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On Some Formulas for the Lauricella Function

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Abstract: Lauricella, G. in 1893 defined four multidimensional hypergeometric functions F_A, F_B, F_C and F_D. These functions depended on three variables but were later generalized to many variables. Lauricella's functions are infinite sums of products of variables and corresponding parameters, each of them has its own parameters. In the present work for Lauricella's function $F_A^{(n)}$, the limit formulas are established, some expansion formulas are obtained that are used to write recurrence relations, and new integral representations and a number of differentiation formulas are obtained that are used to obtain the finite and infinite sums. In the presentation and proof of the obtained formulas, already known expansions and integral representations of the considered $F_A^{(n)}$ function, definitions of gamma and beta functions, and the Gaussian hypergeometric function of one variable are used.

Keywords: Appell functions; Lauricella functions; expansion formulas; integral representation; differentiation formulas; summation formulas

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1. Introduction and Preliminaries

The great success of the theory of hypergeometric functions of a single variable has stimulated the development of a corresponding theory in two or more variables. Multiple hypergeometric functions arise in many areas of modern mathematics, and they enable one to solve constructively many topical problems important for theory and applications [1,2].

In this paper we consider the function $F_A^{(n)}(a, b_1, ..., b_n; c_1, ..., c_n; x_1, ..., x_n)$ introduced by Lauricella [3] as one of the most natural generalizations of the Gauss hypergeometric function $F(a, b; c; x)$ to the case of n complex variables $(x_1, ..., x_n) := \mathbf{x} \in \mathbb{C}^n$ and complex parameters $a \in \mathbb{C}$, $(b_1, ..., b_n) := \mathbf{b} \in \mathbb{C}^n$ and $(c_1, ..., c_n) := \mathbf{c} \in \mathbb{C}^n$. Recall that the Gauss function (of a single complex variable) is defined by the series

$$F(a, b; c; x) := \sum_{k=0}^{\infty} \frac{(a)_k (b)_k}{(c)_k} \frac{x^k}{k!}, \tag{1}$$

which converges in the unit disk $\mathbb{U} : \{x \in \mathbb{C} : |x| < 1\}$. Outside $\mathbb{U}$ this function is an analytic continuation of (1). Here, the expression $(a)_k$, called the Pochhammer symbol, is defined in terms of the gamma function $\Gamma(s)$ by

$$(a)_k := \frac{\Gamma(a+k)}{\Gamma(a)}. \tag{2}$$

For an integer $k \geq 0$ it is a product of the form

$$(a)_0 = 1, \ (a)_k = a(a+1)...(a+k-1), \ k = 1, 2, ...$$

It is assumed in (1) that the parameters a, b, and c can take arbitrary complex values, with the exception that c cannot be a non-positive integer $(c \notin \mathbb{Z}^-)$.

Gauss hypergeometric function (1) is contained in the generalized hypergeometric function ${}_pF_q$ involving p numerator parameters, $a_1, ..., a_p$, and q denominator parameters, $b_1, ..., b_q$, as special case. Following the standard notations and conventions, we define it here as follows [4] p. 182:

$${}_pF_q\left(a_1, ..., a_p; b_1, ..., b_q; x\right) := \sum_{k=0}^{\infty} \frac{\prod_{j=1}^{p} \left(a_j\right)_k}{\prod_{j=1}^{q} \left(b_j\right)_k} \frac{x^k}{k!}.$$

An interesting further generalization of the Gaussian series $F(a, b; c; x)$ is due to Appell who has defined, in 1880, four series, F_1 to F_4, for example, one of which is defined as follows [5]

$$F_2(a, b_1, b_2; c_1, c_2; x, y) = \sum_{m,n=0}^{\infty} \frac{(a)_{m+n}(b_1)_m(b_2)_n}{(c_1)_m(c_2)_n} \frac{x^m}{m!} \frac{y^n}{n!} \ (c_1, c_2 \neq 0, -1, -2, ...). \tag{3}$$

The Appell function $F_2(a, b_1, b_2; c_1, c_2; x, y)$ converges in the squared $|x| + |y| < 1$.

The Appel function F_2 has been studied quite well [6,7] and has found its application in the theory of boundary value problems and potential theory for partial differential equations with singular coefficients [8].

The first Lauricella function, which we denote by $F_A^{(n)}(a, \mathbf{b};\mathbf{c};\mathbf{x})$ for brevity, is defined for $(\mathbf{c} \notin \mathbb{Z}^-)$ by the $n-$variate hypergeometric series

$$F_A^{(n)}(a, \mathbf{b};\mathbf{c};\mathbf{x}) = \sum_{|\mathbf{k}|=0}^{\infty} \frac{(a)_{|\mathbf{k}|}(b_1)_{k_1}...(b_n)_{k_n}}{(c_1)_{k_1}...(c_n)_{k_n}} \frac{x_1^{k_1}}{k_1!}...\frac{x_n^{k_n}}{k_n!} \ (c_j \neq 0, -1, -2, ...j = \overline{1, n}), \tag{4}$$

which converges in $|x_1| + ... + |x_n| < 1$. The sum in (4) is taken over the multi-indices $\mathbf{k} := (k_1, ..., k_n)$ with non-negative integer components $k_1 \geq 0, ..., k_n \geq 0$, and we define $|\mathbf{k}| := k_1 + ... + k_n$.

