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Convergence Analysis for Yosida Variational Inclusion Problem with Its Corresponding Yosida Resolvent Equation Problem through Inertial Extrapolation Scheme

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Abstract: In this paper, we study a Yosida variational inclusion problem with its corresponding Yosida resolvent equation problem. We mention some schemes to solve both the problems, but we focus our study on discussing convergence criteria for the Yosida variational inclusion problem in real Banach space and for the Yosida resolvent equation problem in q-uniformly smooth Banach space. For faster convergence, we apply an inertial extrapolation scheme for both the problems. An example is provided.

Keywords: Yosida; inertial; extrapolation; resolvent; inclusion

MSC: 65J15; 47J25; 65K15

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Citation: Rajpoot, A.K.; Ishtyak, M.; Ahmad, R.; Wang, Y.; Yao, J.C. Convergence Analysis for Yosida Variational Inclusion Problem with Its Corresponding Yosida Resolvent Equation Problem through Inertial Extrapolation Scheme. *Mathematics* **2023**, *11*, 763. https://doi.org/10.3390/math11030763

Academic Editors: Alicia Cordero and Ioannis K. Argyros

Received: 12 December 2022
Revised: 27 January 2023
Accepted: 30 January 2023
Published: 2 February 2023

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1. Introduction

Variational inclusions are the generalized forms of variational inequalties introduced by Hassouni and Moudafi [1]. Many interrelated and unrelated problems of basic and applied sciences can be easily studied via variational inclusions, such as the problems arising in elasticity, structural analysis, oceanography, image processing, physics and engineering sciences, etc., see for example [2–8].

The concept of resolvent equations was introduced by Noor [9,10]. Resolvent equations are generalized forms of Wiener–Hopf equations. The equivalence between the variational inclusions and resolvent equations was shown by many authors. The resolvent operator technique is useful to solve variational inclusion problems, as projection methods fail to solve them. Various generalized resolvent operators involving different monotone operators are available in the literature.

It is a fact that maximal monotone operators are fundamental objects in modern optimization. In addition, set-valued monotone operators can be regularized into a single-valued monotone operator by a process called Yosida approximation. Applications of Yosida approximation operator can be found in a heat equation that describes the distribution of heat over time in a fixed region of space, an initial value problem for the linearized equations of coupled sound and heat flow, and a wave equation in the form of second-order partial differential equation used for the description of waves, see for example [11–15].

Many iterative algorithms were developed using generalized resolvent operators, but it is always beneficial to use an algorithm which accelerates the fast convergence for the sequence generated by the algorithm. Inertial extrapolation schemes are developed by using the inertial extrapolation term $\{\nu(x_n - x_{n-1})\}$ by several authors, where ν is an extrapolating factor that speeds up the convergence rate of the method. Polyak [16] first introduced the inertial-type iterative algorithm to deal with the heavy ball method.

The inertial-type iterative algorithm has two steps in which the consecutive iterations are gained by using the former two terms, see for example [17–20].

In view of the above important discussion, in this paper, we consider a Yosida variational inclusion problem with its corresponding Yosida resolvent equation problem. We mention some schemes for solving Yosida variational inclusion as well as the Yosida resolvent equation problem. We concentrate our study on convergence analysis of both the problems through inertial extrapolation schemes. An example is provided through MATLAB 2015a with a computation table and a convergence graph.

2. Fundamental Tools and Concepts

Suppose that $\widetilde{E}$ is a real Banach space and $\widetilde{E}^*$ is its topological dual equipped with norm $\|\cdot\|$ and duality pairing $\langle\cdot,\cdot\rangle$ between $\widetilde{E}$ and $\widetilde{E}^*$. By $2^{\widetilde{E}}$, we denote the set of all non-empty subsets of $\widetilde{E}$.

For $q > 1$, the generalized duality mapping $J_q : \widetilde{E} \to \widetilde{E}^*$ is defined by

$$J_q(e) = \left\{ f \in \widetilde{E}^* : \langle e, f\rangle = \|e\|^q \text{ and } \|f\| = \|e\|^{q-1} \right\}, \text{ for all } e \in \widetilde{E}.$$

For $q = 2$, J_q becomes normalized duality mapping. Particularly, $J := J_2$ is called normalized duality mapping on $\widetilde{E}$. It is well known that $J_q(e) = \|e\|^{q-2} J_2(e)$ for $e \neq 0$ and $J_q(e)$ is the subdifferential of functional $(1/q)\|\cdot\|^q$ at e, see [21]. The mapping J_q is single-valued if $\widetilde{E}$ is uniformly smooth.

The definition of uniformly smooth Banach space, modulus of smoothness and the following important Lemma can be found in Xu [21].

Lemma 1. *A real uniformly smooth Banach space $\widetilde{E}$ is q-uniformly smooth if and only if there exists a constant $C_q > 0$ such that for all $e, f \in \widetilde{E}$, the following inequality holds :*

$$\|e + f\|^q \leq \|e\|^q + q\langle f, J_q(e)\rangle + C_q\|f\|^q.$$

Before providing essential definitions for the presentation of this paper and for the convenience of readers, we mention the following well-known definitions. For this purpose, we take $\widetilde{E} = H$, a real Hilbert space.

Definition 1. *A single-valued mapping $\widetilde{A} : H \to H$ is called*

(i) *Monotone if*

$$\langle \widetilde{A}(e) - \widetilde{A}(f), e - f\rangle \geq 0, \text{ for all } e, f \in H;$$

(ii) *Strongly monotone if there exists a constant $\delta_{\widetilde{A}} > 0$ such that*

$$\langle \widetilde{A}(e) - \widetilde{A}(f), e - f\rangle \geq \delta_{\widetilde{A}}\|e - f\|^2, \text{ for all } e, f \in H.$$

Definition 2. *A set-valued mapping $M : H \to 2^H$ is called monotone, if*

$$\langle u - v, e - f\rangle \geq 0, \text{ for all } e, f \in H, u \in M(e), v \in M(f).$$

Definition 3. *Let $\widetilde{A} : H \to H$ be a mapping. A set-valued mapping $M : H \to 2^H$ is called $\widetilde{A}$-monotone if M is monotone and*

$$[\widetilde{A} + \xi M](H) = H, \ \xi > 0 \text{ is a constant.}$$

The generalizations of above Definitions 1–3 in q-uniformly smooth Banach space are as follows, which are needed for the presentation of this paper.

Definition 4. *A single-valued mapping $\widetilde{A} : \widetilde{E} \to \widetilde{E}$ is called*

(i) *Accretive if*

$$\langle \widetilde{A}(e) - \widetilde{A}(f), J_q(e-f)\rangle \geq 0, \text{ for all } e, f \in \widetilde{E};$$

(ii) *Strongly accretive if there exists a constant* $\delta_{\widetilde{A}} > 0$ *such that*

$$\langle \widetilde{A}(e) - \widetilde{A}(f), J_q(e-f)\rangle \geq \delta_{\widetilde{A}}\|e-f\|^q, \text{ for all } e, f \in \widetilde{E};$$

(iii) *Lipschitz continuous if there exists a constant* $\lambda_{\widetilde{A}} > 0$ *such that*

$$\|\widetilde{A}(e) - \widetilde{A}(f)\| \leq \lambda_{\widetilde{A}}\|e-f\|, \text{ for all } e, f \in \widetilde{E}.$$

Definition 5. *A set-valued mapping* $M : \widetilde{E} \to 2^{\widetilde{E}}$ *is called accretive if*

$$\langle u - v, J_q(e-f)\rangle \geq 0, \text{ for all } e, f \in \widetilde{E}, u \in M(e), v \in M(f).$$

Definition 6. *Let* $\widetilde{A} : \widetilde{E} \to \widetilde{E}$ *be a mapping. The set-valued mapping* $M : \widetilde{E} \to 2^{\widetilde{E}}$ *is called* $\widetilde{A}$*-accretive if* M *is accretive and*

$$[\widetilde{A} + \xi M](\widetilde{E}) = \widetilde{E}, \ \xi > 0 \text{ is a constant.}$$

It is well known that all the splitting methods are based on the resolvent operator of the form $[I + \xi M]^{-1}$, where M is a set-valued monotone mapping, ξ is a positive constant and I is the identity mapping.

Definition 7. *The resolvent operator* $R^M_{I,\xi} : H \to H$*, where* H *is a Hilbert space, is defined as*

$$R^M_{I,\xi}(e) = [I + \xi M]^{-1}(e), \text{ for all } e \in H.$$

Definition 8. *The Yosida approximation operator* $Y^M_{I,\xi} : H \to H$*, where* H *is a Hilbert space, is defined as*

$$Y^M_{I,\xi}(e) = \frac{1}{\xi}[I - R^M_{I,\xi}](e), \text{ for all } e \in H.$$

The generalized forms of Definitions 7 and 8 are mentioned below.

