{\beta})$ is called an almost Golden semi-Riemannian structure if

$$\check{g}(\check{\beta}\mathbb{X}_1, \mathbb{Y}_1) = \check{g}(\mathbb{X}_1, \check{\beta}\mathbb{Y}_1), \tag{5}$$

where for $\mathbb{X}_1, \mathbb{Y}_1 \in \Gamma(T\check{\mathfrak{M}}^*)$. Therefore, $(\check{\mathfrak{M}}^*, \check{g}, \check{\beta})$ is called an almost Golden semi-Riemannian manifold. In view of (5), we obtain [3]

$$\check{g}(\check{\beta}\mathbb{X}_1, \check{\beta}\mathbb{Y}_1) = \check{g}(\mathbb{X}_1, \check{\beta}\mathbb{Y}_1) + \check{g}(\mathbb{X}_1, \mathbb{Y}_1). \tag{6}$$

Definition 1. *Let $\check{\Im}$ be a $(1,1)$ tensor field on an almost Golden manifold $(\check{\mathfrak{M}}^*, \check{\beta})$ which satisfies*

$$\check{\beta}\check{\Im}^2\mathbb{X}_1 = \check{\beta}\mathbb{X}_1 + \check{\Im}\mathbb{X}_1, \tag{7}$$

for every $\mathbb{X}_1 \in \Gamma(T\check{\mathfrak{M}}^*)$. Then, $\check{\Im}$ is called an almost meta-Golden structure and $(\check{\mathfrak{M}}^*, \check{\beta}, \check{\Im})$ is called an almost meta-Golden manifold [12].

Theorem 1. *A $(1,1)$ tensor field $\check{\Im}$ on an almost Golden manifold $(\check{\mathfrak{M}}^*, \check{\beta})$ is an almost meta-Golden structure if*

$$\check{\Im}^2 = \check{\beta}\check{\Im} - \check{\Im} + I \tag{8}$$

where I is the identity map [12].

We give the following definition inspired by the definition given in [12].

Definition 2. *Let $\check{\Im}$ be an almost meta-Golden structure on $(\check{\mathfrak{M}}^*, \check{\beta}, \check{g})$. If $\check{\Im}$ is compatible with semi-Riemannian metric $\check{g}$ on $\check{\mathfrak{M}}^*$, namely,*

$$\check{g}(\check{\Im}\mathbb{X}_1, \mathbb{Y}_1) = \check{g}(\mathbb{X}_1, \check{\Im}\mathbb{Y}_1), \tag{9}$$

or

$$\check{g}(\check{\Im}\mathbb{X}_1, \check{\Im}\mathbb{Y}_1) = \check{g}(\check{\beta}\mathbb{X}_1, \check{\Im}\mathbb{Y}_1) - \check{g}(\mathbb{X}_1, \check{\Im}\mathbb{Y}_1) + \check{g}(\mathbb{X}_1, \mathbb{Y}_1), \tag{10}$$

then $(\breve{\mathfrak{M}}^*, \breve{\beta}, \breve{\Im}, \breve{g})$ *is called an almost meta-Golden semi-Riemannian manifold where for* $\mathbb{X}_1$, $\mathbb{Y}_1 \in \Gamma(T\breve{\mathfrak{M}}^*)$.

We note that an almost meta-Golden semi-Riemannian manifold is called a meta-Golden semi-Riemannian manifold if $\bar{\nabla}\breve{\Im} = 0$ where $\bar{\nabla}$ is the Levi-Civita connection of $\breve{\mathfrak{M}}^*$. In this case, we also have $\bar{\nabla}\breve{\beta} = 0$.

From here throughout the paper, an almost meta-Golden semi-Riemannian manifold (resp., meta-Golden semi-Riemannian manifold) will be denoted as AMGsR manifold (resp., MGsR manifold).

Let $\breve{\mathfrak{M}}^*$ be an $(n+2)$-dimensional semi-Riemannian manifold with index q, $0 < q < n+1$, and $\mathfrak{M}^*$ be a hypersurface of $\breve{\mathfrak{M}}^*$, with $g = \breve{g} \mid_{\mathfrak{M}^*}$. Then, $\mathfrak{M}^*$ is a lightlike hypersurface of $\breve{\mathfrak{M}}^*$, if the metric g is of rank n and the orthogonal complement $T\mathfrak{M}^{*\perp}$ of $T\mathfrak{M}^*$, given as

$$T\mathfrak{M}^{*\perp} = \bigcup_{p\in\mathfrak{M}^*} \{\mathbb{V}_p \in T_p\breve{\mathfrak{M}}^* : g_p(\mathbb{U}_p, \mathbb{V}_p) = 0, \forall\ \mathbb{U}_p \in \Gamma(T_p\mathfrak{M}^*)\},$$

is a distribution of rank 1 on $\mathfrak{M}^*$ [8]. Here, $T\mathfrak{M}^{*\perp} \subset T\mathfrak{M}^*$ and then it coincides with the distribution called the *radical distribution* given by $Rad(T\mathfrak{M}^*) = T\mathfrak{M}^* \cap T\mathfrak{M}^{*\perp}$.

A complementary bundle of $T\mathfrak{M}^{*\perp}$ in $T\mathfrak{M}^*$ is a non-degenerate distribution of constant rank $(n-1)$ over $\mathfrak{M}^*$, which is known as a *screen distribution* and demonstrated with $S(T\mathfrak{M}^*)$.

Theorem 2 ([8]). *Let* $(\mathfrak{M}^*, g, S(T\mathfrak{M}^*))$ *be a lightlike hypersurface of a semi-Riemannian manifold* $\mathfrak{M}^*$. *Then, there exists a unique rank* 1 *vector sub-bundle* $ltr(T\mathfrak{M}^*)$ *of* $T\breve{\mathfrak{M}}^*$, *with base space* $\mathbb{N}$, *such that for every non-zero section* ξ *of* $Rad(T\mathfrak{M}^*)$ *on a coordinate neighbourhood* $\wp \subset \mathfrak{M}^*$, *there exists a section* $\mathbb{N}$ *of* $ltr(T\mathfrak{M}^*)$ *on* $\wp$ *satisfying:*

$$\breve{g}(\mathbb{N}, \mathbb{W}) = 0, \quad \breve{g}(\mathbb{N}, \mathbb{N}) = 0, \quad \breve{g}(\mathbb{N}, \xi) = 1, \ \textit{for}\ \mathbb{W} \in \Gamma(S(T\mathfrak{M}^*)) \mid_{\wp}.$$

Here, $ltr(T\mathfrak{M}^*)$ *is called the the lightlike transversal vector bundle.*

Via the previous theorem, we obtain:

$$T\mathfrak{M}^* = S(T\mathfrak{M}^*) \perp Rad(T\mathfrak{M}^*), \tag{11}$$

and

$$\begin{aligned} T\breve{\mathfrak{M}}^* &= T\mathfrak{M}^* \oplus ltr(T\mathfrak{M}^*) \\ &= S(T\mathfrak{M}^*) \perp \{Rad(T\mathfrak{M}^*) \oplus ltr(T\mathfrak{M}^*)\}. \end{aligned} \tag{12}$$

For $\mathbb{U}, \mathbb{V} \in \Gamma(T\mathfrak{M}^*)$, $\mathbb{N} \in \Gamma(ltr(T\mathfrak{M}^*))$, from the equations of Gauss and Weingarten formulas, we have

$$\bar{\nabla}_{\mathbb{U}}\mathbb{V} = \nabla_{\mathbb{U}}\mathbb{V} + h(\mathbb{U}, \mathbb{V}), \tag{13}$$

$$\bar{\nabla}_{U}\mathbb{N} = -A_{\mathbb{N}}\mathbb{U} + \nabla^t_{\mathbb{U}}\mathbb{N}. \tag{14}$$

3. Lightlike Hypersurfaces of Almost Meta-Golden Semi-Riemannian Manifolds

In this study, since there are both almost Golden structure and almost meta-Golden structure in AMGsR manifolds, we will obtain two structures that are induced on the lightlike hypersurface.

Throughout this paper, we will consider the structure that is induced from the almost Golden structure on the ambient manifold to the lightlike hypersurface is being an almost Golden structure and invariant, that is, $\breve{\beta}(T\mathfrak{M}^*) \subseteq T\mathfrak{M}^*$ and $\breve{\beta}(T\mathfrak{M}^{*\perp}) \subseteq T\mathfrak{M}^{*\perp}$.

Let $(\breve{\mathfrak{M}}^*, \breve{\beta}, \breve{\Im}, \breve{g})$ be an AMGsR manifold and $\mathfrak{M}^*$ be a lightlike hypersurface of $\breve{\mathfrak{M}}^*$. Consider a $(1,1)$ tensor field $\sharp$ and a 1-form v on $\breve{\mathfrak{M}}^*$. For any $\mathbb{X}_1 \in \Gamma(T\mathfrak{M}^*)$, we have

$$\breve{\Im}\mathbb{X}_1 = \sharp\mathbb{X}_1 + v(\mathbb{X}_1)\mathbb{N}, \qquad \breve{\beta}\mathbb{X}_1 = \beta\mathbb{X}_1 + u(\mathbb{X}_1)\mathbb{N}, \tag{15}$$

and

$$\breve{\Im}\mathbb{N} = \mathbb{V} + v(\mathbb{N})\mathbb{N}, \qquad \breve{\beta}\mathbb{N} = \mathbb{U} + u(\mathbb{N})\mathbb{N}, \tag{16}$$

where $\mathbb{U}, \mathbb{V} \in \Gamma(T\mathfrak{M}^*)$, $\mathbb{N} \in \Gamma(ltr(T\mathfrak{M}^*))$, $v(.) = \breve{g}(., \breve{\Im}\xi)$, $u(.) = \breve{g}(., \breve{\beta}\xi)$ and

$$\sharp : \Gamma(T\mathfrak{M}^*) \to \Gamma(T\mathfrak{M}^*), \quad \sharp\mathbb{X}_1 = (\bar{\Im}\mathbb{X}_1)^{\top}.$$

In this case, the second parts of Equations (15) and (16) are in the form of $\breve{\beta}\mathbb{X}_1 = \beta\mathbb{X}_1$, $u(\mathbb{X}_1) = 0$ and $\mathbb{U} = 0$ due to our assumption. If $\breve{\beta}$ is applied to both sides of the second equation in (16), we have $u(\mathbb{N}) = \dot{\varphi}$ and $u(\mathbb{N}) = 1 - \dot{\varphi}$.

