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Design Efficiency of the Asymmetric Minimum Projection Uniform Designs

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Abstract: Highly efficient designs and uniform designs are widely applied in many fields because of their good properties. The purpose of this paper is to study the issue of design efficiency for asymmetric minimum projection uniform designs. Based on the centered L_2 discrepancy, the uniformity of the designs with mixed levels is defined, which is used to measure the projection uniformity of the designs. The analytical relationship between the uniformity pattern and the design efficiency is established for mixed-level orthogonal arrays with a strength of two. Moreover, a tight lower bound of the uniformity pattern is presented. The research is relevant in the field of experimental design by providing a theoretical basis for constructing the minimum number of projection uniform designs with a high design efficiency under a certain condition. These conclusions are verified by some numerical examples, which illustrate the theoretical results obtained in this paper.

Keywords: uniformity pattern; design efficiency; centered L_2 discrepancy; lower bound; projection uniform design

MSC: 62K15; 62K05

1. Introduction

The uniform designs proposed in [1,2] have been widely used in physical and computer experiments. It requires design points that are uniformly scattered over the experimental domain. Generally, the overall uniformity of the design is often considered, and the projection uniformity of the design in low dimensions is ignored. By the effect sparsity principle, the number of relatively important factors is small in an experiment, so it is necessary to study the projection uniformity of the designs. The authors of [3] first defined the projection discrepancy pattern to measure the projection uniformity of designs based on discrete discrepancy. The authors of [4] proposed the minimum projection uniformity criterion under the centered L_2 discrepancy to measure the projection uniformity of designs with two levels and established the relationship between the generalized minimum aberration criterion [5] and the orthogonality criterion [6]. Similar conclusions are obtained for multi-level and mixed-level ones [7–10]. These theoretical results show that the minimum projection uniformity criterion is equivalent to some other design screening criteria, which provides a theoretical basis for the statistical rationality of the projection uniformity of designs.

According to the maximum estimation capacity of the designs, the design efficiency criterion is proposed, which concerns models included the general mean, i.e., all of the main effects and a selection of two-factor interactions (for more information, one can refer to [11]). Design efficiency criterion are closely associated with minimum aberration or generalized minimum aberration criteria [11–13]. The authors of [14] studied the design efficiency of minimum projection uniform designs with two levels, which shows that the minimum projection uniformity criterion is equivalent to the design efficiency criterion under a certain condition. The authors of [15] transformed the designs in [14] into q-level designs.

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This paper aims at transforming the designs in [15] into mixed-level designs. The relationship between the uniformity pattern, generalized wordlength pattern and design efficiency is established, and the design efficiency of the minimum projection uniform designs are discussed. This paper is organized as follows: Section 2 presents some basic concepts and notations. Section 3 discusses the design efficiency of mixed-level minimum projection uniform designs. Section 4 provides a tight, lower bound uniformity pattern. Some illustrate examples are presented in Section 5. Section 6 presents some concluding remarks.

2. Notations and Preliminaries

Let $\mathcal{U}(n; q_1^{s_1} \times q_2^{s_2})$ be a set of n-run, $s(= s_1 + s_2)$-factor U-type designs with q_p levels from $\{0, 1, \ldots, q_p - 1\}$, $p = 1, 2$. For any design of $d \in \mathcal{U}(n; q_1^{s_1} \times q_2^{s_2})$, design d is called an orthogonal array with the strength t if all of the possible level combinations of any t columns in a design d occur an equal number of times, denoted as $OA(n; q_1^{s_1} \times q_2^{s_2}, t)$. The U-type designs are an orthogonal array with a strength of one. A typical treatment combination of a design d is defined by $z = (z^{(1)}, z^{(2)})$, where $z^{(p)} = (z_1^{(p)}, \ldots, z_{s_p}^{(p)})$, $0 \le z_j^{(p)} \le q_p - 1$, $1 \le j \le s_p$. Let $F^{(1)}, F^{(2)}$ and F, respectively, be the sets of all the $f^{(1)} = q_1^{s_1}, f^{(2)} = q_2^{s_2}$ and $f = q_1^{s_1} \times q_2^{s_2}$ treatment combinations that are lexicographically ordered.

2.1. Generalized Minimum Aberration Criterion

For any design of $d \in \mathcal{U}(n; q_1^{s_1} \times q_2^{s_2})$, $v_1 = 0, \ldots, s_1$, $v_2 = 0, \ldots, s_2$, the distance distribution of d is defined by

$$\begin{aligned} C_{v_1 v_2}(d) =& \frac{1}{n} |\{(i,k) : d_H(i^{(1)}, k^{(1)}) = v_1, d_H(i^{(2)}, k^{(2)}) = v_2, \\ & i = (i^{(1)}, i^{(2)}), k = (k^{(1)}, k^{(2)}) \text{ are two rows of } d\}|, \end{aligned} \tag{1}$$

where $d_H(i, k)$ is the Hamming distance between the i-th and k-th rows (the number of places where they differ), and $|\{(i, k)\}|$ is the cardinality of the set $\{(i, k)\}$. $\delta_{ik} = s - d_H(i, k)$ is the coincidence number between two rows i and k.

The MacWilliams transforms of the distance distribution are

$$A_{j_1 j_2}(d) = \frac{1}{n} \sum_{v_1=0}^{s_1} \sum_{v_2=0}^{s_2} P_{j_1}(v_1; s_1, q_1) P_{j_2}(v_2; s_2, q_2) C_{v_1 v_2}(d), \tag{2}$$

for $0 \le j_1 \le s_1$ and $0 \le j_2 \le s_2$, where $P_{j_p}(v_p; s_p, q_p) = \sum_{r=0}^{j_p} (-1)^r (q_p - 1)^{j_p - r} \binom{v_p}{j_p} \binom{s_p - v_p}{j_p - r}$ is the Krawtchouk polynomial, $p = 1, 2$. For $0 \le j \le s_1 + s_2$, it is defined as

$$A_j(d) = \sum_{j_1 + j_2 = j} A_{j_1 j_2}(d), \tag{3}$$

the vector $(A_1(d), \ldots, A_{s_1+s_2}(d))$ is called the generalized wordlength pattern (GWLP). For two designs, d_1 and $d_2 \in \mathcal{U}(n; q_1^{s_1} \times q_2^{s_2})$, let r be the smallest integer that makes $A_r(d_1) < A_r(d_2)$, $1 \le r \le s_1 + s_2$, and $A_j(d_1) = A_j(d_2)$ for $j = 1, \ldots, r - 1$; this shows that design d_1 has fewer aberrations than design d_2 has. In any design at the same scale, no design has fewer aberrations than d_1 has; design d_1 has a generalized minimum aberration (GMA) (for more information, one can refer to [5]).

2.2. Orthogonality Criterion

For $z \in F$, let $y_d(z)$ be number of treatment combinations z in $d \in \mathcal{U}(n; q_1^{s_1} \times q_2^{s_2})$. For $z^{(1)} \in F^{(1)}$, let $y_d(z^{(1)})$ be a $f^{(2)} \times 1$ vector with elements $y_d(z^{(1)}, z^{(2)})$ for all of the elements $z^{(2)} \in F^{(2)}$ arranged in lexicographic order. Let y_d be a $f \times 1$ vector with elements $y_d(z)$ arranged in lexicographic order.

We denote 1_v as a $v \times 1$ vector with all of the elements in unity and I_v as a $v \times v$ identity matrix, $J_v = 1_v 1_v'$, while the s-fold Kronecker products of 1_v, I_v and J_v are denoted by

$1_v^{(s)}$, $I_v^{(s)}$ and $J_v^{(s)}$, respectively. For $p = 1, 2$, $E^{(p)}(0) = q_p^{-1}J_{q_p}$, $E^{(p)}(1) = I_{q_p} - q_p^{-1}J_{q_p}$, $L^{(p)}(0) = 1'_{q_p}$, $L^{(p)}(1) = I_{q_p}$, let $\Omega^{(p)} = \left\{x^{(p)} = (x_1^{(p)}, \dots, x_{s_p}^{(p)}), x_j^{(p)} \in \{0,1\}, j = 1, \dots, s_p\right\}$, $W^{(p)}(x^{(p)}) = E^{(p)}(x_1^{(p)}) \otimes \cdots \otimes E^{(p)}(x_{s_p}^{(p)})$,$H^{(p)}(x^{(p)}) = L^{(p)}(x_1^{(p)}) \otimes \cdots \otimes L^{(p)}(x_{s_p}^{(p)})$, where $\otimes$ is the Kronecker product. Let $\Omega = \left\{x = (x^{(1)}, x^{(2)}) : x^{(1)} \in \Omega^{(1)}, x^{(2)} \in \Omega^{(2)}\right\}$ and the members of Ω be lexicographically ordered, and the size of Ω is $2^{(s_1+s_2)}$. For $j_1 = 0, \dots, s_1, j_2 = 0, \dots, s_2$, let $\Omega_{j_1 j_2}$ be the subset of Ω consisting of those binary $(s_1 + s_2)$-tuples, which has j_1 ones in $x^{(1)}$ and j_2 ones in $x^{(2)}$. We define the $f \times f$ matrix as $W(x) = W^{(1)}(x^{(1)}) \otimes W^{(2)}(x^{(2)})$. For $j_1 + j_2 = j$, we define it as

$$B_{j_1 j_2}(d) = \sum_{x \in \Omega_{j_1 j_2}} y'_d W(x) y_d,\ 0 \le j_1 \le s_1, 0 \le j_2 \le s_2, (j_1, j_2) \neq (0,0), \tag{4}$$

$$B_j(d) = \sum_{j_1 + j_2 = j} B_{j_1 j_2}(d),\ j = 1, \dots, s. \tag{5}$$

In [6], the vector $(B_1(d), \dots, B_s(d))$ is called the B vector. The difference between d and the orthogonal array with strength t can be measured by $\sum_{j=1}^{t} B_j(d)$. The orthogonality criterion is to sequentially minimize $(B_1(d), \dots, B_s(d))$.