Lauricella, G. gave several elementary properties of this series including, for instance, integral representations of the Eulerian type, transformations and reducible cases, and the system of partial differential equations associated with him. A summary of Lauricella's work is given by Appell and Kampé de Fériet [9]. Note that fundamental solutions and explicit solutions of some boundary value problems for multidimensional singular equations are expressed through the Lauricella function $F_A^{(n)}$ [10].

Clearly, we have

$$F_A^{(1)} = F, \ F_A^{(2)} = F_2,$$

where F and F_2 are the Gauss and Appell series defined by (1) and (3), respectively.

In 2013 for the Appell function $F_2(a, b, b; c, c; w, z)$ and confluent Appell functions, Humbert functions, new relations and transformation formulas were obtained. These relations included limit formulas, integral representations, differentiation and recurrence formulas. Summation formulas for F2, and Humbert functions were derived [11]. In this work, we will try to generalize the results to the Lauricella function $F_A^{(n)}$.

2. The Limit Formulas

The existing experience of the authors in the field of research methods for constructing fundamental solutions for second-order degenerate elliptic equations and solvability of boundary value problems for non-classical partial differential equations shows that hypergeometric Gaussian or Appel's functions arise in many cases [12,13]. The properties of these functions have been very well studied, so reducing hypergeometric functions of many variables to well-known functions is always a pressing problem. In this section, we represent the multidimensional Lauricella's function F_A^n (4) as the product of n generalized Gaussian functions converging to the limit.

$$\lim_{\epsilon\to 0} F_A^{(n)}\left(\frac{a}{\epsilon}, \mathbf{b};\mathbf{c};\epsilon\mathbf{x}\right) = \prod_{j=1}^{n} {}_1F_1\left(b_j; c_j; ax_j\right), \tag{5}$$

$$\begin{aligned}\lim_{\epsilon\to 0} F_A^{(n)}&\left(\frac{a}{\epsilon}, \frac{\mathbf{b}}{\epsilon}; \frac{c_1}{\epsilon}, ..., \frac{c_r}{\epsilon}, c_{r+1}, ..., c_n; \epsilon x_1, ..., \epsilon x_r, \epsilon^2 x_{r+1}, ..., \epsilon^2 x_n\right)\\ &= \prod_{i=1}^{r} e^{ab_i x_i / c_i} \prod_{j=r+1}^{n} {}_0F_1\left(-; c_j; ab_j x_j\right),\end{aligned} \tag{6}$$

$$\begin{aligned}\lim_{\epsilon\to 0} F_A^{(n)}&\left(\frac{a}{\epsilon}, b_1, ..., b_r, \frac{b_{r+1}}{\epsilon}, ..., \frac{b_n}{\epsilon}; \mathbf{c}; \epsilon x_1, ..., \epsilon x_r, \epsilon^2 x_{r+1}, ..., \epsilon^2 x_n\right)\\ &= \prod_{i=1}^{r} {}_1F_1(b_i; c_i; ax_i) \prod_{j=r+1}^{n} {}_0F_1\left(-; c_j; ab_j x_j\right),\end{aligned} \tag{7}$$

$$\lim_{\epsilon\to 0} F_A^{(n)}\left(\frac{a}{\epsilon}, \frac{\mathbf{b}}{\epsilon}; \mathbf{c}; \epsilon^2\mathbf{x}\right) = \prod_{j=1}^{n} {}_0F_1\left(-; c_j; ab_j x_j\right). \tag{8}$$

The obtained limit formulas reduce a function of many variables to the generalized Gaussian functions of one variable ${}_0F_1$ and ${}_1F_1$.

3. Some Decomposition Formulas Associated With the Lauricella Function $F_A^{(n)}$

For a given multiple hypergeometric function, it is useful to find a decomposition formula that would express the multivariable hypergeometric function in terms of products of several simpler hypergeometric functions of fewer variables.

