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On Modulated Lacunary Statistical Convergence of Double Sequences

María del Pilar Romero de la Rosa

Department of Mathematics, University of Cádiz, Avda. de la Universidad s/n, 11405 Jerez de la Frontera, Cádiz, Spain; pilar.romero@uca.es

Abstract: In earlier works, F. León and coworkers discovered a remarkable structure between statistical convergence and strong Cesàro convergence, modulated by a function f (called a modulus function). Such nice structure pivots around the notion of compatible modulus function. In this paper, we will explore such a structure in the framework of lacunary statistical convergence for double sequences and discover that such structure remains true for *lacunary compatible modulus functions*. Thus, we continue the work of Hacer Şenül, Mikail Et and Yavuz Altin, and we fully solve some questions posed by them.

Keywords: double sequences; lacunary convergence; statistical convergence; strong Cesàro convergence; modulus function

MSC: 40H05; 40A35

1. Introduction

The first examples of convergence methods come from the study of divergent series and go back to Euler, Abel and Poisson. Convergence methods continued to be developed in the nineteenth century, and a remarkable contribution was made by Cesàro and some of his contemporaries, such as Frobenius, Hölder and Borel (see [1] for historical background). The term statistical convergence was first presented by Fast [2] and Steinhaus [3] independently in the same year, 1951. A root of the notion of statistical convergence can be detected in the book by Zygmund [4], where he used the term "almost convergence", which turned out to be equivalent to the concept of statistical convergence. These works laid the foundation for summability theory, which is currently a very active field of research involving many researchers (see [5]) and which shows no signs of abatement.

We say that a sequence (x_n) in a normed space X is said to be statistically convergent to L if, for any ε subset, $\{n : \|x_n - L\| > \varepsilon\}$ has zero density on $\mathbb{N}$. On the other hand, a sequence (x_k) on a normed space $(X, \|\cdot\|)$ is said to be strongly Cesàro convergent to L if $\lim_{n\to\infty} \frac{1}{n}\sum_{k=1}^{n} \|x_k - L\| = 0$. Both convergence methods were introduced by different authors and at different times. Although these convergence methods are different, they are intimately related (for instance, for bounded sequences, they are equivalent) thanks to a result discovered by Jeff Connor [6].

A function $f : \mathbb{R}^+ \to \mathbb{R}^+$ is said to be a modulus function if it satisfies the following:

(a) $f(x) = 0$ if and only if $x = 0$.
(b) $f(x+y) \le f(x) + f(y)$ for every $x, y \in \mathbb{R}^+$.
(c) f is increasing.
(d) f is continuous from the right at 0.

From Nakano's work [7], it is known that using modulus functions, it is possible to modulate, in a certain sense, statistical convergence and strong Cesàro convergence in such a way that such convergence methods are more precise. Specifically, fewer elements of the sequence are neglected. The results obtained by F. León et al. [8,9] suggest that only a

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class of modulus functions, namely, the non-compatible modulus functions, are worthy of study, since with convergence methods, compatible modulus functions are basically the classical ones.

Natural extensions of the classical summability methods have been developed by means of lacunary sequences and for double sequences. Interest in summability methods using lacunary sequences first arose with the work of Freeman and Sember [10] and that of Fridy and Orhan [11]. These works were later continued by [12–14]. One of the pioneering results on double sequences is due to Mursaleen ([15]), and his work was later continued by many others ([16–18]).

In [19], Hacer Şenül, Mikail Et and Yavuz Altin made a great effort to understand the relationship between lacunary f-statistical convergence and f-strong Cesàro convergence for double sequences. In this note, we continue the work started in [19] by exploring the role of compatibility in modulus functions, and we fully characterize such connections in terms of the so-called θ''-compatible modulus functions (where θ'' denotes a double lacunary sequence).

This paper is structured as follows: In Section 2, we introduce the notation and some basic notions that will be central to the discussion. For instance, we define the concepts of compatible and θ''-compatible modulus functions and their connections. Two other elements crucial to the discussion are also analyzed: lacunary f-statistical convergence and lacunary f-strong Cesàro convergence for double sequences (introduced in [19]). In Section 3, we explore the connections among several convergence methods for double sequences.

2. Basic Definitions and Preliminary Results

To avoid trivialities, throughout the paper, we assume that f is an unbounded modulus function. Following Patterson and Savaş [20], double sequence $\theta'' = (k_r, l_s)$ is called double lacunary if there exist two increasing sequences of integers (k_r) and (l_s) such that

(a) $k_0 = l_0 = 0$.
(b) $h_{r+1} := k_{r+1} - k_r \to \infty$ as $r \to \infty$.
(c) $\overline{h}_{s+1} := l_{s+1} - l_s \to \infty$ as $s \to \infty$.