2.3. Projection-Centered L_2 Discrepancy

For $g = 1, \dots, s$, we define it as $J_g = \{(g_1, g_2) : g_1 = 0, \dots, s_1, g_2 = 0, \dots, s_2, g_1 + g_2 = g\}$. For any $(g_1, g_2) \in J_g$, $s = s_1 + s_2$, let $I_{g_1 g_2} = \{(u_1, u_2) : u_1 \in \{1, \dots, s_1\}, u_2 \in \{s_1 + 1, \dots, s\}$, $|u_1| = g_1, |u_2| = g_2\}$, $I_g = \cup_{(g_1, g_2) \in J_g} I_{g_1 g_2}$. For any design of $d \in \mathcal{U}(n; q_1^{s_1} \times q_2^{s_2})$, we define it as $u = \{(u_1 \cup u_2), |u| = |u_1| + |u_2| = g\}$ and let d_u denote the projection designs of d onto u. In [16], the projection-centered L_2 discrepancy of design d onto u is denoted by $CD_u(d)$, whose square value can be computed by

$$[CD_u(d)]^2 = \left(\frac{13}{12}\right)^{g_1+g_2} - \frac{2}{n} \sum_{i=1}^{n} \prod_{p=1}^{2} \prod_{j=1}^{g_p} \alpha\left(x_{ij}^{(p)}\right) + \frac{1}{n^2} \sum_{i=1}^{n} \sum_{k=1}^{n} \prod_{p=1}^{2} \prod_{j=1}^{g_p} \beta\left(x_{ij}^{(p)}, x_{kj}^{(p)}\right), \tag{6}$$

where $x_{ij}^{(p)} = \frac{2z_{ij}^{(p)}+1}{2q_p}$ for any fixed i and $\alpha(x_{ij}^{(p)}) = 1 + \frac{1}{2}\left|x_{ij}^{(p)} - \frac{1}{2}\right| - \frac{1}{2}\left|x_{ij}^{(p)} - \frac{1}{2}\right|^2$, $\beta(x_{ij}^{(p)}, x_{kj}^{(p)}) = 1 + \frac{1}{2}\left|x_{ij}^{(p)} - \frac{1}{2}\right| + \frac{1}{2}\left|x_{kj}^{(p)} - \frac{1}{2}\right| - \frac{1}{2}\left|x_{ij}^{(p)} - x_{kj}^{(p)}\right|$.

2.4. Design Efficiency Criterion

For any design of $d \in \mathcal{U}(n; q_1^{s_1} \times q_2^{s_2})$, under the effect sparsity principle, the three- or more factor interactions are ignored, and only the main effects and some two-factor interactions are considered. Let $\mathcal{C}(h)$ be the collection of all of the sets of h two-factor interactions, $1 \le h \le H$, $H = \frac{s(s-1)}{2}$. For $c \in \mathcal{C}(h)$, $M(c)$ denotes the model composed of only the general mean, all of the main effects and the h two-factor interactions in c; $X(c)$ is the model matrix under $M(c)$. The D-criterion aims at maximizing the determinant of the matrix$\{X'(c)X(c)\}$ (i.e., $det\{X'(c)X(c)\}$). If one wishes to include h two-factor interactions in the model, but they have no prior knowledge on which h should be included, then it makes sense to consider the average of $det\{X'(c)X(c)\}$ over all of $c \in \mathcal{C}(h)$. However, it is difficult to handle the D-criterion algebraically, and the minimization of trace of $\{X'(c)X(c)\}^2$ (i.e., $tr[\{X(c)'X(c)\}^2]$) is a good surrogate for the maximization of $det\{X'(c)X(c)\}$. In [11], it is defined as

$$E_h = \binom{H}{h}^{-1} \sum_{c \in \mathcal{C}(h)} tr[\{X(c)'X(c)\}^2],$$

the design efficiency criterion aims at studying designs that keep E_h small for each h, especially for smaller values of h, which are more relevant under the effect sparsity principle.

3. Design Efficiency of Mixed q_1- and q_2-Level Minimum Projection Uniform Designs

In this section, the design efficiency of mixed q_1- and q_2-level minimum projection uniform designs is discussed under the centered L_2 discrepancy.

For any design of $d \in \mathcal{U}(n; q_1^{s_1} \times q_2^{s_2})$, when all of the possible permutations for each factor of a design d are considered, we can obtain $(q_1!)^{s_1} \times (q_2!)^{s_2}$ combinatorially isomorphic designs, and the set of these designs is denoted as $\mathcal{F}(d)$. Similarly, we denote $\mathcal{F}(d_u)$ as the set of the projection designs d_u of $(q_1!)^{s_1} \times (q_2!)^{s_2}$ combinatorially isomorphic designs. The average-centered L_2 discrepancy value of all of the designs in $\mathcal{F}(d_u)$ is denoted by $\overline{[CD_u(d)]^2}$, which is

$$\overline{[CD_u(d)]^2} = \frac{1}{(q_1!)^{g_1} \times (q_2!)^{g_2}} \sum_{d'_u \in \mathcal{F}(d_u)} [CD_u(d)]^2. \tag{7}$$

The relationship between $\overline{[CD_u(d)]^2}$ and the distance distribution $C_{j_1 j_2}(d_u)$ is presented in the following lemma.

Lemma 1. *For any design $d \in \mathcal{U}(n; q_1^{s_1} \times q_2^{s_2})$, $u \in I_g$, $1 \le g (= g_1 + g_2) \le s$. Then*
(i) when q_1 is odd, q_2 is even,

$$\begin{aligned}\overline{[CD_u(d)]^2} =& \left(\frac{13}{12}\right)^g - 2\left(\frac{13q_1^2 - 1}{12q_1^2}\right)^{g_1} \left(\frac{26q_2^2 + 1}{24q_2^2}\right)^{g_2} + \frac{1}{n}\left(\frac{15q_1^2 - 3}{12q_1^2}\right)^{g_1} \left(\frac{5}{4}\right)^{g_2} \\ &\times \sum_{j_1=0}^{g_1} \sum_{j_2=0}^{g_2} \left(\frac{13q_1^2 - 2q_1 - 3}{15q_1^2 - 3}\right)^{j_1} \left(\frac{13q_2 - 2}{15q_2}\right)^{j_2} C_{j_1 j_2}(d_u),\end{aligned} \tag{8}$$

(ii) when q_1 and q_2 are odd,

$$\begin{aligned}\overline{[CD_u(d)]^2} =& \left(\frac{13}{12}\right)^g - 2\left(\frac{13q_1^2 - 1}{12q_1^2}\right)^{g_1} \left(\frac{13q_2^2 - 1}{12q_2^2}\right)^{g_2} + \frac{1}{n}\left(\frac{15q_1^2 - 3}{12q_1^2}\right)^{g_1} \left(\frac{15q_2^2 - 3}{12q_2^2}\right)^{g_2} \\ &\times \sum_{j_1=0}^{g_1} \sum_{j_2=0}^{g_2} \left(\frac{13q_1^2 - 2q_1 - 3}{15q_1^2 - 3}\right)^{j_1} \left(\frac{13q_2^2 - 2q_2 - 3}{15q_2^2 - 3}\right)^{j_2} C_{j_1 j_2}(d_u),\end{aligned} \tag{9}$$

(iii) when q_1 and q_2 are even,

$$\begin{aligned}\overline{[CD_u(d)]^2} =& \left(\frac{13}{12}\right)^g - 2\left(\frac{26q_1^2 + 1}{24q_1^2}\right)^{g_1} \left(\frac{26q_2^2 + 1}{24q_2^2}\right)^{g_2} + \frac{1}{n}\left(\frac{5}{4}\right)^{g} \\ &\times \sum_{j_1=0}^{g_1} \sum_{j_2=0}^{g_2} \left(\frac{13q_1 - 2}{15q_1}\right)^{j_1} \left(\frac{13q_2 - 2}{15q_2}\right)^{j_2} C_{j_1 j_2}(d_u).\end{aligned} \tag{10}$$

Proof of Lemma 1. Similar to the proof of Theorem 3.1 in [17], is found (i) when q_1 is odd and q_2 is even, and we have

$$\begin{aligned}\sum_{d'_u\in\mathcal{F}(d_u)}[CD_u(d)]^2 =&(q_1!)^{g_1}\times(q_2!)^{g_2}\left[\left(\frac{13}{12}\right)^g-2\left(\frac{13q_1^2-1}{12q_1^2}\right)^{g_1}\left(\frac{26q_2^2+1}{24q_2^2}\right)^{g_2}\right]\\&+\frac{(q_1!)^{g_1}\times(q_2!)^{g_2}}{n^2}\left(\frac{13q_1^2-2q_1-3}{12q_1^2}\right)^{g_1}\left(\frac{13q_2-2}{12q_2}\right)^{g_2}\sum_{i,k=1}^{n}\\&\times\left(\frac{15q_1^2-3}{13q_1^2-2q_1-3}\right)^{\delta_{ik}^{u_1}}\left(\frac{15q_2}{13q_2-2}\right)^{\delta_{ik}^{u_2}}.\end{aligned}$$

Combined with the definitions of δ_{ij}^u, $C_{v_1v_2}(d_u)$ and (7), (8) proves (ii) and (iii), which are are similar to (i). □

When design d is an $OA(n;q_1^{s_1}\times q_2^{s_2},t)$, for $u\in I_g$ and $1\le g(=g_1+g_2)\le t$, all of the possible $q_1^{g_1}\times q_2^{g_2}$-level combinations of the projection design d_u occur equally, often in any of the g columns. With row $i_u=(i_u^{(1)},i_u^{(2)})\in d_u$, it is easy to obtain that $\left|\{(i_u,k_u):d_H^{u_1}(i,k)=v_1,d_H^{u_2}(i,k)=v_2,k_u\in d\}\right|=\binom{g_1}{v_1}\binom{g_2}{v_2}\frac{n(q_1-1)^{v_1}(q_2-1)^{v_2}}{q_1^{g_1}q_2^{g_2}}$.