Therefore, we have the following theorem.

Theorem 3. *Let $(\breve{\mathfrak{M}}^*, \breve{\beta}, \breve{\Im}, \breve{g})$ be an AMGsR manifold and $\mathfrak{M}^*$ be a lightlike hypersurface of $\breve{\mathfrak{M}}^*$. In this case, we have a structure $(\beta, g, u, \mathbb{U})$ induced on $\mathfrak{M}^*$ by the almost Golden structure $\breve{\beta}$, satisfies the following equalities:*

$$\beta^2\mathbb{X}_1 = \beta\mathbb{X}_1 + \mathbb{X}_1,$$
$$u(\beta\mathbb{X}_1) = 0,$$
$$\beta\mathbb{U} = 0,$$
$$(u(\mathbb{N}))^2 - u(\mathbb{N}) - 1 = 0,$$
$$g(\beta\mathbb{X}_1, \beta\mathbb{Y}_1) = g(\beta\mathbb{X}_1, \mathbb{Y}_1) + g(\mathbb{X}_1, \mathbb{Y}_1),$$

where for $\mathbb{X}_1, \mathbb{Y}_1 \in \Gamma(T\breve{\mathfrak{M}}^*)$, $\mathbb{N} \in \Gamma(ltr(T\mathfrak{M}^*))$.

Now, we give some characterizations for the structure induced to the lightlike hypersurface from the AMGsR manifold.

Theorem 4. *Let $(\breve{\mathfrak{M}}^*, \breve{\beta}, \breve{\Im}, \breve{g})$ be an AMGsR manifold and $\mathfrak{M}^*$ be a lightlike hypersurface of $\breve{\mathfrak{M}}^*$. In this case, the structure $\Pi = (\sharp, \beta, g, v, \mathbb{V})$ satisfies the following equalities:*

$$\sharp^2\mathbb{X}_1 = \beta\sharp\mathbb{X}_1 - \sharp\mathbb{X}_1 + \mathbb{X}_1 - v(\mathbb{X}_1)\mathbb{V}, \tag{17}$$

$$v(\sharp\mathbb{X}_1) = (u(\mathbb{N}) - v(\mathbb{N}) - I)v(\mathbb{X}_1), \tag{18}$$

$$\sharp\mathbb{V} = \beta V - (1+v(\mathbb{N}))\mathbb{V}, \tag{19}$$

$$(v(\mathbb{N}))^2 = v(\mathbb{N})(u(\mathbb{N}) - 1) + I - v(\mathbb{V}), \tag{20}$$

$$g(\sharp\mathbb{X}_1, \mathbb{Y}_1) = g(\mathbb{X}_1, \sharp\mathbb{Y}_1) + v(\mathbb{Y}_1)\tau(\mathbb{X}_1) - v(\mathbb{X}_1)\tau(\mathbb{Y}_1), \quad \tau(\mathbb{X}_1) = g(\mathbb{X}_1, \mathbb{N}), \tag{21}$$

$$g(\sharp\mathbb{X}_1, \sharp\mathbb{Y}_1) = \begin{pmatrix} g(\beta X_1, \sharp\mathbb{Y}_1) - g(\mathbb{X}_1, \sharp\mathbb{Y}_1) + g(\mathbb{X}_1, \mathbb{Y}_1) \\ +v(\mathbb{Y}_1)\tau(\beta X_1) - v(\mathbb{Y}_1)\tau(\mathbb{X}_1) \\ -v(\mathbb{Y}_1)\zeta(\sharp\mathbb{X}_1) - v(\mathbb{X}_1)\zeta(\sharp\mathbb{Y}_1) \end{pmatrix}, \quad \zeta(\sharp\mathbb{X}_1) = g(\mathbb{X}_1, \sharp\mathbb{N}). \tag{22}$$

Proof. If we apply $\breve{\Im}$ to the first part of Equation (15) and consider Equations (8), (15) and (16), we have

$$\breve{\beta}\breve{\Im}\mathbb{X}_1 - \breve{\Im}\mathbb{X}_1 + \mathbb{X}_1 = \breve{\Im}\sharp\mathbb{X}_1 + v(\mathbb{X}_1)\breve{\Im}\mathbb{N}.$$

By using (15) and (16) in the last equation, we obtain

$$\check{\beta}\sharp\mathbb{X}_1 + v(\mathbb{X}_1)\check{\beta}\mathbb{N} - \sharp\mathbb{X}_1 - v(\mathbb{X}_1)\mathbb{N} + \mathbb{X}_1 = \sharp^2\mathbb{X}_1 + v(\sharp\mathbb{X}_1)\mathbb{N} + v(\mathbb{X}_1)\mathbb{V} + v(\mathbb{N})v(\mathbb{X}_1)\mathbb{N}, \tag{23}$$

which implies

$$\begin{pmatrix} \beta\sharp\mathbb{X}_1 + v(\mathbb{X}_1)u(\mathbb{N})\mathbb{N} \\ -\sharp\mathbb{X}_1 - v(\mathbb{X}_1)\mathbb{N} + \mathbb{X}_1 \end{pmatrix} = \begin{pmatrix} \sharp^2\mathbb{X}_1 + v(\sharp\mathbb{X}_1)\mathbb{N} \\ +v(\mathbb{X}_1)\mathbb{V} + v(\mathbb{N})v(\mathbb{X}_1)\mathbb{N} \end{pmatrix}. \tag{24}$$

If we take the tangential and transversal components of Equation (24), we obtain (17) and (18), respectively.

On the other hand, if we apply $\check{\Im}$ to Equation (16), we have

$$\check{\Im}^2\mathbb{N} = \check{\Im}\mathbb{V} + v(\mathbb{N})\check{\Im}\mathbb{N},$$

which gives

$$\beta(\mathbb{V} + v(\mathbb{N})\mathbb{N}) - (\mathbb{V} + v(\mathbb{N})\mathbb{N}) + \mathbb{N} = \sharp\mathbb{V} + v(\mathbb{V})\mathbb{N} + v(\mathbb{N})\mathbb{V} + (v(\mathbb{N}))^2\mathbb{N},$$

via (8), (15) and (16). Again, equating the tangential and transversal components of the above equation, we obtain (19) and (20), respectively. In addition, if we use (9), (15) and (16), we obtain (21). Applying (8) and (9) in (15), we find (22). □

If we use $\check{\Im}\mathbb{X}_1$ instead of $\mathbb{X}_1$ in Equation (7), we have:

Proposition 1. *Let $(\check{\mathfrak{M}}^*, \check{\beta}, \check{\Im}, \check{g})$ be an MGsR manifold. Then, we have $\bar{\nabla}\check{\beta}\check{\Im} = 0$.*

Theorem 5. *Let $\mathfrak{M}^*$ be a lightlike hypersurface of an MGsR manifold $(\check{\mathfrak{M}}^*, \check{\beta}, \check{\Im}, \check{g})$. Then, we have*

$$(\nabla_{\mathbb{X}_1}\sharp)\mathbb{Y}_1 = v(\mathbb{Y}_1)A_{\mathbb{N}}\mathbb{X}_1 + B(\mathbb{X}_1, \mathbb{Y}_1)\mathbb{V}, \tag{25}$$

$$(\nabla_{\mathbb{X}_1}v)\mathbb{Y}_1 = B(\mathbb{X}_1, \mathbb{Y}_1)v(N) - B(\mathbb{X}_1, \sharp\mathbb{Y}_1)\mathbb{V} - v(\mathbb{Y}_1)\tau(\mathbb{X}_1), \tag{26}$$

$$\nabla_{\mathbb{X}_1}\mathbb{V} = -\sharp A_{\mathbb{N}}\mathbb{X}_1 + \tau(\mathbb{X}_1)\mathbb{V} + v(\mathbb{N})A_{\mathbb{N}}\mathbb{X}_1, \tag{27}$$

$$\mathbb{X}_1(v(\mathbb{N})) = -B(\mathbb{X}_1, \mathbb{V}) - v(A_{\mathbb{N}}\mathbb{X}_1). \tag{28}$$

Proof. Since $\bar{\nabla}\check{\Im} = 0$, by using (15) and (16) and Gauss–Weingarten formulas, we write

$$\begin{pmatrix} \nabla_{\mathbb{X}_1}\sharp\mathbb{Y}_1 + B(\mathbb{X}_1, \sharp\mathbb{Y}_1)\mathbb{N} \\ +\mathbb{X}_1(v(\mathbb{Y}_1))\mathbb{N} - v(\mathbb{X}_1)A_{\mathbb{N}}\mathbb{X}_1 \end{pmatrix} = \begin{pmatrix} \sharp\nabla_{\mathbb{X}_1}\mathbb{Y}_1 + v(\nabla_{\mathbb{X}_1}\mathbb{Y}_1)\mathbb{N} \\ +B(\mathbb{X}_1, \mathbb{Y}_1)\mathbb{V} + B(\mathbb{X}_1, \mathbb{Y}_1)v(\mathbb{N})\mathbb{N} \end{pmatrix},$$

for $\mathbb{X}_1, \mathbb{Y}_1 \in \Gamma(T\check{\mathfrak{M}}^*)$.

