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On Some Solvable Systems of Some Rational Difference Equations of Third Order

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Abstract: Our aim in this paper is to obtain formulas for solutions of rational difference equations such as $x_{n+1} = 1 \pm (x_{n-1}y_n)/(1-y_n)$, $y_{n+1} = 1 \pm (y_{n-1}x_n)/(1-x_n)$, and $x_{n+1} = 1 \pm (x_{n-1}y_{n-2})/(1-y_n)$, $y_{n+1} = 1 \pm (y_{n-1}x_{n-2})/(1-x_n)$, where the initial conditions x_{-2}, x_{-1}, x_0, y_{-2}, y_{-1}, y_0 are non-zero real numbers. In addition, we show that the some of these systems are periodic with different periods. We also verify our theoretical outcomes at the end with some numerical applications and draw it by using some mathematical programs to illustrate the results.

Keywords: difference equations; periodicity nature; system of difference equations

MSC: 39A10

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1. Introduction

Nonlinear difference equations have recently captured the attention of numerous scholars. In fact, during the past ten years, we have encouraged a rapid rise in interest in these kinds of equations. The fact that these types of equations have several applications not just in mathematics but also in related areas, particularly in biological sciences, engineering, ecology, discrete temporal systems, economics, physics, and so on, may have contributed to the desire. We believe that, as more appealing and engaging results are achieved and communicated in studies, this area of research will continue to captivate the minds of more scholars in the years to come. The challenge of solving nonlinear difference equations in closed form has emerged as a common theme in this research area. In reality, a large number of articles attempt to solve nonlinear difference equations in any way they can; for an example, see [1–6]. Evidently, it can be very difficult to obtain the solution form for these kinds of equations in general. However, a number of approaches have lately been proposed to simplify challenging nonlinear difference equations into linear forms with established solution forms. For instance, a sizable class of nonlinear difference equations were solved in closed-form by converting into linear types (see, e.g., [7–12]).

Numerous academics have examined how systems of solved difference equations behave, for instance: Cinar examined the answers to the following system of difference equations in [13].

$$\Psi_{n+1} = \frac{m}{\Theta_n}, \quad \Theta_{n+1} = \frac{p\Theta_n}{\Psi_{n-1}\Theta_{n-1}}.$$

El-Metwally [14] found the solutions form for the following systems of rational difference equations:

$$\mu_{n+1} = \frac{\mu_{n-1}\varpi_n}{\pm\mu_{n-1} \pm \varpi_{n-2}}, \; \varpi_{n+1} = \frac{\varpi_{n-1}\mu_n}{\pm\varpi_{n-1} \pm \mu_{n-2}}.$$

Kara and Yazlik in [15] showed that the following three-dimensional system of difference equations

$$\varpi_{n+1} = \frac{\vartheta_n \vartheta_{n-2}}{b\varpi_{n-1} + a\vartheta_{n-2}}, \ \vartheta_{n+1} = \frac{z_n z_{n-2}}{d\vartheta_{n-1} + cz_{n-2}}, \ z_{n+1} = \frac{\varpi_n \varpi_{n-2}}{fz_{n-1} + e\varpi_{n-2}},$$

can be solved. Furthermore, they determined the forbidden set of the initial conditions by employing acquired formulas. Finally, they provided various applications involving the difference equation system discussed before.

Mansour et al. [16] examined the behavior of the difference equations systems' solutions

$$\varpi_{n+1} = \frac{\varpi_{n-5}}{-1 + \varpi_{n-5} y_{n-2}}, \ y_{n+1} = \frac{y_{n-5}}{\pm 1 \pm y_{n-5} \varpi_{n-2}}.$$

In [17], Ozban studied the positive solutions of the following system of rational difference equations

$$\varpi_{n+1} = \frac{a}{\vartheta_{n-3}}, \ \vartheta_{n+1} = \frac{b\vartheta_{n-3}}{\varpi_{n-q}\vartheta_{n-p}}.$$

Sroysang [18] focused on a system of a rational higher-order difference equation

$$x_{n+1} = \frac{x_{n-m+1}}{A + \vartheta_n \vartheta_{n-1} \ldots \vartheta_{n-m+1}}, \ \vartheta_{n+1} = \frac{\vartheta_{n-m+1}}{A + x_n x_{n-1} \ldots x_{n-m+1}}.$$

Touafek et al. [19] investigated the periodic nature and provided the form of the solutions of the following systems of rational difference equations

$$x_{n+1} = \frac{y_n}{x_{n-1}(\pm 1 \pm y_n)}, \ y_{n+1} = \frac{x_n}{y_{n-1}(\pm 1 \pm x_n)}.$$

Furthermore, Yalçınkaya [20] has obtained the sufficient conditions for the global asymptotic stability of the following system of two nonlinear difference equations

$$x_{n+1} = \frac{x_n + y_{n-1}}{x_n y_{n-1} - 1}, \ y_{n+1} = \frac{y_n + x_{n-1}}{y_n x_{n-1} - 1}.$$

In [21], Zhang et al. studied the boundedness, the persistence, and global asymptotic stability of the positive solutions of the following system

$$x_n = A + \frac{1}{y_{n-p}}, \quad y_n = A + \frac{y_{n-1}}{x_{n-r} y_{n-s}}.$$

Zhang et al. [22] studied the dynamics of a system of the rational third-order difference equation

$$x_{n+1} = \frac{x_{n-2}}{B + y_n y_{n-1} y_{n-2}}, \ y_{n+1} = \frac{y_{n-2}}{A + x_n x_{n-1} x_{n-2}}.$$

For more studies for nonlinear difference equations and systems of rational difference equations see [2,23–32].

Furthermore, difference equations are appropriate models for describing situations where population growth is not continuous but seasonal with overlapping generations.