(i) when q_1 is odd, q_2 is even, and the third term of (8) can be expressed as

$$\begin{aligned}&\frac{1}{n}\left(\frac{15q_1^2-3}{12q_1^2}\right)^{g_1}\left(\frac{5}{4}\right)^{g_2}\sum_{j_1=0}^{g_1}\sum_{j_2=0}^{g_2}\left(\frac{13q_1^2-2q_1-3}{15q_1^2-3}\right)^{j_1}\left(\frac{13q_2-2}{15q_2}\right)^{j_2}C_{j_1j_2}(d_u)\\=&\frac{1}{n}\left(\frac{15q_1^2-3}{12q_1^2}\right)^{g_1}\left(\frac{5}{4}\right)^{g_2}\sum_{j_1=0}^{g_1}\sum_{j_2=0}^{g_2}\left[\left(\frac{13q_1^2-2q_1-3}{15q_1^2-3}\right)^{j_1}\left(\frac{13q_2-2}{15q_2}\right)^{j_2}\right.\\&\left.\times\binom{g_1}{j_1}\binom{g_2}{j_2}\frac{n(q_1-1)^{j_1}(q_2-1)^{j_2}}{q_1^{g_1}q_2^{g_2}}\right]\\=&\left(\frac{13q_1^2-1}{12q_1^2}\right)^{g_1}\left(\frac{13q_2^2+2}{12q_2^2}\right)^{g_2},\end{aligned}$$

so, (8) can be abbreviated as

$$\overline{[CD_u(d)]^2}=\left(\frac{13}{12}\right)^g-2\left(\frac{13q_1^2-1}{12q_1^2}\right)^{g_1}\left(\frac{26q_2^2+1}{24q_2^2}\right)^{g_2}+\left(\frac{13q_1^2-1}{12q_1^2}\right)^{g_1}\left(\frac{13q_2^2+2}{12q_2^2}\right)^{g_2}. \tag{11}$$

Similarly for (ii), when q_1 and q_2 are odd, (9) can be abbreviated as

$$\overline{[CD_u(d)]^2}=\left(\frac{13}{12}\right)^g-2\left(\frac{13q_1^2-1}{12q_1^2}\right)^{g_1}\left(\frac{13q_2^2-1}{12q_2^2}\right)^{g_2}+\left(\frac{13q_1^2-1}{12q_1^2}\right)^{g_1}\left(\frac{13q_2^2-1}{12q_2^2}\right)^{g_2}. \tag{12}$$

(ii) when q_1 and q_2 are even, (10) can be abbreviated as

$$\overline{[CD_u(d)]^2}=\left(\frac{13}{12}\right)^g-2\left(\frac{26q_1^2+1}{24q_1^2}\right)^{g_1}\left(\frac{26q_2^2+1}{24q_2^2}\right)^{g_2}+\left(\frac{13q_1^2+2}{12q_1^2}\right)^{g_1}\left(\frac{13q_2^2+2}{12q_2^2}\right)^{g_2}. \tag{13}$$

The following definition provides the uniformity pattern of design d under the centered L_2 discrepancy.

Definition 1. *For any design $d \in \mathcal{U}(n; q_1^{s_1} \times q_2^{s_2})$, $u \in I_g$, $1 \le g(= g_1 + g_2) \le s$,*
(i) when q_1 is odd, q_2 is even,

$$\overline{MI_g(d)} = \sum_{u \in I_g} \overline{[CD_u(d)]^2} - \sum_{g_1=0}^{g} \left[\left(\frac{13}{12}\right)^g - 2\left(\frac{13q_1^2 - 1}{12q_1^2}\right)^{g_1} \left(\frac{26q_2^2 + 1}{24q_2^2}\right)^{g_2} + \left(\frac{13q_1^2 - 1}{12q_1^2}\right)^{g_1} \left(\frac{13q_2^2 + 2}{12q_2^2}\right)^{g_2} \right],$$

(ii) when q_1 and q_2 are odd,

$$\overline{MI_g(d)} = \sum_{u \in I_g} \overline{[CD_u(d)]^2} - \sum_{g_1=0}^{g} \left[\left(\frac{13}{12}\right)^g - 2\left(\frac{13q_1^2 - 1}{12q_1^2}\right)^{g_1} \left(\frac{13q_2^2 - 1}{12q_2^2}\right)^{g_2} + \left(\frac{13q_1^2 - 1}{12q_1^2}\right)^{g_1} \left(\frac{13q_2^2 - 1}{12q_2^2}\right)^{g_2} \right],$$

(iii) while when q_1 and q_2 are even,

$$\overline{MI_g(d)} = \sum_{u \in I_g} \overline{[CD_u(d)]^2} - \sum_{g_1=0}^{g} \left[\left(\frac{13}{12}\right)^g - 2\left(\frac{26q_1^2 + 1}{24q_1^2}\right)^{g_1} \left(\frac{26q_2^2 + 1}{24q_2^2}\right)^{g_2} + \left(\frac{13q_1^2 + 2}{12q_1^2}\right)^{g_1} \left(\frac{13q_2^2 + 2}{12q_2^2}\right)^{g_2} \right].$$

The vector $(\overline{MI_1(d)}, \ldots, \overline{MI_s(d)})$ is called the uniformity pattern of design d.
According to Definition 1 and (11), (12) and (13), the following theorem can be obtained.

Theorem 1. *For a design $d \in \mathcal{U}(n; q_1^{s_1} \times q_2^{s_2})$, if and only if $\overline{MI_v(d)} = 0$ for $v = 1, \ldots, t$, and $\overline{MI_{t+1}(d)} > 0$, design d is an $OA(n; q_1^{s_1} \times q_2^{s_2}, t)$.*

Theorem 1 indicates that there is a close relationship between $\overline{MI_v(d)}$ of design d and an orthogonal array with a strength of t, which is to say that the closer $\overline{MI_t(d)}$ is to 0, then the closer the projection design is to the orthogonal arrays with a strength of t. It is shown that when the average-centered L_2 discrepancy of the projection designs is small, the orthogonality of the projection designs is also good. From the projection uniformity point of view, the uniformity pattern $\overline{MI_v(d)}$ may be used as a measure used to evaluate and compare the designs.
The definition of the minimum projection uniformity criterion is given below.

Definition 2. *For two designs d_1 and d_2 in $\mathcal{U}(n; q_1^{s_1} \times q_2^{s_2})$, let $\mathcal{R}$ be the smallest integer that makes $\overline{MI_{\mathcal{R}}(d_1)} \ne \overline{MI_{\mathcal{R}}(d_2)}$, and $\overline{MI_g(d_1)} = \overline{MI_g(d_2)}$ for $g = 1, \ldots, \mathcal{R} - 1$, and then d_1 has better projection uniformity than d_2 dies if $\overline{MI_{\mathcal{R}}(d_1)} < \overline{MI_{\mathcal{R}}(d_2)}$. In any design of the same scale, no other design = has better projection uniformity than design d_1 does; design d_1 is called the minimum projection uniform design.*

The following theorem builds a relationship between $\overline{MI_g(d)}$ and $A_{j_1 j_2}(d)$.

Theorem 2. *For any design $d \in \mathcal{U}(n; q_1^{s_1} \times q_2^{s_2})$, $u \in I_g$, $1 \le g(= g_1 + g_2) \le s$. Then* (i) *when q_1 is odd, q_2 is even,*

$$\overline{MI_g(d)} = \sum_{g_1=0}^{g} \left(\frac{13q_1^2 - 1}{12q_1^2}\right)^{g_1} \left(\frac{13q_2^2 + 2}{12q_2^2}\right)^{g_2} \sum_{(j_1,j_2)\in N} \left(\frac{2q_1 + 2}{13q_1^2 - 1}\right)^{j_1} \left(\frac{2q_2 + 2}{13q_2^2 + 2}\right)^{j_2} \times \binom{s_1 - j_1}{s_1 - g_1} \binom{s_2 - j_2}{s_2 - g_2} A_{j_1 j_2}(d), \tag{14}$$

$$A_g(d) = \sum_{g_1=0}^{g} \left(-\frac{13q_1^2 - 1}{2q_1 + 2}\right)^{g_1} \left(-\frac{13q_2^2 + 2}{2q_2 + 2}\right)^{g_2} \sum_{(j_1,j_2)\in N} \left(-\frac{12q_1^2}{13q_1^2 - 1}\right)^{j_1} \left(-\frac{12q_2^2}{13q_2^2 + 2}\right)^{j_2} \times \binom{s_1 - j_1}{s_1 - g_1} \binom{s_2 - j_2}{s_2 - g_2} \overline{MI_{j_1 j_2}(d)}, \tag{15}$$

(ii) *when q_1 and q_2 are odd,*

$$\overline{MI_g(d)} = \sum_{g_1=0}^{g} \left(\frac{13q_1^2 - 1}{12q_1^2}\right)^{g_1} \left(\frac{13q_2^2 - 1}{12q_2^2}\right)^{g_2} \sum_{(j_1,j_2)\in N} \left(\frac{2q_1 + 2}{13q_1^2 - 1}\right)^{j_1} \left(\frac{2q_2 + 2}{13q_2^2 - 1}\right)^{j_2} \times \binom{s_1 - j_1}{s_1 - g_1} \binom{s_2 - j_2}{s_2 - g_2} A_{j_1 j_2}(d), \tag{16}$$

$$A_g(d) = \sum_{g_1=0}^{g} \left(-\frac{13q_1^2 - 1}{2q_1 + 2}\right)^{g_1} \left(-\frac{13q_2^2 - 1}{2q_2 + 2}\right)^{g_2} \sum_{(j_1,j_2)\in N} \left(-\frac{12q_1^2}{13q_1^2 - 1}\right)^{j_1} \left(-\frac{12q_2^2}{13q_2^2 - 1}\right)^{j_2} \times \binom{s_1 - j_1}{s_1 - g_1} \binom{s_2 - j_2}{s_2 - g_2} \overline{MI_{j_1 j_2}(d)}, \tag{17}$$

(iii) *while when q_1 and q_2 are even,*

$$\overline{MI_g(d)} = \sum_{g_1=0}^{g} \left(\frac{13q_1^2 + 2}{12q_1^2}\right)^{g_1} \left(\frac{13q_2^2 + 2}{12q_2^2}\right)^{g_2} \sum_{(j_1,j_2)\in N} \left(\frac{2q_1 + 2}{13q_1^2 + 2}\right)^{j_1} \left(\frac{2q_2 + 2}{13q_2^2 + 2}\right)^{j_2} \times \binom{s_1 - j_1}{s_1 - g_1} \binom{s_2 - j_2}{s_2 - g_2} A_{j_1 j_2}(d), \tag{18}$$

$$A_g(d) = \sum_{g_1=0}^{g} \left(-\frac{13q_1^2 + 2}{2q_1 + 2}\right)^{g_1} \left(-\frac{13q_2^2 + 2}{2q_2 + 2}\right)^{g_2} \sum_{(j_1,j_2)\in N} \left(-\frac{12q_1^2}{13q_1^2 + 2}\right)^{j_1} \left(-\frac{12q_2^2}{13q_2^2 + 2}\right)^{j_2} \times \binom{s_1 - j_1}{s_1 - g_1} \binom{s_2 - j_2}{s_2 - g_2} \overline{MI_{j_1 j_2}(d)}, \tag{19}$$

where $N = \{(j_1, j_2) : j_1 = 0, \ldots, g_1, j_2 = 0, \ldots, g_2, (j_1, j_2) \neq (0,0)\}$, $\overline{MI_j(d)} = \sum_{j_1+j_2=j} \overline{MI_{j_1 j_2}(d)}$.