Burchnall and Chaundy [6,14] systematically presented a number of expansion and decomposition formulas for some double hypergeometric functions in a series of simpler hypergeometric functions. For example, the Appell function F_2 defined by (3) has the expansion:

$$\begin{aligned}F_2(a, b_1, b_2; c_1, c_2; x, y) &= \sum_{i=0}^{\infty} \frac{(a)_i (b_1)_i (b_2)_i}{i!(c_1)_i (c_2)_i} \times\\ &\times x^i y^i F(a+i, b_1+i; c_1+i; x) F(a+i, b_2+i; c_2+i; y).\end{aligned} \tag{9}$$

Following the works [6,14] Hasanov and Srivastava [15] proved that for all $n \in \mathbb{N}\backslash\{1\}$ the following recurring formula is true [16]

$$\begin{aligned}F_A^{(n)}(a, \mathbf{b}; \mathbf{c}; \mathbf{x}) = &\sum_{|\mathbf{k}'|=0}^{\infty} \frac{(a)_{|\mathbf{k}'|}(b_1)_{|\mathbf{k}'|}(b_2)_{k_2}...(b_n)_{k_n}}{k_2!...k_n!(c_1)_{|\mathbf{k}'|}(c_2)_{k_2}...(c_n)_{k_n}} x_1^{|\mathbf{k}'|} x_2^{k_2}...x_n^{k_n} \times\\ &F\left(a+|\mathbf{k}'|, b_1+|\mathbf{k}'|; c_1+|\mathbf{k}'|; x_1\right) \times\\ &\quad F_A^{(n-1)}\left(a+|\mathbf{k}'|, b_2+k_2, ..., b_n+k_n; c_2+k_2, ..., c_n+k_n; x_2, ..., x_n\right),\end{aligned} \tag{10}$$

where $|\mathbf{k}'| := k_2 + ... + k_n$.

However, due to the recurrence of formula (10), additional difficulties may arise in the applications of this expansion. Further study of the properties of the Lauricella function $F_A^{(n)}$ showed that formula (10) can be reduced to a more convenient form.

Before proceeding to the presentation of the main result of this section, we introduce the notations

$$A(k, n) = \sum_{i=2}^{k+1} \sum_{j=i}^{n} m_{i,j}, \ \ B(k, n) = \sum_{i=2}^{k} m_{i,k} + \sum_{i=k+1}^{n} m_{k+1,i},$$

$$|\mathbf{m}_n| := \sum_{i=2}^{n}\sum_{j=i}^{n} m_{i,j}, \quad M_n! := \prod_{i=2}^{n}\prod_{j=i}^{n} m_{i,j}!$$

where k and n are natural numbers with $k \le n$; $m_{i,j}$ are nonnegative integers for $2 \le i \le j \le n$.

Let $a, b_1, \ldots, b_n$ are real numbers with $a \neq 0, -1, -2, \ldots$ and $a > |\mathbf{b}|$, where $|\mathbf{b}| := b_1 + \ldots + b_n$. Then the following decomposition formula holds true at $n \in \mathbb{N}$

$$\begin{aligned} F_A^{(n)}(a, \mathbf{b}; \mathbf{c}; \mathbf{x}) = \sum_{|\mathbf{m}_n|=0}^{\infty} \frac{(a)_{A(n,n)}}{M_n!} \prod_{k=1}^{n} \frac{(b_k)_{B(k,n)}}{(c_k)_{B(k,n)}} \times \\ \times \prod_{k=1}^{n} x_k^{B(k,n)} F(a + A(k,n), b_k + B(k,n); c_k + B(k,n); x_k). \end{aligned} \tag{11}$$

The main result obtained in this section is the expansion Formula (11). We carry out the proof of the equality (11) by the method of mathematical induction.

In the case $n = 1$ the equality (11) is obvious, that is, the Gaussian hypergeometric function (1) is easily derived.

Let $n = 2$. Since $A(1,2) = A(2,2) = B(1,2) = B(2,2) = m_{2,2} := i$, we obtain the expansion Formula (9). So the Formula (11) works for $n = 1$ and $n = 2$. Further, for details, see [17].

4. Integral Representations

The function $F_A^{(n)}$ can be initially represented by the following integral [9] p. 115, Equation (5) (see, also [18] p. 451, Equation 7.2.4 (54))

$$\begin{aligned} F_A^{(n)}(a, \mathbf{b};\mathbf{c};\mathbf{x}) = \prod_{j=1}^{n} \frac{\Gamma(c_j)}{\Gamma(b_j)\Gamma(c_j - b_j)} \times \\ \times \underbrace{\int_0^1 \ldots \int_0^1}_{n \text{ times}} \prod_{j=1}^{n} \left[t_j^{b_j - 1} (1 - t_j)^{c_j - b_j - 1} \right] (1 - t_1 x_1 - \ldots - t_n x_n)^{-a} dt_1 \ldots dt_n, \end{aligned} \tag{12}$$

where $\mathrm{Re}\,(c_j) > \mathrm{Re}\,(b_j) > 0\,(j = \overline{1,n})$. By integrating with respect to $t_{r+1}, \ldots, t_n$ and using the integral representation (12) for $F_A^{(r)}$, $(1 \le r \le n)$, a right side of (12) we can reduce the right side of (12) to:

$$\begin{aligned} F_A^{(n)}(a, \mathbf{b};\mathbf{c};\mathbf{x}) = \prod_{j=1}^{r} \frac{\Gamma(c_j)}{\Gamma(b_j)\Gamma(c_j - b_j)} \underbrace{\int_0^1 \ldots \int_0^1}_{r \text{ times}} \prod_{j=1}^{r} \left[t_j^{b_j - 1} (1 - t_j)^{c_j - b_j - 1} \right] (1 - T_r X_r)^{-a} \times \\ \times F_A^{(n-r)}\left(a, b_{r+1}, \ldots, b_n; c_{r+1}, \ldots, c_n; \frac{x_{r+1}}{1 - T_r X_r}, \ldots, \frac{x_n}{1 - T_r X_r}\right) dt_1 \ldots dt_r, \end{aligned} \tag{13}$$

where $\mathrm{Re}\,(c_j) > \mathrm{Re}\,(b_j) > 0\,(j = \overline{1,r})$, $T_r X_r = t_1 x_1 + \ldots + t_r x_r$.