If the tangential and transversal parts of the above equation are equalized, we find (25) and (26). In a similar way, for $\mathbb{X}_1 \in \Gamma(T\mathfrak{M}^*)$, $\mathbb{N} \in \Gamma(ltrT\mathfrak{M}^*)$, if we use $\bar{\nabla}\check{\Im} = 0$, the Equations (15) and (16) and also Gauss–Weingarten formulas, we obtain

$$\begin{pmatrix} \nabla_{\mathbb{X}_1}\mathbb{V} + B(\mathbb{X}_1, \mathbb{V})\mathbb{N} + \mathbb{X}_1(v(\mathbb{N}))\mathbb{N} \\ -v(\mathbb{N})A_{\mathbb{N}}\mathbb{X}_1 + v(\mathbb{N})\tau(\mathbb{X}_1)\mathbb{V} \end{pmatrix} = \begin{pmatrix} -\sharp A_{\mathbb{N}}\mathbb{X}_1 - v(A_{\mathbb{N}}\mathbb{X}_1)\mathbb{N} \\ +\tau(\mathbb{X}_1)\mathbb{V} + v(\mathbb{N})\tau(\mathbb{X}_1)\mathbb{V} \end{pmatrix}.$$

Therefore, if the tangential and transversal parts of above equation are equalized, we find the Equations (27) and (28). □

Theorem 6. *Let $(\check{\mathfrak{M}}^*, \check{\beta}, \check{\Im}, \check{g})$ be an MGsR manifold and $\mathfrak{M}^*$ be a lightlike hypersurface of $\check{\mathfrak{M}}^*$. Then, we have the following equations:*

$$
\begin{aligned}
\nabla\beta &= 0,\\
B(\mathbb{X}_1,\beta\mathbb{Y}_1) &= B(\mathbb{X}_1,\mathbb{Y}_1)u(\mathbb{N}),\\
\beta A_{\mathbb{N}}\mathbb{X}_1 &= u(\mathbb{N})A_{\mathbb{N}}\mathbb{X}_1,\\
\mathbb{X}_1(u(\mathbb{N})) &= 0.
\end{aligned}
$$

Now, using $\bar{\nabla}\check{\beta}\check{\Im} = 0$, we can give the following theorem regarding the conditions provided by the structures reduced on the lightlike hypersurface of the MGsR manifold $(\check{\mathfrak{M}}^*,\check{\beta},\check{\Im},\check{g})$.

Theorem 7. *Let $(\check{\mathfrak{M}}^*,\check{\beta},\check{\Im},\check{g})$ be an MGsR manifold and $\mathfrak{M}^*$ be a lightlike hypersurface of $\check{\mathfrak{M}}^*$. Then, we have*

$$(\nabla_{\mathbb{X}_1}\beta\sharp)\mathbb{Y}_1 = B(\mathbb{X}_1,\mathbb{Y}_1)\beta\mathbb{V} - v(\mathbb{Y}_1)u(\mathbb{N})A_{\mathbb{N}}\mathbb{X}_1,$$

$$
\begin{aligned}
u(\mathbb{N})(\nabla_{\mathbb{X}_1}v)\mathbb{Y}_1 = &-B(\mathbb{X}_1,\beta\sharp\mathbb{Y}_1) - v(\mathbb{Y}_1)u(\mathbb{N})\tau(\mathbb{X}_1)\\
&+ B(\mathbb{X}_1,\mathbb{Y}_1)v(\mathbb{N})u(\mathbb{N}),
\end{aligned}
$$

$$\nabla_{\mathbb{X}_1}\beta\mathbb{V} = \beta\nabla_{\mathbb{X}_1}\mathbb{V} = v(\mathbb{N})u(\mathbb{N})A_{\mathbb{N}}\mathbb{X}_1 - \beta\sharp A_{\mathbb{N}}\mathbb{X}_1 + \tau(\mathbb{X}_1)\beta\mathbb{V},$$

$$\mathbb{X}_1(v(\mathbb{N}))u(\mathbb{N}) = -v(A_{\mathbb{N}}\mathbb{X}_1)u(\mathbb{N}) - B(\mathbb{X}_1,\beta\mathbb{V}) - v(\mathbb{N})u(\mathbb{N})\tau(\mathbb{X}_1).$$

Proof. For $\mathbb{X}_1,\mathbb{Y}_1 \in \Gamma(T\mathfrak{M}^*)$, $\mathbb{N} \in \Gamma(ltrT\mathfrak{M}^*)$, if we use $\bar{\nabla}\check{\beta}\check{\Im} = 0$ and Equations (15) and (16), we have

$$
\begin{pmatrix}
\nabla_{\mathbb{X}_1}\beta\sharp\mathbb{Y}_1 + B(\mathbb{X}_1,\beta\sharp\mathbb{Y}_1)\mathbb{N}\\
+[\mathbb{X}_1(v(\mathbb{Y}_1))u(\mathbb{N}) + v(\mathbb{Y}_1)\mathbb{X}_1(u(\mathbb{N}))]\mathbb{N}\\
-v(\mathbb{Y}_1)u(\mathbb{N})A_{\mathbb{N}}\mathbb{X}_1\\
+v(\mathbb{Y}_1)u(\mathbb{N})\tau(\mathbb{X}_1)\mathbb{N}
\end{pmatrix}
=
\begin{pmatrix}
\beta\sharp\nabla_{\mathbb{X}_1}\mathbb{Y}_1 + v(\nabla_{\mathbb{X}_1}\mathbb{Y}_1)u(\mathbb{N})\mathbb{N}\\
+B(\mathbb{X}_1,\mathbb{Y}_1)\beta\mathbb{V}\\
+B(\mathbb{X}_1,\mathbb{Y}_1)v(\mathbb{N})u(\mathbb{N})\mathbb{N}
\end{pmatrix}.
$$

By taking the tangential and transversal parts of this equation, the first two of the equations specified in the theorem are obtained. For $\mathbb{X}_1 \in \Gamma(T\mathfrak{M}^*)$, $\mathbb{N} \in \Gamma(ltrT\mathfrak{M}^*)$, by using $\bar{\nabla}\check{\beta}\check{\Im} = 0$ and Equations (15) and (16), we obtain

$$
\begin{pmatrix}
\nabla_{\mathbb{X}_1}\beta\mathbb{V} + B(\mathbb{X}_1,\beta\mathbb{V})\mathbb{N}\\
+[\mathbb{X}_1(v(\mathbb{N}))u(\mathbb{N}) + v(\mathbb{N})\mathbb{X}_1(u(\mathbb{N}))]\mathbb{N}\\
-v(\mathbb{N})u(\mathbb{N})A_{\mathbb{N}}\mathbb{X}_1\\
+v(\mathbb{N})u(\mathbb{N})\tau(\mathbb{X}_1)\mathbb{N}
\end{pmatrix}
=
\begin{pmatrix}
-\beta\sharp A_{\mathbb{N}}\mathbb{X}_1 - v(A_{\mathbb{N}}\mathbb{X}_1)u(\mathbb{N})\mathbb{N}\\
+\tau(\mathbb{X}_1)\beta\mathbb{V} + \tau(\mathbb{X}_1)v(\mathbb{N})\mathbb{V}\\
+\tau(\mathbb{X}_1)v^2(\mathbb{N})\mathbb{N}
\end{pmatrix},
$$

which implies that the last two of the equations specified in the theorem are obtained. Thus, the proof is completed. □

Now, we define some special lightlike hypersurfaces.

Definition 3. *Let $(\check{\mathfrak{M}}^*,\check{\beta},\check{\Im},\check{g})$ be an AMGsR manifold and $\mathfrak{M}^*$ be a lightlike hypersurface of $\check{\mathfrak{M}}^*$. Then,*

1. *if $\check{\beta}\check{\Im}(T\mathfrak{M}^*) \subset T\mathfrak{M}^*$, $\mathfrak{M}^*$ is called as an invariant,*
2. *If $\check{\beta}\check{\Im}(Rad(T\mathfrak{M}^*)) \subset S(T\mathfrak{M}^*)$ and $\check{\beta}\check{\Im}(ltrT\mathfrak{M}^*) \subset S(T\mathfrak{M}^*)$, $\mathfrak{M}^*$ is called a screen semi-invariant,*
3. *If $\check{\beta}\check{\Im}(Rad(T\mathfrak{M}^*)) \subset ltrT\mathfrak{M}^*$, $\mathfrak{M}^*$ is called a radical anti-invariant,*

lightlike hypersurface.

