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Solvability of Sequential Fractional Differential Equation at Resonance

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Abstract: The sequential fractional differential equations at resonance are introduced subject to three-point boundary conditions. The emerged fractional derivative operators in these equations are based on the Caputo derivative of order that lies between 1 and 2. The vital target of the current contribution is to investigate the existence of a solution for the boundary value problem by using the coincidence degree theory due to Mawhin which is basically depending on the Fredholm operator with index zero and two continuous projectors. An example is given to illustrate the deduced theoretical results.

Keywords: coincidence degree theory; resonance; sequential fractional differential equations; three-point boundary conditions

MSC: 26A33; 34A08; 34A12

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1. Introduction

Our fascination with fractional partial differential equations and fractional ordinary differential equations stems from their numerous applications in various fields of research and engineering such as the kinetic theory of gasses, unstable drift waves in plasma, dynamic of viscoelastic materials, coloured noise, signal processing, cyber–physical system continuum and statistical mechanics, biology, solute transport in groundwater and so on [1,2].

The perfect fractional derivative discussion is a long-standing topic that is still being extensively researched. The physical aspect of the fractional derivative has now been shown in several studies. Compared to first-order derivatives, fractional-order derivatives have several advantages. One of the most basic examples of where the fractional derivative has a substantial influence can be observed in diffusion processes. Sub-diffusion is a diffusion process that is well-known in diffusion processes [3]. It is demonstrated that sub-diffusion occurs when the fractional derivative meets the following condition: the order of the fractional derivative is into $(0,1)$. In stability analysis, the fractional derivative has another effect that may be seen. There are several differential equations that are not stable with first-order derivatives, but their fractional equivalents are [4–6].

There are several instances and applications of fractional derivatives, as well as reasons why fractional derivatives were utilized too:

- In economic systems, Traore and Sene [7] clarified the applicability of fractional calculus in economics. They addressed the Ramsey model, which evaluates the growth rate of an economy. They reached the strategy adopted to obtain the growth rate equation with the ordinary derivative is not the same adopted with the fractional model.

- In quantum mechanics, for studying systems under the electrical screening effects in the stationary state, Al-Raeei [8] studied the solving of the fractional Schrodinger equation in the spatial form in the case of the electrical screening potential. He found that the values of the wave functions are not pure real in general and so the function values are physically acceptable because they are limited. Furthermore, he found that the energies which return to the bound states are more probable energies. Darvishi et al. [9,10] introduced soliton solutions for a family of nonlinear Schrodinger-type models with space-time fractional evolution in the sense of a conformable fractional derivative. The Biswas–Milovic equation which describes long-distance optical communications, is the generalized form of the nonlinear Schrodinger equation. This equation was considered by many authors to solve it by obtaining some soliton, periodic soliton and travelling wave solutions (see for example [11–13]).
- In engineering sciences, Jannelli [14] examined a class of fractional mathematical models arising in engineering sciences governed by time-fractional advection-diffusion-reaction equations. He showed that the proposed fractional model is efficient, reliable and easy to implement and can be employed for engineering sciences problems. Besides important engineering science applications, The Riccati differential equation today has many applications in many fields such as random processes, optimal control, robust stabilization, and network synthesis, financial mathematics and diffusion problems [15,16].
- In Biology, Hattaf [17,18] investigated the qualitative properties of solutions of fractional differential equations with the new generalized Hattaf fractional derivative. He applied his idea to a nonlinear system describing the dynamics of an epidemic disease, such as COVID-19.
- In Physics, The nonlinear space-time fractional partial differential symmetric regularized long wave equation can summarize many physical phenomena, for instance: ion-acoustic waves in plasma and solitary waves with shallow water waves, shallow water waves [19,20].

Due to its importance in various fields including engineering science, economy, physics, quantum mechanics biology, etc, fractional calculus has gained considerable popularity and importance. Furthermore, in electrical networks, rheology, chemical physics, fluid flows, etc, fractional differential equations arise (see [21–25] and references cited therein). Fractional differential equations are the subject of a large number of papers today (see [26–30] and references cited therein) due to their various applications.

All this huge amount of applications in various sciences and fields for fractional differential equations and the extent of the impact of fractional derivatives in improving the theoretical results to meet little by little the experimental and laboratory results makes us choose it to be a basic core in the field of our studies without hesitation. Therefore, we discuss the following fractional boundary value problem.

The sequential fractional boundary value problem that is being studied in this article may be expressed as

$$ {}^{c}D^{\alpha}(D+\lambda)x(t) = g(t, x(t), x'(t), {}^{c}D^{\alpha-1}x(t)) + e(t), \quad t \in [0,1] \tag{1} $$

$$ x(0) = 0, \qquad x'(0) = 0, \qquad x(1) = \beta x(\eta), \quad 0 < \eta < 1 \tag{2} $$

where ${}^{c}D^{\alpha}$ represents the Caputo derivative of fractional order $1 < \alpha \leq 2$, while D denotes the first derivative, $g\colon [0,1] \times \mathbb{R}^3 \to \mathbb{R}$ is a functions verifying with the Carathéodory conditions, $e(t) \in L^1[0,1]$, $\lambda \in \mathbb{R}_+$ and $\beta \in \mathbb{R}$ such that

$$ \beta = \frac{\lambda + e^{-\lambda} - 1}{\lambda\eta + e^{-\lambda\eta} - 1}. \tag{3} $$

The sequential fractional differential Equation (1) with three-point boundary value problem (2) occurs to be at resonance in the sense that the related linear homogeneous problem

$$
\begin{aligned}
&{}^{c}D^{\alpha}(D+\lambda)x(t) = 0, \qquad t \in [0,1] \\
&x(0) = 0, \qquad x'(0) = 0, \qquad x(1) = \beta x(\eta)
\end{aligned}
$$

has a nontrivial solution

$$x(t) = c(\lambda t - 1 + e^{-\lambda t}), \qquad c \in \mathbb{R}.$$

It is worth pointing out that Ahmad and Nieto [31] studied the existence results for the following fractional sequential differential equations for nonlocal three-point boundary values,

$$
\begin{aligned}
&{}^{c}D^{\alpha}(D+\lambda)x(t) = g(t, x(t)), \qquad t \in [0,1] \\
&x(0) = 0, \qquad x'(0) = 0, \qquad x(1) = \beta x(\eta), \quad 0 < \eta < 1
\end{aligned}
$$

where $1 < \alpha \leq 2$, $f : [0,1] \times \mathbb{R} \to \mathbb{R}$ is a continuous function, $\lambda > 0$ and $\beta \in \mathbb{R}$ at the non-resonance case

$$\beta \neq \frac{\lambda + e^{-\lambda} - 1}{\lambda\eta + e^{-\lambda\eta} - 1}.$$

Recently, much interest has developed related to the existence of solutions for fractional differential equations when subjected to boundary conditions involving multi-point boundary conditions. The interesting point is the boundary value conditions are imposed on admissible space rather than the functionals. In [32], the existence and multiplicity of solutions to a three-point second-order differential system are proved via Mountain Pass Lemma. In [33], the solvability of three-point boundary value problems at resonance is investigated. In [34], by using a novel efficient iteration method, the existence and uniqueness result of solution for a class of fourth-order p-Laplacian with three-point boundary condition is obtained. For more contributions in the literature (see [35–37] and the references given therein).

