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A Differential Flatness-Based Model Predictive Control Strategy for a Nonlinear Quarter-Car Active Suspension System

Daniel Rodriguez-Guevara [1], Antonio Favela-Contreras [1,*], Francisco Beltran-Carbajal [2], Carlos Sotelo [1] and David Sotelo [1]

[1] Tecnologico de Monterrey, School of Engineering and Sciences, Ave. Eugenio Garza Sada 2501, Monterrey 64849, Mexico
[2] Departamento de Energía, Universidad Autónoma Metropolitana, Unidad Azcapotzalco, Av. San Pablo No. 180, Col. Reynosa Tamaulipas, Mexico City 02200, Mexico
* Correspondence: antonio.favela@tec.mx

Abstract: Controlling an automotive suspension system using an actuator is a complex nonlinear problem that requires both fast and precise solutions in order to achieve optimal performance. In this work, the nonlinear model of a quarter-car active suspension is expressed in terms of a flat output and its derivatives in order to embed the nonlinearities of the system in the flat output. Afterward, a Model Predictive Controller based on the differential flatness derivation (MPC-DF) of the quarter-car is proposed in order to achieve optimal control performance in both passenger comfort and road holding without diminishing the lifespan of the wheel. This formulation results in a linear optimization problem while maintaining the nonlinear behavior of the active suspension system. Afterward, the optimization problem is solved by means of Quadratic Programming (QP), enabling real-time implementation. Simulation results are presented using a realistic road disturbance to show the effectiveness of the proposed control strategy.

Keywords: differential flatness; model predictive control; automotive suspension; nonlinear control; predictive control; optimal control

MSC: 93B45

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1. Introduction

Intelligent systems are now present in almost every section of a car. Automotive suspensions are not the exception. Active suspension systems are present in cars to attenuate disturbances present on the road, such as bumps or holes. Active suspension systems are fundamental for preserving passenger comfort and maintaining safe road-holding conditions. However, harsh road conditions or sudden road disturbances can cause a difficult response for the active suspension controller and an optimal control law needs to be designed which can provide a fast response with proper performance.

Several control strategies have been proposed in the literature in order to handle road disturbances with an active suspension system. Some of these controllers are PID [1–5], H2/H∞ controller [6–9], fuzzy logic controllers [10–12], LQR [13–15], and sliding mode controllers [16–18]. These controllers are often designed using linear active suspension models and they exhibit a tradeoff between passenger comfort and road holding, depending on their design specifications.

Another control strategy widely used in industrial applications is Model Predictive Control (MPC). This kind of controller is based on the prediction of the behavior of the system along a prediction horizon as a function of the control input. Afterward, an optimization problem is solved online at every iteration to define the optimal set of control actions considering the desired performance of the system and a set of constraints that can be both physical or performance-based.

Several MPC solutions have been proposed for the control of active suspension systems. In [19] an MPC is designed for a linear quarter-car active suspension system with a preview of road disturbances. This work examines a linear model of the active suspension which considers ideal springs and dampers. Further, the MPC is solved offline for every possible set of states of the system and disturbances and stored in a set of critical regions in order to allow real-time execution. The results showed an improvement in both comfort and road holding when compared to a Skyhook controller; however, since it is an offline controller, a relatively large memory storage space is needed to store all possible solutions for every set of states and disturbances.

Another MPC work based on a linear representation of an active suspension system is presented in [20]. In the previously mentioned work, a performance comparison between an LQR and an MPC control strategy is made. Both controllers are designed based on a linear ideal active suspension system and exposed to realistic road profiles acting as disturbances. The results showed that the performance of the MPC is better in terms of passenger comfort and with lower power requirements than the LQR controller, which results in less actuator deterioration, extending the actuator life.

Other MPC works for active suspensions based on linear models are presented in [21–24]. These works improved the suspension performance when compared to passive suspensions and other classic controllers; however, since all are based on linear representations of the active suspension system, the prediction of the suspension behavior may not be accurate and the performance on a real suspension may not be as effective as that seen in simulations.

