mathematics

Article

Associated Statistical Parameters' Aggregations in Interactive MADM

Gia Sirbiladze * and Tariel Khvedelidze

Department of Computer Sciences, Ivane Javakhishvili Tbilisi State University, University St. 13, Tbilisi 0186, Georgia
* Correspondence: gia.sirbiladze@tsu.ge; Tel.: +995-599588658

Abstract: From recent studies, the concept of "monotone expectation" (ME) of Interactive Multi-Attribute Decision Making (MADM) is well known, which was developed for the case of different fuzzy sets. This article develops the concept of "monotone expectation" for such statistical parameters as variance, k-order moment and covariance. We investigate the problem of the definition of some statistical parameters, when the uncertainty is represented by a monotone measure—a fuzzy measure—instead of an additive measure. The study presents the concept of the definition of monotone statistical parameters based on the Choquet finite integral for the definition of monotone expectation, monotone variance, monotone k-order moment and monotone covariance. Associated statistical parameters are also presented—expectation, variance, k-order moment and covariance—which are defined in relation to associated probabilities of a fuzzy measure. It is shown that the monotone statistical parameters defined in the study are defined by one particular relevant associated statistical parameter out of the total number $n!$ of such parameters. It is also shown that the aggregations with monotone statistical parameters used in interactive MADM models take into account interactions of the focal elements of only one consonant structure from the $n!$ consonant structures of attributes. In order to take into account the interactions of the focal elements of all $n!$ consonant structures of attributes, the monotone statistical parameters were expanded into the F-associated statistical parameters. Expansion correctness implies that if dual second-order Choquet capacities are taken as the fuzzy measures of aggregation of the F-associated statistical parameters, then the F-associated statistical parameters coincide with the corresponding monotone statistical parameters. A scheme for embedding new aggregation operators, monotone statistical parameters and F-associated statistical parameters into the interactive MADM model has been developed. Specific numerical examples are presented to illustrate the obtained results.

Keywords: fuzzy measure; Choquet integral; monotone statistical parameters; F-associated statistical parameters; aggregation operators; MADM

MSC: 68T37; 68T20; 68T30

Citation: Sirbiladze, G.; Khvedelidze, T. Associated Statistical Parameters' Aggregations in Interactive MADM. *Mathematics* **2023**, *11*, 1061. https://doi.org/10.3390/math11041061

Academic Editor: Vasile Preda

Received: 26 January 2023
Revised: 18 February 2023
Accepted: 19 February 2023
Published: 20 February 2023

1. Introduction

It is known that modern decision-making technologies play an important role in improving almost all aspects of human activity. Along with classical approaches in the construction of decision-making models of complex processes and phenomena, the most important issue is the assumption of fuzziness. This assumption is related to the high complexity of the objects to be studied, which is caused by the lack or absence of objective data. In such cases, expert knowledge and assessments are the only sources of information with which to make reliable decisions. However, the complexity of expert information reduces our ability to make reliable decisions. This is due to the contradictory nature of imprecision and uncertainty of expert assessments. One of the main tasks of researchers is to minimize this complexity.

When dealing with complex decision-making problems where expert judgments are crucial, exact and stochastic models often have less reliability. It becomes quite inconvenient to use them. At this time, the use of fuzzy modeling becomes especially important because a systemic approach for development of information structure of a decision-making system with uncertainty enables us to construct convenient intelligent decision support instruments. Our research deals with the study of fuzzy and stochastic samplings of a dual nature for the decision-making characteristic population. We also deal with how to build such statistics-aggregation tools on these samplings, with which we will be able to reduce the risks of decision making.

For such problems, without limiting the generality, suppose that the decision maker (DM) has a set $D = \{d_1, d_2, \ldots, d_m\}$ of m possible alternatives of an uncertain type, from which the DM must create a ranked list by some preference relation or select the optimal alternative according to some preference relation. The result associated with this problem is a combination of characteristics, activities, attributes, criteria, symptoms and others, affecting the decision procedure. This variable is usually called the state of nature, which affects the payoff, utilities, evaluations, compatibility values, membership levels, etc., of the DM's preferences or subjective activities. This variable is assumed to take its values from some set of attributes $S = \{s_1, s_2, \ldots, s_n\}$. As a result, the DM knows that if he/she selects d_i and the state of nature assumes the value s_j, then his/her payoff (valuation, utility and so on) is c_{ij} (values of some stochastic or fuzzy variable). The objective of the decision is to select the "best" alternative and get the biggest payoff. However, the selection procedure is more difficult. In this case, each alternative can be seen as corresponding to a row vector of possible payoffs. To make a choice, the DM must compare these vectors, a problem which generally does not lead to a compelling solution.

Assume d_i and d_k are two alternatives such that for all j, $j = 1, \ldots, n$; $c_{ij} \geq c_{kj}$. In this case, there is a reason to select d_i. In this situation, we shall say d_i dominates d_k $(d_i \succeq d_k)$. Furthermore, if there exists an alternative (optimal decision) that dominates all the alternatives, then it will be the Pareto-optimal solution. Faced with the general difficulty of comparing payoffs vector, we must provide some means of comparing these vectors. Many studies are focused on the construction of aggregation operator (function) under a fuzzy-probabilistic environment that can take a collection of n values and transform them into a single value.

Aggregations based on fuzzy measure (same capacity, monotone measure) [1,2] in MADM models [3–6] are known for their important roles in representing the uncertainties of interacting attributes [7–9]. Based on the Choquet integral [2] whose special determinant is the fuzzy measure, very interesting aggregation operators [10–26] are constructed for the problems solving of practical value for decision making for different fuzzy environments. The probabilistic interpretation of the Choquet integral is related to the population expectation [27]. It should be noted that the unified theory of aggregation of mathematical statistics parameters based on additive probability measure is effective in MADM aggregation operators and heuristic approaches. However, this cannot be said in the case of a fuzzy measure. For the fuzzy measure as a descriptive tool of uncertainty, there is no unified systematic theory in the aggregation constructions of statistical parameters. This is due to the difficulty associated with the definition of statistical parameters (variance, covariance, correlation, k-order moment, etc.) as aggregation operators, when the additive measure describes the uncertainty in the definition—the probability is replaced by a non-additive but monotone fuzzy measure. The analogy takes place only in the case of expectation when it is replaced by monotone expectation (the Choquet integral) [2,27].

The Choquet integral, or monotone expectation, has been found to have many interesting statistical properties in MADM aggregation schemes. One of the important properties, according to the interests of our research, is the consideration of the influences of interacting attributes on a decision in the MADM aggregation process [7–9,23–25]. It is well known that these interactions to some degree in Choquet aggregations have been demonstrated in a number of studies for different fuzzy environments [23–25]. Let us note here that some similar mathematical foundations of the mathematical statistics tool in the case of fuzzy uncertainty were developed a long time ago. Basically, there are two directions: (1) imprecise probabilities [28,29], by which are constructed the upper and lower expectations and which have a deep semantic interpretation; (2) the Dempster–Shafer belief structure [30,31]. However, the latter can be interpreted as a special case of the imprecise probability approach.

In the statistical modeling of complex phenomena, special progress can be observed in the fuzzy probability and fuzzy statistics approaches [32–34], which mainly develop in two directions. The first direction refers to the methods that develop classical (non-fuzzy) data analysis based on fuzzy set theory. These are: fuzzy clustering [35], fuzzy linear regression [36], fuzzy hypotheses for non-fuzzy-populations [37], fuzzy-logic-based time series forecasting [38], fuzzy-utility-based statistical decision-making theory [39] and others. The second direction includes the following methods: the maximum confidence approach for fuzzy data [40], classification and identification problems in the environment of a Dempster–Shafer belief structure [41], the approach of statistical hypotheses for fuzzy data [42], discriminant analysis [24,25,43] and cluster analysis for fuzzy data [44].