Several works dealing with the boundary value problem at resonance for nonlinear ordinary differential equations of fractional order have been published recently. By using Mawhin's continuation theorem, Zhang and Liu [38] established some sufficient conditions for the existence of a solution for a class of fractional infinite point boundary value problems at resonance. Furthermore, with three-dimensional kernels, they in [39] discussed the existence of solutions for fractional multi-point boundary value problems at resonance. For more studies on this subject (see [40–43] and any mentioned sources therein).

Resonance occurs when the inability to calculate some of the constants results from the differential equation solution. Therefore, we cannot obtain an explicit form of the solution that enables us to define operators through which we can study the properties of the solution. Despite the difficulty of applying the coincidence degree theorem due to Mawhin, which depends mainly on establishing several operators commensurate with the nature of the problem, it is very useful in the case of resonance because it does not depend on studying the form of the explicit solution. Therefore, in this paper, we follow the coincidence degree theorem due to Mawhin to study the existing solution to our boundary value problem (1) and (2) at the resonance case (3).

To the best of our knowledge, nevertheless, no contributions exist, on the possibility of finding solutions to the boundary value problem for sequential fractional differential Equations (1), especially with the boundary condition (2) at the resonance condition (3).

Inspired by the former notice, in this paper, we consider the sequential fractional differential equation boundary value problem (1) and (2) at the resonance case (3). By using the coincidence degree theorem due to Mawhin, we provide an existence of sufficient conditions at resonance case and we organize this paper as follows: In the next section, we present certain preliminaries and basic concepts associated with fractional calculus and linear operator and coincidence degree continuation theorem. In Section 3, we construct some operators and their properties that we need to show our principal results. In section four, the solution's existing property to the boundary value problem (1) and (2) at the resonance condition (3) is investigated. In section five, a numerical example is presented to illustrate our main theorems.

2. Preliminaries and Background Materials

A significant part of this section is dedicated to introducing concepts and definitions related to the coincidence degree continuation theorem and the fractional calculus, including concept, notation and definition.

To understand the coincidence degree continuation theorem, we must first review some background information [44,45].

With the norms $\|\cdot\|_X$ and $\|\cdot\|_Z$, assume that X and Z are two Banach spaces, respectively.

Definition 1. *The Fredholm operator is defined as one that is continuous and can be represented by $L : domL \subset\rightarrow Z$ when the following conditions are satisfied*

1. $\ker L = \{x \in domL | Lx = 0\}$ *is finite dimensional;*
2. $ImL = \{y \in Z | Lx = y, x \in domL\}$ *is closed set;*
3. $CokerL = Z/ImL$ *(the quotient of Z by the ImL) is finite dimensional.*

Codimension, denoted by $codimImL$, is the dimension of the cokernel. When $\dim(\ker L) = codimImL$, *a Fredholm operator is with index zero.*

When $Q^2x = Qx$ for all $x \in X$, a continuous mapping $Q : X \to X$ is a projector.

Definition 2. *Assume that $L\colon domL \subset X \to Z$ is a Fredholm operator with index zero and $P : X \to X$ and $Q\colon Z \to Z$ are continuous projectors such that $ImP = \ker L$, $\ker Q = ImL$ and $X = \ker L \oplus \ker P$, $Z = ImL \oplus ImQ$. Consequently, the operator $L|_{domL\cap\ker P}\colon domL \cap \ker P \to ImL$ is invertible. $K_P\colon ImL \to domL \cap \ker P$ is the inverse of that map. We define Ω as an open bound subject of X such that $domL \cap \Omega \neq \phi$. The operator $N\colon Z \to Z$ can be considered as L-compact on $\overline{\Omega}$ if $QN(\overline{\Omega})$ is bounded and $K_P(I - Q)N\colon \overline{\Omega} \to Z$ is compact.*

In the case where ImQ is isomorphic to $\ker L$, then there exists an isomorphism $J_{NL} : ImQ \to \ker L$. Next, we provide the coincidence degree theorem that it has proven in [44].

Theorem 1. *The operators L and N are called a Fredholm operator with index zero and L-compact on $\overline{\Omega}$, respectively, where Ω can be defined as an open bounded subset of X, if the next assumptions hold:*

(i) $Lx \neq \lambda Nx$ *for all* $x \in (domL\setminus \ker L) \cap \partial\Omega$ *and* $\lambda \in (0, 1)$*;*
(ii) $Nx \notin ImL$ *for all* $x \in \ker L \cap \partial\Omega$*;*
(iii) $deg(J_{NL}QN|_{\ker L}, \Omega \cap \ker L, 0) \neq 0$ *where $Q : Z \to Z$ is a continuous projection as above with* $ImL = \ker Q$*.*

It follows that the equation $Lx = Nx$ has one solution at least in $domL \cap \overline{\Omega}$.

Definition 3. *A function $g : [0, 1] \times \mathbb{R}^3 \to \mathbb{R}$ verifies the Carathéodory assumptions if*

1. *For all $(a, b, c) \in \mathbb{R}^3$, the function $s \mapsto g(s, a, b, c)$ is Lebesgue measurable,*
2. *For almost every $0 \le s \le 1$, the function $(a, b, c) \mapsto g(s, a, b, c)$ is continuous in $\mathbb{R}^3$,*
3. *For all $\ell > 0$, there exists $\psi_\ell(s) \in L^1[0, 1]$ such that $|g(s, a, b, c)| \le \psi_\ell(s)$ for a.e. $s \in [0, 1]$ and all $(a, b, c) \in \mathbb{R}^3$ with $\|(a, b, c)\| \le \ell$, where $\|\cdot\|$ denotes the norm in $\mathbb{R}^3$.*

Let us now introduce fractional calculus concepts, notations and definitions as well as preliminary lemmas for our proofs. In this investigation, we are grateful for the terminology utilized in the books [24,25].

Let $[a,b]$ $(-\infty < a < b < \infty)$ be a finite interval and $\mathcal{AC}^n([a,b]); n \in \mathbb{N}$ be the space of real-valued functions $f(t)$ which have continuous derivatives up to order $n-1$ on $[a,b]$ such that $f^{(n-1)} \in \mathcal{AC}[a,b]$ (the space of primitives of Lebesgue summable functions) with $\mathcal{AC}^1[a,b] = \mathcal{AC}[a,b]$.