Proof of Theorem 2. (i) when q_1 is odd, q_2 is even,

$$\begin{aligned}\overline{MI_g(d)} &= \sum_{u\in I_g}\overline{[CD_u(d)]^2} - \sum_{g_1=0}^{g}\sum_{u\in I_{g_1g_2}}\left[\left(\frac{13}{12}\right)^g - 2\left(\frac{13q_1^2-1}{12q_1^2}\right)^{g_1}\left(\frac{26q_2^2+1}{24q_2^2}\right)^{g_2}\right.\\ &\left.+\left(\frac{13q_1^2-1}{12q_1^2}\right)^{g_1}\left(\frac{13q_2^2+2}{12q_2^2}\right)^{g_2}\right]\\ &= \sum_{g_1=0}^{g}\sum_{u\in I_{g_1g_2}}\left[-\left(\frac{13q_1^2-1}{12q_1^2}\right)^{g_1}\left(\frac{13q_2^2+2}{12q_2^2}\right)^{g_2} + \frac{1}{n}\left(\frac{15q_1^2-3}{12q_1^2}\right)^{g_1}\left(\frac{5}{4}\right)^{g_2}\sum_{v_1=0}^{g_1}\sum_{v_2=0}^{g_2}\right.\\ &\left.\times\left(\frac{13q_1^2-2q_1-3}{15q_1^2-3}\right)^{v_1}\left(\frac{13q_2-2}{15q_2}\right)^{v_2}C_{v_1v_2}(d_u)\right]\end{aligned}$$

if we combine $C_{v_1v_2}(d_u) = n(\frac{1}{q_1})^{g_1}(\frac{1}{q_2})^{g_2}\sum_{j_1=0}^{g_1}\sum_{j_2=0}^{g_2}P_{v_1}(j_1;g_1,q_1)P_{v_2}(j_2;g_2,q_2)A_{j_1j_2}(d_u)$ with $\sum_{u\in I_{g_1g_2}}A_{j_1j_2}(d_u) = \binom{s_1-j_1}{s_1-g_1}\binom{s_2-j_2}{s_2-g_2}A_{j_1j_2}(d)$,

$$\begin{aligned}\overline{MI_g(d)} &= \sum_{g_1=0}^{g}\sum_{u\in I_{g_1g_2}}\left[-\left(\frac{13q_1^2-1}{12q_1^2}\right)^{g_1}\left(\frac{13q_2^2+2}{12q_2^2}\right)^{g_2} + \left(\frac{13q_1^2-1}{12q_1^2}\right)^{g_1}\left(\frac{13q_2^2+2}{12q_2^2}\right)^{g_2}\right.\\ &\left.\times\sum_{j_1=0}^{g_1}\sum_{j_2=0}^{g_2}\left(\frac{2q_1+2}{13q_1^2-1}\right)^{j_1}\left(\frac{2q_2+2}{13q_2^2+2}\right)^{j_2}A_{j_1j_2}(d_u)\right]\\ &= \sum_{g_1=0}^{g}\left(\frac{13q_1^2-1}{12q_1^2}\right)^{g_1}\left(\frac{13q_2^2+2}{12q_2^2}\right)^{g_2}\sum_{(j_1,j_2\in N)}\left(\frac{2q_1+2}{13q_1^2-1}\right)^{j_1}\left(\frac{2q_2+2}{13q_2^2+2}\right)^{j_2}\\ &\times\binom{s_1-j_1}{s_1-g_1}\binom{s_2-j_2}{s_2-g_2}A_{j_1j_2}(d),\end{aligned}$$

(14) remains the same and (15) can be obtained using simple algebra from (14) and mathematical induction; the proofs of (ii) and (iii) are similar to (i), so Theorem 2 is proved. □

When $q_1 = q_2 = q$ in Theorem 2, the following corollary is obtained, which is consistent with the conclusion in [15].

Corollary 1. *For any design $d \in \mathcal{U}(n;q^s)$, $u \in I_g$, $1 \le g \le s$. Then when q is odd,*

$$\overline{MI_g(d)} = \left(\frac{13q^2-1}{12q^2}\right)^g\sum_{i=1}^{g}\left(\frac{2q+2}{13q^2-1}\right)^i\binom{s-i}{s-g}A_i(d),$$

$$A_g(d) = \left(-\frac{13q^2-1}{2q+2}\right)^g\sum_{i=1}^{g}\left(-\frac{12q^2}{13q^2-1}\right)^i\binom{s-i}{s-g}\overline{MI_i(d)},$$

when q is even,

$$\overline{MI_g(d)} = \left(\frac{13q^2+2}{12q^2}\right)^g\sum_{i=1}^{g}\left(\frac{2q+2}{13q^2+2}\right)^i\binom{s-i}{s-g}A_i(d),$$

$$A_g(d) = \left(-\frac{13q^2+2}{2q+2}\right)^g \sum_{i=1}^{g} \left(-\frac{12q^2}{13q^2+2}\right)^i \binom{s-i}{s-g} \overline{MI_i(d)}.$$

In order to discuss the design efficiency of minimum projection uniform designs, the relationship between $\overline{MI_g(d)}$ and $B_{j_1j_2}(d)$ is firstly presented in the following lemma.

Lemma 2. *For any design* $d \in \mathcal{U}(n; q_1^{s_1} \times q_2^{s_2})$, $u \in I_g$, $1 \leq g(= g_1 + g_2) \leq s$. *Then* (i) *when* q_1 *is odd,* q_2 *is even,*

$$\overline{MI_g(d)} = \frac{q_1^{s_1} \times q_2^{s_2}}{n^2} \sum_{g_1=0}^{g} \left(\frac{13q_1^2-1}{12q_1^2}\right)^{g_1} \left(\frac{13q_2^2+2}{12q_2^2}\right)^{g_2} \sum_{(j_1,j_2)\in N} \left(\frac{2q_1+2}{13q_1^2-1}\right)^{j_1} \times \left(\frac{2q_2+2}{13q_2^2+2}\right)^{j_2} \binom{s_1-j_1}{s_1-g_1} \binom{s_2-j_2}{s_2-g_2} B_{j_1j_2}(d), \tag{20}$$

$$B_g(d) = \frac{n^2}{q_1^{s_1} \times q_2^{s_2}} \sum_{g_1=0}^{g} \left(-\frac{13q_1^2-1}{2q_1+2}\right)^{g_1} \left(-\frac{13q_2^2+2}{2q_2+2}\right)^{g_2} \sum_{(j_1,j_2)\in N} \left(-\frac{12q_1^2}{13q_1^2-1}\right)^{j_1} \times \left(-\frac{12q_2^2}{13q_2^2+2}\right)^{j_2} \binom{s_1-j_1}{s_1-g_1} \binom{s_2-j_2}{s_2-g_2} \overline{MI_{j_1j_2}(d)}, \tag{21}$$

(ii) *when* q_1 *and* q_2 *are odd,*

$$\overline{MI_g(d)} = \frac{q_1^{s_1} \times q_2^{s_2}}{n^2} \sum_{g_1=0}^{g} \left(\frac{13q_1^2-1}{12q_1^2}\right)^{g_1} \left(\frac{13q_2^2-1}{12q_2^2}\right)^{g_2} \sum_{(j_1,j_2)\in N} \left(\frac{2q_1+2}{13q_1^2-1}\right)^{j_1} \times \left(\frac{2q_2+2}{13q_2^2-1}\right)^{g_2} \binom{s_1-j_1}{s_1-g_1} \binom{s_2-j_2}{s_2-g_2} B_{j_1j_2}(d), \tag{22}$$

$$B_g(d) = \frac{n^2}{q_1^{s_1} \times q_2^{s_2}} \sum_{g_1=0}^{g} \left(-\frac{13q_1^2-1}{2q_1+2}\right)^{g_1} \left(-\frac{13q_2^2-1}{2q_2+2}\right)^{g_2} \sum_{(j_1,j_2)\in N} \left(-\frac{12q_1^2}{13q_1^2-1}\right)^{j_1} \times \left(-\frac{12q_2^2}{13q_2^2-1}\right)^{j_2} \binom{s_1-j_1}{s_1-g_1} \binom{s_2-j_2}{s_2-g_2} \overline{MI_{j_1j_2}(d)}, \tag{23}$$

(iii) *while when* q_1 *and* q_2 *are even,*

$$\overline{MI_g(d)} = \frac{q_1^{s_1} \times q_2^{s_2}}{n^2} \sum_{g_1=0}^{g} \left(\frac{13q_1^2+2}{12q_1^2}\right)^{g_1} \left(\frac{13q_2^2+2}{12q_2^2}\right)^{g_2} \sum_{(j_1,j_2)\in N} \left(\frac{2q_1+2}{13q_1^2+2}\right)^{j_1} \times \left(\frac{2q_2+2}{13q_2^2+2}\right)^{g_2} \binom{s_1-j_1}{s_1-g_1} \binom{s_2-j_2}{s_2-g_2} B_{j_1j_2}(d), \tag{24}$$

$$B_g(d) = \frac{n^2}{q_1^{s_1} \times q_2^{s_2}} \sum_{g_1=0}^{g} \left(-\frac{13q_1^2+2}{2q_1+2}\right)^{g_1} \left(-\frac{13q_2^2+2}{2q_2+2}\right)^{g_2} \sum_{(j_1,j_2)\in N} \left(-\frac{12q_1^2}{13q_1^2+2}\right)^{j_1}$$

$$\times \left(-\frac{12q_2^2}{13q_2^2+2}\right)^{j_2} \binom{s_1-j_1}{s_1-g_1} \binom{s_2-j_2}{s_2-g_2} \overline{MI_{j_1j_2}(d)}. \tag{25}$$

Proof of Lemma 2. In [18], $B_{j_1j_2}(d) = \frac{n^2}{q_1^{s_1} \times q_2^{s_2}} A_{j_1j_2}(d)$, and if we combine it with Theorem 2, Lemma 2 is proved. □

When design d is an $OA\big(n; q_1^{s_1} \times q_2^{s_2}, 2\big)$, for any $h = 1, \ldots, H$, $H = \frac{s(s-1)}{2}$, $1 \le j < k < l \le s$, we have the relationships between design efficiency E_h^* and $B_3(d), B_4(d)$ in the B vector and the related quantities $B(jkl)$ in [11].

$$E_h^* = c^* + 6f\left[B_3(d) + \frac{h-1}{H-1}\left(B_4(d) - 2B_3(d) + \frac{1}{3}\sum_{jkl\in\Delta(3)}\big(q_j+q_k+q_l\big)B(jkl)\right)\right], \tag{26}$$

where c^* is a constant that may depend on n, s, q_1, q_2 and h. q_j, q_k and q_l are the levels of the jth, kth and lth factors in design d, respectively. $f = q_1^{s_1} \times q_2^{s_2}$, $B_3(d) = \sum_{jkl\in\Delta(3)} B(jkl)$, $B(jkl) = y_d' W(x(jkl)) y_d$, $x(jkl)$ is the binary s-tuple that has ones in the jth, kth and lth levels and zeros elsewhere, while $\Delta(3)$ is the set of all of the ordered triplets j, k and l.