In this paper, our interest is in constructing aggregation operators for interactive MADM, based on statistical parameters, when the degree of uncertainty in aggregations is represented by fuzzy measure. We rely on the concept of "monotone expectation", which has been developed in several studies [18–25,45–49]. Our study is limited by consideration of a discrete universe, which is mainly due to the finite number of attributes in MADM. More specifically: the authors [18–25] developed aggregation operators based on the Choquet integral, in which the fuzzy measure is replaced by its associated probability class [26,27]. The latter represents the direction of imprecise probabilities. In these studies, the concept of "monotone expectation" is considered for different fuzzy environments. In [18], the mentioned concept is developed for immediate associated probabilities and triangular fuzzy arguments environment. In [19], the authors developed the same concept for an intuitionistic fuzzy environment. The extension of fuzzy-weighted aggregation operators with the same concept is presented in [20]. In [21], the concept of associated immediate probability is developed for an intuitionistic fuzzy environment. In [22,25], the concept of monotone expectation is developed for the extension of ordered weighted averaging (OWA) (geometric, OWG)) operators in intuitionistic fuzzy environment. In [23], the same concept for probabilistic averages is developed for the q-Rung orthopair fuzzy discrimination environment. In [24], OWA and OWG operators with the concept of "monotone expectation" are extended for the discrimination q-Rung picture linguistic environment. In this article, we present the concept expansion of "monotone expectation"—not for some fuzzy environment and not for the expansion of any known aggregation operator, but a face-changed model of the concept itself for some statistical parameters: the Choquet-integral-based monotone statistical parameters—monotone variance, monotone k-order moment, and monotone covariance. Additionally, most importantly, as developed in the studies [18–25], the extensions of the mentioned monotonic statistical parameters are presented, the so-called F-associated statistical parameters. From the point of view of MADM, the latter, in contrast to monotone statistical parameters, have the advantage that they take into account all possibilities of attribute interaction, taking into account all variants of attribute consonant structures. This means that their use in interactive MADM models will give us more reliable alternative rankings than using monotonic statistical parameters.

The second section presents definitions of some monotone statistical parameters with respect to fuzzy measures. The definitions of the Choquet second-order extreme capacities and their relations to the associated expectation are given. The third section presents the extensions of monotone statistical parameters in the F-associated statistical parameters. The conditions for correctness of the extensions are presented. The families of specific F-associated statistical parameters are discussed. The advantages of using F-associated statistical parameters in interactive MADM models compared to monotone associated parameters are analyzed. In the fourth section, simple numerical examples are constructed to illustrate the obtained results. A comparative analysis is given. The concluding part and future research perspectives are presented in the fifth section.

2. Preliminaries and Notations

We introduce the definition of a fuzzy measure (or monotone measure) (see Refs. [1–6]) adapted to the case of a finite referential:

Definition 1. *Let* $S = \{s_1, s_2, \ldots, s_n\}$ *be a finite set and* μ *be a set function.* $\mu : 2^S \to [0, 1]$. *We say* μ *is a fuzzy measure on* S *if it satisfies*

$$\begin{array}{l} (i)\ \mu(\varnothing) = 0;\ \ \mu(S) = 1; \\ (ii)\ \forall\, A, B \subseteq S, \quad if\ A \subseteq B,\ then\ \mu(A) \leq \mu(B). \end{array} \tag{1}$$

A fuzzy measure is a normalized and monotone set function. It can be considered as an extension of the probability concept, where additivity is replaced by the weaker condition of monotonicity.

In general, the possible orderings of the elements of S *are given by the permutations of a set with n elements, which form the group* S_n. *Now we consider a definition of associated probabilities induced by a fuzzy measure on the group* S_n.

Definition 2 ([46]). *The probability function* λ_σ *defined by*

$$\begin{array}{l} \lambda_\sigma(s_{\sigma(1)}) = \mu(\left\{s_{\sigma(1)}\right\}), \ldots, \\ \lambda_\sigma(s_{\sigma(i)}) = \mu(\left\{s_{\sigma(1)}, \ldots, s_{\sigma(i)}\right\}) - \mu(\left\{s_{\sigma(1)}, \ldots, s_{\sigma(i-1)}\right\}), \ldots, \\ \lambda_\sigma\left(s_{\sigma(n)}\right) = 1 - \mu(\left\{s_{\sigma(1)}, \ldots, s_{\sigma(n-1)}\right\}), \\ \lambda_\sigma(s_{\sigma(0)}) \equiv 0 \end{array} \tag{2}$$

for each $\sigma = (\sigma(1), \sigma(2), \ldots, \sigma(n)) \in S_n$ *is called the associated probability and the associated probability class (APC)*

$$\left\langle \lambda_\sigma = \left\{\lambda_\sigma(s_{\sigma(1)}), \lambda_\sigma(s_{\sigma(2)}), \ldots, \lambda_\sigma(s_{\sigma(n)})\right\}\right\rangle_{\sigma \in S_n}$$

of a fuzzy measure eμ.

For any associated probability $\lambda_\sigma,\ \sigma \in S_n$ *and any subset of attributes* $A \subset S$, *we have*

$$\lambda_\sigma(A) = \sum_{s \in A} \lambda_\sigma(s) \tag{3}$$

Definition 3 ([2]). *Let* $S = \{s_1, s_2, \ldots, s_n\}$ *be a set of all possible attributes and* $\mu : 2^S \to [0,1]$ *be a fuzzy measure with APC* $\{\lambda_\sigma\}_{\sigma \in S}$, $\lambda_{\sigma(i)} \equiv \lambda_\sigma(s_{\sigma(i)})$. *Let* $h : S \Rightarrow R_0^+$ *be any function on* S, $h_i \equiv h(s_i)$. *Let* $\tau \in S_n$ *be such permutation for which* $h_{\tau(1)} \geq h_{\tau(2)} \geq \ldots \geq h_{\tau(n)}$. *Then, the Choquet integral of* h *with respect to* μ *is*

$$(Ch)\int_S h \odot \mu \equiv \int_0^\infty \mu(s \in S / h(s) \geq \alpha) d\alpha = \sum_{i=1}^n \lambda_{\tau(i)} h_{\tau(i)} \tag{4}$$

It is clear that $(Ch)\int_S h \odot \mu = E_{\lambda_\tau}(h)$, *where* $E_{\lambda_\tau}(h)$ *is an expectation of hwith respect to probability measure* λ_τ *on* S.

2.1. Monotone Expectation

Definition 4 ([46]). *Let μ be a fuzzy measure on a finite set $S = \{s_1, s_2, \ldots, s_n\}$ and $h : S \to [0, +\infty)$ be some function. The monotone expectation of h with respect to the fuzzy measure μ is defined as the Choquet integral*:

$$ME(h) = (Ch)\int_S h \odot \mu. \tag{5}$$

From the definitions of a fuzzy measure, its APC and the Choquet integral, it is clear that the monotone expectation coincides with the mathematical expectation with respect to certain associated probability. Thus,

$$ME(h) = E_{\lambda_\tau}(h) = \sum_{i=1}^{n} h(s_{\tau(i)}) \cdot \lambda_\tau(s_{\tau(i)}) \tag{6}$$

where $h(s_{\tau(1)}) \geq \ldots \geq h(s_{\tau(n)})$.

Definition 5. *For any $\sigma \in S_n$, $E_{\lambda_\sigma}(h)$ is called an associated expectation, and the set of all associated expectations$\{E_{\lambda_\sigma}(h)\}_{\sigma \in S_n}$ is called the class of associated expectations.*

$E_{\lambda_\sigma}(h)$ is associated with the expectation calculated for certain associated probability of the fuzzy measure μ. In general, the relation between monotone expectation $ME(h)$ and associated expectations $\{E_{\lambda_\sigma}(h)\}_{\sigma \in S_n}$ exists as follows.

Proposition 1 ([46]). *Let $S = \{s_1, s_2, \ldots, s_n\}$ be some finite set and μ be any fuzzy measure on S. Let $h : S \to [0, +\infty)$ be some function with associated expectations class $\{E_{\lambda_\sigma}(h)\}_{\sigma \in S_n}$. Then,*

$$\min_{\sigma \in S_n} E_{\lambda_\sigma}(h) \leq ME(h) \leq \max_{\sigma \in S_n} E_{\lambda_\sigma}(h) \tag{7}$$

There is evidence that the equality in inequalities (7) is achieved for a wide class of fuzzy measures, for the Choquet second-order capacities.