MPC solutions for active suspension systems which consider nonlinearities in the model have also been proposed; however, Nonlinear MPCs (NMPC), which consider all nonlinearities of the system in the optimization problem results in a complex optimization problem, which can take long times to be solved online and therefore, restricts real-time implementation. Some works related to NMPC in active suspension systems are presented in [25–28].

To overcome the limitations of NMPC, several MPCs based on the nonlinear model of active suspension systems have been proposed, embedding the nonlinearities of the system in a Linear Parameter Varying model. In this kind of approach, the nonlinearities of the system are embedded in a scheduling parameter in order to make prediction of the system states linear and a function of both the control action and the future scheduling parameters. Some research works regarding this control strategy include [29–32].

This strategy allows the MPC to predict the behavior of the system considering its nonlinearities while solving a linear optimization problem, allowing real-time implementations. However, since the prediction of the future states is a function of the scheduling parameters, these parameters need to be estimated prior to optimization or bounded at a certain rate of change in order to predict the behavior of the system and solve the optimization problem. This often leads to conservative performance due to the uncertainty of the values of the scheduling parameters along the prediction horizon.

Another strategy to find global linearization and solve nonlinear differential systems is Differential Flatness [33]. Flatness is a property of differential systems in which a flat output can be defined and the solution of all the states of the system can be expressed as a function of the flat output and its derivatives. Therefore, knowing the value of the flat output and its derivatives, every state of the system can be derived. Using this property, several flatness-based controllers have been proposed for nonlinear active suspension systems. In [34] a feedforward controller based on a differential flatness representation of the active suspension is proposed. The performance of the controller was defined using the characteristic polynomial of the closed loop differential flatness system and using an L_∞ norm that guarantees the flat output and its derivatives to be globally asymptotically stable. The results proved to be an efficient controller and to mitigate the effects of a realistic road disturbance. However, performance definition is only based on the minimization of the values of the flat outputs, rather than on passenger comfort and road holding.

Some other controllers based on differential flatness have also been proposed for nonlinear active suspension systems, such as LQR [35,36], PI and PID [37,38], sliding mode controllers [39–41] and active disturbance rejection [42,43]. All these controllers have proven to be efficient in mitigating the effects of road disturbances while maintaining both comfort and road holding, however, actuator effort is not considered while designing the controller, which may lead to actuator deterioration in order to achieve the desired performance.

In this research work, a Model Predictive Control strategy based on a differential flatness representation of the nonlinear active suspension system of a quarter car is proposed. This strategy aims to maintain passenger comfort and preserve safe road holding conditions for the driver against realistic road disturbances such as bumps and potholes. The control effort imposed on the actuators is also considered in order to build a smooth control action and avoid sudden force changes. The rest of the work is structured as follow: Section 2 describes the dynamic model of the active suspension system for a quarter car, Section 3 shows the differential flatness representation of the proposed model. Afterwards, Section 4 shows the Model Predictive Control strategy for the differential flatness representation by building a cost function that can be solved by means of QP. Section 5 presents the results of the proposed control strategy applied to a quarter-car active suspension system against realistic road disturbances. Finally, Section 6 concludes the paper.

2. Nonlinear Quarter-Car Active Suspension System

The active suspension system of a quarter-car adds an actuator to the passive elements, consisting of a sprung mass, an unsprung mass, a spring, and a damping element. Figure 1 presents a model of a quarter-car active suspension system. In this model, the objective of the actuator is to create a force F_A to reduce the movement and accelerate the sprung mass m_s and the unsprung mass m_{us}. The sprung mass represents the chassis body, while the unsprung mass is the wheel and suspension unit attached to it. The force F_A is created by an ideal actuator, which can produce forces in both directions in order to reduce the movement of both masses. Reducing the movement of the sprung mass results in more comfort for the passengers. Also, reducing the distance between the sprung and unsprung masses, which is called suspension deflection, is important to maintain proper road-holding conditions.