We denote $k_{r,s} = k_r \overline{l}_s$ and $h_{r,s} = h_r \overline{h}_s$. Moreover, the following intervals are determined by θ'': $I_r = \{k : k_{r-1} < k \le k_r\}$, $I_s = \{l : l_{s-1} < l \le l_s\}$ and $I_{r,s} = \{(k,l) : k_{r-1} < k \le k_r$ and $l_{s-1} < l \le l_s\}$. Ratios $q_r = k_r / k_{r-1}$, $\overline{q}_s = l_s / l_{s-1}$ and $q_{r,s} = q_r \overline{q}_s$.

Let us set θ'', a double lacunary sequence, and let X be a normed space. We denote with $\#A$ the cardinality of a finite subset A. A double sequence $(x_{i,j})$ in X is said to be θ''-lacunary convergent to $L \in X$ if, for any $\varepsilon > 0$,

$$\lim_{r,s} \frac{1}{h_{r,s}} \#\{(k,l) \in I_{r,s} : \|x_{i,j} - L\| > \varepsilon\} = 0.$$

The above limit is defined in the Pringsheim sense. Let us recall that a double sequence $(u_{n,m})$ is convergent to M in the Pringsheim sense if, for any $\varepsilon > 0$, there exists n_0 such that $\|u_{n,m} - M\| < \varepsilon$ for all $n, m \ge n_0$. Let us denote with $S_{\theta''}$ the set of all θ''-lacunary convergent double sequences.

Moreover, sequence $(x_{i,j})$ is said to be θ''-strongly Cesàro convergent to L if

$$\lim_{r,s} \frac{1}{h_{r,s}} \sum_{(k,l) \in I_{r,s}} \|x_{k,l} - L\| = 0.$$

Let us denote with $N_{\theta''}$ the set of all θ''-strongly Cesàro convergent double sequences.

For any unbounded modulus function, following [19], let us define subsets $S^f_{\theta''}$ and $N^f_{\theta''}$.

Definition 1. *A double sequence $(x_{i,j})$ in X is said to be θ'', f-lacunary convergent to $L \in X$ if, for any $\varepsilon > 0$,*

$$\lim_{r,s} \frac{1}{f(h_{r,s})} f\left(\#\{(k,l) \in I_{r,s} : \|x_{i,j} - L\| > \varepsilon\}\right) = 0.$$

Let us denote with $S^f_{\theta''}$ the set of all θ'', f-lacunary convergent sequences.

Following [8,9,21], to obtain a nice structure between f-statistical convergence and f-strong Cesàro convergence, a slight modification of the f-strong Cesàro convergence introduced by Nakano [7] is needed. For double sequences, such slight modification gives the following definition.

Definition 2. *A double sequence $(x_{i,j})$ in X is said to be θ'', f-strongly Cesàro convergent to $L \in X$ if, for any $\varepsilon > 0$,*

$$\lim_{r,s} \frac{1}{f(h_{r,s})} f\left(\sum_{(k,l) \in I_{r,s}} \|x_{k,l} - L\| \right) = 0.$$

Let us denote with $N^f_{\theta''}$ the set of all θ'', f-strongly Cesàro convergent sequences.

In [8], we introduced the notion of compatible modulus function, which plays a central role in this discussion. The compatible modulus function notion has helped to understand different existing mathematical structures and has been used in different research programs (see [9,21–27]).

Definition 3. *Let us denote $\varphi(\varepsilon) = \limsup_n \frac{f(n\varepsilon)}{f(n)}$. A modulus function f is said to be compatible if $\lim_{\varepsilon \to 0} \varphi(\varepsilon) = 0$.*

Remark 1. *Functions $f(x) = x^p + x^q$, $0 < p, q \leq 1$, $f(x) = x^p + \log(x+1)$ and $f(x) = x + \frac{x}{x+1}$ are modulus functions that are compatible. Moreover, $f(x) = \log(x+1)$ and $f(x) = W(x)$ (where W is the W-Lambert function restricted to $\mathbb{R}^+$, that is, the inverse of xe^x) are modulus functions that are not compatible. By taking inverses, there are plenty of non-compatible modulus functions. For instance, we consider the inverse of functions $x^p e^x$, which are generalized W-Lambert functions.*

Indeed, let us show that $f(x) = x + \log(x+1)$ is compatible.

$$\lim_{n \to \infty} \frac{f(n\varepsilon')}{f(n)} = \lim_{n \to \infty} \frac{n\varepsilon' + \log(1 + n\varepsilon')}{n + \log(n+1)} = \varepsilon'$$

On the other hand, if $f(x) = \log(x+1)$, since

$$\lim_{n \to \infty} \frac{\log(1 + n\varepsilon')}{\log(1+n)} = 1$$

we obtain that $f(x) = \log(x+1)$ is not compatible.