Definition 4. *For a function $f \in \mathcal{AC}[0,b]$ and a positive real number a, the R-L fractional integral of order a is presented as follows*

$$I^a f(s) = \frac{1}{\Gamma(a)} \int_0^s (s-t)^{a-1} f(t) dt, \qquad s \in [0,b]$$

where $\Gamma(\cdot)$ indicates the gamma function defined as

$$\Gamma(a) = \int_0^\infty u^{a-1} e^{-u} du, \qquad a > 0.$$

Definition 5. *Based on the definition of fractional R-L integral, in the sense of Caputo, a fractional derivative of a function $f \in \mathcal{AC}^n([0,b])$ is described as*

$${}^cD^a f(s) = I^{n-\alpha} D^n f(s) = \frac{1}{\Gamma(n-a)} \int_0^s (s-t)^{n-a-1} f^{(n)}(t) dt, \qquad s \in [0,b]$$

where n is a positive integer and $n-1 < a \le n$. Looking at the derivative of a constant, we notice that it is zero

Lemma 1. *Suppose a and b are positive reals. Then,*

$$I^a I^b f(s) = I^{a+b} f(s)$$

Lemma 2. *Assume a is a positive real number. Then,*

$$I^a s^\rho = \frac{\Gamma(\rho+1)}{\Gamma(\rho+a+1)} t^{\rho+a}, \qquad \rho > -1.$$

Lemma 3. *Assume $r \in \mathbb{N}$ and $r-1 < a \le r$. For a function $u \in \mathcal{AC}^r([0,b])$, we get*

$$\begin{aligned} &{}^cD^a I^a u(s) = u(s), \\ &I^a\, {}^cD^a u(s) = u(s) + q_0 + q_1 s + \cdots + q_{r-1} s^{r-1}. \end{aligned}$$

where $q_i = u^{(i)}(0)/i!, i = 0,1,\ldots,r-1$.

Suppose $C^n[0,1], n \in \mathbb{N}_0$ is the Banach space of functions on the unit interval $[0,1]$ that possess continuous derivatives of order up to and including n.

Lemma 4. *Let ${}^cD^\gamma(D+\lambda)u(s) \in C^0[0,1]$, then we have $u(s) \in C^n[0,1]$ where $n-1 < \gamma \le n$ and also ${}^cD^{\gamma+1-i}u(s) \in C^0[0,1], i = 0,1,\ldots,n$.*

Proof. Let $y(s) \in C^0[0,1]$ such that

$${}^cD^\gamma(D+\lambda)u(s) = y(s).$$

It follows from Lemma 3 that

$$u'(s) + \lambda u(s) = q_0 + q_1 s + \cdots + q_{n-1} s^{n-1} + I^\gamma y(s)$$

which gives

$$u(s) = \sum_{i=0}^{n-1} q_i \int_0^s t^i e^{-\lambda(s-t)} dt + q_n + \int_0^s e^{-\lambda(s-t)} I^\gamma y(t) dt$$

and

$$u'(s) = q_0 e^{-\lambda s} + \sum_{i=1}^{n-1} i q_i \int_0^s t^{i-1} e^{-\lambda(s-t)} dt - \lambda \int_0^s e^{-\lambda(s-t)} I^\gamma y(t) dt + I^\gamma y(s)$$

which allows us to certify $u'(s) \in C^{n-1}[0,1]$ and so $u(s) \in C^n[0,1]$. The last equation results in the following when ${}^c D^\gamma$ is applied on both sides

$${}^c D^{\gamma+1} u(s) = q_0^c D^\gamma e^{-\lambda s} + \sum_{i=1}^{n-1} i q_i^c D^\gamma \int_0^s t^{i-1} e^{-\lambda(s-t)} dt - \lambda^c D^\gamma \int_0^s e^{-\lambda(s-t)} I^\gamma y(t) dt + y(s)$$

which yields that ${}^c D^{\gamma+1} \in C^0[0,1]$. Definition 5 tells that

$$\begin{aligned} {}^c D^{\gamma+1} u(s) &= I^{n+1-(\gamma+1)} D^{n+1} u(s) = I^{n-\gamma} D^n D u(s) \\ &=^c D^\gamma u'(s) =^c D^{\gamma-1} u''(s) = \cdots =^c D^{\gamma-(n-1)} u^{(n)}(s) \end{aligned}$$

which yields the desired results. □

Lemma 5. *Let β be defined as in (3) and μ be defined as*

$$\mu = \frac{1}{\Gamma(\alpha+1)} \left(\int_0^1 e^{-\lambda} - \beta \int_0^\eta e^{-\lambda\eta} \right) s^\alpha e^{\lambda s} ds. \tag{4}$$

Then we have $\beta > 1$ and $\mu < 0$ if $0 < \alpha < 1$ and $0 < \mu < 1/2$ if $\alpha > 1$ for each $0 < \eta < 1$ and $\lambda > 0$.

Proof. Let the function

$$f(t,\alpha) = \int_0^t s^\alpha e^{-\lambda(t-s)} ds \tag{5}$$

be defined for all $t \in [0,1], \alpha > 0$ and $\lambda > 0$. Partial differentiating with respect to t gives

$$\frac{\partial}{\partial t} f(t,\alpha) = t^\alpha - \lambda \int_0^t s^\alpha e^{-\lambda(t-s)} ds = \alpha \int_0^t s^{\alpha-1} e^{-\lambda(t-s)} ds > 0$$

which means that the function $t \mapsto f(t,\alpha)$ is increasing on $[0,1]$. Since $f(0,\alpha) = 0$ then $f(t,\alpha) \geq 0$ for all $t \in [0,1]$. It follows that $f(1,\alpha) > f(\eta,\alpha) > 0, 0 < \eta < 1$ which yields $\beta = f(1,1)/f(\eta,1) > 1$. It is easy to see that

$$\frac{\partial}{\partial \alpha} \left(\frac{f(1,\alpha)}{f(\eta,\alpha)} \right) = \frac{e^{-\lambda(1+\eta)} g(\eta)}{f^2(\eta,\alpha)}$$

where

$$g(\eta) = \left(\int_0^1 s^\alpha \ln s e^{\lambda s} ds \right) \left(\int_0^\eta s^\alpha e^{\lambda s} ds \right) - \left(\int_0^1 s^\alpha e^{\lambda s} ds \right) \left(\int_0^\eta s^\alpha \ln s e^{\lambda s} ds \right).$$

Differentiation gives

$$g'(\eta) = \eta^\alpha e^{\lambda\eta} \left(\int_0^1 s^\alpha \ln s e^{\lambda s} ds - \ln \eta \int_0^1 s^\alpha e^{\lambda s} ds \right)$$

which follows that $g'(\eta) > 0$ for each $\eta \in (0, \eta_0)$ and $g'(\eta) < 0$ for each $\eta \in (\eta_0, 1)$ where $\eta_0 = \exp\left[\int_0^1 s^\alpha \ln s e^{\lambda s} ds / \int_0^1 s^\alpha e^{\lambda s} ds\right] < 1$ and so the function $g(\eta)$ is increasing on $(0, \eta_0)$ and decreasing on $(\eta_0, 1)$. Since $g(0) = g(1) = 0$, then $g(\eta) > 0$ for all $\eta \in (0, 1)$. This concludes that the function $\alpha \mapsto f(1, \alpha)/f(\eta, \alpha)$ is increasing on $(0, \infty)$ which implies that $f(1, \alpha)/f(\eta, \alpha) < f(1, 1)/f(\eta, 1) = \beta$ if $0 < \alpha < 1$ and $f(1, \alpha)/f(\eta, \alpha) > \beta$ if $\alpha > 1$. This means that $\mu < 0$ if $0 < \alpha < 1$ and $\mu > 0$ if $\alpha > 1$. Now, we are in position to prove that $\mu < 1/2$ for each $\alpha > 1$. In view of (5), μ can be read as

$$\begin{aligned}\mu &= \frac{1}{\Gamma(\alpha+1)}(f(1,\alpha) - \beta f(\eta,\alpha)) \leq \frac{1}{\Gamma(\alpha+1)}(f(1,1) - f(\eta,2)) \\ &\leq \frac{f(1,1)}{\Gamma(\alpha+1)} = \frac{1}{\Gamma(\alpha+1)}\int_0^1 s e^{-\lambda(1-s)} ds\end{aligned}$$