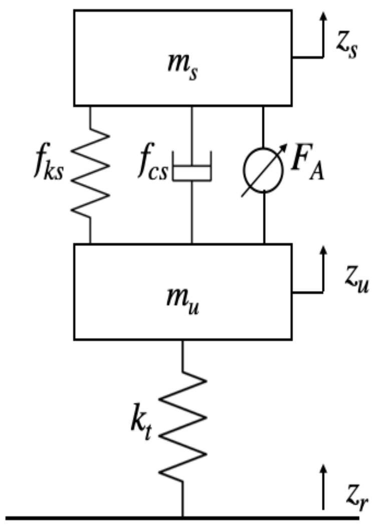

Figure 1. Quarter-Car Active Suspension System.

The mathematical model of the quarter-car active suspension system is described by the coupled nonlinear differential equations presented in (1)

$$
\begin{aligned}
&m_s\ddot{z}_s(t) + f_{cs}(t) + f_{ks}(t) = F_A(t) \\
&m_u\ddot{z}_u(t) + k_t(z_u(t) - z_r(t)) - f_{ks}(t) - f_{cs}(t) = -F_A(t)
\end{aligned}
\tag{1}
$$

where z_s denotes the vertical displacement of the sprung mass, z_u the displacement of the unsprung mass, k_t is the stiffness of the tire. Both, f_{ks} and f_{cs} represent the forces generated by the spring and damping component, respectively. These forces are shown in Equations (2) and (3), respectively.

$$f_{ks}(z_s(t), z_u(t)) = k_s(z_s(t) - z_u(t)) + k_{ns}(z_s(t) - z_u(t))^3 \tag{2}$$

$$f_{cs}(\dot{z}_s(t), \dot{z}_u(t)) = c_s(\dot{z}_s(t) - \dot{z}_u(t)) + c_{ns}(\dot{z}_s(t) - \dot{z}_u(t))^2 \mathbf{sgn}(\dot{z}_s(t) - \dot{z}_u(t)) \tag{3}$$

where **sgn** denotes the signum function. Both forces exhibit a linear and nonlinear component, the spring force depends both linearly and nonlinearly on the suspension deflection $(z_s - z_u)$. k_s is the linear spring constant while k_{ns} the nonlinear. Similarly, the damping force depends linearly on the difference in velocities of both masses by a linear damping constant c_s and a nonlinear c_{ns}.

In order to build a state-space model, the following state variables are defined as $x_1 = z_s$, $x_2 = \dot{z}_s$, $x_3 = z_u$, $x_4 = \dot{z}_u$, and $u = F_A$. This results in the following state-space model

$$\begin{aligned}
\dot{x}_1(t) &= x_2(t) \\
\dot{x}_2(t) &= -\frac{1}{m_s}(F_{ks}(t) + F_{cs})(t) + \frac{1}{m_s}u(t) \\
\dot{x}_3(t) &= x_4(t) \\
\dot{x}_4(t) &= -\frac{k_t}{m_u}x_3(t) + \frac{1}{m_u}(F_{ks}(t) + F_{cs}(t)) - \frac{1}{m_u}u(t) + \frac{k_t}{m_u}z_r(t)
\end{aligned} \tag{4}$$

with $F_{ks}(t)$ and $F_{cs}(t)$ being the state representation of $f_{ks}(t)$ and $f_{cs}(t)$, respectively.

3. Differential Flatness Representation

It is noticeable that the state-space model of the active suspension presented in (4) is completely controllable and observable. However, it exhibits nonlinear behavior, representing a linear controller design challenge. System (4) can be expressed as a differentially flat system. The definition of a differentially flat system is the following:

Definition 1 (Rigatos et al. [44])**.** *A system $\dot{x} = f(x, u)$ with a state vector $x \in \mathbb{R}^n$, input vector $u \in \mathbb{R}^m$, where f is a continuously differentiable function or a smooth vector field, is differentially flat if there exists a vector $L \in \mathbb{R}^m$ in the form:*
$L = h(x, u, \dot{u}, \ddot{u}, \ldots, u^{(r)})$
Such that:
$x = \phi(L, \dot{L}, \ddot{L}, \ldots, L^{(q)})$
$u = \alpha(L, \dot{L}, \ddot{L}, \ldots, L^{(q)})$

The state-space model (4) is differentially flat, with the following flat output as shown in [34]

$$L(t) = m_s x_1(t) + m_u x_3(t) \tag{5}$$

With this flat output, all the state variables and the control input can be expressed in terms of the flat output and its derivatives. Obtaining the derivatives up to the fourth order, the following expressions are obtained.