Researchers have looked at the generalized Beverton–Holt stock recruitment model in [33]

$$x_{n+1} = ax_n + \frac{bx_{n-1}}{1 + cx_{n-1} + dx_n}.$$

Khaliq et al. [34] studied the dynamical analysis of the following system of discrete-time two-predators and the one-prey Lotka–Volterra model

$$\begin{aligned} x_{n+1} &= \frac{\alpha x_n - \beta x_n y_n - \gamma x_n z_n}{1+\delta x_n}, \\ y_{n+1} &= \frac{\zeta y_n + \eta x_n y_n - \mu y_n z_n}{1+\varepsilon y_n}, \\ z_{n+1} &= \frac{\nu z_n + \rho x_n z_n - \sigma y_n z_n}{1+\omega z_n}. \end{aligned}$$

The boundedness character, persistence, local, and global behavior of the following two-directional interacting and invasive species model were examined by Din and Elsayed [35]

$$x_{n+1} = \alpha + \beta x_n + \gamma x_{n-1} e^{-y_n}, \quad y_{n+1} = \delta + \epsilon y_n + \zeta y_{n-1} e^{-x_n}.$$

The authors in [36] explored local dynamics with topological classifications, bifurcation analysis, and chaos control in a discrete-time COVID-19 epidemic model. See also [37–39].

In this paper, we deal with the existence of the form for the solutions of the following systems of difference equations

$$x_{n+1} = 1 + \delta \frac{x_{n-1} y_n}{1-y_n}, \quad y_{n+1} = 1 + \gamma \frac{y_{n-1} x_n}{1-x_n}, \tag{1}$$

and

$$x_{n+1} = 1 + \delta \frac{x_{n-1} y_{n-2}}{1-y_n}, \quad y_{n+1} = 1 + \gamma \frac{y_{n-1} x_{n-2}}{1-x_n}, \tag{2}$$

with the initial conditions $x_{-2},\ x_{-1},\ x_0, y_{-2},\ y_{-1}$, and y_0 are arbitrary non zero real numbers.

2. Main Results

2.1. System (1) When δ = +1 and γ = +1

In this section, we investigate the solutions of the following system of two difference equations

$$x_{n+1} = 1 + \frac{x_{n-1} y_n}{1-y_n}, \quad y_{n+1} = 1 + \frac{y_{n-1} x_n}{1-x_n},$$

where $n \in \mathbb{N}_0$ and the initial conditions are arbitrary non zero real numbers with $x_0 \neq 1$ and $y_0 \neq 1$.

Theorem 1. *Let $\{x_n, y_n\}_{n=-1}^{\infty}$ be a solution of System (1), then*

1. *$\{x_n, y_n\}_{n=-1}^{\infty}$ is a periodic solution with period four, i.e., $x_{n+4} = x_4$, $y_{n+4} = y_n$ for all $n \geq -1$.*
2. *$\{x_n, y_n\}_{n=-2}^{\infty}$ has the following form*

$$\begin{aligned} x_{4n-1} &= x_{-1}, \quad x_{4n} = x_0, \quad x_{4n+1} = 1 + \frac{x_{-1} y_0}{1-y_0}, \quad x_{4n+2} = \frac{(x_0-1)(1-y_{-1})}{y_{-1}}, \\ y_{4n-1} &= y_{-1}, \quad y_{4n} = y_0, \quad y_{4n+1} = 1 + \frac{y_{-1} x_0}{1-x_0}, \ y_{4n+2} = \frac{(x_{-1}-1)(1-y_0)}{x_{-1}}. \end{aligned}$$

Proof. From Equation (1), we have

$$\begin{aligned}
x_{n+1} &= 1+\frac{x_{n-1}y_n}{1-y_n}, \quad y_{n+1}=1+\frac{y_{n-1}x_n}{1-x_n},\\
x_{n+2} &= 1+\frac{x_n y_{n+1}}{1-y_{n+1}} = 1+\frac{x_n\left(1+\frac{y_{n-1}x_n}{1-x_n}\right)}{1-\left(1+\frac{y_{n-1}x_n}{1-x_n}\right)} = 1+\frac{x_n\left(1+\frac{y_{n-1}x_n}{1-x_n}\right)}{\left(\frac{-y_{n-1}x_n}{1-x_n}\right)}\\
&= 1-\frac{x_n(1-x_n+y_{n-1}x_n)}{y_{n-1}x_n} = \frac{y_{n-1}-(1-x_n+y_{n-1}x_n)}{y_{n-1}} = \frac{y_{n-1}-1+x_n-y_{n-1}x_n}{y_{n-1}} = \frac{(x_n-1)(1-y_{n-1})}{y_{n-1}},\\
y_{n+2} &= 1+\frac{y_n x_{n+1}}{1-x_{n+1}} = 1+\frac{y_n\left(1+\frac{x_{n-1}y_n}{1-y_n}\right)}{1-\left(1+\frac{x_{n-1}y_n}{1-y_n}\right)} = 1-\frac{\left(1+\frac{x_{n-1}y_n}{1-y_n}\right)}{\left(\frac{x_{n-1}}{1-y_n}\right)} = 1-\frac{(1-y_n+x_{n-1}y_n)}{x_{n-1}}\\
&= \frac{x_{n-1}-(1-y_n+x_{n-1}y_n)}{x_{n-1}} = \frac{(x_{n-1}-1)(1-y_n)}{x_{n-1}}.
\end{aligned}$$

Furthermore, we see from Equation (1) that

$$\begin{aligned}
x_{n+3} &= 1+\frac{x_{n+1}y_{n+2}}{1-y_{n+2}} = 1+\frac{\left(1+\frac{x_{n-1}y_n}{1-y_n}\right)\left(\frac{(x_{n-1}-1)(1-y_n)}{x_{n-1}}\right)}{1-\left(\frac{(x_{n-1}-1)(1-y_n)}{x_{n-1}}\right)} = 1+\frac{(1-y_n+x_{n-1}y_n)(x_{n-1}-1)}{\{x_{n-1}-(x_{n-1}-1)(1-y_n)\}}\\
&= 1+\frac{(1-y_n+x_{n-1}y_n)(x_{n-1}-1)}{\{x_{n-1}-(x_{n-1}-1+y_n-x_{n-1}y_n)\}} = 1+\frac{(1-y_n+x_{n-1}y_n)(x_{n-1}-1)}{(1-y_n+x_{n-1}y_n)} = 1+(x_{n-1}-1) = x_{n-1},\\
y_{n+3} &= 1+\frac{y_{n+1}x_{n+2}}{1-x_{n+2}} = 1+\frac{\left(1+\frac{y_{n-1}x_n}{1-x_n}\right)\left(\frac{(x_n-1)(1-y_{n-1})}{y_{n-1}}\right)}{1-\left(\frac{(x_n-1)(1-y_{n-1})}{y_{n-1}}\right)} = 1+\frac{(x_n-1-y_{n-1}x_n)(1-y_{n-1})}{y_{n-1}-((x_n-1)(1-y_{n-1}))}\\
&= 1+\frac{(x_n-1-y_{n-1}x_n)(1-y_{n-1})}{y_{n-1}-(x_n-1-y_{n-1}x_n+y_{n-1})} = 1+\frac{(x_n-1-y_{n-1}x_n)(1-y_{n-1})}{-(x_n-1-y_{n-1}x_n)} = 1-(1-y_{n-1}) = y_{n-1}.
\end{aligned}$$