If we put $r = 1$ in the Formula (13), then we obtain

$$\begin{aligned} F_A^{(n)}(a, \mathbf{b};\mathbf{c};\mathbf{x}) = \frac{\Gamma(c_n)}{\Gamma(b_n)\Gamma(c_n - b_n)} \int_0^1 t^{b_n - 1} (1 - t)^{c_n - b_n - 1} (1 - t x_n)^{-a} \times \\ \times F_A^{(n-1)}\left(a, \mathbf{b}'; \mathbf{c}'; \frac{\mathbf{x}'}{1 - t x_n}\right) dt, \end{aligned} \tag{14}$$

where $\mathrm{Re}\,(c_n) > \mathrm{Re}\,(b_n) > 0$. With a change of variables $t \to (t-1)/(t x_n)$ we obtain the following:

$$F_A^{(n)}(a,\mathbf{b};\mathbf{c};\mathbf{x}) = \frac{\Gamma(c_n)}{\Gamma(b_n)\Gamma(c_n-b_n)}\left(\frac{1}{x_n}\right)^{c_n-1}\times$$
$$\times\int_1^{1/(1-x_n)} t^{a-c_n}(t-1)^{b_n-1}(1-t+tx_n)^{c_n-b_n-1}F_A^{(n-1)}(a,\mathbf{b}';\mathbf{c}';t\mathbf{x}')dt, \quad (15)$$

where Re $(c_n) >$ Re $(b_n) > 0$. With a simple change of parameter $t \to 1 - v/t$ followed by change of variable $t \to t/v$, Equation (15) implies

$$F_A^{(n)}\left(a,\mathbf{b};\mathbf{c};v\mathbf{x}',1-\frac{v}{z}\right) = \frac{v^{c_n-b_n-a}z^{b_n}(z-v)^{1-c_n}}{\mathrm{B}(b_n,c_n-b_n)}\times$$
$$\times\int_v^z t^{a-c_n}(t-v)^{b_n-1}(z-t)^{c_n-b_n-1}F_A^{(n-1)}(a,\mathbf{b}';\mathbf{c}';t\mathbf{x}')dt, \quad (16)$$

where Re $(c_n) >$ Re $(b_n) > 0$, $|vx_1| + ... + |vx_{n-1}| + |1 - v/z| < 1$, and $\mathrm{B}(b_n, c_n - b_n)$ is the beta function, whence

$$\int_v^z t^{s-1}(t-v)^{u-1}(z-t)^{a-s-1}F_A^{(n-1)}(a,\mathbf{b}';\mathbf{c}';t\mathbf{x}')dt = \mathrm{B}(u,a-s-u+1)\times$$
$$\times v^{s+u-1}z^{-u}(z-v)^{a-s}F_A^{(n)}\left(a,\mathbf{b}',u;\mathbf{c}',a-s+1;v\mathbf{x}',1-\frac{v}{z}\right) \quad (17)$$

for Re $(a - s + 1) >$ Re $(u) > 0$, $|vx_1| + ... + |vx_{n-1}| + |1 - v/z| < 1$.

Another representation of the Lauricella function $F_A^{(n)}$ has the form

$$F_A^{(n)}(a,\mathbf{b};\mathbf{c};\mathbf{x}) = \frac{1}{\Gamma(a)}\int_0^\infty t^{a-1}e^{-t}\prod_{j=1}^n {}_1F_1(b_j;c_j;tx_j)dt \quad (18)$$

with Re $(a) > 0$ and $|x_1| + ... + |x_n| < 1$. From this formula, we obtain the equality

$$\int_0^\infty t^{s-1}e^{-pt}\prod_{j=1}^n {}_1F_1(b_j;c_j;tx_j)dt = \frac{\Gamma(s)}{p^s}F_A^{(n)}\left(s,\mathbf{b};\mathbf{c};\frac{\mathbf{x}}{p}\right), \quad (19)$$

that can be regarded as the Laplace transform

$$\mathrm{L}[f](p) = \int_0^\infty e^{-pt}f(t)dt, \quad (20)$$

here Re $(s) > 0$, Re $(p) > 0$,

$$\mathrm{M}[f](s) = \int_0^\infty t^{s-1}f(t)dt, \quad (21)$$

of the functions

$$t^{s-1}\prod_{j=1}^n {}_1F_1(b_j;c_j;tx_j) \quad \text{and} \quad e^{-pt}\prod_{j=1}^n {}_1F_1(b_j;c_j;tx_j), \quad (22)$$