Example 1. *Let $\breve{\mathfrak{M}}^* = \mathbb{R}^5_1$ be an almost Golden semi-Riemannian manifold with a coordinate system $(x_1, x_2, x_3, x_4, x_5)$, a semi-Euclidean metric $\breve{g}$ of signature $(-, +, +, +, +)$ and an almost Golden structure defined by*

$$\breve{\beta}(x_1, x_2, x_3, x_4, x_5) = (\dot{\varphi}x_1, \dot{\varphi}x_2, \dot{\varphi}x_3, (1-\dot{\varphi})x_4, (1-\dot{\varphi})x_5).$$

Also, we define a $(1,1)$ tensor field $\breve{\Im}$ on $\breve{\mathfrak{M}}^*$ by

$$\breve{\Im}(x_1, x_2, x_3, x_4, x_5) = (\dot{\chi}x_1, \dot{\chi}x_2, \dot{\chi}x_3, -\tilde{\chi}x_4, -\tilde{\chi}x_5),$$

where $\tilde{\chi} = \frac{\varphi+\sqrt{\varphi^2+4}}{2}$ and $\tilde{\chi}^2 = \varphi\tilde{\chi} + I$, (I is an identity map.)
One can see that $\breve{\Im}$ satisfies (8)–(10) which imply that $(\breve{\mathfrak{M}}^*, \breve{\beta}, \breve{\Im}, \breve{g})$ is an AMGsR manifold. Now, we consider a hypersurface $\mathfrak{M}^*$ of $\breve{\mathfrak{M}}^*$ given by

$$x_1 = u_3, \quad x_2 = -(\sin\alpha)u_1 + (\cos\alpha)u_3,$$

$$x_3 = (\cos\alpha)u_1 + (\sin\alpha)u_3, \quad x_4 = u_2, \quad x_5 = u_4.$$

Then, $T\mathfrak{M}^*$ is spanned by

$$\mathbb{Z}_1 = -\sin\alpha\frac{\partial}{\partial x_2} + \cos\alpha\frac{\partial}{\partial x_3}, \quad \mathbb{Z}_2 = \frac{\partial}{\partial x_4},$$

$$\mathbb{Z}_3 = \frac{\partial}{\partial x_1} + \cos\alpha\frac{\partial}{\partial x_2} + \sin\alpha\frac{\partial}{\partial x_3}, \quad \mathbb{Z}_4 = \frac{\partial}{\partial x_5}.$$

So, $\mathfrak{M}^*$ is a lightlike hypersurface of $\breve{\mathfrak{M}}^*$. In this case, $Rad(T\mathfrak{M}^*)$ and $S(T\mathfrak{M}^*)$ are given by

$$Rad(T\mathfrak{M}^*) = Span\{\mathbb{Z}_3\},$$

and

$$S(T\mathfrak{M}^*) = Span\{\mathbb{Z}_1, \mathbb{Z}_2, \mathbb{Z}_4\},$$

respectively, where $\breve{\beta}\mathbb{Z}_4 = (1-\dot{\varphi})\mathbb{Z}_4 \in \Gamma(S(T\mathfrak{M}^*))$, $\breve{\beta}\mathbb{Z}_3 = \dot{\varphi}\mathbb{Z}_3 \in \Gamma(Rad(T\mathfrak{M}^*))$, $\breve{\beta}\mathbb{Z}_1 = \dot{\varphi}\mathbb{Z}_1 \in \Gamma(S(T\mathfrak{M}^*))$ and $\breve{\beta}\mathbb{Z}_2 = (1-\dot{\varphi})\mathbb{Z}_2 \in \Gamma(S(T\mathfrak{M}^*))$. Thus, $Rad(T\mathfrak{M}^*)$ and $S(T\mathfrak{M}^*)$ are $\breve{\beta}$-invariant distributions. Also, we obtain

$$ltr(T\mathfrak{M}^*) = Span\left\{\mathbb{N} = \frac{1}{2}\left(-\frac{\partial}{\partial x_1} + \cos\alpha\frac{\partial}{\partial x_2} + \sin\alpha\frac{\partial}{\partial x_3}\right)\right\},$$

and $\breve{\beta}\mathbb{N} = \dot{\varphi}\mathbb{N} \in \Gamma(ltr(T\mathfrak{M}^*))$, which imply that $\mathfrak{M}^*$ is a $\breve{\beta}-$invariant lightlike hypersurface of $\breve{\mathfrak{M}}^*$. Since

$$\breve{\Im}\mathbb{Z}_1 = \dot{\chi}\mathbb{Z}_1 \in \Gamma(S(T\mathfrak{M}^*)), \quad \breve{\Im}\mathbb{Z}_2 = -\tilde{\chi}\mathbb{Z}_2 \in \Gamma(S(T\mathfrak{M}^*)),$$

$$\breve{\Im}\mathbb{Z}_4 = -\tilde{\chi}\mathbb{Z}_4 \in \Gamma(S(T\mathfrak{M}^*)), \quad \breve{\Im}\mathbb{Z}_3 = \dot{\chi}\mathbb{Z}_3 \in \Gamma(Rad(T\mathfrak{M}^*)),$$

and

$$\breve{\Im}\mathbb{N} = \dot{\chi}\mathbb{N} \in \Gamma(ltr(T\mathfrak{M}^*)).$$

Then, $\mathfrak{M}^*$ is an invariant lightlike hypersurface of an AMGsR manifold $\breve{\mathfrak{M}}^*$.

Theorem 8. *Let $(\breve{\mathfrak{M}}^*, \breve{\beta}, \breve{\Im}, \breve{g})$ be an AMGsR manifold and $\mathfrak{M}^*$ be a lightlike hypersurface of $\breve{\mathfrak{M}}^*$. Then, the followings are equivalent;*

1. *$\mathfrak{M}^*$ is $\breve{\Im}$-invariant, so $\breve{\beta}\breve{\Im}$ is invariant;*
2. *v vanishes on $\mathfrak{M}^*$;*
3. *$\sharp$ is an almost meta-Golden structure on $\mathfrak{M}^*$.*

Proof. We know that if $\mathfrak{M}^*$ is $\breve{\Im}$-invariant, then for any $\mathbb{X}_1 \in \Gamma(T\mathfrak{M}^*)$ we write $\breve{\Im}\mathbb{X}_1 = \sharp\mathbb{X}_1$. From (15), we obtain $v(\mathbb{X}_1) = 0$. Conversely, if v vanishes on $\mathfrak{M}^*$, then (1) is satisfied. Hence, (1)$\Longleftrightarrow$(2). The necessary and sufficient condition for $v = 0$ on $\mathfrak{M}^*$ is that $\check{\Im}\mathbb{X}_1 = \sharp\mathbb{X}_1$. Then, we obtain

$$\sharp^2\mathbb{X}_1 = \beta\sharp\mathbb{X}_1 - \sharp\mathbb{X}_1 + \mathbb{X}_1.$$

Here, we also have

$$g(\sharp\mathbb{X}_1, \mathbb{Y}_1) = g(\mathbb{X}_1, \sharp\mathbb{Y}_1).$$

Therefore, $\sharp$ is an almost meta-Golden structure on $\mathfrak{M}^*$. □

Theorem 9. *There is no radical anti-invariant lightlike hypersurface of an AMGsR manifold.*

Proof. Let $(\check{\mathfrak{M}}^*, \check{\beta}, \check{\Im}, \check{g})$ be an AMGsR manifold and $\mathfrak{M}^*$ be a radical anti-invariant lightlike hypersurface of $\check{\mathfrak{M}}^*$. From the definition of radical anti-invariant lightlike hypersurface, for any $\xi \in \Gamma(Rad(T\mathfrak{M}^*))$, we have $\check{\Im}\xi \in \Gamma(ltr(T\mathfrak{M}^*))$, which implies

$$\check{g}(\check{\Im}\xi, \check{\Im}\xi) = 0, \quad \check{g}(\check{\Im}\xi, \check{\Im}\mathbb{N}) \neq 0, \quad \check{g}(\check{\Im}\mathbb{N}, \check{\Im}\mathbb{N}) = 0.$$

Therefore, there is no radical anti-invariant lightlike hypersurface. □

4. Screen Semi-Invariant Lightlike Hypersurfaces of Almost Meta-Golden Semi-Riemannian Manifolds

Let $(\check{\mathfrak{M}}^*, \check{\beta}, \check{\Im}, \check{g})$ be an $(m+2)$-dimensional AMGsR manifold and $(\mathfrak{M}^*, g)$ be a screen semi-invariant lightlike hypersurface of $\check{\mathfrak{M}}^*$. Taking $D_T = \sharp Rad(T\mathfrak{M}^*)$, $D_\perp = \sharp ltr(T\mathfrak{M}^*)$ and $D = D_\circ \perp Rad(T\mathfrak{M}^*) \perp \sharp Rad(T\mathfrak{M}^*)$, we have the following decompositions:

$$S(T\mathfrak{M}^*) = D_\circ \perp (D_T \oplus D_\perp), \tag{29}$$

$$T\mathfrak{M}^* = D \oplus D_\perp, \tag{30}$$

$$T\check{\mathfrak{M}}^* = D \oplus D_\perp \oplus ltr(T\mathfrak{M}^*), \tag{31}$$

where $D_\circ$ is an $(m-2)$-dimensional distribution, $\mathbb{V} = \check{\Im}\mathbb{N}$ and $\mathbb{Z} = \check{\Im}\xi$.