Finally we obtain

$$\begin{aligned}
x_{n+4} &= 1+\frac{x_{n+2}y_{n+3}}{1-y_{n+3}} = 1+\frac{\left(\frac{(x_n-1)(1-y_{n-1})}{y_{n-1}}\right)y_{n-1}}{1-y_{n-1}} = 1+\frac{(x_n-1)(1-y_{n-1})}{1-y_{n-1}} = 1+x_n-1 = x_n,\\
y_{n+4} &= 1+\frac{y_{n+2}x_{n+3}}{1-x_{n+3}} = 1+\frac{\left(\frac{(x_{n-1}-1)(1-y_n)}{x_{n-1}}\right)x_{n-1}}{1-x_{n-1}} = 1+\frac{(x_{n-1}-1)(1-y_n)}{1-x_{n-1}} = 1-(1-y_n) = y_n.
\end{aligned}$$

This completes the proof. □

2.2. System (1) When δ = −1 and γ = +1

The system of difference equations' solutions are provided in this section

$$x_{n+1} = 1-\frac{x_{n-1}y_n}{1-y_n}, \quad y_{n+1} = 1+\frac{y_{n-1}x_n}{1-x_n},$$

since $n \in \mathbb{N}_0$ and the initial values are arbitrary nonzero real numbers such that $x_0 \neq 1$ and $y_0 \neq 1$.

Theorem 2. *Assume $\{x_n, y_n\}$ is a solution of System (2). Then, for $n = 0,1,2,\ldots,$*

$$\begin{aligned}
x_{4n-1} &= x_{-1}+(2n), \quad x_{4n} = x_0+(2n),\\
x_{4n+1} &= \frac{x_{-1}y_0}{(y_0-1)}+(2n+1), \quad x_{4n+2} = \frac{1-x_0}{y_{-1}}+x_0+(2n+1),\\
y_{4n-1} &= y_{-1}+\frac{(2n)y_{-1}}{(x_0-1)}, \quad y_{4n} = y_0+\frac{(2n)(y_0-1)}{x_{-1}},\\
y_{4n+1} &= 1-\frac{y_{-1}(2n+x_0)}{x_0-1}, \quad y_{4n+2} = \frac{(1-y_0)(x_{-1}+(2n+1))}{x_{-1}}.
\end{aligned}$$

Proof. The result is true for $n = 0$. Assume n exceeds 1 and that $n - 1$ is consistent with our premise, which is

$$\begin{aligned} x_{4n-5} &= x_{-1} + (2n-2), \quad x_{4n-4} = x_0 + (2n-2), \\ x_{4n-3} &= \frac{x_{-1}y_0}{(y_0-1)} + (2n-1), \quad x_{4n-2} = \frac{1-x_0}{y_{-1}} + x_0 + (2n-1), \end{aligned}$$

$$\begin{aligned} y_{4n-5} &= y_{-1} + \frac{(2n-2)y_{-1}}{(x_0-1)}, \quad y_{4n-4} = y_0 + \frac{(2n-2)(y_0-1)}{x_{-1}}, \\ y_{4n-3} &= 1 - \frac{y_{-1}(2n-2+x_0)}{x_0-1}, \quad y_{4n-2} = \frac{(1-y_0)(x_{-1}+(2n-1))}{x_{-1}}. \end{aligned}$$

It follows from System (2) that

$$\begin{aligned} x_{4n-1} &= 1 - \frac{x_{4n-3}y_{4n-2}}{1-y_{4n-2}} = 1 - \frac{\left(\frac{x_{-1}y_0}{(y_0-1)} + (2n-1)\right)\left(\frac{(1-y_0)(x_{-1}+(2n-1))}{x_{-1}}\right)}{1-\frac{(1-y_0)(x_{-1}+(2n-1))}{x_{-1}}} \\ &= 1 - \frac{(-x_{-1}y_0+(2n-1)(1-y_0))(x_{-1}+(2n-1))}{x_{-1}-(1-y_0)(x_{-1}+(2n-1))} = 1 - \frac{[-x_{-1}y_0+(2n-1)(1-y_0)](x_{-1}+(2n-1))}{x_{-1}-[x_{-1}-y_0x_{-1}+(2n-1)(1-y_0)]} \\ &= 1 - \frac{[-x_{-1}y_0+(2n-1)(1-y_0)](x_{-1}+(2n-1))}{-[-y_0x_{-1}+(2n-1)(1-y_0)]} = 1 + (x_{-1} + (2n-1)) = x_{-1} + 2n, \end{aligned}$$

$$\begin{aligned} y_{4n-1} &= 1 + \frac{y_{4n-3}x_{4n-2}}{1-x_{4n-2}} = 1 + \frac{\left(1-\frac{y_{-1}(2n-2+x_0)}{x_0-1}\right)\left(\frac{1-x_0}{y_{-1}} + x_0 + (2n-1)\right)}{1-\left(\frac{1-x_0}{y_{-1}} + x_0 + (2n-1)\right)} \\ &= 1 + \frac{(x_0 - 1 - y_{-1}(2n-2+x_0))(1 - x_0 + y_{-1}x_0 + y_{-1}(2n-1))}{(x_0-1)y_{-1}\left[1 - \left(\frac{1-x_0}{y_{-1}} + x_0 + (2n-1)\right)\right]} \\ &= 1 + \frac{(x_0-1-y_{-1}(2n-2+x_0))(1-x_0+y_{-1}x_0+y_{-1}(2n-1))}{(x_0-1)[y_{-1}-1+x_0-x_0y_{-1}-y_{-1}(2n-1)]} \\ &= 1 + \frac{(x_0-1-y_{-1}(2n-2+x_0))(1-x_0+y_{-1}x_0+y_{-1}(2n-1))}{(x_0-1)[-1+x_0-x_0y_{-1}-y_{-1}(2n-2)]} \\ &= 1 + \frac{(1-x_0+y_{-1}x_0+y_{-1}(2n-1))}{(x_0-1)} \\ &= 1 - 1 + \frac{(y_{-1}x_0+y_{-1}(2n-1))}{(x_0-1)} = \frac{y_{-1}(x_0+(2n-1))}{(x_0-1)}. \end{aligned}$$