It is obvious that the former integral is decreasing with respect to λ and so it is less than $1/2$. Furthermore, it is known that the gamma function $\Gamma(\alpha+1)$ is increasing for all $\alpha > 1$ and so $1/\Gamma(\alpha+1) \leq 1$. This ends the proof. □

3. Basic Constructions

The classical Banach space $C^0[0,1]$ with the given norm

$$\|x\|_\infty = \max\{|x(t)|, t \in [0,1]\},$$

is used in this study. Define the space

$$X = \{x :^c D^\alpha(D+\lambda)x(t) \in C^0[0,1], x(0) = 0, x'(0) = 0, x(1) = \beta x(\eta)\}$$

equipped with the norm

$$\|x\| = \|x\|_\infty + \|x'\|_\infty + \|^c D^{\alpha-1}x\|_\infty$$

and $Z = L^1[0,1]$ with the norm $\|z\|_1 = \int_0^1 |z(t)| dt$. Similarly as in Lemma 2.5 in [42], one can easily show that X is a Banach space with the norm $\|\cdot\|$.

As Lemma 2.6 in [42] and Lemma 2.2 in [46], we derive the following lemma:

Lemma 6. *If and only if F is uniformly bounded and equicontinuous, the set $F \subset X$ is a sequentially compact set. Uniformly bounded here denotes the existence of $M > 0$, such that for all $x \in F$*

$$\|x\| = \|x\|_\infty + \|x'\|_\infty + \|^c D^{\alpha-1}x\|_\infty < M$$

and equicontinuous denotes that for all $\epsilon > 0$ there exists $\delta > 0$, such that

$$|x^{(i)}(t_1) - x^{(i)}(t_2)| < \epsilon \qquad (\forall t_1, t_2 \in [0,1], |t_1 - t_2| < \delta, x \in F, i = 0, 1)$$

and

$$|^c D^{\alpha-1}x(t_1) -^c D^{\alpha-1}x(t_2)| < \epsilon \qquad (\forall t_1, t_2 \in [0,1], |t_1 - t_2| < \delta, x \in F).$$

Define the linear operator $L : domL \subset X \to Z$ as

$$Lx =^c D^\alpha(D+\lambda)x, \qquad x \in domL$$

with

$$domL = \{x :^c D^\alpha(D+\lambda)x(t) \text{is absolutely continuous in} C^0[0,1]\} \cap X.$$

Also, define the nonlinear operator $N : X \to Z$ as

$$Nx(t) = g(t, x(t), x'(t), ^c D^{\alpha-1}x(t)) + e(t), \qquad t \in [0,1].$$

Then the boundary value problem (1) and (2) can be written as

$$Lx = Nx, \qquad x \in domL.$$

We define $\ker L$ and ImL and some necessary operators in this section as well.

Lemma 7. *Let L represent the linear operator described above. Then, we have*

$$\ker L = \{c(\lambda t - 1 + e^{-\lambda t}), \quad c \in \mathbb{R}\} \subset X \tag{6}$$

and

$$ImL = \left\{ y \in Z | \left(\int_0^1 e^{-\lambda} - \beta \int_0^\eta e^{-\lambda\eta} \right) e^{\lambda s} I^\alpha y(s) ds = 0 \right\} \subset Z. \tag{7}$$

Proof. Let $Lx = 0$ or equivalently ${}^cD^\alpha(D+\lambda)x = 0$. As a result of applying I^α on both sides and using Lemma 3, we can obtain the following first-order linear differential equation: $x'(t) + \lambda x(t) = c_0 + c_1 t$ which has the solution

$$x(t) = \frac{c_0}{\lambda} + \frac{c_1}{\lambda^2}(\lambda t - 1) + c_2 e^{-\lambda t}.$$

Using the boundary conditions $x(0) = 0$ and $x'(0) = 0$ to obtain

$$x(t) = \frac{c_1}{\lambda^2}\left(\lambda t - 1 + e^{-\lambda t}\right)$$

which gives the kernel of the operator L as in (6). In order to prove the relation (7), suppose that $Lx =^c D^\alpha(D+\lambda)x = y$. As above, we get

$$x(t) = \int_0^t e^{-\lambda(t-s)} I^\alpha y(s) ds + \frac{c_1}{\lambda^2}\left(\lambda t - 1 + e^{-\lambda t}\right).$$

Using the boundary condition $x(1) = \beta x(\eta)$ with the relation (3) to get the relation (7). □

Lemma 8. *Suppose condition (3) is met. Then L is a Fredholm operator with index zero and* $dim \ker L = codimImL = 1$.

Proof. Firstly, we construct the following mapping: $Q : Z \to \mathbb{R}$ by

$$Qy = \frac{1}{\mu}\left(\int_0^1 e^{-\lambda} - \beta \int_0^\eta e^{-\lambda\eta} \right) e^{\lambda s} I^\alpha y(s) ds \tag{8}$$

where μ is defined as in (4). It is easy by using Lemma 2 to see that $Q(1) = 1$ which leads to

$$Q^2 y = Q(Qy) = (Qy)Q(1) = Qy$$

then we get the operator $Q : Z \to Z$ is a well-defined projector. It is obvious that $ImL = \ker Q$. Since $Q(y - Qy) = Qy - Q^2 y = 0$, then we have $(I - Q)y \in \ker Q$. Now, let $y \in Z$, then $y = y - Qy + Qy = (I - Q)y + Qy$ and thus $Z = \ker Q + ImQ$. Let $y_0 \in \ker Q \cap ImQ$, then $Qy_0 = 0$ and there exists $y_1 \in Z$ such that $y_0 = Qy_1 = Q^2 y_1 = Qy_0 = 0$ which lead to $\ker Q \cap ImQ = \{0\}$ and thus $Z = \ker Q \oplus ImQ = ImL \oplus ImQ$.

Now, $\dim \ker L = 1 = \dim ImQ = codim \ker Q = codimImL < \infty$, and observing that ImL is closed in Z, so L is a Fredholm operator with index zero. □

Let the mapping $P : X \to X$ be defined by

$$Px(t) = \frac{1}{\lambda^2} x''(0)(\lambda t - 1 + e^{-\lambda t}), \quad t \in [0,1], \quad x \in X.$$

Lemma 4 tells us that the $P : X \to X$ is well-defined and it is easy to see that it is a linear continuous projector with

$$\ker P = \{x \in X : x''(0) = 0\},$$
$$ImP = \{c(\lambda t - 1 + e^{-\lambda t}), \quad c \in \mathbb{R}\} = \ker L.$$

It is easy as above to show that $X = \ker L \oplus \ker P$.