Example 2. *Let $\check{\mathfrak{M}}^* = \mathbb{R}^5_2$ be a semi-Riemannian manifold with coordinate system $(x_1, x_2, x_3, x_4, x_5)$ and signature $(-,+,-,+,+)$. Taking an almost Golden structure*

$$\check{\beta}(x_1, x_2, x_3, x_4, x_5) = (\dot{\varphi}x_1, \dot{\varphi}x_2, \dot{\varphi}x_3, \dot{\varphi}x_4, \dot{\varphi}x_5),$$

with a meta-Golden structure

$$\check{\Im}(x_1, x_2, x_3, x_4, x_5) = (\dot{\chi}x_1, \dot{\chi}x_2, \ddot{\chi}x_3, \dot{\chi}x_4, \ddot{\chi}x_5),$$

then $(\mathbb{R}^5_2, \check{\beta}, \check{\Im}, \check{g})$ is an AMGsR manifold.

Now, we consider a hypersurface $\mathfrak{M}^*$ of $\check{\mathfrak{M}}^*$ given by

$$x_5 = \ddot{\chi}x_1 + \ddot{\chi}x_2 + x_3,$$

Then, $T\mathfrak{M}^*$ is spanned by $\{\mathbb{Z}_1, \mathbb{Z}_2, \mathbb{Z}_3, \mathbb{Z}_4\}$, where

$$\mathbb{Z}_1 = \frac{\partial}{\partial x_1} + \ddot{\chi}\frac{\partial}{\partial x_5}, \quad \mathbb{Z}_2 = \frac{\partial}{\partial x_2} + \ddot{\chi}\frac{\partial}{\partial x_5},$$

$$\mathbb{Z}_3 = \frac{\partial}{\partial x_3} + \frac{\partial}{\partial x_5}, \quad \mathbb{Z}_4 = \frac{\partial}{\partial x_4}.$$

So, $\mathfrak{M}^*$ *is a* 1*-lightlike hypersurface of* $\check{\mathfrak{M}}^*$ *with*

$$Rad(T\mathfrak{M}^*) = Span\{\xi = \ddot{\chi}\frac{\partial}{\partial x_1} - \ddot{\chi}\frac{\partial}{\partial x_2} - \frac{\partial}{\partial x_3} - \frac{\partial}{\partial x_5}\},$$

and

$$S(T\mathfrak{M}^*) = Span\{\mathbb{W}_1, \mathbb{W}_2, \mathbb{W}_3\},$$

where

$$\mathbb{W}_1 = \frac{\partial}{\partial x_4}, \quad \mathbb{W}_2 = -\ddot{\chi}\frac{\partial}{\partial x_1} + \ddot{\chi}\frac{\partial}{\partial x_2} + \frac{\partial}{\partial x_3} + \frac{\partial}{\partial x_5},$$

$$\mathbb{W}_3 = -\ddot{\chi}\frac{\partial}{\partial x_1} - \ddot{\chi}\frac{\partial}{\partial x_2} + \frac{\partial}{\partial x_3} - \frac{\partial}{\partial x_5}.$$

Then, we write $D_T = Span\{\mathbb{W}_2\}$ *and* $D_\perp = Span\{\mathbb{W}_3\}$*. Also, we obtain*

$$ltr(T\mathfrak{M}^*) = Span\left\{\mathbb{N} = \frac{1}{2(1-\ddot{\chi}^2)}\left(\ddot{\chi}\frac{\partial}{\partial x_1} + \ddot{\chi}\frac{\partial}{\partial x_2} - \frac{\partial}{\partial x_3} + \frac{\partial}{\partial x_5}\right)\right\},$$

which implies that $\mathfrak{M}^*$ *is a* $\check{\beta}-$*invariant lightlike hypersurface of* $\check{\mathfrak{M}}^*$*. Furthermore, we obtain*

$$\check{\Im}\xi = \ddot{\chi}\mathbb{W}_2 \in \Gamma(D_T),$$

$$\check{\Im}\mathbb{N} = \frac{\ddot{\chi}}{2(1-\ddot{\chi}^2)}\mathbb{W}_3 \in \Gamma(D_\perp).$$

Therefore, $\mathfrak{M}^*$ *is a screen semi-invariant lightlike hypersurface of* $(\mathbb{R}^5_2, \check{\beta}, \check{\Im}, \breve{g})$*.*

Proposition 2. *Let* $\mathfrak{M}^*$ *be a screen semi-invariant lightlike hypersurface of an AMGsR manifold* $(\check{\mathfrak{M}}^*, \check{\beta}, \check{\Im}, \breve{g})$*. Then, for* $\mathbb{X}_1, \mathbb{Y}_1 \in \Gamma(T\mathfrak{M}^*)$*,* $\mathbb{V} \in \Gamma(D_\perp)$ *and* $\mathbb{Z} \in \Gamma(D_T)$*, we have*

$$v(\sharp\mathbb{X}_1) = v(\mathbb{X}_1)(u(\mathbb{N}) + I),$$

$$\sharp v = \beta v - v,$$

$$v(\mathbb{V}) = 1, \tag{32}$$

$$u(\mathbb{N})(\nabla_{\mathbb{X}_1} v)\mathbb{Y}_1 = -B(\mathbb{X}_1, \beta\sharp\mathbb{Y}_1) - v(\mathbb{Y}_1)u(\mathbb{N})\tau(\mathbb{X}_1),$$

$$\nabla_{\mathbb{X}_1}\beta\mathbb{V} = -\beta\sharp A_{\mathbb{N}}\mathbb{X}_1 + \tau(\mathbb{X}_1)\beta\mathbb{V},$$

$$v(A_{\mathbb{N}}\mathbb{X}_1)u(\mathbb{N}) = -B(\mathbb{X}_1, \beta\mathbb{V}),$$

$$(\nabla_{\mathbb{X}_1} v)\mathbb{Y}_1 = -B(\mathbb{X}_1, \sharp\mathbb{Y}_1) - v(\mathbb{Y}_1)\tau(\mathbb{X}_1),$$

$$B(\mathbb{X}_1, \sharp\mathbb{Y}_1) = \frac{1}{u(\mathbb{N})}B(\mathbb{X}_1, \beta\sharp\mathbb{Y}_1),$$

$$(\nabla_{\mathbb{X}_1}\beta\sharp)\mathbb{Y}_1 = B(\mathbb{X}_1, \mathbb{Y}_1)\beta\mathbb{V},$$

$$u(A_{\mathbb{N}}\mathbb{X}_1)u(\mathbb{N}) = C(\mathbb{X}_1, \sharp\xi)u(\mathbb{N}) = -B(\mathbb{X}_1, \beta\mathbb{V}),$$

$$B(\mathbb{X}_1, \mathbb{V}) = -C(\mathbb{X}_1, \mathbb{Z}), \tag{33}$$

$$\nabla_{\mathbb{X}_1}\mathbb{Z} = -\sharp A^*_{\xi}\mathbb{X}_1 - \tau(\mathbb{X}_1)\mathbb{Z},$$

$$C(\mathbb{X}_1, \mathbb{V}) = 0. \tag{34}$$

Corollary 1. *Let $(\breve{\mathfrak{M}}^*, \breve{\beta}, \breve{\Im}, \breve{g})$ be an AMGsR manifold and $\mathfrak{M}^*$ be a screen semi-invariant lightlike hypersurface of $\breve{\mathfrak{M}}^*$. Then, for $\mathbb{X}_1, \mathbb{Z} \in \Gamma(T\mathfrak{M}^*)$, we have*

$$B(\mathbb{X}_1, \mathbb{Z}) = 0. \tag{35}$$

Corollary 2. *There is no D_T-valued component of A^*_ξ in a screen semi-invariant lightlike hypersurface of an AMGsR manifold.*

Proof. In view of (35), we state

$$B(\mathbb{X}_1, \mathbb{Z}) = -g(A^*_\xi \mathbb{X}_1, \breve{\Im}\xi) = -g(A^*_\xi \mathbb{X}_1, \mathbb{Z}) = 0,$$

for $\mathbb{X}_1, \mathbb{Z} \in \Gamma(T\mathfrak{M}^*)$, which gives our assertion. □

Corollary 3. *There is no $D_\perp-$ valued component of $A_\mathbb{N}$ in a screen semi-invariant lightlike hypersurface of an AMGsR manifold.*

Proof. In view of (34), we write

$$C(\mathbb{X}_1, \mathbb{V}) = -g(A_\mathbb{N}\mathbb{X}_1, \breve{\Im}\mathbb{N}) = -g(A_\mathbb{N}\mathbb{X}_1, \mathbb{V}) = 0,$$

which completes the proof. □

Proposition 3. *Let $(\breve{\mathfrak{M}}^*, \breve{\beta}, \breve{\Im}, \breve{g})$ be an AMGsR manifold and $\mathfrak{M}^*$ be a screen semi-invariant lightlike hypersurface of $\breve{\mathfrak{M}}^*$. Then, for the distribution $D_\circ$, we have $\breve{\Im}D_\circ \subset S(T\mathfrak{M}^*)$.*

Proof. For $\mathbb{X}_1 \in \Gamma(D_\circ)$, $\xi \in \Gamma(Rad(T\mathfrak{M}^*))$ and $\mathbb{N} \in \Gamma(ltr(T\mathfrak{M}^*))$, we obtain

$$\breve{g}(\breve{\Im}\mathbb{X}_1, \xi) = \breve{g}(\mathbb{X}_1, \breve{\Im}\xi) = 0,$$

and

$$\breve{g}(\breve{\Im}\mathbb{X}_1, \mathbb{N}) = \breve{g}(\mathbb{X}_1, \breve{\Im}\mathbb{N}) = 0.$$