Let us assume that θ'' is a double lacunary sequence. Let us denote

$$\varphi_{\theta''}(\varepsilon) = \limsup_{r,s \to \infty} \frac{f(h_{r,s}\varepsilon)}{f(h_{r,s})}.$$

Definition 4. *We say that f is θ''-compatible if $\lim_{\varepsilon \to 0} \varphi_{\theta''}(\varepsilon) = 0$.*

Clearly, when f is compatible, f is θ''-compatible.

3. θ'', f-Statistical Convergence and θ'', f-Strong Cesàro Convergence for Double Sequences

Let us see the connections between the S_θ and S_θ^f spaces.

Theorem 1. *Let us assume that θ'' is a double lacunary sequence:*

(a) For any modulus function f, $S_{\theta''}^f \subset S_{\theta''}$.

(b) If f is θ''-compatible, then $S_{\theta''}^f = S_{\theta''}$

Proof. To prove (a), let us assume that $(x_{i,j})$ is a sequence that is θ'', f-statistically convergent to L. Then, for any $p \in \mathbb{N}$ and for any $\varepsilon > 0$, there exists n_0 such that if $r, s \geq n_0$, then

$$f\big(\#\{(k,l) \in I_{r,s} : \|x_{i,j} - L\| > \varepsilon\}\big) < \frac{1}{p} f(h_{r,s}) \leq f(h_{r,s}/p)$$

but since f is increasing, we obtain

$$\#\{(k,l) \in I_{r,s} : \|x_{i,j} - L\| > \varepsilon\} \leq h_{r,s}/p.$$

Since p is arbitrary, we obtain that $(x_{i,j})$ is θ''-statistically convergent to L, as we desired to prove.

To show (b), let us assume that f is compatible and $(x_{i,j})$ is θ''-statistically convergent to L. Given $\widetilde{\varepsilon}$, since f is θ''-compatible, there exists n_0 such that if $r, s \geq n_0$, then $\frac{f(h_{r,s}\widetilde{\varepsilon})}{f(h_{r,s})} < \varepsilon$.

Then, for any $\varepsilon' > 0, \widetilde{\varepsilon} > 0$, there exists a natural number, which we abusively denote with n_0, such that if $r, s \geq n_0$, then

$$\#\{(k,l) \in I_{r,s} : \|x_{i,j} - L\| > \varepsilon'\} < \widetilde{\varepsilon} h_{r,s}.$$

By using the fact that f is increasing and dividing by $f(h_{r,s})$, we obtain

$$\frac{1}{f(h_{r,s})} f\big(\#\{(k,l) \in I_{r,s} : \|x_{i,j} - L\| > \varepsilon'\}\big) \leq \frac{f(h_{r,s}\widetilde{\varepsilon})}{f(h_{r,s})} < \varepsilon,$$

which yields the desired result. □

A similar result is true for the $N_{\theta''}^f$ and $N_{\theta''}$ spaces.

Theorem 2. *Let us assume that θ'' is a double lacunary sequence.*

(a) For any modulus function f, $N_{\theta''}^f \subset N_{\theta''}$.

(b) If f is θ''-compatible, then $N_{\theta''}^f = N_{\theta''}$

Proof. To show (a), let us assume that $(x_{i,j})$ is θ'', f-strongly Cesàro convergent to L. Then, for any $p \in \mathbb{N}$, there exists n_0 such that if $r, s \geq n_0$, then

$$f\left(\sum_{(k,l) \in I_{r,s}} \|x_{k,l} - L\|\right) \leq \frac{1}{p} f(h_{r,s}) \leq f\left(\frac{h_{r,s}}{p}\right).$$

Again, since f is increasing,

$$\sum_{(k,l) \in I_{r,s}} \|x_{k,l} - L\| \leq \frac{h_{r,s}}{p}$$

which yields the desired result.

To show (b), let us assume that f is θ''-compatible. Thus, for any $\widetilde{\varepsilon} > 0$, there exists n_0 such that if $r, s \geq n_0$, then

$$\frac{f(h_{r,s}\widetilde{\varepsilon})}{f(h_{r,s})} < \varepsilon.$$

On the other hand, if $(x_{i,j})$ is θ''-strongly Cesàro convergent to L, there exists a natural number, which we abusively denote with n_0, such that if $r, s \geq n_0$, then

$$\sum_{(k,l)\in I_{r,s}} \|x_{k,l} - L\| \leq h_{r,s}\widetilde{\varepsilon}.$$

By using the fact that f is increasing and dividing by $f(h_{r,s})$, we obtain

$$\frac{1}{f(h_{r,s})} f\left(\sum_{(k,l)\in I_{r,s}} \|x_{k,l} - L\|\right) \leq \frac{f(h_{r,s}\widetilde{\varepsilon})}{f(h_{r,s})} < \varepsilon.$$

that is, $(x_{i,j})$ is θ'', f-strongly Cesàro convergent to L, as we desired to prove. $\square$

Next, we will see that the hypothesis on θ''-compatibility is necessary to obtain such structure.