Lemma 9. *Let the operator $K_P : ImL \to domL \cap \ker P$ be defined by*

$$(K_P y)(t) = \frac{1}{\Gamma(\alpha)} \int_0^t e^{-\lambda(t-s)} \left(\int_0^s (s-\tau)^{\alpha-1} y(\tau) d\tau \right) ds$$
$$= \int_0^t e^{-\lambda(t-s)} I^\alpha y(s) ds, \qquad y \in ImL.$$

Then, $K_P = L^{-1}_{domL \cap \ker P}$, where $L_{domL \cap \ker P} : domL \cap \ker P \to ImL$.

Proof. In fact, for $y \in ImL$, we have

$$LK_P y =^c D^\alpha (D + \lambda) \left[\int_0^t e^{-\lambda(t-s)} I^\alpha y(s) ds \right]$$
$$=^c D^\alpha \left(e^{-\lambda t} D \int_0^t e^{\lambda s} I^\alpha y(s) ds \right) =^c D^\alpha I^\alpha y(t) = y$$

Furthermore, if $x \in domL \cap \ker P$, then $x \in \ker P$. It yields $x''(0) = 0$, we have

$$K_P L x = \int_0^t e^{-\lambda(t-s)} I^\alpha \, {}^c D^\alpha (D + \lambda) x(s) ds$$
$$= \int_0^t e^{-\lambda(t-s)} \left[(D+\lambda)x(s) - (x'(0) + \lambda x(0)) - s(x''(0) + \lambda x'(0)) \right] ds$$
$$= e^{-\lambda t} \int_0^t D \left[e^{\lambda s} x(s) \right] ds = x.$$

Here, we use the results in Lemma 3. Thus, we obtain the desired result. □

Lemma 10. *We have*

$$\|K_P y\| \le \rho_1 \|y\|_1, \qquad y \in ImL$$

where

$$\rho_1 = 1 + \frac{1 - e^{-\lambda}}{\lambda \Gamma(\alpha)} + \frac{2 - e^{-\lambda}}{\Gamma(\alpha)} + \frac{1 - e^{-\lambda}}{\Gamma(\alpha)\Gamma(3-\alpha)}. \tag{9}$$

Proof. Using the Leibnitz integral rule gives

$$D(K_P y)(t) = D \int_0^t e^{-\lambda(t-s)} I^\alpha y(s) ds$$
$$= I^\alpha y(t) - \lambda \int_0^t e^{-\lambda(t-s)} I^\alpha y(s) ds = I^\alpha y(t) - \lambda (K_P y)(t)$$

which leads to

$${}^c D^{\alpha-1} (K_P y)(t) = I^{2-\alpha} D(K_P y)(t) = I^2 y(t) - \lambda I^{2-\alpha} (K_P y)(t).$$

Now, let $y \in ImL$ and $t \in [0,1]$, then we have

$$|I^{\alpha}y(t)| = \left|\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}y(s)ds\right| \leq \frac{1}{\Gamma(\alpha)}\int_0^1 |y(s)|ds = \frac{\|y\|_1}{\Gamma(\alpha)}$$

which can be used to get

$$\begin{aligned}|(K_P y)(t)| &= \left|\int_0^t e^{-\lambda(t-s)}I^{\alpha}y(s)ds\right| \leq \int_0^t e^{-\lambda(t-s)}|I^{\alpha}y(s)|ds \\ &\leq \frac{\|y\|_1}{\Gamma(\alpha)}\int_0^1 e^{-\lambda(1-s)}ds = \frac{1-e^{-\lambda}}{\lambda\Gamma(\alpha)}\|y\|_1\end{aligned}$$

and

$$\begin{aligned}|I^{2-\alpha}(K_P y)(t)| &\leq |(K_P y)(t)|I^{2-\alpha}(1) = \frac{|(K_P y)(t)|t^{2-\alpha}}{\Gamma(3-\alpha)} \\ &\leq \frac{|(K_P y)(t)|}{\Gamma(3-\alpha)} = \frac{1-e^{-\lambda}}{\lambda\Gamma(\alpha)\Gamma(3-\alpha)}\|y\|_1.\end{aligned}$$

Whence, we can deduce that

$$|D(K_P y)(t)| \leq |I^{\alpha}y(t)| + \lambda|(K_P y)(t)| \leq \frac{2-e^{-\lambda}}{\Gamma(\alpha)}\|y\|_1$$

and

$$|{}^cD^{\alpha-1}(K_P y)(t)| = |I^2 y(t)| + \lambda|I^{2-\alpha}(K_P y)(t)| \leq \left(1 + \frac{1-e^{-\lambda}}{\Gamma(\alpha)\Gamma(3-\alpha)}\right)\|y\|_1.$$

These conclude that

$$\begin{aligned}\|K_P y\| &= \|K_P y\|_{\infty} + \|DK_P y\|_{\infty} + \|{}^cD^{\alpha-1}K_P y\|_{\infty} \\ &= \max_{t\in[0,1]}|K_P y(t)| + \max_{t\in[0,1]}|DK_P y(t)| + \max_{t\in[0,1]}|{}^cD^{\alpha-1}K_P y(t)| \\ &\leq \rho_1\|y\|_1.\end{aligned}$$

It is the intended outcome. □

Lemma 11. *Let $x \in X$, then we have*

$$\begin{aligned}&\|(I-P)x\| \leq \rho_1\|Lx\|_1 \\ &\|Px\| \leq \rho_2|x''(0)|\end{aligned}$$

where ρ_1 is defined in (9) and

$$\rho_2 = \frac{1}{\lambda^2}\left(\lambda + (\lambda+1)(1-e^{-\lambda}) + \frac{\lambda(1-e^{-\lambda})}{\Gamma(3-\alpha)}\right). \tag{10}$$

Proof. Lemma 9 tells that the operator K_P is the inverse of the operator L, then $\|(I-P)x\| = \|K_P L(I-P)x\|$ for each $x \in X$. By using the results in Lemma 10, we have

$$\|(I-P)x\| = \|K_P L(I-P)x\| \leq \rho_1\|L(I-P)x\|_1, \qquad x \in X.$$

Since $Px \in ImP = \ker L$, then $LPx = 0$. This implies the first requirement.