Moreover, for $\mathbb{V} \in \Gamma(D_\perp)$ and $\mathbb{Z} \in \Gamma(D_T)$, we obtain

$$\begin{aligned}
\breve{g}(\breve{\Im}\mathbb{X}_1, \mathbb{V}) &= \breve{g}(\mathbb{X}_1, \breve{\Im}\mathbb{V}) = \breve{g}(\mathbb{X}_1, \breve{\Im}^2\mathbb{N}) \\
&= \breve{g}(\mathbb{X}_1, \breve{\beta}\breve{\Im}\mathbb{N} - \breve{\Im}\mathbb{N} + \mathbb{N}) \\
&= \breve{g}(\breve{\beta}\mathbb{X}_1, \breve{\Im}\mathbb{N})
\end{aligned}$$

and

$$\begin{aligned}
\breve{g}(\breve{\Im}\mathbb{X}_1, \mathbb{Z}) &= \breve{g}(\mathbb{X}_1, \breve{\Im}\mathbb{Z}) = \breve{g}(\mathbb{X}_1, \breve{\Im}^2\xi) \\
&= \breve{g}(\mathbb{X}_1, \breve{\beta}\breve{\Im}\xi - \breve{\Im}\xi + \xi) \\
&= \breve{g}(\breve{\beta}\mathbb{X}_1, \breve{\Im}\xi).
\end{aligned}$$

So, there is no component of $\breve{\Im}\mathbb{X}_1$ on $ltr(T\mathfrak{M}^*)$ and $Rad(T\mathfrak{M}^*)$. □

Corollary 4. *Let $\mathfrak{M}^*$ be a screen semi-invariant lightlike hypersurface of an AMGsR manifold $(\breve{\mathfrak{M}}^*, \breve{\beta}, \breve{\Im}, \breve{g})$. Then, $D_\circ$ is an $\breve{\Im}$-invariant distribution.*

Theorem 10. *Let $(\breve{\mathfrak{M}}^*, \breve{\beta}, \breve{\Im}, \breve{g})$ be an AMGsR manifold and $\mathfrak{M}^*$ be a screen semi-invariant lightlike hypersurface of $\breve{\mathfrak{M}}^*$. Then, the vector field $\mathbb{Z}$ is parallel on $\mathfrak{M}^*$ if $B(\mathbb{X}_1, \mathbb{Y}_1) = -g(\breve{\Im}A^*_\xi\mathbb{X}_1, \breve{\beta}\mathbb{Y}_1)$ and $\tau = 0$.*

Proof. Assume that the vector field $\mathbb{Z}$ is parallel. From (25), for $\mathbb{X}_1 \in \Gamma(T\mathfrak{M}^*)$, we obtain

$$\nabla_{\mathbb{X}_1}\mathbb{Z} = -\check{\Im}A^*_{\xi}\mathbb{X}_1 - \tau(\mathbb{X}_1)\mathbb{Z} = 0. \tag{36}$$

Applying $\check{\Im}$ to (36) and using (15) with (16), we obtain

$$\begin{aligned} -\check{\Im}^2 A^*_{\xi}\mathbb{X}_1 - \tau(\mathbb{X}_1)\check{\Im}^2\xi = & -\check{\beta}\check{\Im}A^*_{\xi}\mathbb{X}_1 + \check{\Im}A^*_{\xi}\mathbb{X}_1 - A^*_{\xi}\mathbb{X}_1 \\ & - \tau(\mathbb{X}_1)\check{\beta}\check{\Im}\xi + \tau(\mathbb{X}_1)\check{\Im}\xi - \tau(\mathbb{X}_1)\xi. \end{aligned} \tag{37}$$

From (36) with (37), we arrive at

$$-\check{\beta}\check{\Im}A^*_{\xi}\mathbb{X}_1 - A^*_{\xi}\mathbb{X}_1 - \tau(\mathbb{X}_1)\check{\beta}\check{\Im}\xi - \tau(\mathbb{X}_1)\xi = 0,$$

which gives $\tau = 0$ and $\check{\beta}\check{\Im}A^*_{\xi}\mathbb{X}_1 = -A^*_{\xi}\mathbb{X}_1$. So, the proof is completed. □

Theorem 11. *Let $\mathfrak{M}^*$ be a screen semi-invariant lightlike hypersurface of an AMGsR manifold $(\check{\mathfrak{M}}^*, \check{\beta}, \check{\Im}, \check{g})$ and the vector field $\mathbb{Z}$ be parallel on $\mathfrak{M}^*$. Then, either $\sharp$ or $\mathbb{V}$ are parallel on $\mathfrak{M}^*$ if $B(\mathbb{X}_1, \mathbb{V}) = 0$ and $A_{\mathbb{N}}\mathbb{X}_1 = 0$.*

Proof. Assume that $\sharp$ is parallel on $\mathfrak{M}^*$. From (25), for $\mathbb{X}_1, \mathbb{Y}_1 \in \Gamma(T\mathfrak{M}^*)$, we obtain

$$0 = (\nabla_{\mathbb{X}_1}\sharp)\mathbb{Y}_1 = v(\mathbb{Y}_1)A_{\mathbb{N}}\mathbb{X}_1 + B(\mathbb{X}_1, \mathbb{Y}_1)\mathbb{V},$$

from which we have

$$B(\mathbb{X}_1, \mathbb{Y}_1)\mathbb{V} = -v(\mathbb{Y}_1)A_{\mathbb{N}}\mathbb{X}_1. \tag{38}$$

Since $\mathbb{Z}$ is parallel, we state

$$\check{g}(\check{\Im}A^*_{\xi}\mathbb{X}_1, \check{\beta}\mathbb{Y}_1)\mathbb{V} = v(\mathbb{Y}_1)A_{\mathbb{N}}\mathbb{X}_1.$$

In the last equation, replacing $\mathbb{Y}_1$ with $\mathbb{V}$ and using (32), we obtain

$$\check{g}(\check{\Im}A^*_{\xi}\mathbb{X}_1, \check{\beta}\mathbb{V})\mathbb{V} = v(\mathbb{V})A_{\mathbb{N}}\mathbb{X}_1 = A_{\mathbb{N}}\mathbb{X}_1,$$

which gives

$$\check{g}(\check{\Im}A^*_{\xi}\mathbb{X}_1, \check{\beta}\mathbb{V}) = g(A_{\mathbb{N}}\mathbb{X}_1, \mathbb{V}) = C(\mathbb{X}_1, \mathbb{V}).$$

By use of (34), we obtain $B(\mathbb{X}_1, \mathbb{V}) = 0$.

Similarly suppose that $\mathbb{V}$ is parallel on $\mathfrak{M}^*$, i.e., $\nabla_{\mathbb{X}_1}\mathbb{V} = 0$. So, we write

$$-\sharp A_{\mathbb{N}}\mathbb{X}_1 + \tau(\mathbb{X}_1)\mathbb{V} = 0,$$

from which

$$-\check{\Im}A_{\mathbb{N}}\mathbb{X}_1 + v(A_{\mathbb{N}}\mathbb{X}_1)\mathbb{N} + \tau(\mathbb{X}_1)\mathbb{V} = 0. \tag{39}$$

Applying $\check{\Im}$ to (39) and using (15), we obtain

$$\begin{aligned} 0 &= -\check{\Im}^2 A_{\mathbb{N}}\mathbb{X}_1 + v(A_{\mathbb{N}}\mathbb{X}_1)\check{\Im}\mathbb{N} + \tau(\mathbb{X}_1)\check{\Im}\mathbb{V} \\ &= -\beta\sharp A_{\mathbb{N}}\mathbb{X}_1 + \sharp A_{\mathbb{N}}\mathbb{X}_1 - A_{\mathbb{N}}\mathbb{X}_1 \\ &\quad + (v(A_{\mathbb{N}}\mathbb{X}_1) - \tau(\mathbb{X}_1))\mathbb{V} \\ &\quad + ((1 - u(\mathbb{N})v(A_{\mathbb{N}}\mathbb{X}_1) + \tau(\mathbb{X}_1))\mathbb{N}. \end{aligned} \tag{40}$$

From (39) with (40), we arrive at

$$-\beta\sharp A_{\mathbb{N}}\mathbb{X}_1 - A_{\mathbb{N}}\mathbb{X}_1 + v(A_{\mathbb{N}}\mathbb{X}_1)\mathbb{V} + ((1 - u(\mathbb{N})v(A_{\mathbb{N}}\mathbb{X}_1) + \tau(\mathbb{X}_1))\mathbb{N} = 0. \tag{41}$$

From (41), we obtain

$$
\begin{aligned}
-\beta\sharp A_{\mathbb{N}}\mathbb{X}_1 - A_{\mathbb{N}}\mathbb{X}_1 + v(A_{\mathbb{N}}\mathbb{X}_1)\mathbb{V} &= 0, \\
(1-u(\mathbb{N}))v(A_{\mathbb{N}}\mathbb{X}_1) + \tau(\mathbb{X}_1) &= 0.
\end{aligned} \tag{42}
$$

Since $\mathbb{Z}$ is parallel, we know that $\tau(\mathbb{X}_1) = 0$, which gives rise to $A_{\mathbb{N}}\mathbb{X}_1 = 0$, via (42). So, the proof is completed. □