Definition 6. *Two fuzzy measures μ_* and μ^* on S are called dual if $\forall A \subseteq S$*

$$\mu_*(A) = 1 - \mu^*(\overline{A}) \tag{8}$$

Proposition 2 ([46]). *APCs of dual fuzzy measures coincide.*

Definition 7 ([46]). *Two dual fuzzy measures μ_* and μ^* on S are called lower and upper Choquet second-order capacities, respectively, if for any two subsets $A, B \subset S$:*

$$\begin{aligned} \mu_*(A \cup B) + \mu_*(A \cap B) &\geq \mu_*(A) + \mu_*(B), \\ \mu^*(A \cup B) + \mu^*(A \cap B) &\leq \mu^*(A) + \mu^*(B). \end{aligned} \tag{9}$$

Proposition 3 ([46]). *Let μ_* and μ^* be the Choquet lower and upper second- order capacities on$S$, h be any function $h : S \to [0, +\infty)$ with associated expectations class $\{E_{\lambda_\sigma}(h)\}_{\sigma \in S_n}$. Then,*

$$\begin{aligned} ME_{\mu_*}(h) &= \min_{\sigma \in S_n} E_{\lambda_\sigma}(h), \\ ME_{\mu^*}(h) &= \max_{\sigma \in S_n} E_{\lambda_\sigma}(h), \end{aligned} \tag{10}$$

$$\begin{aligned} &\text{For any subset } A \subseteq S \\ &\mu_*(A) = \min_{\sigma \in S_n} \lambda_\sigma(A)\ , \\ &\mu^*(A) = \max_{\sigma \in S_n} \lambda_\sigma(A)\ . \end{aligned} \tag{11}$$

The concept of monotone variance of statistical parameters introduced by Campos and Bolanos [46] continued developing in the works [18–25,45,47–49]. Reference [47] presents the concept of monotone variance definition, similar to the concept of monotone expectation presented in [46]. This concept is closely related to the fuzzy measure's associated expectations. In the studies [18–25,45,47–49], the concept is developed for the statistical parameters of the monotone expectation composition.

The notion of monotone variance is introduced as a monotone expectation of quantity $[h - ME(h)]^2$*with some similarity to the definition of the classical variance:*

$$MVar(h) \triangleq ME(h - ME)^2 \tag{12}$$

However, we have some difficulty in the representation (12); h and $(h - ME(h))^2$ *are not comonotonic quantities; and their associated probabilities may be different in the representation of the associated expectation of* $MVar$*. Therefore, in [47], the authors searched for a natural definition of* $MVar$ *by direct classical associated variance.*

Definition 8. *If* $h : S \to [0, +\infty)$ *is some function with associated expectations class* $\{E_{\lambda_\sigma}(h)\}_{\sigma \in S_n}$*, then the associated variance with respect to the given associated probability is called the classical variance*:

$$Var_{\lambda_\sigma}(h) = E_{\lambda_\sigma}(h - E_{\lambda_\sigma}(h))^2\ ,\ \ \forall \sigma \in S_n \tag{13}$$

Definition 9 ([47]). *Let* $h : S \to [0, +\infty)$ *be some function with associated variances class* $\{Var_{\lambda_\sigma}(h)\}_{\sigma \in S_n}$*. Then, a monotone variance of function h is called a value*:

$$MVar(h) = Var_{\lambda_\tau}(h) \tag{14}$$

where $\tau \in S_n$ *is such permutation that* $h(s_{\tau(1)}) \geq h(s_{\tau(2)}) \geq \ldots \geq h(s_{\tau(n)})$.

It is clear that $MVar$ is a non-negative value and if the fuzzy measure is a probability, then $MVar$ coincides with the classical variance, because the associated probabilities of the probability measure coincide and represent this measure itself, i.e., $\{\lambda_\sigma = \lambda\ ,\ \ \sigma \in S_n\}$, where λ is a probability measure on S.

As it turns out, the monotone variance $MVar$ preserves important properties of the classical variance. For example:

Proposition 4 ([47]). *Let* $h : S \to [0, +\infty)$ *be any function and* μ *be fuzzy measure on a set* $S = \{s_1, \ldots, s_n\}$*. Then,*

$$MVar(h) = ME(h^2) - [ME(h)]^2 \tag{15}$$

$$\text{if } a \neq 0 \text{ and } a, b \in R \text{ are constants, then} MVar(ah + b) = a^2 MVar(h) \tag{16}$$

$$\text{if } h = const \text{ then } MVar(h) = 0$$

2.2. Monotone Moments

If we use the same concept that was presented in the previous paragraph, we can extend this concept for the definition of central and non-central moments, when the additive probability measure is replaced by a monotone fuzzy measure.

Definition 10. *Let μ be a fuzzy measure on $S = \{s_1, s_2, \ldots, s_n\}$ and $h : S \to [0, +\infty)$ be some function with associated expectations class $\{E_{\lambda_\sigma}(h)\}_{\sigma \in S_n}$. A (non-central) monotone moment of the function h with respect to the fuzzy measure μ is called the value*

$$M\alpha_k(h) = E_{\lambda_\tau}(h^k)\,,\ k \in N \tag{17}$$

where $\left\{E_{\lambda_\sigma}(h^k) \equiv \alpha_k^{(\lambda_\alpha)}\right\}_{\sigma \in S_n}$ represents the class of associated k-order (non-central) moments, and $\tau \in S_n$ is such permutation for which $h(s_{\tau(1)}) \geq h(s_{\tau(2)}) \geq \ldots \geq h(s_{\tau(n)})$.

Definition 11. *Let μ be a fuzzy measure on $S = \{s_1, s_2, \ldots, s_n\}$ and $h : S \to [0, +\infty)$ be some function with associated expectations class $\{E_{\lambda_\sigma}(h)\}_{\sigma \in S_n}$. A k-th order central monotone moment is called the value*

$$M\gamma_k(h) = E_{\lambda_\tau}(h - ME(h))^k \tag{18}$$

where $\tau \in S_n$ is such a permutation that $h(s_{\tau(1)}) \geq h(s_{\tau(2)}) \geq \ldots \geq h(s_{\tau(n)})$.

One may easily prove the following proposition:

Proposition 5. *Let $h : S = \{s_1, \ldots, s_n\} \to [0, +\infty)$ be some function and μ be a fuzzy measure on S, Then,*

$$M\gamma_k(h) = \sum_{i=o}^{k} (-1)^i \binom{k}{i} [M\alpha_i(h)]^i M\alpha_{k-i}(h) \tag{19}$$

It is easy to prove the propositions for the monotone variance $MVar(h)$, k-order monotone non-central moments $M\alpha_k(h)$, $k = 2, \ldots$, analogous to the Proposition 1 and Proposition 3.

Proposition 6. *Let $h : S = \{s_1, \ldots, s_n\} \to [0, +\infty)$ be some function, and μ_* and μ^* be the second-order dual Choquet capacities on S with associated variances class $\{Var_{\lambda_\sigma}(h)\}_{\sigma \in S_n}$ and associated k-order moments class $\left\{\alpha_k^{(\lambda_\sigma)}(h)\right\}_{\sigma \in S_n}$. Then,*

$$\begin{aligned} MVar_{\mu_*}(h) &= \min_{\sigma \in S_n}\{Var_{\lambda_\sigma}(h)\}\,, \\ MVar_{\mu^*}(h) &= \max_{\sigma \in S_n}\{Var_{\lambda_\sigma}(h)\} \end{aligned} \tag{20}$$

$$\begin{aligned} M\alpha_k^{(\mu_*)}(h) &= \min_{\sigma \in S_n}\left\{\alpha_k^{(\lambda_\sigma)}(h)\right\}, \quad k = 1, 2, \ldots\ , \\ M\alpha_k^{(\mu^*)}(h) &= \max_{\sigma \in S_n}\left\{\alpha_k^{(\lambda_\sigma)}(h)\right\}, \quad k = 1, 2, \ldots \end{aligned} \tag{21}$$

Finally, the concept of monotone expectation $ME(h)$ presented here can be extended to the monotone expectation of any composition $h \circ G$ if $h : S \to [0, +\infty)$ and $G : [0, +\infty) \to [0, +\infty)$ is a non-decreasing function.

Definition 12. *If $h : S \to [0, +\infty)$ is some function and $G : [0, +\infty) \to [0, +\infty)$ is some non-decreasing function, and μ is a fuzzy measure on S,, then a monotone expectation of composition $h \circ G$ is called the value*

$$ME(h \circ G) \triangleq (Ch)\int_S (h \circ G) \odot \mu \tag{22}$$

It is easy to show the following propositions.

Proposition 7. *If $h : S = \{s_1, \ldots, s_n\} \to [0, +\infty)$ is some function and G is non-decreasing function $G : [0, +\infty) \to [0, +\infty)$ with associated expectations class $\{E_{\lambda_\sigma}(h \circ G)\}_{\sigma \in S_n}$, then*

$$ME(h \circ G) = E_{\lambda_\tau}(h \circ G) = \sum_{i=1}^{n} \lambda_\tau(s_{\tau(i)}) \cdot G(h(s_{\tau(i)})) \tag{23}$$

where $\tau \in S_n$ is such permutation that $h(s_{\tau(1)}) \geq h(s_{\tau(2)}) \geq \ldots \geq h(s_{\tau(n)})$.

Here we note that for composition $h \circ G$, we can formulate and prove proposition analogous to Proposition 5.