accordingly. This Equation (19) can be used to derive new formulas of Laplace and Mellin transformations. For example, the Laplace transform of the product of n incomplete gamma functions is

$$\mathrm{L}\left[t^{s-1}\prod_{j=1}^n\gamma(\mu_j,a_jt)\right](p) = \frac{\Gamma(s+\mu)}{p^{s+\mu}}\prod_{j=1}^n\frac{a_j^{\mu_j}}{\mu_j}F_A^{(n)}\left[\begin{array}{l} s+\mu,\mu_1,...,\mu_n; \\ \mu_1+1,...,\mu_n+1; \end{array} -\frac{a_1}{p},...,-\frac{a_n}{p}\right], \quad (23)$$

$\mu := \mu_1 + ... + \mu_n$, Re $(s) > 0$. Re $(p) >$ Re $(a_1) + ... +$ Re (a_n). The Mellin transform of the product of the error functions is

$$\mathrm{M}\left[e^{-ct^2}\prod_{j=1}^{n} erf\left(a_j t\right)\right](s) =$$
$$= \frac{2^{n-1}\Gamma\left(\frac{s}{2}+n-1\right)\prod_{j=1}^{n} a_j}{\pi^{n/2} c^{\frac{s}{2}+n-1}} F_A^{(n)}\left[\begin{array}{l} \frac{s}{2}+n-1, \frac{1}{2}, ..., \frac{1}{2}; \\ \frac{3}{2}, ..., \frac{3}{2}; \end{array} -\frac{a_1^2}{c}, ..., -\frac{a_n^2}{c}\right], \quad (24)$$

where Re $(s) > 2 - 2n$. Re $(c) >$ Re $(a_1^2) + ... +$ Re (a_n^2). By multiplying Equation (4) by $t^{s-1}(w-t)^{b_n-s-1}$ and integrating over the interval $[0, w]$, we obtain

$$\int_0^w t^{s-1}(w-t)^{b_n-s-1} F_A^{(n)}\left(a, \mathbf{b}; \mathbf{c}; \mathbf{x}', t x_n\right) dt =$$
$$= w^{b_n-1}\mathrm{B}(s, b_n - s) F_A^{(n)}\left(a, \mathbf{b}', s; \mathbf{c}; \mathbf{x}', w x_n\right), \quad (25)$$

where $0 <$ Re $(s) <$ Re (b_n), $|x_1| + ... + |x_{n-1}| + |w x_n| < 1$. Similarly,

$$\int_0^w t^{c_n-1}(w-t)^{s-c_n-1} F_A^{(n)}\left(a, \mathbf{b}; \mathbf{c}; \mathbf{x}', t x_n\right) dt =$$
$$= w^{s-1}\mathrm{B}(c_n, s - c_n) F_A^{(n)}\left(a, \mathbf{b}; \mathbf{c}', c_n; \mathbf{x}', w x_n\right), \quad (26)$$

where $0 <$ Re $(c) <$ Re (s), $|x_1| + ... + |x_{n-1}| + |w x_n| < 1$,

$$\int_0^w t^{s-1}(w-t)^{a-s-1} F_A^{(n)}(a, \mathbf{b}; \mathbf{c}; t\mathbf{x}) dt = w^{a-1}\mathrm{B}(s, a-s) F_A^{(n)}(s, \mathbf{b}; \mathbf{c}; w\mathbf{x}), \quad (27)$$

where $0 <$ Re $(s) <$ Re (a), $|x_1| + ... + |x_n| < 1/w$.

The purpose of the study of this section is to obtain a new formula for the integral representation of the function $F_A^{(n)}$, thus equality (18) is formulated and by means of the Laplace and Mellin transforms Formulas (26) and (27) for it are obtained.

5. Differentiation Formulas

This section provides formulas for differentiation of the Lauricella's function $F_A^{(n)}$. Denoting $D_{x_j} f = df/dx_j$, we have

$$D_{x_j}^r\left[F_A^{(n)}(a, \mathbf{b}; \mathbf{c}; \mathbf{x})\right] = \frac{(a)_r (b_j)_r}{(c_j)_r} F_A^{(n)}\left(a, \mathbf{b} + r\mathbf{e}_j; \mathbf{c} + r\mathbf{e}_j; \mathbf{x}\right), \quad (28)$$

$$D_t^r\left[t^{a+r-1} F_A^{(n)}(a, \mathbf{b}; \mathbf{c}; t\mathbf{x})\right] = (a)_r t^{a-1} F_A^{(n)}(a + r, \mathbf{b}; \mathbf{c}; t\mathbf{x}), \quad (29)$$

$$D_{x_j}^r\left[x_j^{b_j+r-1} F_A^{(n)}(a, \mathbf{b}; \mathbf{c}; \mathbf{x})\right] = (b_j)_r x_j^{b_j-1} F_A^{(n)}\left(a, \mathbf{b} + r\mathbf{e}_j; \mathbf{c}; \mathbf{x}\right), \quad (30)$$