Definition 9. *Let* $\widetilde{A} : \widetilde{E} \to \widetilde{E}$ *be a mapping and* $M : \widetilde{E} \to 2^{\widetilde{E}}$ *be a set-valued mapping. The generalized resolvent operator* $R^M_{\widetilde{A},\xi} : \widetilde{E} \to \widetilde{E}$ *associated with* $\widetilde{A}$ *and* M *is defined as:*

$$R^M_{\widetilde{A},\xi}(e) = [\widetilde{A} + \xi M]^{-1}(e), \text{ for all } e \in \widetilde{E} \text{ and } \xi > 0.$$

Definition 10. *The generalized Yosida approximation operator* $Y^M_{\widetilde{A},\xi} : \widetilde{E} \to \widetilde{E}$ *is defined as :*

$$Y^M_{\widetilde{A},\xi}(e) = \frac{1}{\xi}[\widetilde{A} - R^M_{\widetilde{A},\xi}](e), \text{ for all } e \in \widetilde{E} \text{ and } \xi > 0.$$

Note that if $A \equiv I$, the identity mapping, then from Definitions 9 and 10, one can obtain Definitions 7 and 8, respectively.

Proposition 1. [22] *Let $\widetilde{A} : \widetilde{E} \to \widetilde{E}$ be strongly accretive mapping with constant r and $M : \widetilde{E} \to 2^{\widetilde{E}}$ be an $\widetilde{A}$-accretive set-valued mapping. Then, the generalized resolvent operator $R^M_{\widetilde{A},\xi} : \widetilde{E} \to \widetilde{E}$ is Lipschitz continuous with constant $\frac{1}{r}$, that is*

$$\|R^M_{\widetilde{A},\xi}(e) - R^M_{\widetilde{A},\xi}(f)\| \leq \frac{1}{r}\|e - f\|, \text{ for all } e, f \in \widetilde{E}.$$

Lemma 2. [23] *Let $\{S_n\}$ be a sequence of non-negative real numbers such that*

$$S_{n+1} \leq (1 - \beta_n)S_n + \beta_n\sigma_n + \widehat{\xi}_n, \text{ for all } n \geq 1,$$

where

(i) $\{\beta_n\} \subset [0, 1]$, $\sum\limits_{n=1}^{\infty} \beta_n = \infty$;

(ii) $\limsup \sigma_n \leq 0$;

(iii) $\widehat{\xi}_n \geq 0 \ (n \geq 1)$, $\sum\limits_{n=1}^{\infty} \widehat{\xi}_n < \infty$.

Then, $S_n \to 0$, as $n \to \infty$.

Proposition 2. *(i) If $\widetilde{A} : \widetilde{E} \to \widetilde{E}$ is r-strongly accretive, $\beta_{\widetilde{A}}$-expansive, $\lambda_{\widetilde{A}}$-Lipschitz continuous and generalized resolvent operator $R^M_{\widetilde{A},\xi} : \widetilde{E} \to \widetilde{E}$ is $\frac{1}{r}$-Lipschitz continuous, then the generalized Yosida approximation operator $Y^M_{\widetilde{A},\xi} : \widetilde{E} \to \widetilde{E}$ is θ_Y-strongly accretive with respect to $\widetilde{A}$, that is*

$$\langle Y^M_{\widetilde{A},\xi}(e) - Y^M_{\widetilde{A},\xi}(f), J_q\left(\widetilde{A}(e) - \widetilde{A}(f)\right)\rangle \geq \theta_Y\|e - f\|^q,$$

where $\theta_Y = \dfrac{\beta^q_{\widetilde{A}}\, r - \lambda^{q-1}_{\widetilde{A}}}{\xi r}, \xi r \neq 0, \beta^q_{\widetilde{A}}\, r > \lambda^{q-1}_{\widetilde{A}}$, and all the constants are positive.

(ii) If $\widetilde{A}$ is $\lambda_{\widetilde{A}}$-Lipschitz continuous, r-strongly accretive and $R^M_{\widetilde{A},\xi}$ is $\frac{1}{r}$-Lipschitz continuous, then the generalized Yosida approximation operator is λ_Y-Lipschitz continuous, that is

$$\left\|Y^M_{\widetilde{A},\xi}(e) - Y^M_{\widetilde{A},\xi}(f)\right\| \leq \lambda_Y\|e - f\|,$$

where $\lambda_Y = \dfrac{\lambda_{\widetilde{A}}\, r + 1}{\xi r}$.

Proof. (i) Using the definition of generalized duality mapping, expansiveness and Lipschitz continuity of $\widetilde{A}$ and Lipschitz continuity of $R^M_{\widetilde{A},\xi}$, we evaluate

$$
\begin{aligned}
&\left\langle Y^{M}_{\widetilde{A},\xi}(e) - Y^{M}_{\widetilde{A},\xi}(f),\ J_q\left(\widetilde{A}(e) - \widetilde{A}(f)\right)\right\rangle \\
&\quad = \frac{1}{\xi}\left\langle \widetilde{A}(e) - R^{M}_{\widetilde{A},\xi}(e) - \left[\widetilde{A}(f) - R^{M}_{\widetilde{A},\xi}(f)\right], J_q\left(\widetilde{A}(e) - \widetilde{A}(f)\right)\right\rangle \\
&\quad = \frac{1}{\xi}\left\langle \widetilde{A}(e) - \widetilde{A}(f), J_q\left(\widetilde{A}(e) - \widetilde{A}(f)\right)\right\rangle \\
&\qquad - \frac{1}{\xi}\left\langle R^{M}_{\widetilde{A},\xi}((e) - R^{M}_{\widetilde{A},\xi}(f), J_q\left(\widetilde{A}(e) - \widetilde{A}(f)\right)\right\rangle \\
&\quad \geq \frac{1}{\xi}\|\widetilde{A}(e) - \widetilde{A}(f)\|^q - \frac{1}{\xi}\|R^{M}_{\widetilde{A},\xi}(e) - R^{M}_{\widetilde{A},\xi}(f)\|\|\widetilde{A}(e) - \widetilde{A}(f)\|^{q-1} \\
&\quad \geq \frac{1}{\xi}\beta^{q}_{\widetilde{A}}\|e-f\|^q - \frac{1}{\xi}\frac{1}{r}\|e-f\|\|\widetilde{A}(e) - \widetilde{A}(f)\|^{q-1} \\
&\quad \geq \frac{1}{\xi}\beta^{q}_{\widetilde{A}}\|e-f\|^q - \frac{1}{\xi r}\lambda^{q-1}_{\widetilde{A}}\|e-f\|\|e-f\|^{q-1} \\
&\quad = \left[\frac{1}{\xi}\beta^{q}_{\widetilde{A}} - \frac{\lambda^{q-1}_{\widetilde{A}}}{\xi r}\right]\|e-f\|^q \\
&\quad = \theta_Y\|e-f\|^q.
\end{aligned}
$$

That is,

$$
\left\langle Y^{M}_{\widetilde{A},\xi}(e) - Y^{M}_{\widetilde{A},\xi}(f), J_q\left(\widetilde{A}(e) - \widetilde{A}(f)\right)\right\rangle \geq \theta_Y\|e-f\|^q.
$$

Thus, the generalized Yosida approximation operator $Y^{M}_{\widetilde{A},\xi}$ is θ_Y-strongly accretive with respect to $\widetilde{A}$.