2.3. Monotone Covariance

Let us use the same concept and extend this concept to the covariance of two functions, when the additive measure in the covariance definition, the probability, is replaced by a non-additive but monotone measure, a fuzzy measure.

Definition 13. *Let h and g be any functions $h, g : S = \{s_1, \ldots, s_n\} \to [0, +\infty)$ with values $h_i = h(s_i)$ and $g_i = g(s_i)$. Let h and g be comonotonic functions, i.e., there exists such a permutation $\tau \in S_n$ that $h(s_{\tau(1)}) \geq h(s_{\tau(2)}) \geq \ldots \geq h(s_{\tau(n)})$ and $g(s_{\tau(1)}) \geq g(s_{\tau(2)}) \geq \ldots \geq g(s_{\tau(n)})$. Let μ be a fuzzy measure on S. A monotone covariance between comonotonic functions h and g is called the value*

$$\mathrm{Mcov}(h, g) = E_{\lambda_\tau}(h \cdot g) - E_{\lambda_\tau}(h) \cdot E_{\lambda_\tau}(g) = \sum_{i=1}^{n} \lambda_\tau(s_{\tau(i)})(h_{\tau(i)} - E_{\lambda_\tau}(h))(g_{\tau(i)} - E_{\lambda_\tau}(g)) \tag{24}$$

where $Cov_{\lambda_\tau}(h, g) = E_{\lambda_\tau}(h \cdot g) - E_{\lambda_\tau}(h) \cdot E_{\lambda_\tau}(g)$ is called associated covariance.

It is clear that functions h, g and $h \cdot g$ are comonotonic. Then the following proposition is valid (the proof is omitted):

Proposition 8. *Let functions $h, g : S = \{s_1, \ldots, s_n\} \to [0, +\infty)$ be comonotonic, μ be a fuzzy measure on S and $\{e_{\lambda_\sigma}(h)\}_{\sigma \in S_n}$, $\{e_{\lambda_\sigma}(g)\}_{\sigma \in S_n}$ be associated expectations classes for functions h and g, respectively. Then, the monotone covariance between functions h and g is represented by the classical covariance with respect to associated probability λ_σ*

$$\mathrm{Mcov}(h, g) = E_{\lambda_\tau}(h \cdot g) - E_{\lambda_\tau}(h) \cdot E_{\lambda_\tau}(g) = \sum_{i=1}^{n} \lambda_\tau(s_{\tau(i)})(h_{\tau(i)} - E_{\lambda_\tau}(h))(g_{\tau(i)} - E_{\lambda_\tau}(g)) \tag{25}$$

where $\tau \in S_n$ is such a permutation that $h(s_{\tau(1)}) \geq h(s_{\tau(2)}) \geq \ldots \geq h(s_{\tau(n)})$ and $g(s_{\tau(1)}) \geq g(s_{\tau(2)}) \geq \ldots \geq g(s_{\tau(n)})$.

3. Associated Statistical Parameters Based Aggregation Operators in Interactive MADM Models

As we mentioned in the introduction, the values aggregated with monotone statistical parameters of MADM with the Choquet integral describe the interactions between attributes to a certain degree [7–9]. Now, let us develop this idea in the following direction. First, let us recall the formula for definition of monotone expectation:

$$ME(h) = \sum_{i=1}^{n} h(s_{\tau(i)})[\mu(\left\{s_{\tau(i)}, \ldots, s_{\tau(i)}\right\}) - \mu(\{s_{\tau(i)}, \ldots, s_{\tau(i-1)}\})] = \sum_{i=1}^{n} h(s_{\tau(i)}) \cdot \lambda_\tau(s_{\tau(i)}) = E_{\lambda_\tau}(h) \tag{26}$$

where $\tau \in S_n$ is such a permutation that $h(s_{\tau(1)}) \geq h(s_{\tau(2)}) \geq \ldots \geq h(s_{\tau(n)})$. As was mentioned in the introduction, in the works [18–25] for different fuzzy environments, based on the definition of monotone expectation, interactions were revealed between the focal elements of the Dempster–Shafer belief consonant structure of attributes [30]:

$$\left\{s_{\tau(1)}\right\},\ldots,\ \left\{s_{\tau(1)},\ldots,s_{\tau(i-1)}\right\},\left\{s_{\tau(1)},\ldots,s_{\tau(i)}\right\},\ldots,s_{\tau(1)},\ldots,s_{\tau(n)}\right\}$$

which is reflected by the inclusion of a new attribute $s_{\sigma(i)}$ in the previous focal element $\left\{s_{\tau(1)},\ldots,s_{\tau(i-1)}\right\}$, and the difference in (26)

$$\mu(\left\{s_{\tau(1)},\ldots,s_{\tau(i)}\right\})-\mu(\left\{s_{\tau(1)},\ldots,s_{\tau(i-1)}\right\})$$

represents the degree of influence of this inclusion [7,9].

Often, these interactions cannot provide reliable aggregation results, and it becomes necessary to include focal elements of type (26) in aggregations for other permutations $\sigma \in S_n$. *Below, we introduce aggregation operators based on monotone statistical parameters that account for the inclusion of all number of consonant structures* $\left\langle\left\{s_{\sigma(1)}\right\},\ \left\{s_{\sigma(1)},s_{\sigma(2)}\right\},\ldots,\left\{s_{\sigma(1)},\ldots,s_{\sigma(n)}\right\}\right\rangle_{\sigma\in S_n}$.

Such aggregation was first introduced by the authors of this article in [18] for monotone expectation extension, which we present with another interpretation.

Definition 14. *Let* $h : S = \{s_1,\ldots,s_n\} \to [0,+\infty)$ *be some function and* μ *be a fuzzy measure on* S *with the class* $\{E_{\lambda_\sigma}(h)\}_{\sigma\in S_n}$ *of associated expectations to* h*. Suppose that* $F : (R_0^+)^{n!} \to R_0^+$ *is any averaging aggregation operator* [16]. F*-associated expectation of function* h *is called a value*

$$F-AsE(h)=F(E_{\lambda_\sigma}(h),\ldots,E_{\lambda_{n!}}(h)). \tag{27}$$

It follows from the boundedness property of the averaging operator F *that*

$$\min_{\sigma\in S_n}E_{\lambda_\sigma}(h)\leq F-AsE(h)\leq\max_{\sigma\in S_n}E_{\lambda_\sigma}(h). \tag{28}$$

Now let us consider the most important question of our research. This is a question of the correctness of the extension of aggregation operators, when the extended aggregations agree with the basis operators for some essential values of their defining parameters. In particular, for our case, in Section 3, we defined the monotone statistical parameters. At the same time, the relevant associated statistical parameters were also defined. In Section 4, on the basis of associated statistical parameters, we defined F-associated statistical parameters, which, unlike monotone statistical parameters, take into account the consonant structures of all possible combinations of attributes, that is, all their possible interactions, when aggregating in interactive MADM models. What is the relationship between these two classes of parameters? The answer is given in Propositions 9 and 10 below. The gist of these proofs is briefly as follows: these parameters coincide if the second-order dual Choquet capacities are taken as the fuzzy measure, and the operators MIN and MAX are taken as the F-averaging operator.

We can show an analogous proof of Proposition 3 for the F-aggregation operator.

Proposition 9. *Let* μ_* *and* μ^* *be the Choquet lower and upper capacities of order two on* S, h *be any function* $h : S \to [0,+\infty)$ *and* $F : (R_0^+)^{n!} \to R_0^+$ *be any averaging aggregation operator and* $\{E_{\lambda_\sigma}(h)\}_{\sigma\in S_n}$ *be associated expectations class of function* h *with respect to the fuzzy measure* μ*. Then*

(1) *if the* $F = \max$ *operator, then the* $F - AsE(\cdot)$ *operator coincides with the* ME *operator for the* $\mu = \mu^*$ *fuzzy measure:*

$$Max-AsE_{\mu^*}(h)=\max\{E_{\lambda_{\sigma_1}}(h),\ldots,E_{\lambda_{\sigma_{n!}}}(h)\}=E_\tau(h)=(Ch)\int_S f\odot\mu^*=ME(h), \tag{29}$$

(2) *if the $F = \min$ operator, thenthe $F - AsE(\cdot)$ operator coincides with the ME operator for the $\mu = \mu_*$ fuzzy measure:*

$$Min - AsE_{\mu_*}(h) = \min\Big\{E_{\lambda_{\sigma_1}}(h), \ldots, E_{\lambda_{\sigma_{n!}}}(h)\Big\} = E_\tau(h) = (Ch)\int_S f \odot \mu_* = ME(h)$$

where $\tau \in S_n$ is a such permutation for which $h_{\tau(1)} \geq h_{\tau(2)} \geq \ldots \geq h_{\tau(n)}$.