$$D_{x_j}^r\left[x_j^{c_j-1} F_A^{(n)}(a, \mathbf{b}; \mathbf{c}; \mathbf{x})\right] = (-1)^k (1 - c_j)_r x_j^{c_j-r-1} F_A^{(n)}\left(a, \mathbf{b}; \mathbf{c} - r\mathbf{e}_j; \mathbf{x}\right), \quad (31)$$

$$\left(x_j^2 D_{x_j}\right)^r\left[x_j^{b_j} F_A^{(n)}(a, \mathbf{b}; \mathbf{c}; \mathbf{x})\right] = (b_j)_r x_j^{b_j+r} F_A^{(n)}\left(a, \mathbf{b} + r\mathbf{e}_j; \mathbf{c}; \mathbf{x}\right), \quad (32)$$

$$\left(D_{x_j} x_j^2\right)^r\left[x_j^{b_j-2} F_A^{(n)}(a, \mathbf{b}; \mathbf{c}; \mathbf{x})\right] = (b_j)_r x_j^{b_j+r-2} F_A^{(n)}\left(a, \mathbf{b} + r\mathbf{e}_j; \mathbf{c}; \mathbf{x}\right), \quad (33)$$

$$\left(t^2 D_t\right)^r\left[t^a F_A^{(n)}(a, \mathbf{b}; \mathbf{c}; t\mathbf{x})\right] = (a)_r t^{a+r} F_A^{(n)}(a + r, \mathbf{b}; \mathbf{c}; t\mathbf{x}), \quad (34)$$

$$D_{x_j}^{r}\left[x_j^{b_j+r-1}(1+x_j)^{-b_j}F_A^{(n)}\left(a,\mathbf{b};\mathbf{c};x_1,\ldots,x_{j-1},\frac{x_j}{x_j+1},x_{j+1},\ldots,x_n\right)\right]$$
$$=\frac{(b_j)_r x_j^{b_j-1}}{(1+x_j)^{b_j+r}}F_A^{(n)}\left(a,\mathbf{b}+r\mathbf{e}_j;\mathbf{c};x_1,\ldots,x_{j-1},\frac{x_j}{x_j+1},x_{j+1},\ldots,x_n\right), \quad (35)$$

where $\mathbf{e}_j := (0,\ldots,1,\ldots,0)$ denote the vectors with j–th component equal to 1 and the others equal to 0.

$$\left(D_t t^2\right)^r\left[t^{a-2}F_A^{(n)}(a,\mathbf{b};\mathbf{c};t\mathbf{x})\right]=(a)_r t^{a+r-2}F_A^{(n)}(a+r,\mathbf{b};\mathbf{c};t\mathbf{x}). \quad (36)$$

These new differentiation Formulas (28)–(36) can be proved by using the definition (4) or the expansion Formula (11).

6. Finite Sums

The differentiation formulas derived in the preceding section can be applied to derive new finite sums that involve the Lauricella function $F_A^{(n)}$. Utilizing the generalized Leibnitz formula for differentiating a product of two functions, the differentiation formula (28) leads to the following relationship:

$$D_{x_j}^{r}\left[x_j^{1-c_j}x_j^{c_j-1}F_A^{(n)}(a,\mathbf{b};\mathbf{c};\mathbf{x})\right]=\sum_{k=0}^{r}\binom{r}{k}D_{x_j}^{r-k}\left[x_j^{1-c_j}\right]D_{x_j}^{k}\left[x_j^{c_j-1}F_A^{(n)}(a,\mathbf{b};\mathbf{c};\mathbf{x})\right]$$
$$=\sum_{k=0}^{r}\binom{r}{k}\frac{(c_j-1)_r(1-c_j)_k}{(2-c_j-r)_k}x_j^{-r}F_A^{(n)}(a,\mathbf{b};\mathbf{c}-k\mathbf{e}_j;\mathbf{x}),$$

whence

$$\sum_{k=0}^{r}(-1)^k\binom{r}{k}\frac{(1-c_j)_k}{(2-c_j-r)_k}F_A^{(n)}(a,\mathbf{b};\mathbf{c}-k\mathbf{e}_j;\mathbf{x})$$
$$=(-1)^r\frac{(a)_r(b_j)_r}{(c_j)_r(c_j-1)_r}x_j^rF_A^{(n)}(a+r,\mathbf{b}+r\mathbf{e}_j;\mathbf{c}+r\mathbf{e}_j;\mathbf{x}), \quad (37)$$

$$\sum_{k=0}^{r}(-1)^k\binom{r}{k}\frac{(c_j+r-1)_k}{(c_j)_k}F_A^{(n)}(a,\mathbf{b};\mathbf{c}+k\mathbf{e}_j;\mathbf{x})$$
$$=(-1)^r\frac{(a)_r(b_j)_r}{(c_j)_{2r}}x_j^rF_A^{(n)}(a+r,\mathbf{b}+r\mathbf{e}_j;\mathbf{c}+2r\mathbf{e}_j;\mathbf{x}), \quad (38)$$