(ii) Using Lipschitz continuity of $\widetilde{A}$ and $R^{M}_{\widetilde{A},\xi}$, we evaluate

$$
\begin{aligned}
\|Y^{M}_{\widetilde{A},\xi}(e) - Y^{M}_{\widetilde{A},\xi}(f)\| &= \|\frac{1}{\xi}\left(\widetilde{A}(e) - R^{M}_{\widetilde{A},\xi}(e)\right) - \frac{1}{\xi}\left(\widetilde{A}(f) - R^{M}_{\widetilde{A},\xi}(f)\right)\| \\
&= \frac{1}{\xi}\|\widetilde{A}(e) - \widetilde{A}(f) - \left[R^{M}_{\widetilde{A},\xi}(e) - R^{M}_{\widetilde{A},\xi}(f)\right]\| \\
&\leq \frac{1}{\xi}\|\widetilde{A}(e) - \widetilde{A}(f)\| + \frac{1}{\xi}\|R^{M}_{\widetilde{A},\xi}(e) - R^{M}_{\widetilde{A},\xi}(f)\| \\
&\leq \frac{1}{\xi}\lambda_{\widetilde{A}}\|e-f\| + \frac{1}{\xi}\frac{1}{r}\|e-f\| \\
&= \left[\frac{\lambda_{\widetilde{A}}}{\xi} + \frac{1}{\xi r}\right]\|e-f\| \\
&= \lambda_Y\|e-f\|.
\end{aligned}
$$

That is,

$$
\|Y^{M}_{\widetilde{A},\xi}(e) - Y^{M}_{\widetilde{A},\xi}(f)\| \leq \lambda_Y\|e-f\|.
$$

Thus, the genearlized Yosida approximation operator $Y^{M}_{\widetilde{A},\xi}$ is λ_Y-Lipschitz continuous. □

3. Yosida Variational Inclusion Problem and Yosida Resolvent Equation Problem

Let $\widetilde{A} : \widetilde{E} \to \widetilde{E}$ be single-valued mapping and $M : \widetilde{E} \to 2^{\widetilde{E}}$ be set-valued mapping. Let $Y^{M}_{\widetilde{A},\xi}$ be a generalized Yosida approximation operator. We consider the following Yosida variational inclusion problem.

Find $e \in \widetilde{E}$ such that

$$0 \in Y^{M}_{\widetilde{A},\xi}(e) + M(e). \tag{1}$$

If $Y^{M}_{\widetilde{A},\xi}(e) = 0$, then problem (1) reduces to the problem of finding $e \in \widetilde{E}$ such that

$$0 \in M(e),$$

which is a fundamental problem of analysis that has been considered by Rockafellar [24].

Lemma 3. *The Yosida variational inclusion problem (1) has a solution $e \in \widetilde{E}$ if and only if the following equation is satisfied :*

$$e = R^{M}_{\widetilde{A},\xi}\left[\widetilde{A}(e) - \xi Y^{M}_{\widetilde{A},\xi}(e)\right]. \tag{2}$$

Proof. One can prove it easily by using the definition of generalized resolvent operator $R^{M}_{\widetilde{A},\xi}$. □

Based on Lemma 3, we mention an iterative scheme as well as an inertial extrapolation scheme for solving Yosida variational inclusion problem (1).

Iterative Scheme 1. *For any $e_0 \in \widetilde{E}$, compute the sequence $\{e_n\}$ by the following scheme:*

$$e_{n+1} = R^{M}_{\widetilde{A},\xi}\left[\widetilde{A}(e_n) - \xi Y^{M}_{\widetilde{A},\xi}(e_n)\right], \text{ where } n = 0, 1, 2, \cdots \tag{3}$$

and $\xi > 0$ is a constant.

Equation (2) can also be written as

$$e = R^{M}_{\widetilde{A},\xi}\left[\frac{\widetilde{A}(e) + \widetilde{A}(e)}{2} - \xi Y^{M}_{\widetilde{A},\xi}(e)\right]. \tag{4}$$

Based on (4), we suggest the following iterative scheme.

Iterative Scheme 2. *For any $e_0 \in \widetilde{E}$, compute e_{n+1} by the recurrance relation,*

$$e_{n+1} = (1-\alpha_n)e_n + \alpha_n R^{M}_{\widetilde{A},\xi}\left[\frac{\widetilde{A}(e_n) + \widetilde{A}(e_{n+1})}{2} - \xi Y^{M}_{\widetilde{A},\xi}(e_{n+1})\right], \text{where } n = 0, 1, 2, \cdots, \tag{5}$$

$\alpha_n \in [0,1]$ and $\xi > 0$ is a constant.

We mention the following inertial extrapolation scheme using the predictor–corrector approach.

Iterative Scheme 3. *For any $e_0 \in \widetilde{E}$, compute e_{n+1} by the recurrance relation,*

$$w_n = e_n + \nu_n(e_n - e_{n-1}), \tag{6}$$

$$e_{n+1} = (1-\alpha_n)e_n + \alpha_n R^M_{\widetilde{A},\xi}\left[\frac{\widetilde{A}(e_n) + \widetilde{A}(w_n)}{2} - \xi Y^M_{\widetilde{A},\xi}(w_n)\right], \tag{7}$$

where $\alpha_n, \nu_n \in [0,1]$, ν_n is the extrapolating term for all $n \geq 1$ and $\xi > 0$ is a constant.

Note that one can use the above-mentioned schemes 1 and 2 to obtain the existence and convergence result for Yosida variational inclusion problem (1). Using inertial extrapolation scheme 3, we prove a convergence result for Yosida variational inclusion problem (1) in the sequel.

In connection with Yosida variational inclusion problem (1), we state the following Yosida resolvent equation problem.

Find $e, \widehat{z} \in \widetilde{E}$, such that

$$Y^M_{\widetilde{A},\xi}(e) + \xi^{-1} T^M_{\widetilde{A},\xi}(\widehat{z}) = 0, \tag{8}$$

where $T^M_{\widetilde{A},\xi}(\widehat{z}) = \left[I - \widetilde{A}\left(R^M_{\widetilde{A},\xi}\right)\right](\widehat{z})$ and $\widetilde{A}\left[R^M_{\widetilde{A},\xi}(\widehat{z})\right] = \left[\widetilde{A}\left(R^M_{\widetilde{A},\xi}\right)\right](\widehat{z})$.

The following Lemma ensures that Yosida variational inclusion problem (1) is equivalent to Yosida resolvent equation problem (8).

Lemma 4. *The Yosida variational inclusion problem (1) has a solution $e \in \widetilde{E}$ if and only if Yosida resolvent equation problem (8) has a solution $e, \widehat{z} \in \widetilde{E}$, provided $\widetilde{A}$ is one-one and*

$$e = R^M_{\widetilde{A},\xi}(\widehat{z}), \tag{9}$$

$$\widehat{z} = \widetilde{A}(e) - \xi Y^M_{\widetilde{A},\xi}(e), \tag{10}$$

where $\xi > 0$ is a constant.

Proof. Let $e \in \widetilde{E}$ be a solution of Yosida variational inclusion problem (1). Then, by Lemma 3, it satisfies the equation:

$$e = R^M_{\widetilde{A},\xi}\left[\widetilde{A}(e) - \xi Y^M_{\widetilde{A},\xi}(e)\right].$$

$$e = R^M_{\widetilde{A},\xi}(\widehat{z}), \text{ since } \widehat{z} = \widetilde{A}(e) - \xi Y^M_{\widetilde{A},\xi}(e).$$

Using (9), (10) becomes

$$\begin{aligned}
\widehat{z} = &\widetilde{A}\left(R^M_{\widetilde{A},\xi}(\widehat{z})\right) - \xi Y^M_{\widetilde{A},\xi}(e), \\
\widehat{z} - &\widetilde{A}\left(R^M_{\widetilde{A},\xi}(\widehat{z})\right) = -\xi Y^M_{\widetilde{A},\xi}(e), \\
\widehat{z} - &\left[\widetilde{A}\left(R^M_{\widetilde{A},\xi}\right)\right](\widehat{z}) = -\xi Y^M_{\widetilde{A},\xi}(e), \\
\Big[I - &\left[\widetilde{A}\left(R^M_{\widetilde{A},\xi}\right)\right]\Big](\widehat{z}) = -\xi Y^M_{\widetilde{A},\xi}(e), \\
&T^M_{\widetilde{A},\xi}(\widehat{z}) = -\xi Y^M_{\widetilde{A},\xi}(e), \text{ since } I - \widetilde{A}\left(R^M_{\widetilde{A},\xi}\right) = T^M_{\widetilde{A},\xi}.
\end{aligned}$$

Thus, we have

$$Y^M_{\widetilde{A},\xi}(e) + \xi^{-1} T^M_{\widetilde{A},\xi}(\widehat{z}) = 0,$$

which is required for Yosida resolvent equation problem (8).