The following theorem builds a relationship between E_h^*, $B(jkl)$ and $\overline{MI_{j_1j_2}(d)}$, where $j_1 + j_2 = 3, 4$.

Theorem 3. *Let design d be an $OA\big(n; q_1^{s_1} \times q_2^{s_2}, 2\big)$. Then*
(i) when q_1 is odd, q_2 is even,

$$E_h^* = c^* + 6n^2\left[\left(1 - 2\frac{h-1}{H-1}\right)z_1 + \frac{h-1}{H-1}z_2\right] + 2f\frac{h-1}{H-1}\sum_{jkl\in\Delta(3)}\big(q_j+q_k+q_l\big)B(jkl), \tag{27}$$

(ii) when q_1 and q_2 are odd,

$$E_h^* = c^* + 6n^2\left[\left(1 - 2\frac{h-1}{H-1}\right)z_3 + \frac{h-1}{H-1}z_4\right] + 2f\frac{h-1}{H-1}\sum_{jkl\in\Delta(3)}\big(q_j+q_k+q_l\big)B(jkl), \tag{28}$$

(iii) while when q_1 and q_2 are even,

$$E_h^* = c^* + 6n^2\left[\left(1 - 2\frac{h-1}{H-1}\right)z_5 + \frac{h-1}{H-1}z_6\right] + 2f\frac{h-1}{H-1}\sum_{jkl\in\Delta(3)}\big(q_j+q_k+q_l\big)B(jkl), \tag{29}$$

where

$$\begin{aligned} z_1 = z_3 = z_5 = &\frac{216q_2^6}{(q_2+1)^3}\overline{MI_{03}(d)} + \frac{216q_1^2q_2^4}{(q_1+1)(q_2+1)^2}\overline{MI_{12}(d)} \\ &+ \frac{216q_1^4q_2^2}{(q_1+1)^2(q_2+1)}\overline{MI_{21}(d)} + \frac{216q_1^6}{(q_1+1)^3}\overline{MI_{30}(d)}, \end{aligned}$$

$$
\begin{aligned}
z_2 = & \frac{1296q_2^8}{(q_2+1)^4}\overline{MI_{04}(d)} + \frac{1296q_1^8}{(q_1+1)^4}\overline{MI_{40}(d)} + \frac{1296q_1^2q_2^6}{(q_1+1)(q_2+1)^3}\overline{MI_{13}(d)} + \frac{1296q_1^4q_2^4}{(q_1+1)^2(q_2+1)^2} \\
& \times \overline{MI_{22}(d)} + \frac{1296q_1^6q_2^2}{(q_1+1)^3(q_2+1)}\overline{MI_{31}(d)} - \left(\frac{1404q_2^8+216q_2^6}{(q_2+1)^4}(s_2-3) + \frac{1404q_1^2q_2^6-108q_2^6}{(q_1+1)(q_2+1)^3}s_1\right) \\
& \overline{MI_{03}(d)} - \left(\frac{1404q_1^2q_2^6+216q_1^2q_2^4}{(q_1+1)(q_2+1)^3}(s_2-2) + \frac{1404q_1^4q_2^4-108q_1^2q_2^4}{(q_1+1)^2(q_2+1)^2}(s_1-1)\right)\overline{MI_{12}(d)} \\
& - \left(\frac{1404q_1^4q_2^4+216q_1^4q_2^2}{(q_1+1)^2(q_2+1)^2}(s_2-1) + \frac{1404q_1^6q_2^2-108q_1^4q_2^2}{(q_1+1)^3(q_2+1)}(s_1-2)\right)\overline{MI_{21}(d)} \\
& - \left(\frac{1404q_1^6q_2^2+216q_1^6}{(q_1+1)^3(q_2+1)}s_2 + \frac{1404q_1^8-108q_1^6}{(q_1+1)^4}(s_1-3)\right)\overline{MI_{30}(d)},
\end{aligned}
$$

$$
\begin{aligned}
z_4 = & \frac{1296q_2^8}{(q_2+1)^4}\overline{MI_{04}(d)} + \frac{1296q_1^8}{(q_1+1)^4}\overline{MI_{40}(d)} + \frac{1296q_1^2q_2^6}{(q_1+1)(q_2+1)^3}\overline{MI_{13}(d)} + \frac{1296q_1^4q_2^4}{(q_1+1)^2(q_2+1)^2} \\
& \times \overline{MI_{22}(d)} + \frac{1296q_1^6q_2^2}{(q_1+1)^3(q_2+1)}\overline{MI_{31}(d)} - \left(\frac{1404q_2^8-108q_2^6}{(q_2+1)^4}(s_2-3) + \frac{1404q_1^2q_2^6-108q_2^6}{(q_1+1)(q_2+1)^3}s_1\right) \\
& \overline{MI_{03}(d)} - \left(\frac{1404q_1^2q_2^6-108q_1^2q_2^4}{(q_1+1)(q_2+1)^3}(s_2-2) + \frac{1404q_1^4q_2^4-108q_1^2q_2^4}{(q_1+1)^2(q_2+1)^2}(s_1-1)\right)\overline{MI_{12}(d)} \\
& - \left(\frac{1404q_1^4q_2^4-108q_1^4q_2^2}{(q_1+1)^2(q_2+1)^2}(s_2-1) + \frac{1404q_1^6q_2^2-108q_1^4q_2^2}{(q_1+1)^3(q_2+1)}(s_1-2)\right)\overline{MI_{21}(d)} \\
& - \left(\frac{1404q_1^6q_2^2-108q_1^6}{(q_1+1)^3(q_2+1)}s_2 + \frac{1404q_1^8-108q_1^6}{(q_1+1)^4}(s_1-3)\right)\overline{MI_{30}(d)},
\end{aligned}
$$

$$
\begin{aligned}
z_6 = & \frac{1296q_2^8}{(q_2+1)^4}\overline{MI_{04}(d)} + \frac{1296q_1^8}{(q_1+1)^4}\overline{MI_{40}(d)} + \frac{1296q_1^2q_2^6}{(q_1+1)(q_2+1)^3}\overline{MI_{13}(d)} + \frac{1296q_1^4q_2^4}{(q_1+1)^2(q_2+1)^2} \\
& \times \overline{MI_{22}(d)} + \frac{1296q_1^6q_2^2}{(q_1+1)^3(q_2+1)}\overline{MI_{31}(d)} - \left(\frac{1404q_2^8+216q_2^6}{(q_2+1)^4}(s_2-3) + \frac{1404q_1^2q_2^6+216q_2^6}{(q_1+1)(q_2+1)^3}s_1\right) \\
& \overline{MI_{03}(d)} - \left(\frac{1404q_1^2q_2^6+216q_1^2q_2^4}{(q_1+1)(q_2+1)^3}(s_2-2) + \frac{1404q_1^4q_2^4+216q_1^2q_2^4}{(q_1+1)^2(q_2+1)^2}(s_1-1)\right)\overline{MI_{12}(d)} \\
& - \left(\frac{1404q_1^4q_2^4+216q_1^4q_2^2}{(q_1+1)^2(q_2+1)^2}(s_2-1) + \frac{1404q_1^6q_2^2+216q_1^4q_2^2}{(q_1+1)^3(q_2+1)}(s_1-2)\right)\overline{MI_{21}(d)} \\
& - \left(\frac{1404q_1^6q_2^2+216q_1^6}{(q_1+1)^3(q_2+1)}s_2 + \frac{1404q_1^8+216q_1^6}{(q_1+1)^4}(s_1-3)\right)\overline{MI_{30}(d)}.
\end{aligned}
$$

Proof of Theorem 3. The proof of Theorem 3 is obtained by combining Lemma 2 and (26). □

Theorem 3 shows that the design efficiency of an orthogonal array with a strength of two depends on $\overline{MI_{j_1j_2}(d)}$, where $j_1 + j_2 = 3, 4$. In particular, when design d is an $OA(n; q_1^{s_1} \times q_2^{s_2}, 3)$,
$\overline{MI_{j_1j_2}(d)} = 0$, for $j_1 + j_2 = 3$, we can obtain the following corollary.

Corollary 2. *Let design d be an $OA(n; q_1^{s_1} \times q_2^{s_2}, 3)$. Then*
(i) when q_1 is odd, q_2 is even,

$$
E_h^* = c^* + 6n^2\left[\frac{h-1}{H-1}z_2^*\right],
$$

(ii) *when* q_1 *and* q_2 *are odd,*

$$E_h^* = c^* + 6n^2\left[\frac{h-1}{H-1}z_4^*\right],$$

(iii) *while when* q_1 *and* q_2 *are even,*

$$E_h^* = c^* + 6n^2\left[\frac{h-1}{H-1}z_6^*\right],$$

where z_2^*, z_4^* *and* z_6^* *are* z_2, z_4 *and* z_6, *which satisfy* $\overline{MI_{j_1j_2}(d)} = 0$ $(j_1+j_2=3)$ *in Theorem 3.*

Corollary 2 indicates that for an orthogonal array with a strength of three, the MPU criterion is completely equivalent to the design efficiency criterion.