From this proof, we can draw the following conclusion: operator $F - AsE(\cdot)$ represents an extension of monotone expectation $ME(\cdot)$, and in the case of $F - Max$ and $F - Min$, this operator coincides with monotone expectation $ME(\cdot)$ for the Choquet second-order extremal capacities. Analogous proof can be stated with respect to other monotone statistical parameters too.

Definition 15. *Let functions h and g, $h, g : S = \{s_1, \ldots, s_n\} \to [0, +\infty)$ be comonotonic functions for some $\tau \in S_n$ permutation $h(s_{\tau(1)}) \geq h(s_{\tau(2)}) \geq \ldots \geq h(s_{\tau(n)})$ and $g(s_{\tau(1)}) \geq g(s_{\tau(2)}) \geq \ldots \geq g(s_{\tau(n)})$, μ be a fuzzy measure on S, $\{Var_{\lambda_\sigma}(h)\}_{\sigma\in S_n}$ be an associated variances class, $\Big\{\alpha_k^{(\lambda_\sigma)}(h)\Big\}_{\sigma\in S_n}$ be an associated k-order associated moments class and $\{Cov_{\lambda_\sigma}(f,g)\}_{\sigma\in S_n}$ be an associated covariances class and $F : (R_0^+)^{n!} \to R_0^+$ be some averaging operator. Then*

(1) *F-associated variance of function h is called the operator*

$$F - AsVar(h) = F(Var_{\lambda_{\sigma_1}}(h), \ldots, Var_{\lambda_{\sigma_{n!}}}(h)) \tag{30}$$

(2) *F-associated k-order moment of function h is called the operator*

$$F - As\alpha_k(h) = F(\alpha_k^{(\lambda_{\sigma_1})}(h), \ldots, \alpha_k^{(\lambda_{\sigma_{n!}})}(h)),\ k = 1, 2, \ldots \tag{31}$$

(3) *F-associated covariance of comonotonic functions h and g is called the operator*

$$F - AsCov(h, g) = F(Cov_{\lambda_{\sigma_1}}(h, g), \ldots, Cov_{\lambda_{\sigma_{n!}}}(h, g)) \tag{32}$$

For the new aggregation operators introduced in Definition 14, we can easily show proofs similar to Proposition 8, which indicates the correctness of the distribution of monotone statistical parameters.

Proposition 10. *Let μ_* and μ^* be the Choquet dual lower and upper capacities of order two on $S = \{s_1, \ldots, s_n\}$, h and g be comonotonic functions for some permutation $\tau \in S_n$, $h(s_{\tau(1)}) \geq h(s_{\tau(2)}) \geq \ldots \geq h(s_{\tau(n)})$ and $g(s_{\tau(1)}) \geq g(s_{\tau(2)}) \geq \ldots \geq g(s_{\tau(n)})$, $F : (R_0^+)^{n!} \to R_0^+$ be some averaging operator, $\{Var_{\lambda_\sigma}(h)\}_{\sigma\in S_n}$ be associated variances class of function h, $\Big\{\alpha_k^{(\lambda_\sigma)}(h)\Big\}_{\sigma\in S_n}$ be the associated non-central k -order moments class and $\{Cov_{\lambda_\sigma}(f,g)\}_{\sigma\in S_n}$ be the associated covariances class of functions h and g. Then,*

1. *If F is a* max *operator and $\mu = \mu^*$ is an upper capacity of order two, then*
   (a) *the $F - As\,Var(\cdot)$ operator coincides with the monotone variance parameter of h:*

$$Max - AsVar(h) = \max\{Var_{\lambda_{\sigma_1}}(h), \ldots, Var_{\lambda_{\sigma_{n!}}}(h)\} = Var_{\lambda_\tau}(h) = MVar(h) = (Ch)\int_S (h - ME(h))^2 \odot \mu^* \tag{33}$$

   (b) *the $F = As\alpha_k(\cdot)$ operator coincides with monotone k-order non-central moment parameter of h:*

$$Max - As\alpha_k(h) = \max\{\alpha_k^{(\lambda_{\sigma_1})}(h), \ldots, \alpha_k^{(\lambda_{\sigma_{n!}})}(h)\} = \alpha_k^{(\lambda_\tau)}(h) = M\alpha_k(h) = (Ch)\int_S (h^k) \odot \mu^* \tag{34}$$

(c) *the $F - AsCov(\cdot)$ operator coincides with monotone covariance parameter of comonotonic functions, h and g:*

$$\begin{aligned} &Max - AcCov(h,g) = \max\left\{Cov_{\lambda_{\sigma_1}}(h,g),\ldots,Cov_{\lambda_{\sigma_{n!}}}(h,g)\right\} = Cov_{\lambda_\tau}(h,g) = \\ &= MCov(h,g) = (Ch)\int_S (h - ME(h))(g - ME(g)) \odot \mu^* \end{aligned} \tag{35}$$

2. *If Fis a* min *operator and $\mu = \mu_*$ has a lower capacity of order two, then*

(a′) *The $F - AsVar(\cdot)$ operator coincides with monotone variance parameter of h:*

$$Min - AsVar(h) = \min\{Var_{\lambda_{\sigma_1}}(h),\ldots,Var_{\lambda_{\sigma_{n!}}}(h)\} = Var_{\lambda_\tau}(h) = MVar(h) = (Ch)\int_S (h - ME(h))^2 \odot \mu_* \tag{36}$$

(b′) *the $F - As\alpha_k(\cdot)$ operator coincides with the monotone k-order non-central moment parameter of h:*

$$Min - As\alpha_k(h) = \min\{\alpha_k^{(\lambda_{\sigma_1})}(h),\ldots,\alpha_k^{(\lambda_{\sigma_{n!}})}(h)\} = \alpha_k^{(\lambda_\tau)}(h) = M\alpha_k(h) = (Ch)\int_S (h^k) \odot \mu_*$$

(c′) *the $F - AsCov(\cdot)$ operator coincides with the monotone covariance parameter of hand g functions:*

$$\begin{aligned} &Min - AcCov(h,g) = \min\left\{Cov_{\lambda_{\sigma_1}}(h,g),\ldots,Cov_{\lambda_{\sigma_{n!}}}(h,g)\right\} = Cov_{\lambda_\tau}(h,g) = \\ &= MCov(h,g) = (Ch)\int_S (h - ME(h))(g - ME(g)) \odot \mu_* \end{aligned} \tag{37}$$

In conclusion, we note that the operations of aggregation of F-associated statistical parameters defined here represent some kind correct extensions of monotone statistical parameters. If the extreme second order Choquet capacities are taken as the fuzzy measure, then the associated and monotone statistical parameters coincide.

In the new aggregations, all $n!$ quantities of consonant structure attribute focal element interactions are taken into account

$$\left\langle \left\{s_{\sigma(1)}\right\}, \left\{s_{\sigma(1)}, s_{\sigma(2)}\right\}, \ldots, \left\{s_{\sigma(1)}, \ldots, s_{\sigma(n)}\right\} \right\rangle_{\sigma \in S_n} \tag{38}$$

4. Using Operators $F - As(\cdot)$ in the MADM Problem (Illustrating Examples)

Let us consider a numerical example in which the new aggregation operator's calculation techniques and the possibility of its use are shown. Suppose the following decision-making matrix is given with $S = \{s_1, s_2, s_3\}$-three attributes and four possible alternatives, $D = \{d_1, d_2, d_3, d_4\}$ (see Table 1).

Table 1. Decision-making matrix.

D \ S	s_1	s_2	s_3
d_1	0.5	0.4	0.7
d_2	0.3	0.8	0.6
d_3	0.6	0.5	0.3
d_4	0.4	0.3	0.8

Note that the evaluations of alternative d_i in relation to attributes S in aggregations should be considered as possible arguments of functions h and g.