Theorem 12. *Let $\mathfrak{M}^*$ be a screen semi-invariant lightlike hypersurface of an AMGsR manifold $(\breve{\mathfrak{M}}^*, \breve{\beta}, \breve{\Im}, \breve{g})$. If $\sharp$ is parallel with respect to the induced connection ∇ on $\mathfrak{M}^*$, then D is parallel with respect to ∇. Furthermore, $\mathfrak{M}^*$ has $\mathfrak{M}_1 \times \mathfrak{M}_2$ local product structure, where $\mathfrak{M}_1$ is a null curve tangent to $\breve{\Im}ltr(T\mathfrak{M}^*)$ and $\mathfrak{M}_2$ is a leaf of distribution D.*

Proof. Assume that $\sharp$ is parallel with respect to the induced connection ∇ on $\mathfrak{M}^*$. D is parallel with respect to ∇ if and only if

$$
g(\nabla_{\mathbb{X}_1}\xi, \breve{\Im}\xi) = g(\nabla_{\mathbb{X}_1}\breve{\Im}\xi, \breve{\Im}\xi) = g(\nabla_{\mathbb{X}_1}\mathbb{Y}_1, \breve{\Im}\xi) = 0, \tag{43}
$$

for $\mathbb{X}_1 \in \Gamma(T\mathfrak{M}^*)$ and $\mathbb{Y}_1 \in \Gamma(D_\circ)$. From (13)–(15), we obtain

$$
g(\nabla_{\mathbb{X}_1}\xi, \breve{\Im}\xi) = g(\breve{\Im}\nabla_{\mathbb{X}_1}\xi, \xi) = g(\nabla_{\mathbb{X}_1}\breve{\Im}\xi, \xi) = B(\mathbb{X}_1, \mathbb{Z}), \quad g(\nabla_{\mathbb{X}_1}\breve{\Im}\xi, \breve{\Im}\xi) = 0, \tag{44}
$$

and

$$
\begin{aligned}
g(\nabla_{\mathbb{X}_1}\mathbb{Y}_1, \breve{\Im}\xi) &= g(\bar{\nabla}_{\mathbb{X}_1}\mathbb{Y}_1, \breve{\Im}\xi) \\
&= g(\breve{\Im}\bar{\nabla}_{\mathbb{X}_1}\mathbb{Y}_1, \xi) \\
&= g(\bar{\nabla}_{\mathbb{X}_1}\breve{\Im}\mathbb{Y}_1, \xi) \\
&= -g(\breve{\Im}\mathbb{Y}_1, \bar{\nabla}_{\mathbb{X}_1}\xi) \\
&= g(\breve{\Im}\mathbb{Y}_1, A^*_{\mathbb{X}_1}\xi) \\
&= B(\mathbb{X}_1, \breve{\Im}\mathbb{Y}_1).
\end{aligned} \tag{45}
$$

From (35), we know that $B(\mathbb{X}_1, \mathbb{Z}) = 0$. By use of (38), we obtain

$$
B(\mathbb{X}_1, \breve{\Im}\mathbb{Y}_1)\mathbb{V} = -v(\breve{\Im}\mathbb{Y}_1)A_{\mathbb{N}}\mathbb{X}_1 = 0.
$$

If we consider this equation in (45), we obtain the proof of our assertion. □

Definition 4. *Let $(\breve{\mathfrak{M}}^*, \breve{\beta}, \breve{\Im}, \breve{g})$ be an AMGsR manifold and $\mathfrak{M}^*$ be a lightlike hypersurface of $\breve{\mathfrak{M}}^*$. If the second fundamental form B of $\mathfrak{M}^*$ satisfies*

$$
B(\mathbb{X}_1, \mathbb{Y}_1) = 0, \quad \mathbb{X}_1, \mathbb{Y}_1 \in \Gamma(D_\perp),
$$

then we say that $\mathfrak{M}^*$ is a $D_\perp$- totally geodesic lightlike hypersurface.

Definition 5. *Let $(\breve{\mathfrak{M}}^*, \breve{\beta}, \breve{\Im}, \breve{g})$ be an AMGsR manifold and $\mathfrak{M}^*$ be a screen semi-invariant lightlike hypersurface of $\breve{\mathfrak{M}}^*$. If the second fundamental form B of $\mathfrak{M}^*$ satisfies*

$$
B(\mathbb{X}_1, \mathbb{Y}_1) = 0, \quad \mathbb{X}_1 \in \Gamma(D), \ \mathbb{Y}_1 \in \Gamma(D_\perp),
$$

then $\mathfrak{M}^*$ is called a mixed geodesic lightlike hypersurface.

Theorem 13. *Let $\mathfrak{M}^*$ be a screen semi-invariant lightlike hypersurface of an AMGsR manifold $(\breve{\mathfrak{M}}^*, \breve{\beta}, \breve{\Im}, \breve{g})$. Then, the following assertions are equivalent:*

(i) *$\mathfrak{M}^*$ is a mixed geodesic lightlike hypersurface.*

(ii) *There is no D_T-valued component of A_N.*
(iii) *There is no $D_\perp$-valued component of A^*_ξ.*

Proof. Suppose that $\mathfrak{M}^*$ is a mixed geodesic lightlike hypersurface. Then from (33), for $\mathbb{X}_1 \in \Gamma(D), \mathbb{V} \in \Gamma(D_\perp)$ and $\mathbb{Z} \in \Gamma(D_T)$, we have

$$B(\mathbb{X}_1, \mathbb{V}) = -C(\mathbb{X}_1, \mathbb{Z}) = -g(A_N\mathbb{X}_1, \mathbb{Z}) = 0$$

which implies the equivalence of (i) and (ii).

The equivalence of (ii) and (iii) follows from

$$B(\mathbb{X}_1, \mathbb{V}) = -C(\mathbb{X}_1, \mathbb{Z}) \Rightarrow -g(A_N\mathbb{X}_1, \mathbb{Z}) = g(A^*_\xi\mathbb{X}_1, \mathbb{V}) = 0,$$

which completes the proof. □

Theorem 14. *Let $(\check{\mathfrak{M}}^*, \check{\beta}, \check{\Im}, \check{g})$ be an AMGsR manifold and $\mathfrak{M}^*$ be a screen semi-invariant lightlike hypersurface of $\check{\mathfrak{M}}^*$. Then, the distribution D is integrable if and only if*

$$B(\check{\Im}\mathbb{Y}_1, \check{\Im}\mathbb{X}_1) = B(\mathbb{X}_1, \check{\Im}\check{\beta}\mathbb{Y}_1) - B(\mathbb{X}_1, \check{\Im}\mathbb{Y}_1) + B(\mathbb{X}_1, \mathbb{Y}_1), \tag{46}$$

for any $\mathbb{X}_1, \mathbb{Y}_1 \in \Gamma(D)$.

Proof. It is known that, for $\mathbb{X}_1 \in \Gamma(D)$, if the D is invariant then $\check{\Im}\mathbb{X}_1 \in \Gamma(D)$. So the D is integrable if and only if

$$v([\check{\Im}\mathbb{X}_1, \mathbb{Y}_1]) = 0.$$

From the above equation, we obtain

$$\begin{aligned}
v([\check{\Im}\mathbb{X}_1, \mathbb{Y}_1]) &= g([\check{\Im}\mathbb{X}_1, \mathbb{Y}_1], \check{\Im}\xi) \\
&= g(\bar{\nabla}_{\check{\Im}\mathbb{X}_1}\mathbb{Y}_1, \check{\Im}\xi) - g(\bar{\nabla}_{\mathbb{Y}_1}\check{\Im}\mathbb{X}_1, \check{\Im}\xi) \\
&= g(\bar{\nabla}_{\check{\Im}\mathbb{X}_1}\check{\Im}\mathbb{Y}_1, \xi) - g(\check{\Im}\bar{\nabla}_{\mathbb{Y}_1}\mathbb{X}_1, \check{\Im}\xi) \\
&= g(\nabla_{\check{\Im}\mathbb{X}_1}\check{\Im}\mathbb{Y}_1 + B(\check{\Im}\mathbb{X}_1, \check{\Im}\mathbb{Y}_1)N, \xi) \\
&\quad - g(\nabla_{\mathbb{X}_1}\check{\Im}\check{\beta}\mathbb{Y}_1, \xi) + g(\nabla_{\mathbb{X}_1}\check{\Im}\mathbb{Y}_1, \xi) - g(\nabla_{\mathbb{X}_1}\mathbb{Y}_1, \xi) \\
&= B(\check{\Im}\mathbb{Y}_1, \check{\Im}\mathbb{X}_1) - B(\mathbb{X}_1, \check{\Im}\check{\beta}\mathbb{Y}_1) \\
&\quad + B(\mathbb{X}_1, \check{\Im}\mathbb{Y}_1) - B(\mathbb{X}_1, \mathbb{Y}_1),
\end{aligned}$$

which gives (46). □

Theorem 15. *Let $(\check{\mathfrak{M}}^*, \check{\beta}, \check{\Im}, \check{g})$ be an AMGsR manifold and $\mathfrak{M}^*$ be a screen semi-invariant lightlike hypersurface of $\check{\mathfrak{M}}^*$. Then, the following assertions are equivalent:*

(i) *The distribution D is parallel.*
(ii) *The distribution D is totally geodesic.*
(iii) $(\nabla_{\mathbb{X}_1}\sharp)\mathbb{Y}_1 = 0$, *for any $\mathbb{X}_1, \mathbb{Y}_1 \in \Gamma(D)$.*

Proof. The distribution D is parallel if for any $\mathbb{X}_1, \mathbb{Y}_1 \in \Gamma(D)$ and $\mathbb{Z} \in \Gamma(D_T)$

$$v(\nabla_{\mathbb{X}_1}\mathbb{Y}_1) = 0.$$

From the above equation, we obtain

$$\begin{aligned} v(\nabla_{\mathbb{X}_1}\mathbb{Y}_1) &= g(\nabla_{\mathbb{X}_1}\mathbb{Y}_1, \check{\Im}\xi) \\ &= g(\bar{\nabla}_{\mathbb{X}_1}\mathbb{Y}_1, \check{\Im}\xi) \\ &= g(\check{\Im}\bar{\nabla}_{\mathbb{X}_1}\mathbb{Y}_1, \xi) \\ &= g(\bar{\nabla}_{\mathbb{X}_1}\check{\Im}\mathbb{Y}_1, \xi) \\ &= B(\mathbb{X}_1, \check{\Im}\mathbb{Y}_1) \end{aligned}$$

which gives the equivalence of (i) and (ii).