$$\begin{aligned}
&L(t) = m_s x_1(t) + m_u x_3(t) \\
&\dot{L}(t) = m_s x_2(t) + m_u x_4(t) \\
&\ddot{L}(t) = k_t(z_r - x_3(t)) \\
&L^{(3)}(t) = k_t(\dot{z}_r - x_4(t)) \\
&L^{(4)}(t) = \frac{k_t}{m_u}u(t) + \frac{k_t^2}{m_u}x_3(t) - \frac{k_t}{m_u}(F_{ks}(t) + F_{cs}(t)) - \frac{k_t^2}{m_u}z_r(t) + k_t\ddot{z}_r(t)
\end{aligned} \tag{6}$$

Having the flat output and its derivatives (6), the following parametrization of the states can be made

$$\begin{aligned}
x_1(t) &= \frac{m_u}{k_t m_s}\ddot{L}(t) + \frac{1}{m_s}L(t) - \frac{m_u}{m_s}z_r(t) \\
x_2(t) &= \frac{m_u}{k_t m_s}L^{(3)}(t) + \frac{1}{m_s}\dot{L}(t) - \frac{m_u}{m_s}\dot{z}_r(t) \\
x_3(t) &= -\frac{1}{k_t}\ddot{L}(t) + z_r(t) \\
x_4(t) &= \frac{1}{k_t}L^{(3)}(t) + \dot{z}_r(t)
\end{aligned} \tag{7}$$

Further, renaming system (6) derivatives by the following: $L = L_1$, $\dot{L} = L_2$, $\ddot{L} = L_3$, $L^{(3)} = L_4$, and $L^{(4)} = v$; the following matrix representation can be performed.

$$\begin{bmatrix} \dot{L}_1(t) \\ \dot{L}_2(t) \\ \dot{L}_3(t) \\ \dot{L}_4(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} L_1(t) \\ L_2(t) \\ L_3(t) \\ L_4(t) \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} v(t) \tag{8}$$

where v represents an auxiliary control action to the flat-system.

With the equations presented in (7) and (8), a linear representation of the nonlinear active suspension system shown in (4) can be derived by knowing the values of the flat output L and its derivatives.

The matrix representation of the flat-output and its derivatives presented in (8) can be expressed in a compact form by defining matrices $\mathbf{A}$ and $\mathbf{B}$. Therefore, system (8) can be written as:

$$\dot{L}(t) = \mathbf{A}L(t) + \mathbf{B}v(t) \tag{9}$$

for continuous-time and as:

$$L(k+1) = \mathbf{A}_d L(k) + \mathbf{B}_d v(k) \tag{10}$$

for discrete-time representation where $\mathbf{A}_d$ and $\mathbf{B}_d$ are the discrete representation of matrices $\mathbf{A}$ and $\mathbf{B}$, respectively, using sampling time T_s.

4. Model Predictive Control Based on Differential Flatness

In order to design an MPC controller based on the differential flatness representation presented in (10), a prediction of the future values of the flat output and its derivatives must be performed in order to predict the value of the future states as a function of the control input. The prediction of the future values of the flat output and its derivatives along the prediction horizon N_p is shown in the following matrix equation.

$$\mathbf{L}_p(k) = \mathbf{\Phi} * L(k) + \mathbf{\Psi} * \mathbf{V}(k) \tag{11}$$

With:

$$\mathbf{L}_p(k) = \begin{bmatrix} L(k+1) \\ L(k+2) \\ \vdots \\ L(k+N_p) \end{bmatrix}$$

$$\mathbf{\Phi} = \begin{bmatrix} A_d \\ A_d^2 \\ \vdots \\ A_d^{N_p} \end{bmatrix}$$

$$\mathbf{\Psi} = \begin{bmatrix} B_d & 0_{nx \cdot n_u} & \dots & 0_{nx \cdot n_u} & 0_{nx \cdot n_u} \\ A_d B_d & B_d & \dots & 0_{nx \cdot n_u} & 0_{nx \cdot n_u} \\ A_d^2 B_d & A_d B_d & \dots & 0_{nx \cdot n_u} & 0_{nx \cdot n_u} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ A_d^{N_p-1} B_d & A_d^{N_p-2} B_d & \dots & A_d B_d & B_d \end{bmatrix}$$

$$\mathbf{V}(k) = \begin{bmatrix} v(k) \\ v(k+1) \\ \vdots \\ v(k+N_p-1) \end{bmatrix}$$