4. A Lower Bound of Uniformity Pattern

This section gives a lower bound of the uniformity pattern in Definition 1; the lower bound provides a basis for measuring the uniformity of the projection designs. Two lemmas are given below, which are important to obtain the lower bound of the uniformity pattern.

Lemma 3. *For any design* $d \in \mathcal{U}(n; q_1^{s_1} \times q_2^{s_2})$, $u \in I_g$, $1 \le g(= g_1 + g_2) \le s$, *we have*
(i) *when* q_1 *is odd,* q_2 *is even,*

$$\sum_{i=1}^{n}\sum_{k(\neq i)=1}^{n}\delta_{ik}^{u_1} = \frac{n(n-q_1)g_1}{q_1},\quad \sum_{i=1}^{n}\sum_{k(\neq i)=1}^{n}\delta_{ik}^{u_2} = \frac{n(n-q_2)g_2}{q_2},$$
$$\sum_{i=1}^{n}\sum_{k(\neq i)=1}^{n}\phi_{ik}^{u} = \ln\left(\frac{15q_1^2-3}{13q_1^2-2q_1-3}\right)\frac{n(n-q_1)g_1}{q_1} + \ln\left(\frac{15q_2}{13q_2-2}\right)\frac{n(n-q_2)g_2}{q_2} \triangleq \bigtriangledown_i,$$

(ii) *when* q_1 *and* q_2 *are odd,*

$$\sum_{i=1}^{n}\sum_{k(\neq i)=1}^{n}\phi_{ik}^{u} = \ln\left(\frac{15q_1^2-3}{13q_1^2-2q_1-3}\right)\frac{n(n-q_1)g_1}{q_1} + \ln\left(\frac{15q_2^2-3}{13q_2^2-2q_2-3}\right)\frac{n(n-q_2)g_2}{q_2} \triangleq \bigtriangledown_{ii},$$

(iii) *while when* q_1 *and* q_2 *are even,*

$$\sum_{i=1}^{n}\sum_{k(\neq i)=1}^{n}\phi_{ik}^{u} = \ln\left(\frac{15q_1}{13q_1-2}\right)\frac{n(n-q_1)g_1}{q_1} + \ln\left(\frac{15q_2}{13q_2-2}\right)\frac{n(n-q_2)g_2}{q_2} \triangleq \bigtriangledown_{iii}.$$

Lemma 3 is obvious, so the proof is omitted.

Lemma 4 ([19]). *Let* $a_1, a_2, \ldots, a_n$, $b_1, b_2, \ldots, b_n$ *be two sets of non-negative real numbers and satisfy* $\sum_{i=1}^n a_i = r_1$, $\sum_{i=1}^n b_i = r_2$. *For* $i = 1, 2, \ldots, n$, *let* $\Gamma_i = Aa_i + Bb_i$, *and* $c' = Ar_1 + Br_2$, *where* $A > 0, B > 0$. *Denote* $\Gamma_{(1)}, \ldots, \Gamma_{(i)}$ *as the ordered arrangements of the distinct possible values of* $\Gamma_1, \ldots, \Gamma_n$, *where* $1 \le i \le n$. *For any integer* r,

$$\sum_{i=1}^{n}\Gamma_i^r \ge P\Gamma_{(v)}^r + Q\Gamma_{(v+1)}^r,$$

where v *is the largest integer, such that* $\Gamma_{(v)} \le c'/n < \Gamma_{(v+1)}$, P *and* Q *are non-negative integers, such that* $P + Q = n$ *and* $P\Gamma_{(v)} + Q\Gamma_{(v+1)} = c'$.

A lower bound of the uniformity pattern $\overline{MI_g(d)}$ of design d is given in the following lemma.

Lemma 5. *For any design $d \in \mathcal{U}(n; q_1^{s_1} \times q_2^{s_2})$, $u \in I_g$, $1 \leq g(= g_1 + g_2) \leq s$, we have*

$$\overline{MI_g(d)} \geq LB_1\left(\overline{MI_g(d)}\right),$$

(i) when q_1 is odd, q_2 is even,

$$LB_1\left(\overline{MI_g(d)}\right) = \frac{1}{n^2} \sum_{g_1=0}^{g} \sum_{u \in I_{g_1 g_2}} \left(\frac{13q_1^2 - 2q_1 - 3}{12q_1^2}\right)^{g_1} \left(\frac{13q_2 - 2}{12q_2}\right)^{g_2} \times \left[P_u e^{\phi^u_{(m_u)}} + Q_u e^{\phi^u_{(m_u+1)}}\right] + c_i, \tag{30}$$

where m_u is the largest integer, such that $\phi^u_{(m_u)} \leq \frac{\nabla i}{n(n-1)} < \phi^u_{(m_u+1)}$, P_u and Q_u are non-negative real numbers, such that $P_u + Q_u = n(n-1)$ and $P_u\phi^u_{(m_u)} + Q_u\phi^u_{(m_u+1)} = \nabla i$, $c_i = \sum_{g_1=0}^{g}\left[\frac{1}{n}\left(\frac{15q_1^2-3}{12q_1^2}\right)^{g_1}\left(\frac{5}{4}\right)^{g_2} - \left(\frac{13q_1^2-1}{12q_1^2}\right)^{g_1}\left(\frac{13q_2^2+2}{12q_2^2}\right)^{g_2}\right]$;
(ii) when q_1 and q_2 are odd,

$$LB_1\left(\overline{MI_g(d)}\right) = \frac{1}{n^2} \sum_{g_1=0}^{g} \sum_{u \in I_{g_1 g_2}} \left(\frac{13q_1^2 - 2q_1 - 3}{12q_1^2}\right)^{g_1} \left(\frac{13q_2^2 - 2q_2 - 3}{12q_2^2}\right)^{g_2} \times \left[P_u e^{\phi^u_{(m_u)}} + Q_u e^{\phi^u_{(m_u+1)}}\right] + c_{ii}, \tag{31}$$

where m_u is the largest integer, such that $\phi^u_{(m_u)} \leq \frac{\nabla ii}{n(n-1)} < \phi^u_{(m_u+1)}$, P_u and Q_u are non-negative real numbers, such that $P_u + Q_u = n(n-1)$ and $P_u\phi^u_{(m_u)} + Q_u\phi^u_{(m_u+1)} = \nabla ii$, $c_{ii} = \sum_{g_1=0}^{g}\left[\frac{1}{n}\left(\frac{15q_1^2-3}{12q_1^2}\right)^{g_1}\left(\frac{15q_2^2-3}{12q_2^2}\right)^{g_2} - \left(\frac{13q_1^2-1}{12q_1^2}\right)^{g_1}\left(\frac{13q_2^2-1}{12q_2^2}\right)^{g_2}\right]$;
(iii) when q_1 and q_2 are even,

$$LB_1\left(\overline{MI_g(d)}\right) = \frac{1}{n^2} \sum_{g_1=0}^{g} \sum_{u \in I_{g_1 g_2}} \left(\frac{13q_1 - 2}{12q_1}\right)^{g_1} \left(\frac{13q_2 - 2}{12q_2}\right)^{g_2} \times \left[P_u e^{\phi^u_{(m_u)}} + Q_u e^{\phi^u_{(m_u+1)}}\right] + c_{iii}, \tag{32}$$

where m_u is the largest integer, such that $\phi^u_{(m_u)} \leq \frac{\nabla iii}{n(n-1)} < \phi^u_{(m_u+1)}$, P_u and Q_u are non-negative real numbers, such that $P_u + Q_u = n(n-1)$ and $P_u\phi^u_{(m_u)} + Q_u\phi^u_{(m_u+1)} = \nabla iii$, $c_{iii} = \sum_{g_1=0}^{g}\left[\frac{1}{n}\left(\frac{5}{4}\right)^{g} - \left(\frac{13q_1^2+2}{12q_1^2}\right)^{g_1}\left(\frac{13q_2^2+2}{12q_2^2}\right)^{g_2}\right]$.

Proof of Lemma 5. (i) when q_1 is odd, q_2 is even, and from Definition 1 and Lemma 3,

$$\begin{aligned}
\overline{MI_g(d)} &= \sum_{u \in I_g} \overline{[CD_u(d)]^2} - \sum_{g_1=0}^{g} \sum_{u \in I_{g_1 g_2}} \left[\left(\frac{13}{12}\right)^{g} - 2\left(\frac{13q_1^2 - 1}{12q_1^2}\right)^{g_1}\left(\frac{26q_2^2+1}{24q_2^2}\right)^{g_2} \right. \\
&\quad \left. + \left(\frac{13q_1^2 - 1}{12q_1^2}\right)^{g_1}\left(\frac{13q_2^2+2}{12q_2^2}\right)^{g_2}\right] \\
&= c_i + \frac{1}{n^2} \sum_{g_1=0}^{g} \sum_{u \in I_{g_1 g_2}} \left(\frac{13q_1^2 - 2q_1 - 3}{12q_1^2}\right)^{g_1} \left(\frac{13q_2 - 2}{12q_2}\right)^{g_2} \sum_{i=1}^{n} \sum_{k(\neq i)=1}^{n} e^{\phi^u_{ik}} \\
&= c_i + \frac{1}{n^2} \sum_{g_1=0}^{g} \sum_{u \in I_{g_1 g_2}} \left(\frac{13q_1^2 - 2q_1 - 3}{12q_1^2}\right)^{g_1} \left(\frac{13q_2 - 2}{12q_2}\right)^{g_2} \sum_{i=1}^{n} \sum_{k(\neq i)=1}^{n} \left[\sum_{t=0}^{\infty} \frac{(\phi^u_{ik})^t}{t!}\right],
\end{aligned}$$

and by Lemma 4,

$$\overline{MI_g(d)} \geq c_i + \frac{1}{n^2}\sum_{g_1=0}^{g}\sum_{u\in I_{g_1g_2}}\left(\frac{13q_1^2-2q_1-3}{12q_1^2}\right)^{g_1}\left(\frac{13q_2-2}{12q_2}\right)^{g_2}\left[P_u e^{\phi^u_{(m_u)}} + Q_u e^{\phi^u_{(m_u+1)}}\right].$$

The proofs of (ii) and (iii) are similar to (i), so Lemma 5 is proved. $\square$

Next, another lower bound of the uniformity pattern $\overline{MI_g(d)}$ is obtained.