Furthermore, we obtain from System (2) that

$$\begin{aligned} x_{4n} &= 1 - \frac{x_{4n-2}y_{4n-1}}{1-y_{4n-1}} = 1 - \frac{\left(\frac{1-x_0}{y_{-1}} + x_0 + (2n-1)\right)\left(\frac{y_{-1}(x_0+(2n-1))}{(x_0-1)}\right)}{1-\frac{y_{-1}(x_0+(2n-1))}{(x_0-1)}} \\ &= 1 - \frac{(1-x_0+x_0y_{-1}+y_{-1}(2n-1))(x_0+(2n-1))}{(x_0-1)-y_{-1}(x_0+(2n-1))} = 1 + (x_0 + (2n-1)) = x_0 + 2n, \\ y_{4n} &= 1 + \frac{y_{4n-2}x_{4n-1}}{1-x_{4n-1}} = 1 + \frac{\left(\frac{(1-y_0)(x_{-1}+(2n-1))}{x_{-1}}\right)(x_{-1}+2n)}{1-(x_{-1}+2n)} = 1 - \frac{(1-y_0)(x_{-1}+2n)}{x_{-1}} = y_0 + \frac{(2n)(y_0-1)}{x_{-1}}. \end{aligned}$$

Similarly, we can prove the other relations. This completes the proof. □

Lemma 1. *Let $\{x_n, y_n\}$ be a solution of System (2), then $\{x_n\}$, $\{y_n\}$ are unbounded solutions.*

Proof. The proof follows from the expressions of solutions of System (2). □

The following theorems can be proved similar to the previous theorem so it will be omitted.

Theorem 3. *Assume that the sequences $\{x_n, y_n\}$ are a solution of the system*

$$x_{n+1} = 1 + \frac{x_{n-1}y_n}{1-y_n}, \quad y_{n+1} = 1 - \frac{y_{n-1}x_n}{1-x_n}.$$

Then, all solutions are unbounded and provided by the following formulas

$$\begin{aligned}
x_{4n-1} &= x_{-1}\left(\frac{y_0+(2n-1)}{y_0-1}\right), \quad x_{4n} = x_0 + \frac{(2n)(x_0-1)}{y_{-1}}, \\
x_{4n+1} &= 1 + \frac{x_{-1}(y_0+(2n))}{(1-y_0)}, \quad x_{4n+2} = \frac{(1-x_0)(y_{-1}+(2n+1))}{y_{-1}}, \\
y_{4n-1} &= y_{-1} + (2n), \quad y_{4n} = y_0 + (2n), \\
y_{4n+1} &= (2n+1) + \frac{y_{-1}x_0}{x_0-1}, \quad y_{4n+2} = \frac{(1-y_0)}{x_{-1}} + y_0 + (2n+1),
\end{aligned}$$

where $n = 0, 1, 2, \ldots$ *and* $x_0, y_0 \neq 1$.

Theorem 4. *If $x_0, y_0 \neq 1$, then the solutions of the system*

$$x_{n+1} = 1 - \frac{x_{n-1}y_n}{1-y_n}, \quad y_{n+1} = 1 - \frac{y_{n-1}x_n}{1-x_n},$$

are unbounded and for $n = 0, 1, 2, \ldots$

$$\begin{aligned}
x_{4n-1} &= (2n) + x_{-1}\left(\frac{y_0+(2n-1)}{y_0-1}\right), \quad x_{4n} = x_0 + (2n) + \frac{(2n)(x_0-1)}{y_{-1}}, \\
x_{4n+1} &= (2n+1) + \frac{x_{-1}(y_0+(2n))}{(y_0-1)}, \quad x_{4n+2} = (2n+1) + x_0 + \frac{(x_0-1)(2n+1)}{y_{-1}}, \\
y_{4n-1} &= (2n) + \frac{y_{-1}(x_0+2n-1)}{(x_0-1)}, \quad y_{4n} = y_0 + (2n) + \frac{(2n)(y_0-1)}{x_{-1}}, \\
y_{4n+1} &= (2n+1) + \frac{y_{-1}(x_0+2n)}{x_0-1}, \quad y_{4n+2} = y_0 + (2n+1) + \frac{(2n+1)(y_0-1)}{x_{-1}}.
\end{aligned}$$

2.3. System (2) When $\delta = +1$ and $\gamma = +1$

In this section, we obtain the form of the solutions of the following system of difference equations

$$x_{n+1} = 1 + \frac{x_{n-1}y_{n-2}}{1-y_n}, \quad y_{n+1} = 1 + \frac{y_{n-1}x_{n-2}}{1-x_n}, \tag{3}$$

where $n \in \mathbb{N}_0$ and the initial conditions are arbitrary non-zero real numbers such that $x_0, y_0 \neq 1$.

Theorem 5. *If $\{x_n\}_{n=-2}^{\infty}$, $\{y_n\}_{n=-2}^{\infty}$ are solutions of System (3). Then, for $n = 0, 1, \ldots$*

$$\begin{aligned}
x_{2n-1} &= \sum_{i=0}^{n-1}\left(\frac{y_{-2}}{1-y_0}\right)^i + x_{-1}\left(\frac{y_{-2}}{1-y_0}\right)^n, \quad x_{2n} = \sum_{i=0}^{n-1}\left(\frac{x_0-1}{x_{-2}}\right)^i + x_0\left(\frac{x_0-1}{x_{-2}}\right)^n, \\
y_{2n-1} &= \sum_{i=0}^{n-1}\left(\frac{x_{-2}}{1-x_0}\right)^i + y_{-1}\left(\frac{x_{-2}}{1-x_0}\right)^n, \quad y_{2n} = \sum_{i=0}^{n-1}\left(\frac{y_0-1}{y_{-2}}\right)^i + y_0\left(\frac{y_0-1}{y_{-2}}\right)^n,
\end{aligned}$$

where $\sum_{i=0}^{-1}(A)^i = 0$.