Theorem 3. *Let f be a modulus function and $\theta'' = (k_r, l_s)$ a lacunary sequence.*

(a) If $S_{\theta''} = S^f_{\theta''}$, then f must be θ''-compatible.

(b) If $N_{\theta''} = N^f_{\theta''}$, then f must be θ''-compatible.

Proof. Let us assume that f is not a θ''-compatible modulus function. Let us construct, on normed space $X = \mathbb{R}$, a sequence $(x_n) \in S_{\theta''} \setminus S^f_{\theta''}$. An easy modification provides a counterexample on any normed space.

Since $\varphi_{\theta''}(\varepsilon)$ is an increasing function, if f is not θ''-compatible, then there exists $c > 0$ such that, for any $\varepsilon > 0$, $\varphi_{\theta''}(\varepsilon) > c$.

Let us fix $\varepsilon_k \to 0$. Thus, for each k, there exist h_{r_k} and $\overline{h}_{s_k}$ large enough such that $f(h_{r_k,s_k}\varepsilon_k) \geq cf(h_{r_k,s_k})$. Moreover, since h_{r_k} and $\overline{h}_{s_k}$ are increasing, we can suppose that

$$h_{r_k}(1 - \varepsilon_k) - 1 > 0. \tag{1}$$

and

$$\overline{h}_{s_k}(1 - \varepsilon_k) - 1 > 0. \tag{2}$$

Let us denote with $\lfloor x \rfloor$ the integer part of x. We set $m_k = \lfloor h_{r_k}\sqrt{\varepsilon_k} \rfloor + 1$ and $n_k = \lfloor \overline{h}_{s_k}\sqrt{\varepsilon_k} \rfloor + 1$. According to Equations (1) and (2), we obtain that $h_{r_k} - m_k > 0$ and $\overline{h}_{s_k} - n_k > 0$. Let us define subset $A_k = [k_{r_k} - m_k, k_{r_k}] \times [l_{s_k} - n_k, l_{s_k}] \cap \mathbb{N} \times \mathbb{N} \subset I_{r_k,s_k}$, with $A = \bigcup_k A_k$. If we denote with $\chi_A(\cdot)$ the characteristic function of A, we claim that sequence $x_{i,j} = \chi_A(i,j)$ is θ''-statistically convergent to 0 but not θ'', f-statistically convergent, which is a contradiction.

Indeed, if $(r,s) \neq (r_k, s_k)$ for any k, then

$$\frac{\#\{(i,j) \in I_{r,s} : |x_{i,j}| > \varepsilon\}}{h_r\overline{h}_s} \leq \frac{0}{h_r\overline{h}_s} = 0.$$

Moreover, for $(r,s) = (r_k, s_k)$,

$$\frac{\#\{(i,j) \in I_{r_k,s_k} : |x_{i,j}| > \varepsilon\}}{h_{r_k}\overline{h}_{s_k}} = \frac{n_k m_k}{h_{r_k}\overline{h}_{s_k}} \to 0$$

as $k \to \infty$.

On the other hand,

$$\frac{f(\#\{(i,j)\in I_{r_k,s_k}\ :\ |x_{i,j}|>1/2\})}{f(h_{r_k}\overline{h}_{s_k})}=\frac{f(n_k m_k)}{f(h_{r_k}\overline{h}_{s_k})}\geq\frac{f(h_{r_k}\overline{h}_{s_k}\varepsilon_k)}{f(h_{r_k}\overline{h}_{s_k})}\geq c$$

which proves part (a). To show part (b), let us see that if f is not θ''-compatible, the previously constructed sequence $(x_{i,j})$ satisfies that $(x_{i,j})\in N_{\theta''}$, but $(x_{i,j})\notin N^f_{\theta''}$ is a contradiction. □

Next, we will obtain Connor–Khan–Orhan-type results by relating θ'', f-lacunary strong Cesàro convergence ($N^f_{\theta''}$) with θ'', f-lacunary statistical convergence ($S^f_{\theta''}$).