Conversely, let $e, \widehat{z}$ be the solution of Yosida resolvent equation problem (8). Then, we have

$$\begin{aligned}
\xi Y^M_{\widetilde{A},\xi}(e) &= -T^M_{\widetilde{A},\xi}(\widehat{z}) \\
&= -\left[I - \widetilde{A}\left(R^M_{\widetilde{A},\xi}\right)\right](\widehat{z}) \\
&= \left[\widetilde{A}\left(R^M_{\widetilde{A},\xi}\right)\right](\widehat{z}) - \widehat{z} \\
&= \widetilde{A}\left[R^M_{\widetilde{A},\xi}(\widehat{z})\right] - \widehat{z} \\
&= \widetilde{A}\left[R^M_{\widetilde{A},\xi}\left(\widetilde{A}(e) - \xi Y^M_{\widetilde{A},\xi}(e)\right)\right] - \left[\widetilde{A}(e) - \xi Y^M_{\widetilde{A},\xi}(e)\right],
\end{aligned}$$

which implies that

$$\widetilde{A}(e) = \widetilde{A}\left[R^M_{\widetilde{A},\xi}\left(\widetilde{A}(e) - \xi Y^M_{\widetilde{A},\xi}(e)\right)\right].$$

Since $\widetilde{A}$ is one–one, we have

$$e = R^M_{\widetilde{A},\xi}\left(\widetilde{A}(e) - \xi Y^M_{\widetilde{A},\xi}(e)\right).$$

By Lemma 3, it follows that $e \in \widetilde{E}$ is the solution of Yosida variational inclusion problem (1). □

Alternative Proof. Let

$$\widehat{z} = \widetilde{A}(e) - \xi Y^M_{\widetilde{A},\xi}(e).$$

Using (9), we have

$$\widehat{z} = \widetilde{A}\left(R^M_{\widetilde{A},\xi}(\widehat{z})\right) - \xi Y^M_{\widetilde{A},\xi}(e),$$

which implies that

$$Y^M_{\widetilde{A},\xi}(e) + \xi^{-1} T^M_{\widetilde{A},\xi}(\widehat{z}) = 0,$$

the required Yosida resolvent equation problem (8). □

Based on Lemma 4, we state the following scheme for solving Yosida resolvent equation problem (8).

Iterative Scheme 4. *For any $e_0, \widehat{z}_0 \in \widetilde{E}$, compute the sequences $\{e_n\}$ and $\{z_n\}$ by the following scheme:*

$$e_n = R^M_{\widetilde{A},\xi}(\widehat{z}_n), \tag{11}$$

$$\widehat{z}_{n+1} = \widetilde{A}(e_n) - \xi Y^M_{\widetilde{A},\xi}(e_n), \tag{12}$$

where $n = 0, 1, 2, \cdots$ and $\xi > 0$ is a constant. □

The Yosida resolvent equation problem (8) can be restated as:

$$\widehat{z} = \widetilde{A}(e) - Y^M_{\widetilde{A},\xi}(e) + (I - \xi^{-1}) T^M_{\widetilde{A},\xi}(\widehat{z}). \tag{13}$$

Verification. Using (9)

$$\widehat{z} = \widetilde{A}\Big(R^M_{\widetilde{A},\xi}(\widehat{z})\Big) - Y^M_{\widetilde{A},\xi}(e) + T^M_{\widetilde{A},\xi}(\widehat{z}) - \xi^{-1}T^M_{\widetilde{A},\xi}(\widehat{z})$$
$$\Big[I - \widetilde{A}\Big(R^M_{\widetilde{A},\xi}\Big)\Big](\widehat{z}) = -Y^M_{\widetilde{A},\xi}(e) + T^M_{\widetilde{A},\xi}(\widehat{z}) - \xi^{-1}T^M_{\widetilde{A},\xi}(\widehat{z}).$$

Since $\Big[I - \widetilde{A}\Big(R^M_{\widetilde{A},\xi}\Big)\Big] = T^M_{\widetilde{A},\xi}$, we have

$$T^M_{\widetilde{A},\xi}(\widehat{z}) = -Y^M_{\widetilde{A},\xi}(e) + T^M_{\widetilde{A},\xi}(\widehat{z}) - \xi^{-1}T^M_{\widetilde{A},\xi}(\widehat{z}).$$

It follows that

$$Y^M_{\widetilde{A},\xi}(e) + \xi^{-1}T^M_{\widetilde{A},\xi}(\widehat{z}) = 0.$$

□

Using fixed point formulation (13), we suggest the following iterative scheme.

Iterative Scheme 5. *For given $e_0, \widehat{z}_0 \in \widetilde{E}$, compute the sequences $\{e_n\}$ and $\{z_n\}$ by the following scheme:*

$$e_n = R^M_{\widetilde{A},\xi}(\widehat{z}_n),$$
$$\widehat{z}_{n+1} = \widetilde{A}(e_n) - Y^M_{\widetilde{A},\xi}(e_n) + (I - \xi^{-1})T^M_{\widetilde{A},\xi}(\widehat{z}_n),$$

where $n = 0, 1, 2, \cdots$ and $\xi > 0$ is a constant. □

For positive step size δ, the Yosida resolvent equation problem (8) can also be written as:

$$e = e - \delta\Big[\widehat{z} - \widetilde{A}\Big(R^M_{\widetilde{A},\xi}(\widehat{z})\Big) + \xi Y^M_{\widetilde{A},\xi}(e)\Big]. \tag{14}$$

Verification.

$$e = e - \delta\Big[\Big[I - \widetilde{A}\Big(R^M_{\widetilde{A},\xi}\Big)\Big](\widehat{z}) + \xi Y^M_{\widetilde{A},\xi}(e)\Big],$$
$$e = e - \delta\Big[T^M_{\widetilde{A},\xi}(\widehat{z}) + \xi Y^M_{\widetilde{A},\xi}(e)\Big]$$
$$e = e - \delta\Big[Y^M_{\widetilde{A},\xi}(e) + \xi^{-1}T^M_{\widetilde{A},\xi}(\widehat{z})\Big],$$

which gives

$$Y^M_{\widetilde{A},\xi}(e) + \xi^{-1}T^M_{\widetilde{A},\xi}(\widehat{z}) = 0.$$

□

The fixed point formulation (14) enables us to suggest the following iterative scheme.

Iterative Scheme 6. *For $e_0, \widehat{z}_0 \in \widetilde{E}$, compute the sequences $\{e_n\}$ and $\{\widehat{z}_n\}$ by the following scheme:*

$$e_{n+1} = e_n - \delta\Big[\widehat{z}_n - \widetilde{A}\Big(R^M_{\widetilde{A},\xi}(\widehat{z}_n)\Big) + \xi Y^M_{\widetilde{A},\xi}(e_n)\Big],$$

where $\xi, \delta > 0$ are constants and $n = 0, 1, 2, \cdots$.

One can apply schemes 4–6 to obtain existence and convergence results for Yosida resolvent equation problem (8).

In order to accelerate the convergence rate, we suggest the following inertial extrapolation scheme for solving Yosida resolvent equation problem (8).

Equation (10) can also be written as

$$\widehat{z} = \frac{\widetilde{A}(e) + \widetilde{A}(e)}{2} - \xi Y^{M}_{\widetilde{A},\xi}(e). \tag{15}$$

Based on (15), we establish the following implicit scheme for solving Yosida resolvent equation problem (8).

Iterative Scheme 7. *For $e_0 \in \widetilde{E}$, compute the sequences $\{e_n\}$ and $\{\widehat{z}_n\}$ by the recurrance relation*

$$e_n = R^{M}_{\widetilde{A},\xi}(\widehat{z}_n),$$

$$\widehat{z}_{n+1} = (1-\alpha_n)\widehat{z}_n + \alpha_n\left[\frac{\widetilde{A}(e_n) + \widetilde{A}(e_{n+1})}{2} - \xi Y^{M}_{\widetilde{A},\xi}(e_{n+1})\right],$$

where $n = 0, 1, 2, \cdots$ and $\alpha_n \in [0,1]$.

We design the following inertial extrapolation scheme for solving Yosida resolvent equation problem (8) applying the predictor–corrector technique.

Iterative Scheme 8. *For $e_0, \widehat{z}_0 \in \widetilde{E}$, compute sequences $\{e_n\}$ and $\{z_n\}$ by the recurrance relation:*

$$w_n = \widehat{z}_n + \nu_n(\widehat{z}_n - \widehat{z}_{n-1}), \tag{16}$$

$$\widehat{z}_{n+1} = (1-\alpha_n)\widehat{z}_n + \alpha_n\left[\frac{\widetilde{A}(\widehat{z}_n) + \widetilde{A}(w_n)}{2} - \xi Y^{M}_{\widetilde{A},\xi}(w_n)\right], \tag{17}$$

where $\xi > 0$ is a constant, $\alpha_n, \nu_n \in [0,1]$ such that $\sum\limits_{n=1}^{\infty} \alpha_n = \infty$ and ν_n is an extrapolating term for all $n \geq 1$.

4. Convergence Analysis

First, we discuss the convergence of scheme 3 for Yosida variational inclusion problem (1) in real Banach space. Thenceforth, we demonstrate convergence of scheme 8 for Yosida resolvent equation problem (8) in real q-uniformly smooth Banach space.