Now, let $t \in [0,1]$, then we have

$$\left|\lambda t - 1 + e^{-\lambda t}\right| \le \lambda |t| + \left|1 - e^{-\lambda t}\right| \le \lambda + 1 - e^{-\lambda},$$
$$\left|D\left(\lambda t - 1 + e^{-\lambda t}\right)\right| = \lambda\left|1 - e^{-\lambda t}\right| \le \lambda\left(1 - e^{-\lambda}\right)$$

and

$$\begin{aligned}\left|{}^{c}D^{\alpha-1}\left(\lambda t - 1 + e^{-\lambda t}\right)\right| &= \left|I^{2-\alpha} D\left(\lambda t - 1 + e^{-\lambda t}\right)\right| \\ &= \lambda\left|I^{2-\alpha}\left(1 - e^{-\lambda t}\right)\right| \le \lambda\left|1 - e^{-\lambda t}\right| I^{2-\alpha}(1) \\ &\le \frac{\lambda(1 - e^{-\lambda})}{\Gamma(3-\alpha)}.\end{aligned}$$

Whence

$$\begin{aligned}\|Px\| &= \frac{|x''(0)|}{\lambda^2}\left[\left\|\lambda t - 1 + e^{-\lambda t}\right\|_\infty + \left\|D\left(\lambda t - 1 + e^{-\lambda t}\right)\right\|_\infty + \left\|{}^{c}D^{\alpha-1}\left(\lambda t - 1 + e^{-\lambda t}\right)\right\|_\infty\right] \\ &\le \rho_2 |x''(0)|.\end{aligned}$$

This ends the proof. □

4. Main Results

Assume that function $g : [0.1] \times \mathbb{R}^3 \to \mathbb{R}$ meets the Carathédory assumptions. Consider the following conditions:

$(\mathcal{H}_1)$ There exist the functions $a_i \in Z, i = 1,2,3,4$ where for almost every $t \in [0,1]$, $a_i(t) \ge 0$ and for almost every $t \in [0,1]$ and $(x,y,z) \in \mathbb{R}^3$, we get

$$|g(t,x,y,z)| \le a_1(t)|x| + a_2(t)|y| + a_3(t)|z| + a_4(t)$$

$(\mathcal{H}_2)$ The constant $\ell_1 > 0$ exists such that

$$QNx \neq 0 \qquad \text{for all} \qquad x \in domL \setminus \ker L$$

satisfying $|x''(0)| > \ell_1$.

$(\mathcal{H}_3)$ There exists a constant

$$\rho_3 = \int_0^1 |a_1(s)|ds + \int_0^1 |a_2(s)|ds + \int_0^1 |a_3(s)|ds \tag{11}$$

such that $0 < \rho_1\rho_3 < 1$ where ρ_1 is defined as in (9)

$(\mathcal{H}_4)$ For any $c \in \mathbb{R}$, if $|c| > \ell_2$, such that $\ell_2 > 0$ is a constant, then either

$$cQNx < 0, \qquad x \in \ker L, \tag{12}$$

or else

$$cQNx > 0, \qquad x \in \ker L, \tag{13}$$

Lemma 12. *The set*

$$\Omega_1 = \{x \in domL \setminus \ker L : Lx = \gamma Nx, \gamma \in [0,1]\} \tag{14}$$

is bounded.

Proof. Let $x \in \Omega_1$, then $x \in domL$ and $x \notin \ker L$ with $Lx = \gamma Nx$ which means that $Lx \neq 0$ and $\gamma Nx \neq 0$, so $\gamma \neq 0$ and $Nx \in ImL = \ker Q$. Hence $QNx = 0$. According to the assumption $(\mathcal{H}_2)$, there exists a constant $\ell_1 > 0$ such that $|x''(0)| \leq \ell_1$. In view of Lemma 11, it follows that

$$\|x\| = \|(I-P)x + Px\| \leq \|(I-P)x\| + \|Px\| \leq \rho_1\|Lx\|_1 + \rho_2|x''(0)| \leq \rho_1\|Nx\|_1 + \rho_2\ell_1$$

where ρ_1 and ρ_2 are defined in (9) and (10), respectively. In the sense of the assumption $(\mathcal{H}_1)$, for each $x \in \Omega_1$, we have

$$\begin{aligned}
\|Nx\|_1 &= \int_0^1 \Big| g(s, x(s), x'(s), {}^c D^{\alpha-1}x(s)) + e(s) \Big| ds \\
&\leq \int_0^1 |a_1(s)| \cdot |x(s)| ds + \int_0^1 |a_2(s)| \cdot |x'(s)| ds \\
&+ \int_0^1 |a_3(s)| \cdot |{}^c D^{\alpha-1}x(s))| ds + \int_0^1 |a_4| ds + \int_0^1 |e(s)| ds \\
&\leq \|x\| \int_0^1 |a_1(s)| ds + \|x\| \int_0^1 |a_2(s)| ds + \|x\| \int_0^1 |a_3(s)| ds \\
&+ \int_0^1 |a_4(s)| ds + \int_0^1 |e(s)| ds = \rho_3\|x\| + \rho_4
\end{aligned}$$

where

$$\rho_4 = \int_0^1 |a_4| ds + \int_0^1 |e(s)| ds.$$

Therefore, we have

$$\|x\| \leq \frac{\rho_1\rho_4 + \rho_2\ell_1}{1 - \rho_1\rho_3}.$$

In view of the assumption $(\mathcal{H}_3)$, it implies that Ω_1 is bounded. □

Lemma 13. *The set*

$$\Omega_2 = \{x \in \ker L : Nx \in ImL\} \tag{15}$$

is bounded.

Proof. In the case of $x \in \ker L$ and $Nx \in ImL = \ker Q$, then $x = c(\lambda t - 1 + e^{-\lambda t}), c \in \mathbb{R}$ and $QNx = 0$. Based on the assumption $(\mathcal{H}_4)$, there exists a constant ℓ_2 such that $|c| \leq \ell_2$. Therefore as in Lemma 11

$$\|x\| \leq |c|\lambda^2\rho_2 \leq \lambda^2\ell_2\rho_2$$

where ρ_2 is defined in (10), and thus the set Ω_2 is bounded. □

Lemma 14. *The set*

$$\Omega_3 = \{x \in \ker L : -\gamma x + (1-\gamma)J_{NL}QNx = 0, \gamma \in [0,1]\} \tag{16}$$

if (12) holds, or

$$\Omega_3 = \{x \in \ker L : \gamma x + (1-\gamma)J_{NL}QNx = 0, \gamma \in [0,1]\} \tag{17}$$

if (13) holds, is bounded, where the linear isomorphism $J_{NL} : ImQ \to \ker L$ has the following definition:

$$J_{NL}(c) = c\Big(\lambda t - 1 + e^{-\lambda t}\Big), \quad c \in \mathbb{R}, t \in [0,1]$$

Proof. By taking $x \in \Omega_3$ (defined in (16)), we get $x = c\big(\lambda t - 1 + e^{-\lambda t}\big)$ and

$$\gamma c\Big(\lambda t - 1 + e^{-\lambda t}\Big) = (1-\gamma)\Big(\lambda t - 1 + e^{-\lambda t}\Big) QN\Big(c(\lambda t - 1 + e^{-\lambda t})\Big).$$

Our proof can be divided into three cases:

Case 1. If $\gamma = 0$, then $QNx = 0$ which yields with the assumption $(\mathcal{H}_4)$ that there exists a constant $\ell_2 > 0$ such that $|c| \leq \ell_2$. In view of Lemma 11, we find that $\|x\| \leq \lambda^2 \ell_2 \rho_2$, so the set Ω_3 is bounded.

Case 2. If $\gamma = 1$, then $c = 0$, so $x = 0$ and the set Ω_3 is bounded.

Case 3. If $\gamma = (0,1)$, according to Lemma 5 and the expression (12), then we have a contradicts with $\gamma c^2 > 0$. Thus, there exists $\ell_2 > 0$ such that $|c| \leq \ell_2$, so the set Ω_3 is bounded.