Theorem 4. *Let us assume that $\theta''=(k_t,l_s)$ is a double lacunary sequence and f is a modulus function. Then, $N^f_{\theta''}\subset S^f_{\theta''}$.*

Proof. First of all, in order to prove that $(x_{i,j})$ is θ''-lacunary f-statistically convergent to L, it is sufficient to show that for all $m\in\mathbb{N}$,

$$\lim_{(r,s)\to\infty}\frac{f(\#\{(i,j)\in I_{r,s}\ :\ \|x_{i,j}-L\|>\frac{1}{m}\})}{f(h_r\overline{h}_s)}=0. \tag{3}$$

Indeed, let us fix $\varepsilon>0$ and let us consider m such that $\frac{1}{m+1}\leq\varepsilon\leq\frac{1}{m}$. Then, we obtain

$$\#\{(i,j)\in I_{r,s}\ :\|x_{i,j}-L\|>\varepsilon\}\leq\#\left\{(i,j)\in I_{r,s}\ :\|x_{i,j}-L\|>\frac{1}{m+1}\right\},$$

therefore, since f is increasing

$$\lim_{(r,s)\to\infty}\frac{f(\#\{(i,j)\in I_{r,s}\ :\|x_{i,j}-L\|>\varepsilon\})}{f(h_r\overline{h}_s)}\leq\lim_{(r,s)\to\infty}\frac{f\left(\#\{(i,j)\in I_{r,s}\ :\|x_{i,j}-L\|>\frac{1}{m+1}\}\right)}{f(h_r\overline{h}_s)}$$

thus by taking the limits as $(r,s)\to\infty$, we obtain what we desired.

Therefore, let $m\in\mathbb{N}$, and let us show equation (3).

$$\begin{aligned}
f\left(\sum_{(i,j)\in I_{r,s}}\|x_{i,j}-L\|\right) &\geq f\left(\sum_{\substack{(i,j)\in I_{r,s}\\ \|x_{i,j}-L\|\geq\frac{1}{m}}}\|x_{i,j}-L\|\right) && (4)\\
&\geq f\left(\sum_{\substack{(i,j)\in I_{r,s}\\ \|x_{i,j}-L\|\geq\frac{1}{m}}}\frac{1}{m}\right)\geq\frac{1}{m}f\left(\sum_{\substack{(i,j)\in I_{r,s}\\ \|x_{i,j}-L\|\geq\frac{1}{m}}}1\right) && (5)\\
&= \frac{1}{m}f\left(\#\left\{(i,j)\in I_{r,s}\ :\ \|x_{i,j}-L\|>\frac{1}{m}\right\}\right). && (6)
\end{aligned}$$

Since $(x_{i,j})$ is θ''-lacunary f-strongly Cesàro convergent to L, we have that

$$\lim_{r,s\to\infty}\frac{f\left(\sum_{(i,j)\in I_{r,s}}\|x_{i,j}-L\|\right)}{f(h_r\overline{h}_s)}=0,$$

therefore, by dividing by $f(h_r\overline{h}_s)$ equation (4) and taking the limits as $r,s \to \infty$, we obtain that for each $m \in \mathbb{N}$,

$$\lim_{r,s\to\infty} \frac{f(\#\{(i,j) \in I_{r,s} \ : \ \|x_{i,j} - L\| > \frac{1}{m}\})}{f(h_r\overline{h}_s)} = 0$$

which implies that $(x_{i,j})$ is θ''-lacunary f-statistically convergent to L, as we desired. $\square$

For classical convergences, it is known that statistical convergence implies strong Cesàro convergence in the realm of all bounded sequences. This is the 1988 result obtained by Connor [6]. This result was improved by Khan and Orhan [28] by proving that a sequence is statistically convergent if and only if it is strongly Cesàro convergent and uniformly integrable.

The reader should take into account that a sequence $(x_{i,j})$ that is convergent to L in the Pringsheim sense is not necessarily bounded. Therefore, a weakening of the boundedness notion is needed for double sequences.

Next, we can obtain a result analogous to that of Khan and Orhan, which is new for θ''-lacunary statistical convergence. Here, it is crucial to optimally measure the integrability (in a θ''-lacunary form) of a sequence.

Definition 5. *Let $\theta'' = (k_t, l_s)$ be a double lacunary sequence. A sequence $(x_{i,j})$ is said to be θ''-lacunary uniformly integrable if*

$$\lim_{M\to\infty} \sup_{r,s} \sum_{\substack{(i,j)\in I_{r,s} \\ \|x_{i,j}\|\geq M}} \|x_{i,j}\| = 0.$$

Let us denote with $I_{\theta''}$ the space of all lacunary uniformly integrable sequences. Let us observe that if a sequence $(x_{i,j})$ is bounded, then $(x_{i,j})$ is θ''-lacunary uniformly integrable, that is, $\ell_\infty(X) \subset I_{\theta''}$. On the other hand, if a sequence $(x_{i,j})$ is uniformly integrable, then any translation $(x_{i,j} - L)$ is also uniformly integrable for every $L \in X$.