Theorem 1. *Let $\widetilde{E}$ be real Banach space and $\widetilde{A} : \widetilde{E} \to \widetilde{E}$ be single-valued mapping such that $\widetilde{A}$ is r-strongly accretive and $\lambda_{\widetilde{A}}$-Lipschitz continuous. Let $M : \widetilde{E} \to 2^{\widetilde{E}}$ be $\widetilde{A}$-accretive set-valued mapping. Suppose that $R^{M}_{\widetilde{A},\xi} : \widetilde{E} \to \widetilde{E}$ is a generalized resolvent operator such that $R^{M}_{\widetilde{A},\xi}$ is $\frac{1}{r}$-Lipschitz continuous and $Y^{M}_{\widetilde{A},\xi} : \widetilde{E} \to \widetilde{E}$ is a generalized Yosida approximation operator such that $Y^{M}_{\widetilde{A},\xi}$ is λ_Y-Lipschitz continuous. Suppose that the following conditions are satisfied:*

$$\left|r - \frac{3\lambda_{\widetilde{A}}}{4}\right| > \frac{\sqrt{9\lambda^2_{\widetilde{A}} + 16}}{4}, \tag{18}$$

$$|r - \lambda_{\widetilde{A}}| > \sqrt{\lambda^2_{\widetilde{A}} + 1}, \tag{19}$$

$$\lambda_{\widetilde{A}} < r + \xi\lambda_Y, \tag{20}$$

where $\lambda_Y = \dfrac{\lambda_{\widetilde{A}} r + 1}{\xi r}, r \neq 0, \xi r \neq 0.$

Let $\alpha_n, \nu_n \in [0,1]$*, for all* $n \geq 1$ *such that*

$$\sum_{n=1}^{\infty} \alpha_n = \infty \text{ as well as } \sum_{n=1}^{\infty} \nu_n (e_n - e_{n-1}) < \infty, \tag{21}$$

where all contants are positive and ν_n *is an extrapolating term.*

Then, sequence $\{e_n\}$ *generated by the scheme 3 strongly converges to the unique solution* $e^* \in \widetilde{E}$ *of Yosida variational inclusion problem (1).*

Proof. Let $e \in \widetilde{E}$ be the solution of Yosida variational inclusion problem (1). Using (4), we have

$$e^* = (1-\alpha_n)e^* + \alpha_n R^M_{\widetilde{A},\xi}\left[\frac{\widetilde{A}(e^*) + \widetilde{A}(e^*)}{2} - \xi Y^M_{\widetilde{A},\xi}(e^*)\right], \tag{22}$$

where $\alpha_n \in [0,1]$, for all $n \geq 1$. Using (7) and (22), we evaluate

$$\begin{aligned}
\|e_{n+1} - e^*\| &= \Bigg\|(1-\alpha_n)e_n + \alpha_n R^M_{\widetilde{A},\xi}\left[\frac{\widetilde{A}(e_n) + \widetilde{A}(w_n)}{2} - \xi Y^M_{\widetilde{A},\xi}(w_n)\right] \\
&\quad - (1-\alpha_n)e^* + \alpha_n R^M_{\widetilde{A},\xi}\left[\frac{\widetilde{A}(e^*) + \widetilde{A}(e^*)}{2} - \xi Y^M_{\widetilde{A},\xi}(e^*)\right]\Bigg\| \\
&= \Bigg\|(1-\alpha_n)(e_n - e^*) + \alpha_n \Bigg[R^M_{\widetilde{A},\xi}\left[\frac{\widetilde{A}(e_n) + \widetilde{A}(w_n)}{2} - \xi Y^M_{\widetilde{A},\xi}(w_n)\right] \\
&\quad - R^M_{\widetilde{A},\xi}\left[\frac{\widetilde{A}(e^*) + \widetilde{A}(e^*)}{2} - \xi Y^M_{\widetilde{A},\xi}(e^*)\right]\Bigg]\Bigg\| \\
&\leq (1-\alpha_n)\|e_n - e^*\| + \alpha_n \Bigg\|R^M_{\widetilde{A},\xi}\left[\frac{\widetilde{A}(e_n) + \widetilde{A}(w_n)}{2} - \xi Y^M_{\widetilde{A},\xi}(w_n)\right] \\
&\quad - R^M_{\widetilde{A},\xi}\left[\frac{\widetilde{A}(e^*) + \widetilde{A}(e^*)}{2} - \xi Y^M_{\widetilde{A},\xi}(e^*)\right]\Bigg\|.
\end{aligned} \tag{23}$$

Applying the Lipschitz continuity of generalized resolvent operator $R^M_{\widetilde{A},\xi}$, $\widetilde{A}$ and generalized Yosida approximation operator $Y^M_{\widetilde{A},\xi}$, from (23), we obtain

$$\begin{aligned}
\|e_{n+1}-e^*\| &\leq (1-\alpha_n)\|e_n-e^*\| + \frac{\alpha_n}{r}\Bigg\|\left[\frac{\widetilde{A}(e_n)+\widetilde{A}(w_n)}{2}\right]-\left[\frac{\widetilde{A}(e^*)-\widetilde{A}(e^*)}{2}\right]\\
&\quad -\xi\left[Y^M_{\widetilde{A},\xi}(w_n)-Y^M_{\widetilde{A},\xi}(e^*)\right]\Bigg\|\\
&\leq (1-\alpha_n)\|e_n-e^*\| + \frac{\alpha_n}{2r}\|\widetilde{A}(e_n)-\widetilde{A}(e^*)\| + \frac{\alpha_n}{2r}\|\widetilde{A}(w_n)-\widetilde{A}(e^*)\|\\
&\quad + \frac{\alpha_n}{r}\xi\left\|Y^M_{\widetilde{A},\xi}(w_n)-Y^M_{\widetilde{A},\xi}(e^*)\right\|\\
&\leq (1-\alpha_n)\|e_n-e^*\| + \frac{\alpha_n}{2r}\lambda_{\widetilde{A}}\|e_n-e^*\| + \frac{\alpha_n}{2r}\lambda_{\widetilde{A}}\|w_n-e^*\|\\
&\quad + \frac{\alpha_n}{r}\xi\left\|Y^M_{\widetilde{A},\xi}(w_n)-Y^M_{\widetilde{A},\xi}(e^*)\right\|\\
&\leq (1-\alpha_n)\|e_n-e^*\| + \frac{\alpha_n}{2r}\lambda_{\widetilde{A}}\|e_n-e^*\| + \frac{\alpha_n}{2r}\lambda_{\widetilde{A}}\|w_n-e^*\|\\
&\quad + \frac{\alpha_n}{r}\xi\lambda_Y\|w_n-e^*\|\\
&= \left[(1-\alpha_n)+\frac{\alpha_n}{2r}\lambda_{\widetilde{A}}\right]\|e_n-e^*\| + \left[\frac{\alpha_n}{2r}\lambda_{\widetilde{A}}+\frac{\alpha_n}{r}\xi\lambda_Y\right]\|w_n-e^*\|.
\end{aligned} \tag{24}$$

From (6), we have

$$\begin{aligned}
\|w_n-e^*\| &= \|e_n-e^*+\nu_n(e_n-e_{n-1})\|\\
&\leq \|e_n-e^*\|+\nu_n\|e_n-e_{n-1}\|.
\end{aligned} \tag{25}$$

Combining (24) and (25), we obtain

$$\begin{aligned}
\|e_{n+1}-e^*\| &\leq \left[(1-\alpha_n)+\frac{\alpha_n\lambda_{\widetilde{A}}}{2r}\right]\|e_n-e^*\| + \frac{\alpha_n(\lambda_{\widetilde{A}}+2\xi\lambda_Y)}{2r}\|e_n-e^*\|\\
&\quad + \frac{\alpha_n(\lambda_{\widetilde{A}}+2\xi\lambda_Y)}{2r}\nu_n\|e_n-e_{n-1}\|.
\end{aligned}$$

Thus, we have

$$\begin{aligned}
\|e_{n+1}-e^*\| &\leq \left[(1-\alpha_n)+\alpha_n P_1+\alpha_n P_2\right]\|e_n-e^*\|+\alpha_n P_2\nu_n\|e_n-e_{n-1}\|\\
&= \left[(1-\alpha_n)(1-(P_1+P_2))\right]\|e_n-e^*\|+\nu_n\|e_n-e_{n-1}\|,
\end{aligned} \tag{26}$$

where

$$\begin{aligned}
P_1 &= \frac{\lambda_{\widetilde{A}}}{2r},\\
P_2 &= \frac{\lambda_{\widetilde{A}}+2\xi\lambda_Y}{2r} < 1, \text{ (By condition (18))}\\
P_1+P_2 &= \frac{\lambda_{\widetilde{A}}}{r}+\frac{\xi\lambda_Y}{r}.
\end{aligned}$$

Letting $P = P_1+P_2$, $P<1$ from condition (19). By condition (21), we have $\sum_{n=1}^{\infty}\alpha_n=\infty$ and $\sum_{n=1}^{\infty}\nu_n\|e_n-e_{n-1}\|<\infty$. Setting $\sigma_n=0$ and $\tilde{\xi}_n=\sum_{n=1}^{\infty}\nu_n\|e_n-e_{n-1}\|<\infty$. Then, by Lemma 2 and (26), we have $e_n\to e^*$, as $n\to\infty$. Thus, the sequence $\{e_n\}$ generated by scheme 3 strongly converges to the unique solution $e^*\in\widetilde{E}$ of Yosida variational inclusion problem (1). Furthermore, we show that the solution of Yosida variational inclusion problem (1) is unique.