Lemma 6. *For any design $d \in \mathcal{U}(n; q_1^{s_1} \times q_2^{s_2})$, $u \in I_g$, $1 \leq g(= g_1 + g_2) \leq s$, we have*

$$\overline{MI_g(d)} \geq LB_2\left(\overline{MI_g(d)}\right),$$

(i) when q_1 is odd, q_2 is even,

$$\begin{aligned} LB_2(\overline{MI_g(d)}) = \sum_{g_1=0}^{g}\Bigg[&\frac{1}{n^2}\left(\frac{13q_1^2-2q_1-3}{12q_1^2}\right)^{g_1}\left(\frac{13q_2-2}{12q_2}\right)^{g_2}\sum_{j_1=0}^{g_1}\sum_{j_2=0}^{g_2}\binom{g_1}{j_1}\binom{g_2}{j_2} \\ &\times\left(\frac{2q_1^2+2q_1}{13q_1^2-2q_1-3}\right)^{j_1}\left(\frac{2q_2+2}{13q_2-2}\right)^{j_2}\theta_{j_1j_2} - \left(\frac{13q_1^2-1}{12q_1^2}\right)^{g_1}\left(\frac{13q_2^2+2}{12q_2^2}\right)^{g_2}\Bigg], \end{aligned} \tag{33}$$

(ii) when q_1 and q_2 are odd,

$$\begin{aligned} LB_2(\overline{MI_g(d)}) = \sum_{g_1=0}^{g}\Bigg[&\frac{1}{n^2}\left(\frac{13q_1^2-2q_1-3}{12q_1^2}\right)^{g_1}\left(\frac{13q_2^2-2q_2-3}{12q_2^2}\right)^{g_2}\sum_{j_1=0}^{g_1}\sum_{j_2=0}^{g_2}\binom{g_1}{j_1}\binom{g_2}{j_2} \\ &\times\left(\frac{2q_1^2+2q_1}{13q_1^2-2q_1-3}\right)^{j_1}\left(\frac{2q_2^2+2q_2}{13q_2^2-2q_2-3}\right)^{j_2}\theta_{j_1j_2} - \left(\frac{13q_1^2-1}{12q_1^2}\right)^{g_1}\left(\frac{13q_2^2-1}{12q_2^2}\right)^{g_2}\Bigg], \end{aligned} \tag{34}$$

(iii) while when q_1 and q_2 are even,

$$\begin{aligned} LB_2(\overline{MI_g(d)}) = \sum_{g_1=0}^{g}\Bigg[&\frac{1}{n^2}\left(\frac{13q_1-2}{12q_1}\right)^{g_1}\left(\frac{13q_2-2}{12q_2}\right)^{g_2}\sum_{j_1=0}^{g_1}\sum_{j_2=0}^{g_2}\binom{g_1}{j_1}\binom{g_2}{j_2} \\ &\times\left(\frac{2q_1+2}{13q_1-2}\right)^{j_1}\left(\frac{2q_2+2}{13q_2-2}\right)^{j_2}\theta_{j_1j_2} - \left(\frac{13q_1^2+2}{12q_1^2}\right)^{g_1}\left(\frac{13q_2^2+2}{12q_2^2}\right)^{g_2}\Bigg], \end{aligned} \tag{35}$$

where $\theta_{j_1j_2} = n\lambda_{j_1j_2} + \mu_{j_1j_2}(1+\lambda_{j_1j_2})$, $\mu_{j_1j_2} = n - q_1^{j_1}q_2^{j_2}\lambda_{j_1j_2}$, $\lambda_{j_1j_2} = \left\lfloor n/\left(q_1^{j_1}q_2^{j_2}\right)\right\rfloor$ is the largest integer that is less than or equal to $n/\left(q_1^{j_1}q_2^{j_2}\right)$.

Proof of Lemma 6. (i) when q_1 is odd, q_2 is even, let

$$D_0^{(1)} = \begin{pmatrix} X_{11} & X_{12} & \cdots & X_{1q_1} \\ X_{21} & X_{22} & \cdots & X_{2q_1} \\ \cdots & \cdots & \cdots & \cdots \\ X_{q_11} & X_{q_12} & \cdots & X_{q_1q_1} \end{pmatrix},\ D_0^{(2)} = \begin{pmatrix} Y_{11} & Y_{12} & \cdots & Y_{1q_2} \\ Y_{21} & Y_{22} & \cdots & Y_{2q_2} \\ \cdots & \cdots & \cdots & \cdots \\ Y_{q_21} & Y_{q_22} & \cdots & Y_{q_2q_2} \end{pmatrix},$$

where the diagonal elements in $D_0^{(1)}$ and $D_0^{(2)}$ are $\frac{15q_1^2-3}{12q_1^2}$ and $\frac{5}{4}$ respectively, and the rest of them are $\frac{13q_1^2-2q_1-3}{12q_1^2}$ and $\frac{13q_2-2}{12q_2}$ respectively. We denote $D_{g_1}^{(1)} = \bigotimes_{j_1=1}^{g_1} D_0^{(1)}$, $D_{g_2}^{(2)} = \bigotimes_{j_2=1}^{g_2} D_0^{(2)}$, where

$$
\begin{aligned}
D_0^{(1)} &= \frac{13q_1^2-2q_1-3}{12q_1^2} L^{(1)}(0)' L^{(1)}(0) + \frac{q_1+1}{6q_1} L^{(1)}(1)' L^{(1)}(1), \\
D_0^{(2)} &= \frac{13q_2-2}{12q_2} L^{(2)}(0)' L^{(2)}(0) + \frac{q_2+1}{6q_2} L^{(2)}(1)' L^{(2)}(1),
\end{aligned}
$$

Let $D = D_{g_1}^{(1)} \bigotimes D_{g_2}^{(2)}$, so

$$
\begin{aligned}
D = & \left(\frac{13q_1^2-2q_1-3}{12q_1^2}\right)^{g_1} \left(\frac{13q_2-2}{12q_2}\right)^{g_2} \sum_{x^{(1)}\in\Omega^{(1)}} \sum_{x^{(2)}\in\Omega^{(2)}} \left(\frac{2q_1^2+2q_1}{13q_1^2-2q_1-3}\right)^{\sum x_j^{(1)}} \\
& \left(\frac{2q_2+2}{13q_2-2}\right)^{\sum x_j^{(2)}} H'(x)H(x),
\end{aligned}
\tag{36}
$$

$$
\begin{aligned}
y_d' D y_d = & \left(\frac{13q_1^2-2q_1-3}{12q_1^2}\right)^{g_1} \left(\frac{13q_2-2}{12q_2}\right)^{g_2} \sum_{j_1=0}^{g_1} \sum_{j_2=0}^{g_2} \left(\frac{2q_1^2+2q_1}{13q_1^2-2q_1-3}\right)^{j_1} \\
& \left(\frac{2q_2+2}{13q_2-2}\right)^{j_2} \left(\sum_{x\in\Omega_{j_1 j_2}} y_d' H'(x) H(x) y_d \right),
\end{aligned}
\tag{37}
$$

for any $\sum_{x\in\Omega_{j_1 j_2}}$, the elements of $(q_1^{j_1} q_2^{j_2}) \times 1$ vector $H(x)y_d$ are non-negative integers with a sum of n. So, from [18], we have

$$
\begin{aligned}
y_d' H'(X) H(x) y_d &\geq \lambda_{j_1 j_2}^2 \left(q_1^{j_1} q_2^{j_2} - \mu_{j_1 j_2}\right) + (\lambda_{j_1 j_2}+1)^2 \mu_{j_1 j_2} \\
&= n\lambda_{j_1 j_2} + \mu_{j_1 j_2}(\lambda_{j_1 j_2}+1).
\end{aligned}
\tag{38}
$$

So

$$
\begin{aligned}
\overline{MI_g(d)} = & \sum_{g_1=0}^{g} \sum_{u\in I_{g_1 g_2}} \left[\frac{1}{n^2} \left(\frac{13q_1^2-2q_1-3}{12q_1^2}\right)^{g_1} \left(\frac{13q_2-2}{12q_2}\right)^{g_2} \sum_{i=1}^{n} \sum_{k=1}^{n} \left(\frac{15q_1^2-3}{13q_1^2-2q_1-3}\right)^{\delta_{ik}^{u_1}} \right. \\
& \left. \times \left(\frac{15q_2}{13q_2-2}\right)^{\delta_{ik}^{u_2}} - \left(\frac{13q_1^2-1}{12q_1^2}\right)^{g_1} \left(\frac{13q_2^2+2}{12q_2^2}\right)^{g_2} \right] \\
= & \sum_{g_1=0}^{g} \left[\frac{1}{n^2} y_d' D y_d - \left(\frac{13q_1^2-1}{12q_1^2}\right)^{g_1} \left(\frac{13q_2^2+2}{12q_2^2}\right)^{g_2} \right],
\end{aligned}
$$

if we combine (37) and (38), (i) is proved, and the proofs of (ii) and (iii) are similar to (i), so Lemma 6 is proved. □

By combining Lemmas 5 and 6, we can give a more general lower bound of the uniformity pattern $\overline{MI_g(d)}$ for any design $d \in \mathcal{U}(n; q_1^{s_1} \times q_2^{s_2})$, as follows.

Theorem 4. *For any design $d \in \mathcal{U}\left(n; q_1^{s_1} \times q_2^{s_2}\right)$, $u \in I_g$, $1 \le g(= g_1 + g_2) \le s$, we have*

$$\overline{MI_g(d)} \ge LB\left(\overline{MI_g(d)}\right),$$

where

$$LB(\overline{MI_g(d)}) = max\left\{LB_1(\overline{MI_g(d)}), LB_2(\overline{MI_g(d)})\right\}.$$

Theorem 4 gives a lower bound of the uniformity pattern of a design d. The lower bound can be used to measure the uniformity of the projection designs.

5. Numerical Examples

In this section, some numerical examples are provided to illustrate our theoretical results.