Theorem 5. *Let us assume that $\theta'' = (k_r, l_s)$ is a double lacunary sequence and f is a θ''-compatible modulus function. Then, $S^f_{\theta''} \cap I_{\theta''} \subset N^f_{\theta''}$. Moreover, if for some modulus function f we have $S^f_{\theta''} \cap I_{\theta''} \subset N^f_{\theta''}$, then modulus f must be θ''-compatible.*

Proof. Let us assume that $(x_{i,j})$ is a sequence such that $(x_{i,j})$ is θ''-lacunary f-statistically convergent to L and θ''-lacunary uniformly integrable. Let us consider $\varepsilon > 0$. Since f is θ''-compatible, there exists $\varepsilon' > 0$ such that

$$\frac{f\left(h_r\overline{h}_s\varepsilon'\right)}{f(h_r\overline{h}_s)} < \frac{\varepsilon}{3} \tag{7}$$

for all $r,s \geq n_0(\varepsilon)$.

Now, since $(x_{i,j})$ is θ''-lacunary uniformly integrable, there exists a natural number $M > 0$ large enough satisfying $\frac{1}{M} < \varepsilon'$, and for all $r,s \in \mathbb{N}$,

$$\frac{1}{h_r\overline{h}_s} \sum_{\substack{(i,j)\in I_{r,s} \\ \|x_{i,j}-L\|\geq M}} \|x_{i,j} - L\| < \varepsilon'. \tag{8}$$

Moreover, since $(x_{i,j})$ is θ''-lacunary f-statistically convergent to L, there exists a natural number, which we abusively denote with $n_0(\varepsilon)$, such that for all $r,s \geq t_0(\varepsilon)$,

$$\frac{1}{f(h_t\overline{h}_s)} f(\#\{(i,j) \in I_{r,s} \ : \ \|x_{i,j} - L\| > \varepsilon'\}) < \frac{\varepsilon}{3M}. \tag{9}$$

Therefore,

$$\begin{aligned}\frac{f\left(\sum_{(i,j)\in I_{r,s}} \|x_{i,j} - L\|\right)}{f(h_r\overline{h}_s)} \leq & \frac{1}{f(h_r\overline{h}_s)} f\left(\sum_{\substack{(i,j)\in I_{r,s} \\ M > \|x_{i,j}-L\| \geq \varepsilon'}} \|x_{i,j} - L\|\right) + \\ & \frac{1}{f(h_r\overline{h}_s)} f\left(\sum_{\substack{(i,j)\in I_{r,s} \\ \|x_{i,j}-L\| \geq M}} \|x_{i,j} - L\|\right) + \\ & \frac{1}{f(h_r\overline{h}_s)} f\left(\sum_{\substack{(i,j)\in I_{r,s} \\ \|x_{i,j}-L\| < \varepsilon'}} \|x_{i,j} - L\|\right)\end{aligned} \tag{10}$$

Since f is increasing, according to (9), we obtain that for all $r.s \geq n_0(\varepsilon)$, the first term of (10) is

$$\begin{aligned}\frac{1}{f(h_r\overline{h}_s)} f\left(\sum_{\substack{(i,j)\in I_{r,s} \\ M > \|x_{i,j}-L\| \geq \varepsilon'}} \|x_{i,j} - L\|\right) &< \frac{f\left(\#\{(i,j) \in I_{r,s} \ : \ \|x_{i,j} - L\| > \varepsilon'\} \cdot M\right)}{f(h_r\overline{h}_s)} \\ &\leq M \frac{1}{f(h_r\overline{h}_s)} f\left(\#\{(i,j) \in I_{r,s} \ : \ \|x_{i,j} - L\| > \varepsilon'\}\right) \\ &< M\frac{\varepsilon}{3M} = \frac{\varepsilon}{3}.\end{aligned} \tag{11}$$

On the other hand, let us estimate the second summand of inequality (10). By using the fact that f is increasing and applying inequality (8) first and inequality (7) afterwards, we have that for $r,s \geq n_0(\varepsilon)$,

$$\begin{aligned}\frac{1}{f(h_r\overline{h}_s)} f\left(\sum_{\substack{(i,j)\in I_{r,s} \\ \|x_{i,j}-L\| \geq M}} \|x_{i,j} - L\|\right) &= \frac{1}{f(h_r\overline{h}_s)} f\left(h_r\overline{h}_s \frac{1}{h_r\overline{h}_s} \sum_{\substack{(i,j)\in I_{r,s} \\ \|x_{i,j}-L\| \geq M}} \|x_{i,j} - L\|\right) \\ &\leq \frac{1}{f(h_r\overline{h}_s)} f\left(h_r\overline{h}_s\varepsilon'\right) \leq \frac{\varepsilon}{3}.\end{aligned} \tag{12}$$