If $\mu = 1$, we can prove that the set Ω_3 (defined in (17)) is also bounded if the expression (13) is true (based on Lemma 5). □

Theorem 2. *Suppose a function $g : [0.1] \times \mathbb{R}^3 \to \mathbb{R}$ fulfils theCarathédory assumptions. Assume that the assumptions $(\mathcal{H}_1)$-$(\mathcal{H}_4)$ and the condition (3) hold. At least one solution exists in domL for the given boundary value problem. Let the function $f : [0.1] \times \mathbb{R}^3 \to \mathbb{R}$ satisfy the Carathédory conditions. Assume that the assumptions $(\mathcal{H}_1)$-$(\mathcal{H}_4)$ and the condition (3) all hold. Then the boundary value problem (1) and (2) has at least one solution in domL.*

Proof. Set Ω to be a bounded open set of X such that $\Omega \supset \overline{\Omega}_1 \cup \overline{\Omega}_2 \cup \overline{\Omega}_3$. Define the operator $K_{P,Q} : \overline{\Omega} \to X$ by

$$K_{P,Q}x = K_P(I-Q)Nx, \qquad x \in \overline{\Omega}.$$

In view of Lemma 10, we obtain

$$\|K_{P,Q}\| = \|K_P(I-Q)N\| \leq \rho_1 \|(I-Q)N\|_1$$

where ρ_1 is defined in (9), which leads to the operator $K_{P,Q}$ is bounded almost everywhere on $[0,1]$. When $f : [0,1] \times \mathbb{R}^3 \to \mathbb{R}$ satisfies the Carathéodory conditions (Definition 3), then $(I-Q)N(\overline{\Omega})$ must bounded almost everywhere on $[0,1]$, i.e., there exists constant $M > 0$ such that $|(I-Q)Nx(t)| \leq M, x \in \overline{\Omega}$ almost everywhere on $[0,1]$. Next, we show that $K_{P,Q} : \overline{\Omega} \to X$ is compact. To do this, let $0 \leq t_1 \leq t_2 \leq 1$ and $t_2 - t_1 \leq \delta$, then we have

$$\begin{aligned}
&|K_{P,Q}x(t_2) - K_{P,Q}x(t_1)| = |K_P(I-Q)Nx(t_2) - K_P(I-Q)Nx(t_1)| \\
&= \left| \int_0^{t_2} e^{-\lambda(t_2-s)} I^\alpha (I-Q)Nx(s)ds - \int_0^{t_1} e^{-\lambda(t_1-s)} I^\alpha (I-Q)Nx(s)ds \right| \\
&= \int_0^{t_1} \left[e^{-\lambda(t_1-s)} - e^{-\lambda(t_2-s)} \right] |I^\alpha (I-Q)Nx(s)|ds + \int_{t_1}^{t_2} e^{-\lambda(t_2-s)} |I^\alpha (I-Q)Nx(s)ds| \\
&\leq \frac{2\|(I-Q)N\|_1}{\Gamma(\alpha)} \Big(1 - e^{-\lambda(t_2-t_1)}\Big) \leq \frac{2\|(I-Q)N\|_1}{\Gamma(\alpha)} \Big(1 - e^{-\lambda\delta}\Big) < \epsilon.
\end{aligned}$$

It follows that the operator $K_{P,Q}$ is equicontinuous on $\overline{\Omega}$. Using the results obtained in Lemma 10, we fined that

$$
\begin{aligned}
&|K'_{P,Q}x(t_2) - K'_{P,Q}x(t_1)| \\
&= |[I^{\alpha}(I-Q)Nx(t_2) - I^{\alpha}(I-Q)Nx(t_1)] - \lambda[K_P(I-Q)Nx(t_2) - K_P(I-Q)Nx(t_1)]| \\
&\leq \frac{M}{\Gamma(\alpha)}\int_0^{t_1}\Big[(t_2-s)^{\alpha-1} - (t_1-s)^{\alpha-1}\Big]ds + \frac{M}{\Gamma(\alpha)}\int_{t_1}^{t_2}(t_2-s)^{\alpha-1}ds \\
&+ \frac{2\|(I-Q)N\|_1}{\Gamma(\alpha)}\Big(1-e^{-\lambda\delta}\Big) \\
&= \frac{M}{\Gamma(\alpha)}(t_2^{\alpha} - t_1^{\alpha}) + \frac{2\|(I-Q)N\|_1}{\Gamma(\alpha)}\Big(1-e^{-\lambda\delta}\Big)
\end{aligned}
$$

Since t^{α} is uniformly continuous on $[0,1]$, then $K'_P(I-Q)N(\overline{\Omega})$ is equicontinuous. In addition, we have

$$
\begin{aligned}
&|{}^cD^{\alpha-1}K_{P,Q}x(t_2) - {}^cD^{\alpha-1}K_{P,Q}x(t_1)| \\
&= \Big|[I^2(I-Q)Nx(t_2) - I^2(I-Q)Nx(t_1)] \\
&-\lambda[I^{2-\alpha}K_P(I-Q)Nx(t_2) - I^{2-\alpha}K_P(I-Q)Nx(t_1)]\Big| \\
&\leq M(t_2^{\alpha}-t_1^{\alpha}) + \frac{\|K_P(I-Q)N\|}{\Gamma(2-\alpha)}\left(\int_0^{t_1}\Big[(t_1-s)^{1-\alpha} - (t_2-s)^{1-\alpha}\Big]ds + \int_{t_1}^{t_2}(t_2-s)^{1-\alpha}ds\right) \\
&\leq M(t_2^{\alpha}-t_1^{\alpha}) + \frac{2\rho_1\|(I-Q)N\|_1}{\Gamma(3-\alpha)}(t_2-t_1)^{2-\alpha} \leq M(t_2^{\alpha}-t_1^{\alpha}) + \frac{2\rho_1\|(I-Q)N\|_1}{\Gamma(3-\alpha)}\delta^{2-\alpha}.
\end{aligned}
$$

Thus, ${}^cD^{\alpha-1}K_{P,Q}$ is equicontinuous. By Lemma 6, we see that $K_{P,Q} : \overline{\Omega} \to X$ is compact. Therefore, the operator N is L-compact on Ω. Then by Lemmas 12 and 13, we have

(i) $Lx \neq \gamma Nx$ for all $x \in (domL \setminus \ker L) \cap \partial\Omega$ and $\gamma \in (0,1)$,
(ii) $Nx \notin ImL$ for all $x \in \ker L \cap \partial\Omega$.