Proof. The conclusion is true for $n = 0$. Assume n exceeds 1 and that $n - 1$ is covered by our supposition, so that

$$\begin{aligned}
x_{2n-4} &= \sum_{i=0}^{n-3}\left(\frac{x_0-1}{x_{-2}}\right)^i + x_0\left(\frac{x_0-1}{x_{-2}}\right)^{n-2}, \; x_{2n-3} = \sum_{i=0}^{n-2}\left(\frac{y_{-2}}{1-y_0}\right)^i + x_{-1}\left(\frac{y_{-2}}{1-y_0}\right)^{n-1}, \\
x_{2n-2} &= \sum_{i=0}^{n-2}\left(\frac{x_0-1}{x_{-2}}\right)^i + x_0\left(\frac{x_0-1}{x_{-2}}\right)^{n-1}, \\
y_{2n-4} &= \sum_{i=0}^{n-3}\left(\frac{y_0-1}{y_{-2}}\right)^i + y_0\left(\frac{y_0-1}{y_{-2}}\right)^{n-2}, \; y_{2n-3} = \sum_{i=0}^{n-2}\left(\frac{x_{-2}}{1-x_0}\right)^i + y_{-1}\left(\frac{x_{-2}}{1-x_0}\right)^{n-1}, \\
y_{2n-2} &= \sum_{i=0}^{n-2}\left(\frac{y_0-1}{y_{-2}}\right)^i + y_0\left(\frac{y_0-1}{y_{-2}}\right)^{n-1}.
\end{aligned}$$

Now, it follows from System (3) that

$$\begin{aligned}
x_{2n-1} &= 1 + \frac{x_{2n-3}y_{2n-4}}{1-y_{2n-2}} \\
&= 1 + \frac{\left(\sum_{i=0}^{n-2}\left(\frac{y_{-2}}{1-y_0}\right)^i + x_{-1}\left(\frac{y_{-2}}{1-y_0}\right)^{n-1}\right)\left(\sum_{i=0}^{n-3}\left(\frac{y_0-1}{y_{-2}}\right)^i + y_0\left(\frac{y_0-1}{y_{-2}}\right)^{n-2}\right)}{1-\left(\sum_{i=0}^{n-2}\left(\frac{y_0-1}{y_{-2}}\right)^i + y_0\left(\frac{y_0-1}{y_{-2}}\right)^{n-1}\right)} \\
&= 1 + \frac{\left(\sum_{i=0}^{n-2}\left(\frac{y_{-2}}{1-y_0}\right)^i + x_{-1}\left(\frac{y_{-2}}{1-y_0}\right)^{n-1}\right)\left(\sum_{i=0}^{n-3}\left(\frac{y_0-1}{y_{-2}}\right)^i + y_0\left(\frac{y_0-1}{y_{-2}}\right)^{n-2}\right)}{1-\left(1+\left(\frac{y_0-1}{y_{-2}}\right)+\left(\frac{y_0-1}{y_{-2}}\right)^2+\cdots+\left(\frac{y_0-1}{y_{-2}}\right)^{n-2} + y_0\left(\frac{y_0-1}{y_{-2}}\right)^{n-1}\right)} \\
&= 1 - \frac{\left(\sum_{i=0}^{n-2}\left(\frac{y_{-2}}{1-y_0}\right)^i + x_{-1}\left(\frac{y_{-2}}{1-y_0}\right)^{n-1}\right)\left(\sum_{i=0}^{n-3}\left(\frac{x_{-2}}{1-x_0}\right)^i + y_0\left(\frac{x_{-2}}{1-x_0}\right)^{n-2}\right)}{\left(\frac{y_0-1}{y_{-2}}\right)\left(1+\left(\frac{y_0-1}{y_{-2}}\right)+\left(\frac{y_0-1}{y_{-2}}\right)^2+\cdots+\left(\frac{y_0-1}{y_{-2}}\right)^{n-3} + y_0\left(\frac{y_0-1}{y_{-2}}\right)^{n-2}\right)} \\
&= 1 - \left(\frac{y_{-2}}{y_0-1}\right)\left(\sum_{i=0}^{n-2}\left(\frac{y_{-2}}{1-y_0}\right)^i + x_{-1}\left(\frac{y_{-2}}{1-y_0}\right)^{n-1}\right) \\
&= 1 + \left(\frac{y_{-2}}{1-y_0}\right)\left(\sum_{i=0}^{n-2}\left(\frac{y_{-2}}{1-y_0}\right)^i + x_{-1}\left(\frac{y_{-2}}{1-y_0}\right)^{n-1}\right) \\
&= \sum_{i=0}^{n-1}\left(\frac{y_{-2}}{1-y_0}\right)^i + x_{-1}\left(\frac{y_{-2}}{1-y_0}\right)^{n}. \\
y_{2n-1} &= 1 + \frac{y_{2n-3}x_{2n-4}}{1-x_{2n-2}} \\
&= 1 + \frac{\left(\sum_{i=0}^{n-2}\left(\frac{x_{-2}}{1-x_0}\right)^i + y_{-1}\left(\frac{x_{-2}}{1-x_0}\right)^{n-1}\right)\left(\sum_{i=0}^{n-3}\left(\frac{x_0-1}{x_{-2}}\right)^i + x_0\left(\frac{x_0-1}{x_{-2}}\right)^{n-2}\right)}{1-\left(\sum_{i=0}^{n-2}\left(\frac{x_0-1}{x_{-2}}\right)^i + x_0\left(\frac{x_0-1}{x_{-2}}\right)^{n-1}\right)} \\
&= 1 + \frac{\left(\sum_{i=0}^{n-2}\left(\frac{x_{-2}}{1-x_0}\right)^i + y_{-1}\left(\frac{x_{-2}}{1-x_0}\right)^{n-1}\right)}{\left(\frac{1-x_0}{x_{-2}}\right)} \\
&= 1 + \left(\frac{x_{-2}}{1-x_0}\right)\left(\sum_{i=0}^{n-2}\left(\frac{x_{-2}}{1-x_0}\right)^i + y_{-1}\left(\frac{x_{-2}}{1-x_0}\right)^{n-1}\right) \\
&= \sum_{i=0}^{n-1}\left(\frac{x_{-2}}{1-x_0}\right)^i + y_{-1}\left(\frac{x_{-2}}{1-x_0}\right)^{n}.
\end{aligned}$$

We can prove the other relations similarly. The proof is complete. □

Lemma 2. *Let $\{x_n, y_n\}$ be a solution of System (3), then $\{x_n\}$, $\{y_n\}$ are unbounded solutions.*

The following theorems can be treated similarly to the previous results.