Let $e, e^* \in \widetilde{E}$ be the two solutions of Yosida variational inclusion problem (1). Then, by Lemma 3, we have

$$e = R^M_{\widetilde{A},\xi}\Big[\widetilde{A}(e) - \xi Y^M_{\widetilde{A},\xi}(e)\Big]$$
$$\text{and} \qquad e^* = R^M_{\widetilde{A},\xi}\Big[\widetilde{A}(e^*) - \xi Y^M_{\widetilde{A},\xi}(e^*)\Big].$$

Using the Lipschitz continuity of the generalized resolvent operator $R^M_{\widetilde{A},\xi}$, generalized Yosida approximation operator $Y^M_{\widetilde{A},\xi}$ and mapping $\widetilde{A}$, we evaluate

$$\begin{aligned}
\|e - e^*\| &= \Big\|R^M_{\widetilde{A},\xi}\Big[\widetilde{A}(e) - \xi Y^M_{\widetilde{A},\xi}(e)\Big] - R^M_{\widetilde{A},\xi}\Big[\widetilde{A}(e^*) - \xi Y^M_{\widetilde{A},\xi}(e^*)\Big]\Big\| \\
&\leq \frac{1}{r}\Big\|\widetilde{A}(e) - \widetilde{A}(e^*) - \xi\Big[Y^M_{\widetilde{A},\xi}(e) - Y^M_{\widetilde{A},\xi}(e^*)\Big]\Big\| \\
&\leq \frac{1}{r}\lambda_{\widetilde{A}}\|e - e^*\| + \frac{\xi\lambda_Y}{r}\|e - e^*\| \qquad (27)\\
&= \left(\frac{\lambda_{\widetilde{A}}}{r} - \frac{\xi\lambda_Y}{r}\right)\|e - e^*\| \\
&= A(\theta)\|e - e^*\|,
\end{aligned}$$

where $A(\theta) = \left(\frac{\lambda_{\widetilde{A}}}{r} - \frac{\xi\lambda_Y}{r}\right), \lambda_Y = \left(\frac{\lambda_{\widetilde{A}}\, r + 1}{\xi\, r}\right)$. It follows from condition (20) that $0 < A(\theta) < 1$. Thus, from (27), we have $e = e^*$. That is, e^* is the unique solution of Yosida variational inclusion problem (1). □

Now, we study the convergence analysis of scheme 8 for Yosida resolvent equation problem (8) in the setting of q-uniformly smooth Banach space. Note that by Lemma 4, Yosida variational inclusion problem (1) is equivalent to Yosida resolvent equation problem (8). As Yosida variational inclusion problem (1) admits a unique solution, Yosida resolvent equation problem (8) also admits a unique solution.

Theorem 2. *Let $\widetilde{E}$ be q-uniformly smooth Banach space and $\widetilde{A} : \widetilde{E} \to \widetilde{E}$ be single-valued mapping such that $\widetilde{A}$ is one–one, $\lambda_{\widetilde{A}}$-Lipschitz continuous, $\beta_{\widetilde{A}}$-expansive and r-strongly accretive. Let $M : \widetilde{E} \to \widetilde{E}$ be $\widetilde{A}$-accretive set-valued mapping and $R^M_{\widetilde{A},\xi} : \widetilde{E} \to \widetilde{E}$ be a generalized resolvent operator such that $R^M_{\widetilde{A},\xi}$ is $\frac{1}{r}$-Lipschitz continuous. Let $Y^M_{\widetilde{A},\xi} : \widetilde{E} \to \widetilde{E}$ be a generalized Yosida approximation operator such that $Y^M_{\widetilde{A},\xi}$ is θ_Y-strongly accretive with respect to $\widetilde{A}$ and λ_Y-Lipschitz continuous. Let $T^M_{\widetilde{A},\xi}(\widehat{z}) = \Big[I - \widetilde{A}\Big(R^M_{\widetilde{A},\xi}\Big)\Big](\widehat{z})$, where $\widetilde{A}\Big(R^M_{\widetilde{A},\xi}(\widehat{z})\Big) = \Big[\widetilde{A}\Big(R^M_{\widetilde{A},\xi}\Big)\Big](\widehat{z}), \widehat{z} \in \widetilde{E}$. Suppose that the following conditions are satisfied.*

$$\left(\lambda^q_{\widetilde{A}} - 2q\xi\theta_Y + 2^q\xi^q C_q \lambda^q_Y\right) < 2^q, \qquad (28)$$

$$\lambda_{\widetilde{A}} < 2. \qquad (29)$$

Let $\alpha_n, \nu_n \in [0,1]$, for all $n \geq 1$ such that

$$\sum_{n=1}^{\infty} \alpha_n = \infty \text{ and } \sum_{n=1}^{\infty} \left(\|\widetilde{z}_n - w^*\| + \nu_n\|\widehat{z}_n - \widehat{z}_{n-1}\|\right) < \infty, \qquad (30)$$

where $\xi > 0$ is a constant, C_q is the same as in Lemma 1, $\theta_Y = \dfrac{\beta_{\widetilde{A}}^q r - \lambda_{\widetilde{A}}^{q-1}}{\xi r}$ and $\lambda_Y = \dfrac{\lambda_{\widetilde{A}} r + 1}{\xi r}, \xi r \neq 0, r \neq 0, \beta_{\widetilde{A}}^q r > \lambda_{\widetilde{A}}^{q-1}$, and all the constants are positive.

Then, sequences $\{e_n\}$ and $\{\widehat{z}_n\}$ generated by scheme 8 strongly converge to the unique solution e^* and z^* of Yosida resolvent equation problem (8).

Proof. Applying (17) of scheme 8 and Lipschitz continuity of $\widetilde{A}$, we evaluate

$$\begin{aligned}
\|\widehat{z}_{n+1} - \widehat{z^*}\| &= \Bigg\|(1-\alpha_n)\widehat{z}_n + \alpha_n\left[\frac{\widetilde{A}(\widehat{z}_n) + \widetilde{A}(w_n)}{2} - \xi Y^M_{\widetilde{A},\xi}(w_n)\right] \\
&\quad - \left[(1-\alpha_n)z^* + \alpha_n\left[\frac{\widetilde{A}(\widehat{z^*}) + \widetilde{A}(w^*)}{2} - \xi Y^M_{\widetilde{A},\xi}(w^*)\right]\right]\Bigg\| \\
&= \Bigg\|(1-\alpha_n)(\widehat{z}_n - \widehat{z^*}) + \alpha_n\Bigg[\frac{\widetilde{A}(\widehat{z}_n) - \widetilde{A}(\widehat{z^*})}{2} + \frac{\widetilde{A}(w_n) - \widetilde{A}(w^*)}{2} \\
&\quad - \xi\left[Y^M_{\widetilde{A},\xi}(w_n) - Y^M_{\widetilde{A},\xi}(w^*)\right]\Bigg]\Bigg\| \\
&\le (1-\alpha_n)\|\widehat{z}_n - \widehat{z^*}\| + \frac{\alpha_n}{2}\left\|\widetilde{A}(\widehat{z}_n) - \widetilde{A}(\widehat{z^*})\right\| \\
&\quad + \frac{\alpha_n}{2}\left\|\widetilde{A}(w_n) - \widetilde{A}(w^*) - 2\xi\left[Y^M_{\widetilde{A},\xi}(w_n) - Y^M_{\widetilde{A},\xi}(w^*)\right]\right\| \\
&\le (1-\alpha_n)\|\widehat{z}_n - \widehat{z^*}\| + \frac{\alpha_n}{2}\lambda_{\widetilde{A}}\|\widehat{z}_n - \widehat{z^*}\| \\
&\quad + \frac{\alpha_n}{2}\left\|\widetilde{A}(w_n) - \widetilde{A}(w^*) - 2\xi\left[Y^M_{\widetilde{A},\xi}(w_n) - Y^M_{\widetilde{A},\xi}(w^*)\right]\right\|.
\end{aligned} \tag{31}$$