In order to better understand the calculations presented below, we offer a classic scheme of MADM in the case of our data:

Calculation scheme 1:

1. Create a MADM model by forming sets of attributes S and possible alternatives D.

2. Form expert evaluations of alternatives in relation to attributes. Build a decision-making matrix $\{h_{ij}\},\ \ i=1,\ldots,m;\ \ j=1,\ldots,n$. For each alternative, create from this matrix evaluation functions h and g.
3. Perform a formation-identification of fuzzy measure μ on the set of attributes S.
4. Construct the class of associated probabilities $\{\lambda_\sigma\}_{\sigma\in S}$ of the fuzzy measure μ.
5. Construct the class of associated expectations $\{E_{\lambda_\sigma}(h)\}_{\sigma\in S_n}$.
6. For each alternative calculate the monotone expectation ME of h with respect to the fuzzy measure μ.
7. Construct the associated variances class $\{Var_{\lambda_\sigma}(h)\}_{\sigma\in S_n}$.
8. For each alternative, calculate the monotone variance $MVar$ of h with respect to the fuzzy measure μ.
9. Construct the class of associated k-order (non-central) moments $\left\{\alpha_k^{(\lambda_\alpha)}\right\}_{\sigma\in S_n}$.
10. For each alternative calculate $M\alpha_k$—the k-th order non-central monotone moment of h with respect to the fuzzy measure μ.
11. For each alternative, calculate Mcov-the monotone covariance between comonotonic functions h and g with respect to the fuzzy measure μ.
12. For each alternative, calculate $F-AsE(h)$—the associated expectation of function h with respect to the fuzzy measure $\mu(F=\min,\ \ F=Max)$.
13. For each alternative, calculate $F-AsVar$—the associated variance of function h with respect to the fuzzy measure $\mu(F=\min,\ \ F=Max)$.
14. For each alternative calculate $F-As\alpha_k$—the associated k-order moment of function h with respect to the fuzzy measure $\mu(F=\min,\ \ F=Max)$.
15. For each alternative calculate $F-AsCov$ —the associated covariance of comonotonic functions h and g with respect to the fuzzy measure $\mu(F=\min,\ \ F=Max)$.
16. Rank the alternatives with monotone and F-associated aggregation statistical parameters.

Suppose that it is given a fuzzy measure on S μ (the first column of Table 2) and all possible associated probabilities obtained by corresponded associated probabilities (Def. 2, Formula (2)). Permutations are obtained by shifting the elements of the set of indices $\{1,2,3\}$(see Table 2). The associated probabilities are calculated by formula:

$$\lambda_{(\sigma(1),\sigma(2),\sigma(3))}\left(s_{\sigma(i)}\right)=\mu\left(\left\{s_{\sigma(1)},\ldots,s_{\sigma(i)}\right\}\right)-\mu\left(\left\{s_{\sigma(1)},\ldots,s_{\sigma(i-1)}\right\}\right),\ \ i=1,2,3$$

Table 2. Fuzzy measure μ on S and its associated probabilities. $\{\lambda_\sigma\}$, $\sigma=(\sigma(1),\sigma(2),\sigma(3))\in S_3$.

2^S	μ	$\lambda_{(1,2,3)}$	$\lambda_{(1,3,2)}$	$\lambda_{(2,1,3)}$	$\lambda_{(2,3,1)}$	$\lambda_{(3,1,2)}$	$\lambda_{(3,2,1)}$
$\{s_1\}$	0.2	0.2	0.2	0.4	0.6	0.3	0.6
$\{s_2\}$	0.1	0.3	0.4	0.1	0.1	0.4	0.1
$\{s_3\}$	0.3	0.5	0.4	0.5	0.3	0.3	0.3
$\{s_1,s_2\}$	0.5	0.5	0.6	0.5	0.7	0.7	0.7
$\{s_1,s_3\}$	0.6	0.7	0.6	0.9	0.9	0.6	0.9
$\{s_2,s_3\}$	0.4	0.8	0.8	0.6	0.4	0.7	0.4
$\{s_1,s_2,s_3\}$	1.0	1.0	1.0	1.0	1.0	1.0	1.0

Note that the values of the probabilities $\lambda_{(\sigma(1),\sigma(2),\sigma(3))}\left(s_{\sigma(i)}\right)$ are represented in the corresponded column.

Using results of Tables 1 and 2, we calculated monotone expectations (Formula (5)), monotone variances (Formula (14)), monotone covariances (Formula (24)) and statistical-parameters-associated expectations (Formula (6)), associated variances (Formula (13)) for possible alternatives $d_i,\ i=1,\ldots,4$ and associated covariances (Definition 13) for the following pairs of alternatives (d_1,d_2), (d_1,d_3) and (d_2,d_3) (Tables 3–5).

Table 3. Associated expectations and monotone expectations for the alternatives d_i, $i = 1, \ldots, 4$.

D \ $E_{\lambda_\sigma}(\cdot)$	$E_{\lambda_{(1,2,3)}}$	$E_{\lambda_{(1,3,2)}}$	$E_{\lambda_{(2,1,3)}}$	$E_{\lambda_{(2,3,1)}}$	$E_{\lambda_{(3,1,2)}}$	$E_{\lambda_{(3,2,1)}}$	$ME(d_i)$
d_1	0.57	0.54	0.63	0.59	0.52	0.59	0.52
d_2	0.60	0.62	0.50	0.44	0.35	0.38	0.44
d_3	0.42	0.44	0.44	0.50	0.47	0.50	0.42
d_4	0.51	0.52	0.59	0.51	0.48	0.51	0.48

Table 4. Associated variances and monotone variances for the alternatives d_i, $i = 1, \ldots, 4$.

D \ $Var_{\lambda_\sigma}(\cdot)$	$Var_{\lambda_{(1,2,3)}}$	$Var_{\lambda_{(1,3,2)}}$	$Var_{\lambda_{(2,1,3)}}$	$Var_{\lambda_{(2,3,1)}}$	$Var_{\lambda_{(3,1,2)}}$	$Var_{\lambda_{(3,2,1)}}$	$MVar(d_i)$
d_1	0.018	0.067	0.013	0.010	0.016	0.010	0.016
d_2	0.030	0.034	0.030	0.032	0.028	0.032	0.032
d_3	0.016	0.014	0.020	0.018	0.014	0.018	0.016
d_4	0.054	0.053	0.045	0.037	0.046	0.037	0.046

Table 5. Associated covariances and monotone covariances for the pairs of alternatives. $\left\langle d_i, d_j \right\rangle$, $i, j = 1, \ldots, 4$, $i < j$.

$\langle d_i, d_j \rangle$ \ $Cov_{\lambda_\sigma}(\cdot)$	$Cov_{\lambda_{(1,2,3)}}$	$Cov_{\lambda_{(1,3,2)}}$	$Cov_{\lambda_{(2,1,3)}}$	$Cov_{\lambda_{(2,3,1)}}$	$Cov_{\lambda_{(3,1,2)}}$	$Cov_{\lambda_{(3,2,1)}}$	$MCov(d_i, d_j)$
$\langle d_1, d_2 \rangle$	−0.048	−0.009	0.007	0.006	0.017	0.006	non-comonotonic
$\langle d_1, d_3 \rangle$	−0.014	0.046	−0.015	−0.012	−0.011	−0.012	non-comonotonic
$\langle d_1, d_4 \rangle$	−0.129	−0.097	−0.130	−0.077	−0.070	−0.077	comonotonic −0.070
$\langle d_2, d_3 \rangle$	−0.006	−0.005	−0.018	−0.018	0.023	−0.018	non-comonotonic
$\langle d_2, d_4 \rangle$	−0.006	−0.010	0.017	−0.016	0.008	0.016	non-comonotonic
$\langle d_3, d_4 \rangle$	−0.006	−0.025	−0.029	−0.024	0.022	−0.024	non-comonotonic

Based on the results presented in the Tables 1–5, we calculate F-associated statistical parameters' aggregations for the alternatives d_i, $i = 1, \ldots, 4$ (Table 6) for $F = \max$ or $F = \min$.

Table 6. Maxima and minima associated and monotone statistical parameters.

D \ $F{-}As(\cdot)$	$Max{-}AsE(\cdot)$	$Min{-}AsE(\cdot)$	$ME(\cdot)$	$Max{-}AsVar(\cdot)$	$Min{-}AsVar(\cdot)$	$MVar$
d_1	0.59	0.52	0.52	0.067	0.010	0.016
d_2	0.62	0.35	0.44	0.034	0.028	0.032
d_3	0.50	0.42	0.50	0.020	0.014	0.016
d_4	0.59	0.48	0.42	0.054	0.037	0.046

$\left\langle d_i, d_j \right\rangle$ \ $F - As(\cdot)$	$Max - AsCov(\cdot)$	$Min - AsCov(\cdot)$	$MCov$
$\langle d_1, d_2 \rangle$	0.017	−0.048	-
$\langle d_1, d_3 \rangle$	0.046	−0.014	-
$\langle d_1, d_4 \rangle$	−0.070	−0.129	−0.070
$\langle d_2, d_3 \rangle$	0.023	−0.018	-
$\langle d_2, d_4 \rangle$	0.016	−0.016	-
$\langle d_3, d_4 \rangle$	0.022	−0.025	-

Based on the new aggregation operators' calculation results for the possible alternative set D, we introduce total ranking binary relations. We can say that

$$\begin{aligned}
&d_i \succeq_{F-AsE} d_j \Leftrightarrow F - AsE(d_i) \geq F - AsE(d_j), \\
&d_i \succeq_{F-AsVar} d_j \Leftrightarrow F - AsVar(d_i) \leq F - AsVar(d_j), \\
&d_i \succeq_{ME} d_j \Leftrightarrow ME(d_i) \geq ME(d_j), \\
&d_i \succeq_{MVar} d_j \Leftrightarrow MVar(d_i) \leq MVare(d_j).
\end{aligned}$$

Using these ranking relations and results represented in the previous Tables, we rank the alternatives by their aggregation values using F-associated and monotone statistical parameters (see Table 7).