Theorem 6. *The solutions of the system*

$$x_{n+1} = 1 - \frac{x_{n-1}y_{n-2}}{1-y_n},\ y_{n+1} = 1 - \frac{y_{n-1}x_{n-2}}{1-x_n}, \tag{4}$$

where $x_0, y_0 \neq 1$ are provided as follows

$$\begin{aligned} x_{2n-1} &= \sum_{i=0}^{n-1}\left(\frac{y_{-2}}{y_0-1}\right)^i + x_{-1}\left(\frac{y_{-2}}{y_0-1}\right)^n,\ x_{2n} = \sum_{i=0}^{n-1}\left(\frac{x_0-1}{x_{-2}}\right)^i + x_0\left(\frac{x_0-1}{x_{-2}}\right)^n, \\ y_{2n-1} &= \sum_{i=0}^{n-1}\left(\frac{x_{-2}}{x_0-1}\right)^i + y_{-1}\left(\frac{x_{-2}}{x_0-1}\right)^n,\ y_{2n} = \sum_{i=0}^{n-1}\left(\frac{y_0-1}{y_{-2}}\right)^i + y_0\left(\frac{y_0-1}{y_{-2}}\right)^n. \end{aligned}$$

Lemma 3. *If $x_0 + x_{-2} = 1$, and $y_0 + y_{-2} = 1$, then the solutions are bounded and periodic with period four and adopt the form*

$$\begin{aligned} \{x_n\}_{n=-2}^{\infty} &= \{x_{-2}, x_{-1}, x_0, 1-x_{-1}, x_{-2}, x_{-1}, \dots\}, \\ \{y_n\}_{n=-2}^{\infty} &= \{y_{-2}, y_{-1}, y_0, 1-y_{-1}, y_{-2}, y_{-1}, \dots\}. \end{aligned}$$

Otherwise, every solution of System (4) is unbounded.

Theorem 7. *Every solution $\{x_n, y_n\}$ of the following system*

$$x_{n+1} = 1 + \frac{x_{n-1}y_{n-2}}{1-y_n},\quad y_{n+1} = 1 - \frac{y_{n-1}x_{n-2}}{1-x_n}, \tag{5}$$

with non-zero real numbers, the initial conditions satisfies $x_0, y_0 \neq 1$ adopts the form

$$\begin{aligned} x_{4n-1} &= \sum_{i=0}^{n-1}(-1)^i\left\{\left(\frac{y_{-2}}{y_0-1}\right)^{2i} + \left(\frac{y_{-2}}{y_0-1}\right)^{2i+1}\right\} + (-1)^n x_{-1}\left(\frac{y_{-2}}{y_0-1}\right)^{2n}, \\ x_{4n} &= \sum_{i=0}^{n-1}(-1)^i\left\{\left(\frac{x_0-1}{x_{-2}}\right)^{2i} + \left(\frac{x_0-1}{x_{-2}}\right)^{2i+1}\right\} + (-1)^n x_0\left(\frac{x_0-1}{x_{-2}}\right)^{2n}, \\ x_{4n+1} &= 1 + \sum_{i=0}^{n-1}(-1)^{i+1}\left\{\left(\frac{y_{-2}}{y_0-1}\right)^{2i+1} + \left(\frac{y_{-2}}{y_0-1}\right)^{2i+2}\right\} + (-1)^{n+1} x_{-1}\left(\frac{y_{-2}}{y_0-1}\right)^{2n+1}, \\ x_{4n+2} &= 1 + \sum_{i=0}^{n-1}(-1)^{i+1}\left\{\left(\frac{x_0-1}{x_{-2}}\right)^{2i+1} + \left(\frac{x_0-1}{x_{-2}}\right)^{2i+2}\right\} + (-1)^{n+1} x_0\left(\frac{x_0-1}{x_{-2}}\right)^{2n+1}, \\ y_{4n-1} &= \sum_{i=0}^{n-1}(-1)^i\left\{\left(\frac{x_{-2}}{x_0-1}\right)^{2i} - \left(\frac{x_{-2}}{x_0-1}\right)^{2i+1}\right\} + (-1)^n y_{-1}\left(\frac{x_{-2}}{x_0-1}\right)^{2n}, \\ y_{4n} &= \sum_{i=0}^{n-1}(-1)^i\left\{\left(\frac{y_0-1}{y_{-2}}\right)^{2i} + \left(\frac{y_0-1}{y_{-2}}\right)^{2i+1}\right\} + (-1)^n y_0\left(\frac{y_0-1}{y_{-2}}\right)^{2n}, \\ y_{4n+1} &= 1 + \sum_{i=0}^{n-1}(-1)^i\left\{\left(\frac{x_{-2}}{x_0-1}\right)^{2i+1} - \left(\frac{x_{-2}}{x_0-1}\right)^{2i+2}\right\} + (-1)^n y_{-1}\left(\frac{x_{-2}}{x_0-1}\right)^{2n+1}, \end{aligned}$$

$$y_{4n+2} = 1 + \sum_{i=0}^{n-1}(-1)^{i+1}\left\{\left(\frac{y_0-1}{y_{-2}}\right)^{2i+1} + \left(\frac{y_0-1}{y_{-2}}\right)^{2i+2}\right\} + (-1)^{n+1} y_0\left(\frac{y_0-1}{y_{-2}}\right)^{2n+1}.$$

Lemma 4. *If $x_0 + x_{-2} = 1$ and $y_0 + y_{-2} = 1$, then the solutions of System (5) are bounded and periodic with period eight as follows*

$$\begin{aligned} \{x_n\}_{n=-2}^{\infty} &= \{x_{-2}, x_{-1}, x_0, 1 + x_{-1}, 1 + x_0, -x_{-1}, -x_0, 1 - x_{-1}, x_{-2}, x_{-1}, \dots\}, \\ \{y_n\}_{n=-2}^{\infty} &= \{y_{-2}, y_{-1}, y_0, 1 - y_{-1}, 1 + y_0, 2 - y_{-1}, -y_0, -1 + y_{-1}, y_{-2}, y_{-1}, \dots\}. \end{aligned}$$

Otherwise, every solution of System (5) is unbounded.