$$\sum_{k=0}^{r}(-1)^k\binom{r}{k}F_A^{(n)}(a,\mathbf{b}',-k;\mathbf{c};\mathbf{x})=\frac{(a)_r}{(c_n)_r}x_n^rF_A^{(n-1)}(a+r,\mathbf{b}';\mathbf{c}';\mathbf{x}'), \quad (39)$$

$$\sum_{k=0}^{r}F_A^{(n)}(\mathbf{b}',-k;\mathbf{c};\mathbf{x})$$
$$=\frac{c_n-1}{(a-1)x_n}\left[F_A^{(n-1)}(a-1,\mathbf{b}';\mathbf{c}';\mathbf{x}')-F_A^{(n)}(a-1,\mathbf{b}',-r-1;\mathbf{c}',c_n-1;\mathbf{x})\right]. \quad (40)$$

7. Infinite Sums

Various infinite series with $F_A^{(n)}$ can be obtained using the definition (4) and the expansion Formula (11).

$$\begin{aligned}\sum_{k=0}^{\infty}\frac{(a)_k}{k!}t^k F_A^{(n)}(a+k,\mathbf{b};\mathbf{c};\mathbf{x}) &= \sum_{k=0}^{\infty}\frac{(a)_k}{k!}t^k\sum_{|\mathbf{k}|=0}^{\infty}\frac{(a+k)_{|\mathbf{k}|}(b_1)_{k_1}\dots(b_n)_{k_n}}{(c_1)_{k_1}\dots(c_n)_{k_n}}\frac{x_1^{k_1}}{k_1!}\dots\frac{x_n^{k_n}}{k_n!}\\ &= \sum_{|\mathbf{k}|=0}^{\infty}\sum_{k=0}^{\infty}\frac{(a)_k}{k!}t^k\frac{(a+k)_{|\mathbf{k}|}(b_1)_{k_1}\dots(b_n)_{k_n}}{(c_1)_{k_1}\dots(c_n)_{k_n}}\frac{x_1^{k_1}}{k_1!}\dots\frac{x_n^{k_n}}{k_n!}\\ &= \sum_{|\mathbf{k}|=0}^{\infty}(1-t)^{-a-|\mathbf{k}|}\frac{(a)_{|\mathbf{k}|}(b_1)_{k_1}\dots(b_n)_{k_n}}{(c_1)_{k_1}\dots(c_n)_{k_n}}\frac{x_1^{k_1}}{k_1!}\dots\frac{x_n^{k_n}}{k_n!}\\ &= (1-t)^{-a}F_A^{(n)}\left(a,\mathbf{b};\mathbf{c};\frac{\mathbf{x}}{1-t}\right),\end{aligned}$$

whence

$$\sum_{k=0}^{\infty}\frac{(a)_k}{k!}t^k F_A^{(n)}(a+k,\mathbf{b};\mathbf{c};\mathbf{x}) = (1-t)^{-a}F_A^{(n)}\left(a,\mathbf{b};\mathbf{c};\frac{\mathbf{x}}{1-t}\right), \tag{41}$$

where $|t|<1$, $\sum_{s=1}^{n}|x_s|<|1-t|$. Similarly,

$$\sum_{k=0}^{\infty}\frac{(b_j)_k}{k!}t^k F_A^{(n)}(a,\mathbf{b}+k\mathbf{e}_j;\mathbf{c};\mathbf{x}) = (1-t)^{-b_j}F_A^{(n)}\left(a,\mathbf{b};\mathbf{c};x_1,\dots,x_{j-1},\frac{x_j}{1-t},x_{j+1},\dots,x_n\right), \tag{42}$$

where $|t|<1$, $\sum_{s=1,s\neq j}^{n}|x_s|+\dfrac{x_j}{1-t}<1$,

$$\sum_{k=0}^{\infty}\frac{(a)_k(b_j)_k}{k!(c_j)_k}t^k F_A^{(n)}(a+k,\mathbf{b}+k\mathbf{e}_j;\mathbf{c}+k\mathbf{e}_j;\mathbf{x}) = F_A^{(n)}(a,\mathbf{b};\mathbf{c};x_1,\dots,x_{j-1},x_j+t,x_{j+1},\dots,x_n), \tag{43}$$

where $|t|<1$, $\sum_{s=1,s\neq j}^{n}|x_s|+|t+x_j|<1$,

$$\begin{aligned}&\sum_{k=0}^{\infty}\frac{(a)_k(b_j)_k}{(2k)!}(-x_j)^k F_A^{(n)}(a+k,\mathbf{b};c_1,\dots,c_{j-1},2k+1,c_{j+1},\dots,c_n;\mathbf{x})\\ &\qquad = F_A^{(n-1)}(a,\mathbf{b}_j;\mathbf{c}_j;\mathbf{x}_j)+F_A^{(n)}(a,\mathbf{b};c_1,\dots,c_{j-1},1,c_{j+1},\dots,c_n;\mathbf{x}),\end{aligned} \tag{44}$$