Table 7. Ranking of MADM alternatives by the F -associated and monotone expectations and variations.

$Max - AsE(\cdot)$	$d_2 \succeq d_1 \succeq d_4 \succeq d_3$
$Min - AsE(\cdot)$	$d_1 \succeq d_4 \succeq d_3 \succeq d_2$
$Max - AsVar(\cdot)$	$d_3 \succeq d_2 \succeq d_4 \succeq d_1$
$Min - AsVar(\cdot)$	$d_1 \succeq d_3 \succeq d_2 \succeq d_4$
$ME(\cdot)$	$d_1 \succeq d_3 \succeq d_2 \succeq d_4$
$MVar(\cdot)$	$d_1 \succeq d_3 \succeq d_2 \succeq d_4$

For comparison of a statistical independence of alternatives, we introduce a total $F - Cov$ -relation on $D \times D$. Introduce a total relation $\succeq_{AsCov}$ on the pairs of alternatives. We say that pair $\langle d_i, d_i \rangle$ is more statistical dependent than pair $\langle d_k, d_l \rangle$ if

$$\langle d_i, d_j \rangle \succeq_{AsCov} \langle d_k, d_l \rangle \Leftrightarrow \left| F - AsCov(d_i, d_j) \right| \geq \left| F - AsCov(d_k, d_l) \right|$$

Using this ranking relations and results of the previous tables, we obtain (Table 8):

Table 8. Ranking of alternatives pairs by the statistical independency relation $\succeq_{AsCov}$, using the functions $F - AsCov(\cdot)$ ($F = Max$ or $F = Min$).

$Max - AsCov(\cdot)$	$\langle d_1, d_4 \rangle \succeq \langle d_1, d_3 \rangle \succeq \langle d_2, d_3 \rangle \succeq \langle d_3, d_4 \rangle \succeq \langle d_1, d_2 \rangle \succeq \langle d_2, d_4 \rangle$
$Min - AsCov(\cdot)$	$\langle d_1, d_4 \rangle \succeq \langle d_1, d_2 \rangle \succeq \langle d_3, d_4 \rangle \succeq \langle d_2, d_3 \rangle \succeq \langle d_2, d_4 \rangle \succeq \langle d_1, d_3 \rangle$

Comparative analysis. From Table 7, we can see that monotone aggregations ME and $MVar$ make optimal choices of an alternative d_1. pessimistic operator $Min - AsE(\cdot)$ and $Min - AsVar(\cdot)$ from first four aggregations from $F - As$, and make an optimal choice also of the same alternative d_1,, but the further choice of the first of them is different alternative d_4. In these selections, the alternative d_1 is thus dominant.

If we consider optimistic aggregations of $F - As(\cdot)$, $Max - AsE(\cdot)$ and $Max - AsVar(\cdot)$,, their optimal choices are different from the choices of other pessimistic $F - As(\cdot)$ and monotone aggregation operators. If the alternative d_2 is optimal for $F - AsE(\cdot)$, the alternative d_3 is optimal for $F - AsVar(\cdot)$. Probably, this difference arose from the circumstances that the phenomenon of interaction of focal sets of consonant structures of attributes was reflected to a higher degree in optimistic aggregations than was observed in other operators.

Now, about the results of covariate aggregations: According to the operators $Max - AsCov(\cdot)$ and $Min - AsCov(\cdot)$, the most statistically dependent alternatives are d_1 and d_4, and the second statistical dependence is observed in such pairs where the alternative d_1 is considered, although this alternative is with the alternative d_3 in the case of $Max - AsCov(\cdot)$, and with the alternative d_2 in the case of $Min - AsCov(\cdot)$. From these aggregations, it can be seen that the pair $\langle d_1, d_4 \rangle$ has the least statistical dependence.

5. Conclusions

The construction of minimal risk decision-making aggregation operators in interactive MADM models is a very important problem for many researchers today. It is clear that the main problem here is the conflicting dual phenomenon arising in the model data in the form of uncertainty-imprecision. The art of mastering this phenomenon as much as possible determines the construction of reliable and believable decision support systems. Our research addresses these issues. How can one aggregate evaluations on interactive attributes of alternatives with non-additive, Choquet integral-type aggregation operators, so as to achieve reduction and minimization of decision-making risks? In this direction, the F-associated statistical parameters (F-associated expectation, F-associated variance, F-associated moment of order k and F-associated covariance) based on the Choquet integral are constructed in the work. In their aggregations, they take into account all possible variants of consonant structures of attributes—that is, they combine all variants. The way to define them is as follows. F-associated statistical parameters in interactive MADM models are based on the respective monotone statistical parameters. The case of fuzzy uncertainty is considered when the additive probability measure is replaced by a monotone fuzzy measure. F-associated statistical parameters represent a kind of extensions of monotone statistical parameters. These monotone parameters are monotone expectation, monotone variance, monotone moment of order k, monotone covariance. Extension correctness implies that if the second-order extremal Choquet capacities are taken as the fuzzy measure and the F-mean aggregation operator is a max or min operator, then the F-associated statistical and monotone statistical parameters coincide. The difference is that a monotone statistical parameter can be calculated with only one relevant associated statistical parameter, but at least all $n!$ associated statistical parameters participate in the definition of the F-associated statistical parameter. Therefore, in the F-aggregations, the interactions of all focal sets of attributes of all consonant structures of MADM attributes are taken into account. This increases the credibility of decision making. Numerical examples are given to illustrate the obtained results.

A limitation of the application of our new approach in interactive MADM problems of practical value is that it will be necessary to solve a rather difficult fuzzy measure identification problem.

In the future studies, it is envisaged to use the developed concept for the definition of new monotone statistical parameters and to construct the extensions of these parameters for the F-associated statistical parameters. A new constructed concept will be developed for different fuzzy environments. The obtained results will be illustrated in high value decision-making problems. A deep machine learning approach will be developed for fuzzy measure identification problems.

Author Contributions: Conceptualization, G.S.; Methodology, T.K. All authors have read and agreed to the published version of the manuscript.

Funding: This research was funded by Shota Rustaveli National Science Foundation of Georgia (SRNFG), grant number (FR-21-2015).

Data Availability Statement: No new data are created.

Conflicts of Interest: The authors declare no conflict of interest.

References

1. Sugeno, M. Theory of Fuzzy Integrals and Its Applications. Ph.D. Thesis, Tokyo Institute of Technology, Tokyo, Japan, 1974.
2. Choquet, G. Theory of capacities. *Ann. De L'institut Fourier* **1954**, *5*, 131–295. [CrossRef]
3. Denneberg, D. *Non-Additive Measure and Integral*; Kluwer Academic: Norwell, MA, USA, 1994.
4. Li, J. On null-continuity of monotone measures. *Mathematics* **2020**, *8*, 205. [CrossRef]
5. Beliakov, G.; James, S.; Wu, J. *Discrete Fuzzy Measures*; Studies in Fuzziness and Soft Computing; Springer: Berlin/Heidelberg, Germany, 2020; Volume 382.
6. Sirbiladze, G. *Extremal Fuzzy Dynamic Systems: Theory and Applications*; IFSR International Series on Systems Science and Engineering 28; Springer: New York, NY, USA; Berlin/Heidelberg, Germany; Dordrecht, The Netherlands; London, UK, 2013.
7. Grabisch, M. The representation of importance and interaction of features by fuzzy measures. *Pattern Recogn. Lett.* **1996**, *17*, 567–575. [CrossRef]
8. Roubens, M. Interaction between criteria and definition of weights in MCDA problems. In Proceedings of the 44th Meeting of the European Working Group "Multicriteria Aid for Decisions, Brussels, Belgium, 3–4 October 1996.