Remark 1. *As in the previous theorem, we can obtain the solutions of the following system*

$$x_{n+1} = 1 - \frac{x_{n-1}y_{n-2}}{1 - y_n}, \; y_{n+1} = 1 + \frac{y_{n-1}x_{n-2}}{1 - x_n}.$$

3. Numerical Examples

In this section, we present some numerical examples that support the above theoretical results.

Example 1. *Suppose System (1) under the initial conditions $x_{-1} = 0.5$, $x_0 = -2$, $y_{-1} = 7$ and $y_0 = -0.8$. See Figure 1 below.*

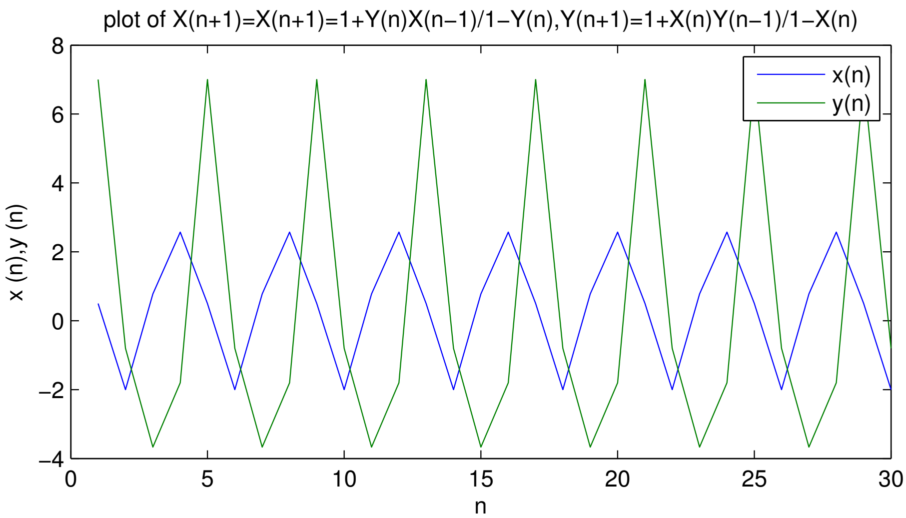

Figure 1. Represents behavior of System (1) when $x_{-1} = 0.5$, $x_0 = -2$, $y_{-1} = 7$ and $y_0 = -0.8$.

Example 2. *See Figure 2 below as an example for System (2) with the initial values $x_{-1} = 5$, $x_0 = 0.4$, $y_{-1} = 0.3$, and $y_0 = 0.7$.*

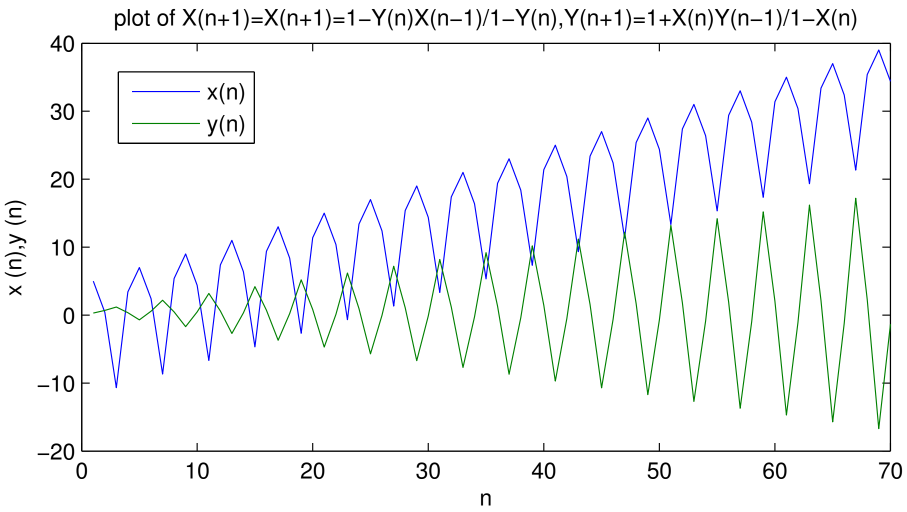

Figure 2. Shows the behavior for System (2) with the initial values $x_{-1} = 5$, $x_0 = 0.4$, $y_{-1} = 0.3$ and $y_0 = 0.7$.

Example 3. *Figure 3 below shows the behaviour of the solution of System (3) when the initial values $x_{-2} = 0.5$, $x_{-1} = 1.2$, $x_0 = 0.6$, $y_{-2} = 0.7$, $y_{-1} = 0.8$, and $y_0 = 0.3$.*

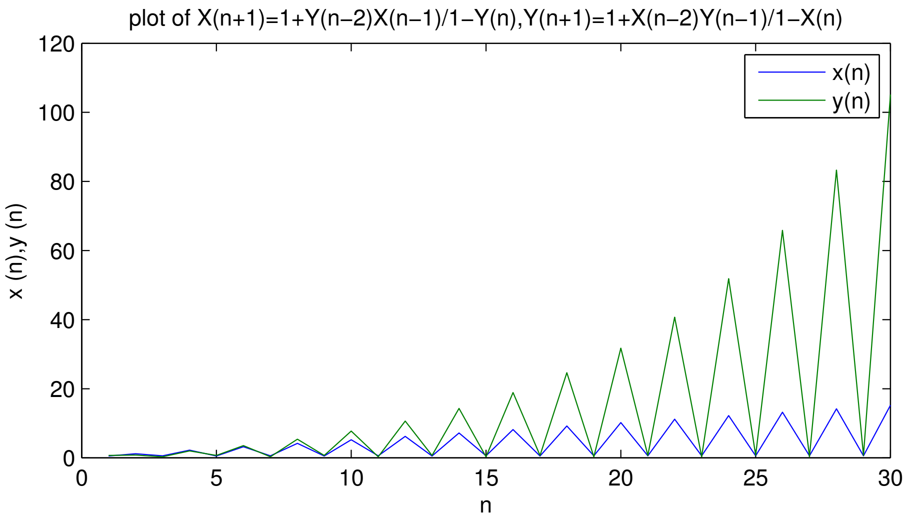

Figure 3. Expresses the solution of System (3) when the initial values $x_{-2} = 0.5$, $x_{-1} = 1.2$, $x_0 = 0.6$, $y_{-2} = 0.7$, $y_{-1} = 0.8$ and $y_0 = 0.3$.