Finally, for the third summand in (10), by applying inequality (7), we obtain that if $r.s \geq n_0(\varepsilon)$,

$$\frac{1}{f(h_r\overline{h}_s)} f\left(\sum_{\substack{(i,j)\in I_{r,s} \\ \|x_{i,j}-L\| \leq \varepsilon'}} \|x_{i,j} - L\|\right) \leq \frac{1}{f(h_r\overline{h}_s)} f\left(h_r\overline{h}_s \frac{1}{M}\right) < \frac{\varepsilon}{3}. \tag{13}$$

Thus, by using inequalities (11), (12) and (13) in inequality (10), we obtain that if $r, s \geq n_0(\varepsilon)$,

$$\frac{f\left(\sum_{(i,j)\in I_{r,s}} \|x_{i,j} - L\|\right)}{f(h_r \overline{h}_s)} \leq \varepsilon,$$

that is, $(x_{i,j})$ is θ''-lacunary f-strongly Cesàro convergent to L, as we desired.

For the converse, let us assume that f is not θ''-compatible. We can construct, without loss, a counterexample normed space $X = \mathbb{R}$. Since f is not θ''-compatible, given (ε_k), a decreasing sequence converging to zero, there exist subsequences (r_k) and (s_k) such that $f(h_{r_k}\overline{h}_{s_k}\varepsilon_k) \geq cf(h_{r_k}\overline{h}_{s_k})$ for some $c > 0$. We consider $A_k = (k_{r_k} - 1, k_{r_k}] \times (l_{s_k} - 1, l_{s_k}] \cap \mathbb{N} \times \mathbb{N}$.

Sequence $x_{i,j} = \sum_{k=1}^{\infty} \varepsilon_k \chi_{A_k}(i,j)$. Sequence $(x_{i,j})$ is clearly bounded, and since ε_k is decreasing, an easy check shows that $(x_{i,j})$ is θ''-lacunary f-statistically convergent to 0. On the other hand,

$$\frac{1}{f(h_{r_k}\overline{h}_{s_k})} f\left(\sum_{(i,j)\in I_{r_k,s_k}} |x_{i,j}|\right) = \frac{f(h_{r_k}\overline{h}_{s_k}\varepsilon_k)}{f(h_{r_k}\overline{h}_{s_k})} \geq c$$

which proves that $(x_{i,j})$ is not θ''-lacunary f-strongly Cesàro convergent, as we desired to prove. □

4. Concluding Remarks and Open Questions

The novelty of this paper lies in discovering new compatible modulus functions, namely, lacunary compatible modulus functions, which allows us to understand the existing structure between two convergence methods: lacunary f-statistical convergence and f-strong Cesàro convergence for double sequences.

Paraphrasing Theorems 4 and 5, one of the main result of this paper is the following: Given each double lacunary sequence θ'' and any modulus function f, if a sequence $(x_{i,j})$ is θ''-lacunary f-strongly Cesàro convergent, then $(x_{i,j})$ is θ''-lacunary f-strongly statistically convergent. However, the converse is not true, even for bounded sequences.

Efforts were then directed to determine when θ''-lacunary f-strongly statistically convergent functions are θ''-lacunary f-strongly Cesàro convergent. Moreover, this is true when sequence $(x_{i,j})$ is bounded and f is θ''-compatible. Additionally, we can relax the hypothesis on boundedness on $(x_{i,j})$ using the weaker condition of θ''-lacunary uniformly integrable.

Finally, surprisingly enough, the above result is quite appropriate, in the sense that the condition on θ''-compatibility is necessary.

Although compatible modulus functions do not allow us to establish or clarify the relationship between f-strong Cesàro convergence and f-statistical convergence for double sequences, we would like to clarify the relationship between compatible modulus functions and θ''-lacunary compatible modulus functions.

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References

1. Tucciarone, J. The development of the theory of summable divergent series from 1880 to 1925. *Arch. Hist. Exact Sci.* **1973**, *10*, 1–40. [CrossRef]
2. Fast, H. Sur la convergence statistique. *Colloq. Math.* **1951**, *2*, 241–244. [CrossRef]