Using Lemma 1, Lipschitz continuity of $\widetilde{A}$, strong accretiveness of $Y^M_{\widetilde{A},\xi}$ with respect to $\widetilde{A}$ and Lipschitz continuity of $Y^M_{\widetilde{A},\xi}$, we have

$$\begin{aligned}
&\left\|\widetilde{A}(w_n) - \widetilde{A}(w^*) - 2\xi\left[Y^M_{\widetilde{A},\xi}(w_n) - Y^M_{\widetilde{A},\xi}(w^*)\right]\right\|^q \\
&\quad \le \|\widetilde{A}(w_n) - \widetilde{A}(w^*)\|^q - q2\xi\langle Y^M_{\widetilde{A},\xi}(w_n) - Y^M_{\widetilde{A},\xi}(w^*), J_q(\widetilde{A}(w_n) - \widetilde{A}(w^*))\rangle \\
&\qquad + 2^q\xi^q C_q\left\|Y^M_{\widetilde{A},\xi}(w_n) - Y^M_{\widetilde{A},\xi}(w^*)\right\|^q \\
&\quad \le \lambda^q_{\widetilde{A}}\|w_n - w^*\|^q - 2q\xi\theta_Y\|w_n - w^*\|^q + 2^q\xi^q C_q\lambda^q_Y\|w_n - w^*\|^q \\
&\quad \le \left[\lambda^q_{\widetilde{A}} - 2q\xi\theta_Y + 2^q\xi^q C_q\lambda^q_Y\right]\|w_n - w^*\|^q,
\end{aligned} \tag{32}$$

where $\theta_Y = \dfrac{\beta_{\widetilde{A}}^q r - \lambda_{\widetilde{A}}^{q-1}}{\xi r}, \xi r \neq 0, \lambda_Y = \dfrac{\lambda_{\widetilde{A}} r + 1}{\xi r}$ and $\lambda_{\widetilde{A}} + 2^q\xi^q C_q\lambda^q_Y > 2q\xi\theta_Y$.

It follows from (32) that

$$\|\widetilde{A}(w_n) - \widetilde{A}(w^*) - 2\xi\left[Y^M_{\widetilde{A},\xi}(w_n) - Y^M_{\widetilde{A},\xi}x(w^*)\right]\| \le \sqrt[q]{\lambda^q_{\widetilde{A}} - 2q\xi\theta_Y + 2^q\xi^q C_q\lambda^q_Y}\|w_n - w^*\|. \tag{33}$$

Combining (31) and (33), we obtain

$$\|\widehat{z}_{n+1}-\widehat{z^*}\| \leq (1-\alpha_n)\|\widehat{z}_n-\widehat{z^*}\| + \frac{\alpha_n}{2}\lambda_{\widetilde{A}}\|\widehat{z}_n-\widehat{z^*}\| + \frac{\alpha_n}{2}\sqrt[q]{\lambda_{\widetilde{A}}^q - 2q\xi\theta_Y + 2^q\xi^q C_q\lambda_Y^q}\|w_n-w^*\|. \tag{34}$$

Applying (16) of scheme 8, we have

$$\begin{aligned}\|w_n-w^*\| &= \|\widehat{z}_n - w^* + \nu_n(\widehat{z}_n-\widehat{z}_{n-1}\| \\ &\leq \|\widehat{z}_n-w^*\| + \nu_n\|\widehat{z}_n-\widehat{z}_{n-1}\|.\end{aligned} \tag{35}$$

Combining (34) with (35), we have

$$\begin{aligned}\|\widehat{z}_{n+1}-\widehat{z^*}\| &\leq (1-\alpha_n)\|\widehat{z}_n-\widehat{z^*}\| + \frac{\alpha_n}{2}\lambda_{\widetilde{A}}\|\widehat{z}_n-\widehat{z^*}\| \\ &+ \frac{\alpha_n}{2}\sqrt[q]{\lambda_{\widetilde{A}}^q - 2q\xi\theta_Y + 2^q\xi^q C_q\lambda_Y^q}\Big[\|\widehat{z}_n-w^*\| + \nu_n\|\widehat{z}_n-\widehat{z}_{n-1}\|\Big],\end{aligned}$$

that is

$$\begin{aligned}\|\widehat{z}_{n+1}-\widehat{z^*}\| &\leq \big[(1-\alpha_n(1-\widehat{\Pi}_1)\big]\|\widehat{z}_n-\widehat{z^*}\| + \alpha_n\Pi_2\big[\|\widehat{z}_n-w^*\| + \nu_n\|\widehat{z}_n-\widehat{z}_{n-1}\|\big] \\ &\leq \big[1-\alpha_n(1-\widehat{\Pi}_1)\big]\|\widehat{z}_n-\widehat{z^*}\| + \alpha_n\big[\|\widehat{z}_n-w^*\| + \nu_n\|\widehat{z}_n-\widehat{z}_{n-1}\|\big],\end{aligned} \tag{36}$$

where $\widehat{\Pi}_1 = \frac{\lambda_{\widetilde{A}}}{2}$ and $\widehat{\Pi}_2 = \frac{\sqrt[q]{\lambda_{\widetilde{A}}^q - 2q\xi\theta_Y + 2^q\xi^q C_q\lambda_Y^q}}{2}$. By condition (28), $\widehat{\Pi}_2 < 1$ and by condition (29), $\widehat{\Pi}_2 < 1$.

Applying condition (30), we have $\sum_{n=1}^{\infty}\alpha_n = \infty$ and $\sum_{n=1}^{\infty}\big(\|\widehat{z}_n-w^*\| + \nu_n\|\widehat{z}_n-\widehat{z}_{n-1}\|\big) < \infty$. We set $\sigma_n = 0$ and $\widehat{\xi}_n = \sum_{n=1}^{\infty}\big(\|\widehat{z}_n-w^*\| + \nu_n\|\widehat{z}_n-\widehat{z}_{n-1}\|\big) < \infty$. Applying Lemma 2 and (36), we have $\widehat{z}_n \to \widehat{z^*}$, as $n\to\infty$.

In addition,

$$\|e_n-e^*\| = \|R^M_{\widetilde{A},\xi}(\widehat{z}_n) - R^M_{\widetilde{A},\xi}(\widehat{z^*})\| \leq \frac{1}{r}\|\widehat{z}_n-\widehat{z^*}\|. \tag{37}$$

Since $\widehat{z}_n \to \widehat{z^*} \in \widetilde{E}$, it follows from (37) that $e_n \to e^* \in \widetilde{E}$. Hence, the sequences $\{\widehat{z}_n\}$ and $\{e_n\}$ generated by scheme 8 converge strongly to $\widehat{z^*}$ and e^*, providing the unique solution of Yosida resolvent equation problem (8). □

5. Numerical Experiment

In support of Theorem 1, construct the following numerical example using MATLAB 2015a with a computation table and convergence graph.

Example 1. *Let $\widetilde{E} = \mathbb{R}$ with usual inner product and norm, $\widetilde{A} : \widetilde{E} \to \widetilde{E}$ be single-valued mapping and $M : \widetilde{E} \to 2^{\widetilde{E}}$ be set-valued mapping such that*

$$\begin{aligned} A(e) &= \frac{6}{5}e, \\ \text{and} \quad M(e) &= \{\frac{1}{10}e\},\ \text{for all } e\in\widetilde{E}.\end{aligned}$$

(i) $\widetilde{A}$ *is r-strongly accretive and* $\lambda_{\widetilde{A}}$*-Lipschtiz continuous.*

$$\begin{aligned}\langle \widetilde{A}(e) - \widetilde{A}(f), e-f\rangle &= \left\langle \frac{6}{5}e - \frac{6}{5}f, e-f \right\rangle \\ &= \frac{6}{5}\|e-f\|^2 \geq \frac{11}{10}\|e-f\|^2.\end{aligned}$$

Thus, $\widetilde{A}$ is $r = \frac{11}{10}$-strongly accretive mapping.
In addition,

$$\begin{aligned}\|\widetilde{A}(e) - \widetilde{A}(f)\| &= \left\| \frac{6}{5}e - \frac{6}{5}f \right\| \\ &= \frac{6}{5}\|e-f\| \leq \frac{13}{10}\|e-f\|.\end{aligned}$$

Thus, $\widetilde{A}$ is $\lambda_{\widetilde{A}} = \frac{13}{10}$-Lipschitz continuous mapping.