Thus, we only need to show that the third condition of Theorem 1 is satisfied. Consider the operator

$$H(x,\gamma) = \pm\gamma x + (1-\gamma)J_{NL}QNx.$$

According to the arguments in Lemma 14, we have

$$H(x,\gamma) \neq 0 \qquad \forall x \in \ker L \cap \partial\Omega$$

and therefore, by means of using the homotopy property of degree, we obtain that

$$
\begin{aligned}
deg(J_{NL}QN|_{\ker L}, \Omega\cap\ker L, 0) &= deg(H(\cdot,0), \Omega\cap\ker L, 0) \\
&= deg(H(\cdot,1), \Omega\cap\ker L, 0) \\
&= deg(\pm I, \Omega\cap\ker L, 0) \neq 0.
\end{aligned}
$$

Based on Theorem 1, we can conclude that the operator function $Lx = Nx$ has at least one solution in $domL \cap \partial\Omega$, which, equivalently, implies that the problem (1) and (2) has at least one solution in X in light of the condition (3). The proof is finished. □

5. Examples

We give three examples to demonstrate the benefits of our key results. We look at the boundary value problem for fractional Langevin equations:

$$
\begin{cases}
{}^cD^{\frac{3}{2}}(D+5)x(t) = g(t,x(t),x'(t),{}^cD^{\frac{1}{2}}x(t)) + e(t), & 0<t<1 \\
x(0)=0, \quad x'(0)=0, \quad x(1)=\beta x(\frac{1}{2})
\end{cases}
\tag{18}
$$

where $\alpha = \frac{3}{2}$, $\lambda = 5$, $\eta = \frac{1}{2}$. Consider

$$g(t,x,y,z) = a_1(t)x + a_2(t)y + a_3(t)z + 10t\sin(\pi t), \qquad \text{and} \qquad e(t) = 1 + \cos^2(\pi t)$$

where $a_i(t); i = 1,2,3$ $a_4 = 10t\sin(\pi t)$ are non-negative functions for all $t \in [0,1]$.

By carrying out Mathematica software and using the given values, it is found that

$$\beta = 2.53257, \qquad \mu = 0.035937,$$
$$\rho_1 = 0.23966, \qquad \rho_2 = 0.662538.$$

It is clear that the value of $\mu \neq 0$ which verify Lemma 5. Furthermore, the assumptions $(\mathcal{H}_1)$ and $(\mathcal{H}_4)$ hold as follow.

$(\mathcal{H}_1)$ It is easy to see that the function above satisfies

$$|g(t,x,y,z)| \leq a_1(t)|x| + a_2(t)|y| + a_3(t)|z| + 10t\sin(\pi t).$$

$(\mathcal{H}_4)$ Let $x \in \ker L$, then $x = c(\lambda t - 1 + e^{-\lambda t})$ and so

$$\begin{aligned} cQNx =& \frac{c}{\mu\Gamma(\alpha)}\left(\int_0^1 e^{-\lambda} - \beta\int_0^\eta e^{-\lambda\eta}\right)e^{\lambda u}\int_0^u (u-s)^{\alpha-1}[a_1(s)x(s) \\ &+ a_2(s)x'(s) + a_3(s)^c D^{\alpha-1}x(s) + 10t\sin(\pi s) + 1 + \cos^2(\pi s)\Big]dsdu. \end{aligned}$$

Example 1. *Let us suppose that*

$$a_1(t) = 0.5t^2, \quad a_2(t) = 0.5e^{-t}, \quad a_3(t) = 7t.$$

By carrying out Mathematica software and using the given values, it is found that

$$\rho_3 = 3.98273, \qquad \rho_4 = 5.82551.$$

All assumptions $(\mathcal{H}_2)$-$(\mathcal{H}_4)$ hold as follow*

$(\mathcal{H}_2)$ *To prove the boundedness of the set Ω_1 defined in (11), we arrived at*

$$\|x\| \leq \frac{\rho_1\rho_4 + \rho_2\ell_1}{1 - \rho_1\rho_3} = 30.6858 + 14.5619\ell_1$$

and this is true for all $\ell_1 > 0$.

$(\mathcal{H}_3)$ *In view of the calculations above $(0 < \rho_1\rho_3 = 0.954502 < 1)$ which verifies the third condition.*

$(\mathcal{H}_4)$ *Through the values above, $cQNx$ can be read as*

$$cQNx = \frac{c}{\mu\Gamma(\alpha)}(0.747365c + 0.321111)$$

and thus we can choose $|c| > \ell_2 = 0.321111/0.747365 = 0.429658$ which yields that $cQNx > 0$.

Therefore all assumptions of Theorem 11 hold.

Example 2. *Let us suppose that*

$$a_1(t) = 0.5t^2, \quad a_2(t) = 0, \quad a_3(t) = 0.$$

By carrying out Mathematica software and using the given values, it is found that

$$\rho_3 = 0.166667, \qquad \rho_4 = 5.82551.$$

All assumptions ($\mathcal{H}_2$)-($\mathcal{H}_4$) *hold as follow*

($\mathcal{H}_2$) *To prove the boundedness of the set* Ω_1 *defined in* (11), *we arrived at*

$$\|x\| \leq \frac{\rho_1\rho_4 + \rho_2\ell_1}{1-\rho_1\rho_3} = 1.45423 + 0.690103\ell_1$$

and this is true for all $\ell_1 > 0$.

($\mathcal{H}_3$) *In view of the calculations above* $(0 < \rho_1\rho_3 = 0.0399434 < 1)$ *which verifies the third condition.*

($\mathcal{H}_4$) *Through the values above,* $cQNx$ *can be read as*

$$cQNx = \frac{c}{\mu\Gamma(\alpha)}(0.0166261c + 0.321111)$$

and thus we can choose $|c| > \ell_2 = 0.321111/0.0166261 = 19.3137$ *which yields that* $cQNx > 0$.

Therefore all assumptions of Theorem 11 hold.

Example 3. *Let us suppose that*

$$a_1(t) = 0, \quad a_2(t) = e^{-t}, \quad a_3(t) = 6t.$$

By carrying out Mathematica software and using the given values, it is found that

$$\rho_3 = 3.13212, \qquad \rho_4 = 5.82551.$$

All assumptions ($\mathcal{H}_2$)-($\mathcal{H}_4$) *hold as follow*

($\mathcal{H}_2$) *To prove the boundedness of the set* Ω_1 *defined in* (11), *we arrived at*

$$\|x\| \leq \frac{\rho_1\rho_4 + \rho_2\ell_1}{1-\rho_1\rho_3} = 5.59903 + 2.65701\ell_1$$

and this is true for all $\ell_1 > 0$.

($\mathcal{H}_3$) *In view of the calculations above* $(0 < \rho_1\rho_3 = 0.750645 < 1)$ *which verifies the third condition.*

($\mathcal{H}_4$) *Through the values above,* $cQNx$ *can be read as*

$$cQNx = \frac{c}{\mu\Gamma(\alpha)}(0.584641c + 0.321111)$$

and thus we can choose $|c| > \ell_2 = 0.321111/0.584641 = 0.549246$ *which yields that* $cQNx > 0$.

Therefore all assumptions of Theorem 11 hold.

6. Conclusions

In the resonance case, we presented some sufficient conditions to investigate the existence of a solution to the three-point boundary value problem involving a sequential fractional differential equation. We applied the coincidence degree theory due to Mawhin to discuss the existence results at resonance which depends on the Fredholm operator with index zero L and two continuous projectors P and Q. We proved that the operator L is invertible and deduced its inverse $K_P\colon ImL \to domL \cap \ker P$ which was the main tool to complete the idea. To illustrate our results, we provided an example. In addition, our approach was simple to understand and can be applied to a variety of real-life problems. Through the previous examples, we emphasize that the norm that we are working on in our paper does not depend on the form of the function g whether it depends on x, x' or

${}^{c}D^{\alpha-1}x$. The form of a norm depends basically on if it is appropriate to make the space a Banach space.

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