3. Steinhaus, H. Sur la convergence ordinaire et la convergence asymptotique. *Colloq. Math.* **1951**, *2*, 73–74.
4. Zygmund., A. *Trigonometrical Series*; Dover Publications: New York, NY, USA, 1955; p. vii+329.
5. Mursaleen, M. *Applied Summability Methods*; Springer Briefs in Mathematics; Springer: Cham, Switzerland, 2014; p. x+124. [CrossRef]
6. Connor, J. The statistical and strong p-Cesàro convergence of sequences. *Analysis* **1988**, *8*, 47–63. [CrossRef]
7. Nakano, H. Concave modulars. *J. Math. Soc. Jpn.* **1953**, *5*, 29–49. [CrossRef]
8. León-Saavedra, F.; Listán-García, M.d.C.; Pérez Fernández, F.J.; Romero de la Rosa, M.P. On statistical convergence and strong Cesàro convergence by moduli. *J. Inequal. Appl.* **2019**, *2019*, 298. [CrossRef]
9. León-Saavedra, F.; Listán-García, M.d.C.; Romero de la Rosa, M.d.P. On statistical convergence and strong Cesàro convergence by moduli for double sequences. *J. Inequal. Appl.* **2022**, *2022*, 62. [CrossRef]
10. Freedman, A.R.; Sember, J.J.; Raphael, M. Some Cesàro-type summability spaces. *Proc. Lond. Math. Soc.* **1978**, *37*, 508–520. [CrossRef]
11. Fridy, J.; Orhan, C. Lacunary statistical convergence. *Pac. J. Math.* **1993**, *160*, 43–51. [CrossRef]
12. Gökhan, A.; Et, M.; Mursaleen, M. Almost lacunary statistical and strongly almost lacunary convergence of sequences of fuzzy numbers. *Math. Comput. Model.* **2009**, *49*, 548–555. [CrossRef]
13. Mursaleen, M.; Mohiuddine, S.A. On lacunary statistical convergence with respect to the intuitionistic fuzzy normed space. *J. Comput. Appl. Math.* **2009**, *233*, 142–149. [CrossRef]
14. Mursaleen, M.; Alotaibi, A. Statistical lacunary summability and a Korovkin type approximation theorem. *Ann. Univ. Ferrara Sez. VII Sci. Mat.* **2011**, *57*, 373–381. [CrossRef]
15. Mursaleen, M.; Edely, O. Statistical convergence of double sequences. *J. Math. Anal. Appl.* **2003**, *288*, 223–231. [CrossRef]
16. Çinar, M.; Yilmaz, E.; Altin, Y.; Gulsen, T. Statistical convergence of double sequences on product time scales. *Analysis* **2019**, *39*, 71–77. [CrossRef]
17. Das, B.; Tripathy, B.C.; Debnath, P.; Bhattacharya, B. Characterization of statistical convergence of complex uncertain double sequence. *Anal. Math. Phys.* **2020**, *10*, 71. [CrossRef]
18. Pancaroğlu Akın, N. Regularly ideal invariant convergence of double sequences. *J. Inequal. Appl.* **2021**, *2021*, 131. [CrossRef]
19. Şengül, H.; Et, M.; Altin, Y. f-lacunary statistical convergence and strong f-lacunary summability of order α of double sequences. *Facta Univ. Ser. Math. Inform.* **2020**, *35*, 495–506.
20. Patterson, R.F.; Savaş, E. Lacunary statistical convergence of double sequences. *Math. Commun.* **2005**, *10*, 55–61.
21. Romero de la Rosa, M.d.P. On lacunary statistical convergence and lacunary strong Cesàro convergence by moduli. *Preprint* **2022**, 1–11.
22. García-Pacheco, F.J.; Kama, R. f-statistical convergence on topological modules. *Electron. Res. Arch.* **2022**, *30*, 2183–2195. [CrossRef]
23. Altay, B.; García-Pacheco, F.J.; Kama, R. On f-strongly Cesàro and f-statistical derivable functions. *AIMS Math.* **2022**, *7*, 11276–11291. [CrossRef]
24. Belen, C.; Yıldırım, M.; Sümbül, C. On statistical and strong convergence with respect to a modulus function and a power series method. *Filomat* **2020**, *34*, 3981–3993. [CrossRef]
25. León-Saavedra, F.; de la Rosa, M.; Sala, A. Orlicz-Pettis theorem through summability methods. *Mathematics* **2019**, *7*, 895. [CrossRef]
26. León-Saavedra, F.; Moreno-Pulido, S.; Sala, A. Orlicz-Pettis Type Theorems via Strong p-Cesàro Convergence. *Numer. Funct. Anal. Optim.* **2019**, *40*, 798–802. [CrossRef]
27. León-Saavedra, F.; Moreno-Pulido, S.; Sala-Pérez, A. Completeness of a normed space via strong p-cesàro summability. *Filomat* **2019**, *33*, 3013–3022. [CrossRef]
28. Gadjiev, A.D.; Orhan, C. Some approximation theorems via statistical convergence. *Rocky Mt. J. Math.* **2002**, *32*, 129–138. [CrossRef]