Example 4. *Suppose the initial conditions for the system $x_{n+1} = 1 - \dfrac{x_{n-1}y_{n-2}}{1 - y_n}$, $y_{n+1} = 1 - \dfrac{y_{n-1}x_{n-2}}{1 - x_n}$ are $x_{-2} = 0.9$, $x_{-1} = 1.2$, $x_0 = 0.1$, $y_{-2} = 0.3$, $y_{-1} = -2$, and $y_0 = 0.7$. See Figure 4.*

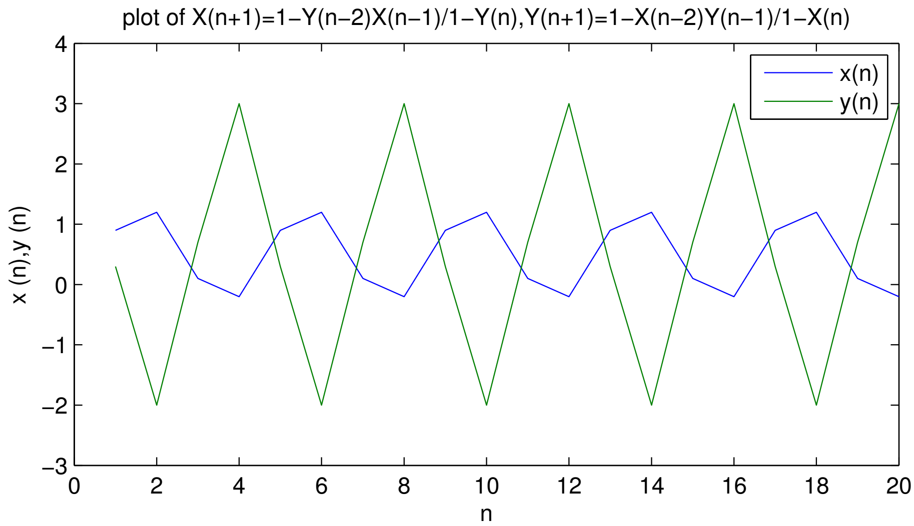

Figure 4. Plot of solutions of System (4) when the initial conditions are $x_{-2} = 0.9$, $x_{-1} = 1.2$, $x_0 = 0.1$, $y_{-2} = 0.3$, $y_{-1} = -2$ and $y_0 = 0.7$.

Example 5. *Figure 5 shows the periodic nature of the solution of System (5) with the initial conditions* $x_{-2} = 0.4,\ x_{-1} = -1,\ x_0 = 0.6,\ y_{-2} = 0.2,\ y_{-1} = -3$, *and* $y_0 = 0.8$.

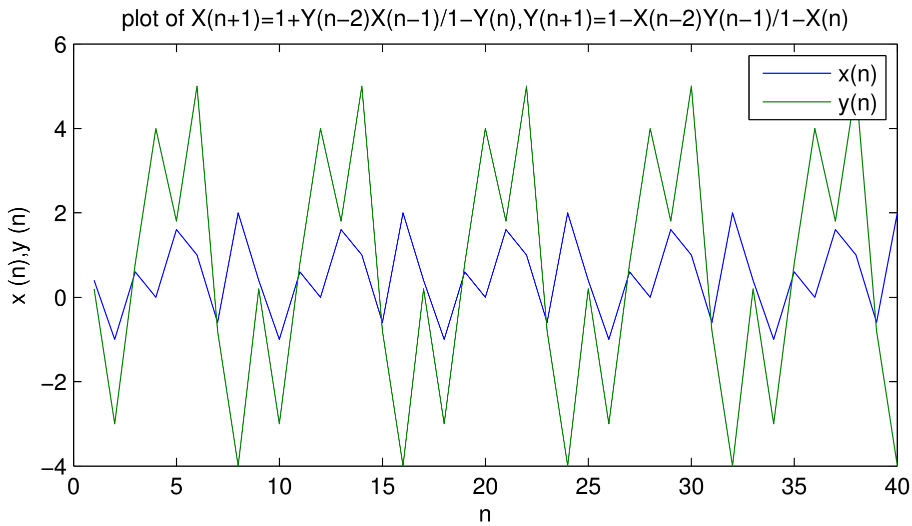

Figure 5. Shows the periodicity of the solution of System (5) with $x_{-2} = 0.4,\ x_{-1} = -1,\ x_0 = 0.6,\ y_{-2} = 0.2,\ y_{-1} = -3$ and $y_0 = 0.8$.

4. Conclusions

In this paper, we obtained the expressions of the solutions of different classes of third-order rational systems of difference equations. In Section 1, the work of the authors on the same side of the difference equations, whether they are equations, systems of equations, or some applications of difference equations, is presented. After the introduction in Section 2, we have solved the first system of second order rational difference equations $x_{n+1} = 1 + \dfrac{x_{n-1}y_n}{1-y_n}$, $y_{n+1} = 1 + \dfrac{y_{n-1}x_n}{1-x_n}$. After finding the solutions, we provided numerical examples to illustrate the results. In Section 3, we obtained the form of the solution of the second system $x_{n+1} = 1 - \dfrac{x_{n-1}y_n}{1-y_n}$, $y_{n+1} = 1 + \dfrac{y_{n-1}x_n}{1-x_n}$; we also mentioned the solutions of the other systems $x_{n+1} = 1 + \dfrac{x_{n-1}y_n}{1-y_n}$, $y_{n+1} = 1 - \dfrac{y_{n-1}x_n}{1-x_n}$, and $x_{n+1} = 1 - \dfrac{x_{n-1}y_n}{1-y_n}$, $y_{n+1} = 1 - \dfrac{y_{n-1}x_n}{1-x_n}$. Finally, Section 4 was devoted to the form of the solutions of the main system of third order fractional difference equations $x_{n+1} = 1 + \frac{x_{n-1}y_{n-2}}{1-y_n}$, $y_{n+1} = 1 + \frac{y_{n-1}x_{n-2}}{1-x_n}$ and some other systems that we obtained the expressions of the solutions for and studied the periodicity nature of the solutions. Moreover, we confirmed our results using numerical simulations and drew them using Matlab program.

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