9. Kojadinovic, I. Modeling interaction phenomena using fuzzy measures: On the notions of interaction and independence. *Fuzzy Sets Syst.* **2002**, *135*, 317–340. [CrossRef]
10. Tan, C.Q. A multi-criteria interval-valued intuitionistic fuzzy group decision making with Choquet integral-based TOPSIS. *Expert. Syst. Appl.* **2011**, *38*, 3023–3033. [CrossRef]
11. Tan, C.Q.; Chen, X. Intuitionistic fuzzy Choquet integral operator for multi-criteria decision-making. *Expert. Syst. Appl.* **2010**, *37*, 149–157. [CrossRef]
12. Xu, Z.S. Choquet integrals of weighted intuitionistic fuzzy information. *Inf. Sci.* **2010**, *180*, 726–736. [CrossRef]
13. Wu, J.; Chen, F.; Nie, C.; Zhang, Q. Intuitionistic fuzzy-valued Choquet integral and its application in multicriteria decision making. *Inf. Sci.* **2013**, *222*, 509–527. [CrossRef]
14. Peng, X.; Yang, Y. Pythagorean Fuzzy Choquet Integral Based MABAC Method for Multiple Attribute Group Decision Making. *Int. J. Intell. Syst.* **2016**, *31*, 989–1020. [CrossRef]
15. Khan, M.S.A.; Abdullah, S.; Ali, M.Y.; Hussain, I.; Farooq, M. Extension of TOPSIS method base on Choquet integral under interval-valued Pythagorean fuzzy environment. *J. Intell. Fuzzy Syst.* **2018**, *34*, 267–282. [CrossRef]
16. Beliakov, G.; Pradera, A.; Calvo, I. *Aggregation Functions: A Guide for Practitioners*; Springer: Berlin/Heidelberg, Germany, 2007.
17. Beliakov, G.; Divakov, D. Aggregation with dependencies: Capacities and fuzzy integrals. *Fuzzy Sets Syst.* **2021**, *446*, 222–232. [CrossRef]
18. Sirbiladze, G. New Fuzzy Aggregation Operators Based on the Finite Choquet Integral—Application in the MADM Problem. *Int. J. Inf. Technol. Decis. Mak.* **2016**, *15*, 517–551. [CrossRef]
19. Sirbiladze, G.; Badagadze, O. Intuitionistic Fuzzy Probabilistic Aggregation Operators Based on the Choquet Integral: Application in Multicriteria Decision-Making. *Int. J. Inf. Technol. Decis. Mak.* **2017**, *16*, 245–279. [CrossRef]
20. Sirbiladze, G.; Khutsishvili, I.; Badagadze, O.; Tsulaia, G. Associated Probability Intuitionistic Fuzzy Weighted Operators in Business Start-up Decision Making. *Iran. J. Fuzzy Syst.* **2018**, *15*, 1–25.
21. Sirbiladze, G.; Khutsishvili, I.; Midodashvili, B. Associated Immediate Probability Intuitionistic Fuzzy Aggregations in MCDM. *Comput. Ind. Eng.* **2018**, *123*, 1–8. [CrossRef]
22. Sirbiladze, G.; Sikharulidze, A. Extentions of Probability Intuitionistic Fuzzy Aggregation Operators in Fuzzy Environmet. *Int. J. Inf. Tech. Decis.* **2018**, *17*, 621–655. [CrossRef]
23. Sirbiladze, G. Associated Probabilities' Aggregations in Interactive MADM for q-Rung Orthopair Fuzzy Discrimination Environment. *Int. J. Intell. Syst.* **2020**, *35*, 335–372. [CrossRef]
24. Sirbiladze, G. Associated Probabilities in Interactive MADM under Discrimination q-Rung Picture Linguistic Environment. *Mathematics* **2021**, *9*, 2337. [CrossRef]
25. Sirbiladze, G.; Garg, H.; Khutsishvili, I.; Ghvaberidze, B.; Midodashvili, B. Associated Probabilities Aggregations in Multistage Investment Decision-Making. *Kybernetes* **2022**. *ahead-of-print*. [CrossRef]
26. Sirbiladze, G.; Manjafarashvili, T. Connections between Campos-Bolanos and Murofushi–Sugeno Representations of a Fuzzy Measure. *Mathematics* **2022**, *10*, 516. [CrossRef]
27. Campos, L.M.; Bolanos, M.N. Representation of fuzzy measures through probabilities. *Fuzzy Set Syst.* **1989**, *31*, 23–36.
28. Walley, P. *Statistical Reasoning with Imprecise Probabilities*; Chapman and Hall: London, UK, 1991.
29. Walley, P. *BI Statistical Methods. Volume I: Foundations*; Prescience Press: New York, NY, USA, 2015.
30. Dempster, A.P. Upper and lower probabilities induced by a multivalued mapping. *Ann. Math. Stat.* **1967**, *38*, 325–339. [CrossRef]
31. Shafer, G. *A Mathematical Theory of Evidence*; Princeton University Press: Princeton, NJ, USA, 1976.
32. Buckley, J. *Fuzzy Probability and Statistics*; Studies in Fuzziness and Soft Computing; Springer: Berlin/Heidelberg, Germany, 2006; Volume 196.
33. Vierti, R. *Statistical Methods for Fuzzy Data*; Wiley: Chichester, UK, 2011.
34. Kruse, R.; Held, P.; Moewes, C. On fuzzy data analysis. In *On Fuzziness—A Homage to Lotfi A. Zadeh, Volume 1*; Studies in Fuzziness and Soft Computing; Springer: Berlin/Heidelberg, Germany, 2013; Volume 298, pp. 343–347.
35. D'Urso, P. Informational paradigm, management of uncertainty and theoretical formalisms in the clustering framework: A review. *Inf. Sci.* **2017**, *400–401*, 30–62.
36. Tanaka, H.; Uejima, S.; Asai, K. Linear regression analysis with fuzzy model. *IEEE Trans. Syst. Man Cybern.* **1982**, *12*, 903–907.
37. Parchami, A.; Taheri, S.M.; Mashinchi, M. Fuzzy p-value in testing fuzzy hypotheses with crisp data. *Stat. Pap.* **2010**, *51*, 209. [CrossRef]
38. Zhang, R.; Ashuri, B.; Deng, Y. A novel method for forecasting time series based on fuzzy logic and visibility graph. *Adv. Data Anal. Classif.* **2017**, *11*, 759–783. [CrossRef]
39. Gil, M.A.; López-Díaz, M. Fundamentals and Bayesian analyses of decision problems with fuzzy-valued utilities. *Int. J. Approx. Reason.* **1996**, *15*, 95–115. [CrossRef]
40. Denoeux, T. Maximum likelihood estimation from fuzzy data using the EM algorithm. *Fuzzy Sets Syst.* **2011**, *183*, 72–91. [CrossRef]
41. Quost, B.; Denoeux, T.; Li, S. Parametric classification with soft labels using the evidential EM algorithm: Linear discriminant analysis versus logistic regression. *Adv. Data Anal. Classif.* **2017**, *11*, 659–690. [CrossRef]
42. Wu, H.C. Statistical hypotheses testing for fuzzy data. *Inf. Sci.* **2005**, *279*, 446–459. [CrossRef]
43. Colubi, A.; González-Rodríguez, G.; Gil, M.A.; Trutschnig, W. Nonparametric criteria for supervised classification of fuzzy data. *Int. J. Approx. Reason.* **2011**, *52*, 1272–1282. [CrossRef]

44. Coppi, R.; D'Urso, P.; Giordani, P. Fuzzy and possibilistic clustering for fuzzy data. *Comput. Stat. Data Anal.* **2012**, *56*, 915–927. [CrossRef]
45. Bolanos, M.J.; De Campos, L.M.; González, A. Convergence properties on monotone expectation and its applications to the extension of fuzzy measures. *Fuzzy Sets Syst.* **1989**, *33*, 201–212. [CrossRef]
46. Sirbiladze, G.; Gachechiladze, T. Restored Fuzzy Measures in Expert Decision-Making. *Inf. Sci.* **2005**, *169*, 71–95. [CrossRef]
47. Reche, F.; María Morales, M.; Salmerón, A. Statistical Parameters Based on Fuzzy Measures. *Mathematics* **2020**, *8*, 2015. [CrossRef]
48. Reche, F.; Morales, M.; Salmerón, A. Construction of fuzzy measures over product spaces. *Mathematics* **2020**, *8*, 1605. [CrossRef]
49. Reche, F.; Salmerón, A. Operational approach to general fuzzy measures. *Int. J. Uncertain. Fuzziness-Knowl.-Based Syst.* **2000**, *8*, 369–382. [CrossRef]