Example 1. *Consider the two designs d_1 and d_2, which are $OA\left(18; 3^5 \times 2, 2\right)$ as follows. By calculating the degrees of freedom, when the model contains four or more two-factor interactions, the matrix $X(c)'X(c)$ is singular. Hence, we consider E_h^*, $h = 1, 2, 3$. Using (3), (5), (27) and Definition 1, the GWLP, the B vector, the design efficiency and the uniformity pattern of d_1 and d_2 are obtained, and specific numerical results are listed in Table 1.*

$$d_1 = \begin{bmatrix} 0&0&0&1&1&1&2&2&2&0&0&0&1&1&1&2&2&2 \\ 0&0&1&0&2&2&1&1&2&1&2&2&0&1&1&0&0&2 \\ 0&1&0&2&0&2&1&2&1&2&1&2&1&0&1&0&2&0 \\ 0&1&1&2&2&1&2&0&0&2&2&0&0&0&1&2&1&1 \\ 0&1&2&2&1&0&0&1&2&0&2&1&0&2&1&1&2&0 \\ 0&0&0&0&0&0&0&0&0&1&1&1&1&1&1&1&1&1 \end{bmatrix}',$$

$$d_2 = \begin{bmatrix} 0&1&2&0&1&2&0&1&2&0&1&2&0&1&2&0&1&2 \\ 0&1&2&0&1&2&1&2&0&2&0&1&1&2&0&2&0&1 \\ 0&1&2&1&2&0&0&1&2&2&0&1&2&0&1&1&2&0 \\ 0&1&2&1&2&0&2&0&1&1&2&0&0&1&2&2&0&1 \\ 0&1&2&2&0&1&2&0&1&0&1&2&1&2&0&1&2&0 \\ 0&0&0&0&0&0&0&0&0&1&1&1&1&1&1&1&1&1 \end{bmatrix}'.$$

From Table 1, it is shown that d_1 is better than d_2 in terms of the GMA, orthogonality, design efficiency and MPU criteria.

Table 1. Numerical results of Example 1.

		d_1				
$A_g(d_1)$	0	0	8.50000	12	3	2.50000
$B_g(d_1)$	0	0	5.66700	8	2	1.66700
$\overline{MI_g(d_1)}$	0	0	0.00401	0.01358	0.01531	0.00574
E_h^*	16,524.97000	19,278.83000	22,032.69000			
		d_2				
$A_g(d_2)$	0	0	9	10.50000	4.50000	2
$B_g(d_2)$	0	0	6	7.50000	3	1.33300
$\overline{MI_g(d_2)}$	0	0	0.00478	0.01587	0.01759	0.00650
E_h^*	17,496	20,122.71000	22,749.43000			

Example 2. *Considerthe two designs d_3 and d_4, which are orthogonal arrays $OA\left(81; 9 \times 3^4, 2\right)$ from http://pietereendebak.nl/oapage/ (accessed on 10 December 2022). Using (3), (5), (28) and*

Definition 1, the GWLP, the B vector, the design efficiency and the uniformity pattern of d_3 and d_4 are obtained, and specific numerical results are listed in Table 2.

Table 2. Numerical results of Example 2.

			d_3		
$A_g(d_3)$	0	0	4	2	2
$B_g(d_3)$	0	0	36	18	18
$MI_g(d_3)$	0	0	0.00104	0.00225	0.00122
E_h^*	157,464	201,204	244,944	288,684	
			d_4		
$A_g(d_4)$	0	0	8	0	0
$B_g(d_4)$	0	0	72	0	0
$MI_g(d_4)$	0	0	0.00325	0.00701	0.00378
E_h^*	314,928	349,920	384,912	419,904	

From Table 2, it is shown that d_3 is better than d_4 in terms of the GMA, orthogonality, design efficiency and MPU criteria.

Example 3. *Consider the two designs, d_5 and d_6, which are $OA(16; 4 \times 2^5, 2)$ as follows. Using (3), (5), (29) and Definition 1, the GWLP, the B vector, the design efficiency and the uniformity pattern of d_5 and d_6 are obtained, and specific numerical results are listed in Table 3 .*

$$d_5 = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 2 & 2 & 2 & 2 & 3 & 3 & 3 & 3 \\ 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 \end{bmatrix}',$$

$$d_6 = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 2 & 2 & 2 & 2 & 3 & 3 & 3 & 3 \\ 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 \end{bmatrix}'.$$

Table 3. Numerical results of Example 3.

		d_5				
$A_g(d_5)$	0	0	3	3	1	0
$B_g(d_5)$	0	0	6	6	2	0
$MI_g(d_5)$	0	0	0.00358	0.01232	0.01416	0.00542
E_h^*	4608	4864				
		d_6				
$A_g(d_6)$	0	0	4	3	0	0
$B_g(d_6)$	0	0	8	6	0	0
$MI_g(d_6)$	0	0	0.00553	0.01900	0.02174	0.00828
E_h^*	6144	6180.57100				

From Table 3, it is shown that d_5 is better than d_6 in terms of the GMA, orthogonality, design efficiency and MPU criteria.

Example 4. *Considerthe following designs* $d_7 \in \mathcal{U}(18; 3^2 \times 6)$ *and* $d_8 \in \mathcal{U}(20; 5 \times 2^3)$.

$$d_7 = \begin{bmatrix} 0 & 1 & 2 & 0 & 1 & 2 & 0 & 1 & 2 & 0 & 1 & 2 & 0 & 1 & 2 & 0 & 1 & 2 \\ 0 & 1 & 2 & 0 & 1 & 2 & 1 & 2 & 0 & 1 & 2 & 0 & 2 & 0 & 1 & 2 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 & 2 & 2 & 2 & 3 & 3 & 3 & 4 & 4 & 4 & 5 & 5 & 5 \end{bmatrix}',$$

$$d_8 = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 2 & 2 & 2 & 2 & 3 & 3 & 3 & 3 & 4 & 4 & 4 & 4 \\ 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 1 \end{bmatrix}'.$$

By Definition 1 and Theorem 4, the uniformity patterns and lower bounds of d_7 and d_8 are listed in Table 4.

Table 4. Numerical results of Example 4.

g	1	2	3	4
$\overline{MI_g(d_7)}$	0	0	0.00036	
$LB(\overline{MI_g(d_7)})$	0	0	0.00036	
$\overline{MI_g(d_8)}$	0	0	0	0.00016
$LB(\overline{MI_g(d_8)})$	0	0	0	0.00016

From Table 4, it is obvious that $\overline{MI_g(d_7)} = LB(\overline{MI_g(d_7)})$ for $1 \leq g \leq 3$; $\overline{MI_g(d_8)} = LB(\overline{MI_g(d_8)})$, while for $1 \leq g \leq 4$. Designs d_7 and d_8 are the minimum projection uniform designs under the centered L_2 discrepancy.

Example 5. *Consider the following designs* $d_9 \in \mathcal{U}(9; 9 \times 3^2)$ *and* $d_{10} \in \mathcal{U}(27; 9 \times 3^2)$.

$$d_9 = \begin{bmatrix} 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 0 & 1 & 2 & 0 & 1 & 2 & 0 & 1 & 2 \\ 0 & 1 & 2 & 2 & 0 & 1 & 1 & 2 & 0 \end{bmatrix}',$$

$$d_{10} = \begin{bmatrix} 0 & 0 & 0 & 1 & 1 & 1 & 2 & 2 & 2 & 3 & 3 & 3 & 4 & 4 & 4 & 5 & 5 & 5 & 6 & 6 & 6 & 7 & 7 & 7 & 8 & 8 & 8 \\ 0 & 1 & 2 & 0 & 1 & 2 & 0 & 1 & 2 & 0 & 1 & 2 & 0 & 1 & 2 & 0 & 1 & 2 & 0 & 1 & 2 & 0 & 1 & 2 & 0 & 1 & 2 \\ 0 & 1 & 2 & 1 & 2 & 0 & 2 & 0 & 1 & 0 & 1 & 2 & 1 & 2 & 0 & 2 & 0 & 1 & 0 & 1 & 2 & 1 & 2 & 0 & 2 & 0 & 1 \end{bmatrix}'.$$

By Definition 1 and Theorem 4, the uniformity patterns and their lower bounds of d_9 and d_{10} are listed in Table 5.

Table 5. Numerical results of Example 5.

g	1	2	3
$\overline{MI_g(d_9)}$	0	0.00609663	0.00699984
$LB(\overline{MI_g(d_9)})$	0	0.00609663	0.00699984
$\overline{MI_g(d_{10})}$	0	0	0.00022580
$LB(\overline{MI_g(d_{10})})$	0	0	0.00022580

From Table 5, it is obvious that $\overline{MI_g(d_9)} = LB(\overline{MI_g(d_9)})$, $\overline{MI_g(d_{10})} = LB(\overline{MI_g(d_{10})})$ for $1 \leq g \leq 3$. Designs d_9 and d_{10} are the minimum projection uniform designs under the centered L_2 discrepancy.

Example 6. *Considerthe following designs* $d_{11} \in \mathcal{U}(8; 2^2 \times 4)$ *and* $d_{12} \in \mathcal{U}(12; 2^2 \times 6)$.

$$d_{11} = \begin{bmatrix} 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 2 & 2 & 3 & 3 \end{bmatrix}',$$

$$d_{12} = \begin{bmatrix} 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 & 1 & 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 2 & 2 & 3 & 3 & 4 & 4 & 5 & 5 \end{bmatrix}'.$$

By Definition 1 and Theorem 4, the uniformity patterns and their lower bounds of d_{11} and d_{12} are listed in Table 6.

Table 6. Numerical results of Example 6.

g	1	2	3
$\overline{MI_g(d_{11})}$	0	0	0.0008138
$LB(\overline{MI_g(d_{11})})$	0	0	0.0008138
$\overline{MI_g(d_{12})}$	0	0	0.0005064
$LB(\overline{MI_g(d_{12})})$	0	0	0.0005064

Table 6 shows that $\overline{MI_g(d_{11})} = LB(\overline{MI_g(d_{11})})$, $\overline{MI_g(d_{12})} = LB(\overline{MI_g(d_{12})})$, $1 \le g \le 3$. It shows that designs d_{11} and d_{12} are the minimum projection uniform designs under the centered L_2 discrepancy.

6. Conclusions

In this paper, the relationship between the projection uniformity and design efficiency of mixed q_1- and q_2-level designs is explored under the centered L_2 discrepancy. The results show that when the design is an orthogonal array with a strength of three, the projection uniformity is equivalent to the design efficiency. Furthermore, a tight lower bound of the uniformity pattern is also obtained, which can serve as a benchmark for measuring the minimum projection uniform designs.

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