In view of (25), the equivalence of (ii) and (iii) follows from

$$(\nabla_{\mathbb{X}_1}\sharp)\mathbb{Y}_1 = v(\mathbb{Y}_1)A_N\mathbb{X}_1 + B(\mathbb{X}_1, \mathbb{Y}_1)\mathbb{V} \Rightarrow (\nabla_{\mathbb{X}_1}\sharp)\mathbb{Y}_1 = B(\mathbb{X}_1, \mathbb{Y}_1)\mathbb{V},$$

which completes the proof. □

Theorem 16. *Let $(\breve{\mathfrak{M}}^*, \breve{\beta}, \check{\Im}, \breve{g})$ be an AMGsR manifold and $\mathfrak{M}^*$ be a screen semi-invariant lightlike hypersurface of $\breve{\mathfrak{M}}^*$. Then, $\mathfrak{M}^*$ is totally geodesic if for any $\mathbb{X}_1 \in \Gamma(T\mathfrak{M}^*)$, $\mathbb{Y}_1 \in \Gamma(D)$ and $\mathbb{V} \in \Gamma(D_\perp)$*

$$(\nabla_{\mathbb{X}_1}\sharp)\mathbb{Y}_1 = 0, \tag{47}$$

$$(\nabla_{\mathbb{X}_1}\sharp)\mathbb{V} = A_{\mathbb{N}}\mathbb{X}_1. \tag{48}$$

Proof. Suppose that $\mathfrak{M}^*$ is totally geodesic; then for $\mathbb{Y}_1 \in \Gamma(D)$, we obtain

$$v(\mathbb{Y}_1) = g(\mathbb{Y}_1, \check{\Im}\xi) = g(\check{\Im}\mathbb{Y}_1, \xi) = 0.$$

From (25), we have

$$(\nabla_{\mathbb{X}_1}\sharp)\mathbb{Y}_1 = v(\mathbb{Y}_1)A_{\mathbb{N}}\mathbb{X}_1 + B(\mathbb{X}_1, \mathbb{Y}_1)\mathbb{V} = 0.$$

Similarly, for $\mathbb{V} \in \Gamma(D_\perp)$, we have $v(\mathbb{V}) = 1$. In Equation (25), replacing $\mathbb{Y}_1$ by $\mathbb{V}$, we obtain

$$(\nabla_{\mathbb{X}_1}\sharp)\mathbb{V} = v(\mathbb{V})A_{\mathbb{N}}\mathbb{X}_1 + B(\mathbb{X}_1, \mathbb{V})\mathbb{V} = A_{\mathbb{N}}\mathbb{X}_1.$$

Conversely, we suppose that Equations (47) and (48) are satisfied. In view of decomposition (30), for any $\mathbb{Y}_1 \in \Gamma(T\mathfrak{M}^*)$ we find a function f such that $\mathbb{Y}_1 = \mathbb{Y}_d + f\mathbb{V}$, where $\mathbb{Y}_d \in \Gamma(D)$. So we write

$$B(\mathbb{X}_1, \mathbb{Y}_1) = B(\mathbb{X}_1, \mathbb{Y}_d) + fB(\mathbb{X}_1, \mathbb{V}). \tag{49}$$

In (25), replacing $\mathbb{Y}$ with $\mathbb{Y}_d$ and using (47), we obtain

$$\begin{aligned} 0 &= (\nabla_{\mathbb{X}_1}\sharp)\mathbb{Y}_d \\ &= v(\mathbb{Y}_d)A_{\mathbb{N}}\mathbb{X}_1 + B(\mathbb{X}_1, \mathbb{Y}_d)\mathbb{V}, \end{aligned}$$

which gives $B(\mathbb{X}_1, \mathbb{Y}_d) = 0$.

Similarly in (25), replacing $\mathbb{Y}$ with $\mathbb{V}$ and using (48), we have

$$\begin{aligned} 0 &= (\nabla_{\mathbb{X}_1}\sharp)\mathbb{V} \\ &= v(\mathbb{V})A_{\mathbb{N}}\mathbb{X}_1 + B(\mathbb{X}_1, \mathbb{V})\mathbb{V} \\ &= A_{\mathbb{N}}\mathbb{X}_1 + B(\mathbb{X}_1, \mathbb{V})\mathbb{V} \end{aligned}$$

which implies $B(\mathbb{X}_1, \mathbb{V}) = 0$. So, from (49) we arrive at $B(\mathbb{X}_1, \mathbb{Y}_1) = 0$. This completes the proof. □

Theorem 17. *Let $\mathfrak{M}^*$ be a totally umbilic screen semi-invariant lightlike hypersurface of an AMGsR manifold $(\breve{\mathfrak{M}}^*, \breve{\beta}, \breve{\Im}, \breve{g})$. Then, $\mathfrak{M}^*$ is totally geodesic on $\breve{\mathfrak{M}}^*$.*

Proof. Suppose that $\mathfrak{M}^*$ is a totally umbilic screen semi-invariant lightlike hypersurface of $\breve{\mathfrak{M}}^*$. From (35), for any $\mathbb{X}_1 \in \Gamma(T\mathfrak{M}^*)$ we have

$$B(\mathbb{X}_1, \mathbb{Z}) = \lambda g(\mathbb{X}_1, \mathbb{Z}).$$

Replacing $\mathbb{X}_1$ with $\mathbb{V}$ in the last equation, we obtain

$$B(\mathbb{V}, \mathbb{Z}) = \lambda g(\mathbb{V}, \mathbb{Z}) = 0,$$

which yields $\lambda = 0$. In this case, we obtain $B = 0$. □

Theorem 18. *Let $(\breve{\mathfrak{M}}^*, \breve{\beta}, \breve{\Im}, \breve{g})$ be an AMGsR manifold and $\mathfrak{M}^*$ be a screen semi-invariant lightlike hypersurface of $\breve{\mathfrak{M}}^*$. Then, if screen distribution $S(T\mathfrak{M}^*)$ is totally umbilic, then $S(T\mathfrak{M}^*)$ is totally geodesic.*

Proof. Suppose that $S(T\mathfrak{M}^*)$ is totally umbilic. From (34), for any $\mathbb{X}_1 \in \Gamma(T\mathfrak{M}^*)$, we have

$$C(\mathbb{X}_1, \mathbb{V}) = \delta g(\mathbb{X}_1, \mathbb{V}).$$

Replacing $\mathbb{X}_1$ with $\mathbb{Z}$ in the above equation, we obtain

$$C(\mathbb{Z}, \mathbb{V}) = \delta g(\mathbb{Z}, \mathbb{V}) = 0,$$

which gives $\delta = 0$. In this case, we obtain $C = 0$. So, the proof is completed. □

5. Conclusions

In this study, we found structures reduced from the meta-Golden structure of an almost meta-Golden semi-Riemannian manifold onto the tangent and transversal bundles of the lightlike hypersurface. We gave the definitions of invariant, anti-invariant and screen semi-invariant lightlike hypersurfaces of the meta-Golden semi-Riemannian manifold. We have obtained the necessary and sufficient conditions for the distributions of these hypersurfaces to be integrable and totally geodesic.

Working with manifolds with a polynomial structure with constant coefficients allows the definition of many results from classical algebra and geometry, tools that make calculations and proofs simpler (tensor fields, 1-forms, reduced structures, etc.). For example, the fundamental theorem of algebra states that any polynomial with complex coefficients is factored by linear equations, and this result is used to prove that certain manifolds are topologically equivalent to a sphere. These types of manifolds are also important for the differential geometry of manifolds because the properties offered by these structures make geometric structures and curves much easier to examine and understand.

Hypersurfaces and submanifolds are special types of general manifolds and have certain geometric properties. These structures allow the achievement of more specific and meaningful results in mathematical analysis. Submanifolds represent situations where certain parts of the manifold have flatter and simpler geometry. This is important for its flattenability and minimalism properties. Submanifolds provide the ability to better model and understand physical phenomena. For example, they can be used to model physical quantities such as time, which is a submanifold of spacetime. Hypersurfaces and submanifolds are widely used in physics, engineering, computer science and other applications. For example, these structures are frequently encountered concepts in fields such as image processing, graphic design and data analysis. In physics, the integrability of distributions of submanifolds of a manifold provides the ability to better model and understand physical phenomena. It is particularly important for the analysis of physical quantities such as energy distributions or currents on submanifolds of spacetime. An integrable surface provides advantages in calculating center of gravity, moment and similar quantities. Such calculations are important in engineering and physics, especially for analyzing the geometric properties of objects. The concept of parallelism plays an important role in fields such as differential geometry on manifolds and general relativity. For example, it is important in physics, especially in the general theory of relativity, for describing the trajectories of objects in a gravitational field.

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