where $\sum_{j=1}^{n}|x_j|<1$, $\mathbf{x}_j := (x_1,\dots,x_{j-1},x_{j+1},\dots,x_n)$,

$$\begin{aligned}&\sum_{k=0}^{\infty}\frac{(a)_k(b_j)_k}{(2k+1)!}x_j^k F_A^{(n)}(a+k,b_1,\dots,b_{j-1},b_j+k,b_{j+1},\dots,b_n;c_1,\dots,c_{j-1},2k+3,c_{j+1},\dots,c_n;\mathbf{x})\\ &\qquad = F_A^{(n)}(a,\mathbf{b};c_1,\dots,c_{j-1},2,c_{j+1},\dots,c_n;\mathbf{x}),\end{aligned} \tag{45}$$

where $\sum_{j=1}^{n}|x_j|<1$.

8. Recurrence-Type Relations

Corresponding relations for the Gauss hypergeometric function [4] and the expansion Formula (11) can be used to obtain the following recurrence formulas:

$$F_A^{(n)}(a,\mathbf{b};\mathbf{c};\mathbf{x}) = F_A^{(n)}(a,\mathbf{b};\mathbf{c}-\mathbf{e}_j;\mathbf{x}) - \frac{ab_jx_j}{c_j(c_j-1)}F_A^{(n)}(a+1,\mathbf{b}+\mathbf{e}_j;\mathbf{c}+\mathbf{e}_j;\mathbf{x}), \tag{46}$$

$$F_A^{(n)}(a,\mathbf{b};\mathbf{c};\mathbf{x}) = \frac{b_j}{b_j - c_j + 1} F_A^{(n)}(a,\mathbf{b}+\mathbf{e}_j;\mathbf{c};\mathbf{x}) - \frac{c_j - 1}{b_j - c_j + 1} F_A^{(n)}(a,\mathbf{b};\mathbf{c}-\mathbf{e}_j;\mathbf{x}), \tag{47}$$

$$F_A^{(n)}(a,\mathbf{b};\mathbf{c};\mathbf{x}) = \frac{b_j}{c_j} F_A^{(n)}(a,\mathbf{b}+\mathbf{e}_j;\mathbf{c}+\mathbf{e}_j;\mathbf{x}) - \frac{b_j - c_j}{c_j} F_A^{(n)}(a,\mathbf{b};\mathbf{c}+\mathbf{e}_j;\mathbf{x}), \tag{48}$$

$$\begin{aligned} F_A^{(n)}(a,\mathbf{b};\mathbf{c};\mathbf{x}) &= F_A^{(n)}(a,\mathbf{b};\mathbf{c}+\mathbf{e}_j;\mathbf{x}) \\ &+ \frac{a x_j}{c_j} F_A^{(n)}(a+1,\mathbf{b};\mathbf{c}+\mathbf{e}_j;\mathbf{x}) + \frac{a(b_j - c_j - 1)x_j}{c_j(c_j+1)} F_A^{(n)}(a+1,\mathbf{b};\mathbf{c}+2\mathbf{e}_j;\mathbf{x}), \end{aligned} \tag{49}$$

$$F_A^{(n)}(a+r,\mathbf{b};\mathbf{c};\mathbf{x}) = F_A^{(n)}(a,\mathbf{b};\mathbf{c};\mathbf{x}) + \sum_{k=0}^{r-1}\sum_{j=1}^{n} \frac{b_j x_j}{c_j} F_A^{(n)}(a+1+k,\mathbf{b}+\mathbf{e}_j;\mathbf{c}+\mathbf{e}_j;\mathbf{x}). \tag{50}$$

The resulting formulas differ in coefficients and parameters and can be applied in many different cases. Recurrence relations allow us to obtain analytic continuation formulas for hypergeometric functions.

9. Conclusions

The paper focuses on generalizing the results of a function F_2 and applying it to one of Lauricella's functions $F_A^{(n)}$. By using the definition of the function $F_A^{(n)}$, the limit formulas are derived, which are expressed by the generalized Gaussian function.

The Burchnall–Chaundy and Shrivastava–Hasanov operators are used to obtain an expansion formula that decomposes the function $F_A^{(n)}$ as the sum of products of a one-dimensional hypergeometric function. The expansion formula is then used to prove several differentiation formulas. Additionally, an integral representation of Lauricella's function $F_A^{(n)}$ is derived, leading to new formulas for the Mellin and Laplace transformations. The paper also introduces finite sum formulas using the generalized Leibniz function and the differentiation formula. Infinite sum formulas are determined by applying the expansion formula.

Furthermore, the paper establishes various recurrence relations for the multidimensional function $F_A^{(n)}$. The methods presented in this work can potentially be applied to obtain similar results for other hypergeometric series in future research.

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