(ii) M is $\widetilde{A}$-accretive.

$$\begin{aligned}\|M(e) - M(f)\| &= \left\| \frac{1}{10}e - \frac{1}{10}f \right\| \\ &= \frac{1}{10}\|e-f\| \geq 0.\end{aligned}$$

That is, M is accretive and also for $\xi = 1$, it is easy to verify that

$$[\widetilde{A} + \xi M](\widetilde{E}) = \widetilde{E}.$$

Thus, M is $\widetilde{A}$-accretive mapping.

(iii) For $\xi = 1$, we define a generalized resolvent operator

$$R^M_{\tilde{A},\xi}(e) = [\widetilde{A} + \xi M]^{-1}(e) = \frac{10}{13}e, \text{ for all } e \in \widetilde{E}.$$

In addition,

$$\begin{aligned}\|R^M_{\tilde{A},\xi}(e) - R^M_{\tilde{A},\xi}(f)\| &= \left\| \frac{10}{13}e - \frac{10}{13}f \right\| \\ &= \frac{10}{13}\|e-f\| \leq \frac{1}{(11/10)}\|e-f\|.\end{aligned}$$

Thus, the generalized resolvent operator $R^M_{\tilde{A},\xi}$ is $\frac{1}{r} = \frac{1}{(11/10)}$-Lipschitz continuous.

(iv) Based on step (iii), we calculate the generalized Yosida approximation operator

$$Y^M_{\tilde{A},\xi}(e) = \frac{1}{\xi}[\tilde{A} - R^M_{\tilde{A},\xi}](e) = \frac{28}{65}e, \text{ for all } e \in \widetilde{E}.$$

In addition,

$$\begin{aligned}\|Y^M_{\tilde{A},\xi}(e) - Y^M_{\tilde{A},\xi}(f)\| &= \left\| \frac{28}{65}e - \frac{28}{65}f \right\| \\ &= \frac{28}{65}\|e-f\| \leq \frac{243}{110}\|e-f\|,\end{aligned}$$

Thus, the generalized Yosida approximation operator $Y^M_{\tilde{A},\xi}$ is $\lambda_Y = \frac{\lambda_{\tilde{A}} r + 1}{\xi r} = \frac{243}{110}$-Lipschitz continuous.

(v) *In view of constants calculated above, the conditions* (18)–(20) *of Theorem 1 are fulfilled.*

(vi) *For* $\alpha_n = \frac{5}{n+1}$ *and* $\nu_n = \frac{1}{n}$, *inertial extrapolation scheme 8 has the following model:*

$$
\begin{aligned}
w_n &= e_n + \nu_n(e_n - e_{n-1}) \\
e_{n+1} &= (1-\alpha_n)e_n + \alpha_n R^{M}_{\widetilde{A},\xi}\left[\frac{\widetilde{A}(e_n) + \widetilde{A}(w_n)}{2} - \xi Y^{M}_{\widetilde{A},\xi}(w_n)\right] \\
&= (1-\alpha_n)e_n + \alpha_n R^{M}_{\widetilde{A},\xi}\left[\frac{3}{5}e_n + \frac{11}{65}w_n\right] \\
&= (1-\alpha_n)e_n + \alpha_n\left[\frac{6}{13}e_n + \frac{110}{845}w_n\right].
\end{aligned}
$$

It is shown through a computation table (Table 1) and convergence graph (Figure 1) that for different initial values $e_0 = 1, -5.0, 5.0$, *the sequence* $\{e_n\}$ *converges to* $e^* = 0$, *which is the solution of Yosida variational inclusion problem* (1). *For the composition of a computation table and convergence graph, we use the tools of MATLAB 2015a.*

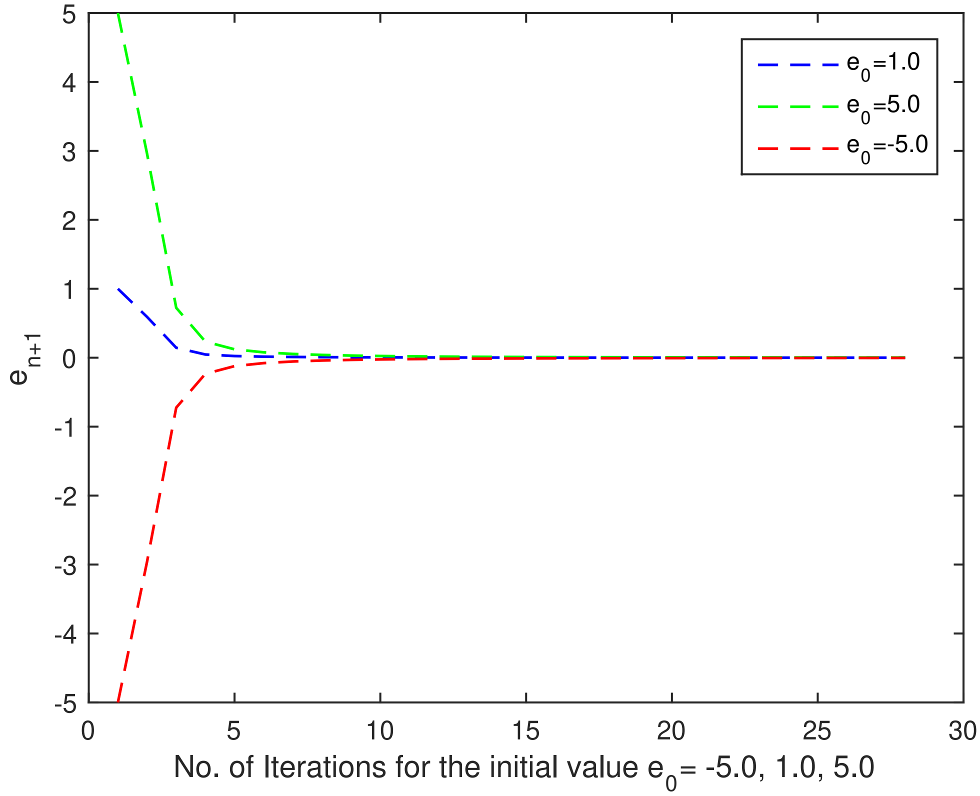

Figure 1. Graphical representation of convergence of sequence $\{e_n\}$ with different initial values $e_0 = 1, -5.0$ and 5.0.

Table 1. Computation outputs for different initial values $e_0 = 1, -5.0$ and 5.0.

No. of Iterations	$e_0 = 1.0$ e_n	No. of Iterations	$e_0 = -5.0$ e_n	No. of Iterations	$e_0 = 5.0$ e_n
1	0.5917	1	−2.9586	1	2.9586
2	0.1448	2	−0.7239	2	0.7239
3	0.0466	3	−0.2332	3	0.2332
4	0.0244	4	−0.1220	4	0.1220
5	0.0156	5	−0.0781	5	0.0781
15	0.0018	15	0.0000	15	0.0000
20	0.0010	20	0.0000	20	0.0000
25	6.4947×10^{-04}	25	0.0000	25	0.0000
26	6.0032×10^{-04}	26	0.0000	26	0.0000
27	5.5651×10^{-04}	27	0.0000	27	0.0000
28	5.1730×10^{-04}	28	0.0000	28	0.0000
29	4.8207×10^{-04}	29	0.0000	29	0.0000
30	4.1327×10^{-04}	30	0.0000	30	0.0000

6. Conclusions

In this paper, we have considered and studied a Yosida variational inclusion problem with its corresponding Yosida resolvent equation problem. Using the resolvent operator technique, it is shown that both the problems are equivalent under appropriate conditions. We mention the number of iterative schemes for solving both the problems. We concentrate our study on the convergence analysis of both the problems applying an inertial extrapolation scheme. For illustration, an example is constructed.

One can extend our results in higher-dimensional spaces. Engineers, physicists and other scientists may use our result for their practical purposes.

Author Contributions: Conceptualization, A.K.R. and R.A.; methodology, Y.W.; software, A.K.R.; validation, M.I. and Y.W.; formal analysis, M.I.; investigation, R.A.; resources, Y.W.; data curation, M.I.; writing—original draft preparation, A.K.R.; writing—review and editing, R.A. and J.-C.Y.; visualization, M.I.; supervision, R.A. and J.-C.Y.; project administration, J.-C.Y.; funding acquisition, Y.W. All authors have read and agreed to the published version of the manuscript.

Funding: This research was funded by the National Natural Science Foundation of China (Grant no. 12171435).

Institutional Review Board Statement: Not applicable.

Informed Consent Statement: Not applicable.

Data Availability Statement: Not applicable.

Acknowledgments: All authors are thankful to all referees for their valuable suggestions which improve this paper a lot.

Conflicts of Interest: The authors declare no conflict of interest.

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