Having the prediction of the future values of the flat output and its derivatives, a prediction of the future states can be performed. The states parametrization shown in (7) can be expressed compactly in a discrete matrix form as the following:

$$\mathbf{x}(k) = \mathbf{S} * L(k) + \mathbf{D}_r * \begin{bmatrix} z_r(k) \\ z_{rd}(k) \end{bmatrix} \tag{12}$$

with $z_{rd}(k) = (z_r(k) - z_r(k-1))/T_s$ and:

$$\mathbf{x}(k) = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$$

$$\mathbf{S} = \begin{bmatrix} \frac{1}{m_s} & 0 & \frac{m_u}{k_t m_s} & 0 \\ 0 & \frac{1}{m_s} & 0 & \frac{m_u}{k_t m_s} \\ 0 & 0 & -\frac{1}{k_t} & 0 \\ 0 & 0 & 0 & -\frac{1}{k_t} \end{bmatrix}$$

$$\mathbf{D}_r = \begin{bmatrix} -\frac{m_u}{m_s} & 0 \\ 0 & -\frac{m_u}{m_s} \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Therefore, a conversion from the flat output and its derivatives can be performed at every sampling time k.

The objective of the MPC controller is to minimize the deviation of the system states when encountering a disturbance $z_r(k)$ along a prediction horizon N_p. In order to achieve this, an optimization problem is performed in order to minimize a cost function defined as the following:

$$\begin{aligned} J =& (\sigma * (\mathbf{\Phi} * L(k) + \mathbf{\Psi} * \mathbf{V}))^T * \mathbf{Q} * (\sigma * (\mathbf{\Phi} * L(k) + \mathbf{\Psi} * \mathbf{V})) \\ &+ \mathbf{V}^T * \mathbf{R} * \mathbf{V} \end{aligned} \tag{13}$$

With $\mathbf{Q}$ and $\mathbf{R}$ being weight matrices of dimensions $N_p \cdot n_x \times N_p \cdot n_x$ and $N_p \cdot n_u \times N_p \cdot n_u$, respectively, with n_x being the number of states and n_u the number of controllable control inputs. The disturbance matrix $\mathbf{D}_r$ is omitted, since the future disturbance and its derivative is known along the prediction horizon; thus, it is assumed to be zero. Matrix σ is an augmented version of the matrix $\mathbf{S}$ for predicting the future values of the states and is defined as the following:

$$\sigma = \begin{bmatrix} \mathbf{S} & \mathbf{0}_{nx \cdot n_x} & \cdots & \mathbf{0}_{nx \cdot n_u} \\ \mathbf{0}_{nx \cdot n_u} & \mathbf{S} & \cdots & \mathbf{0}_{nx \cdot n_u} \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{0}_{nx \cdot n_u} & \mathbf{0}_{nx \cdot n_u} & \cdots & \mathbf{S} \end{bmatrix}$$

Therefore, the objective function is defined as the following:

$$\begin{gathered} \min_V (k) J \\ s.t. \\ v_{min} \leq \mathbf{V}(k) \leq v_{max} \end{gathered} \tag{14}$$

Therefore, solving the optimization problem (14) will get the optimal set of auxiliary control actions $v(k)$ along the prediction horizon N_p. In order to obtain the optimal control action $u(k)$ to be inputted into the active suspension system, a transformation using the expression of $L^{(4)} = v(k)$ from Equation (6) needs to be performed. Therefore, the control action $u(k)$ can be obtained by $v(k)$ using the following feedforward control law:

$$u(k) = \frac{m_u}{k_t} v(k) - m_u z_{r2d}(k) - k_t x_3(k) + F_{cs}(k) + F_{ks}(k) + k_t z_r(k) \tag{15}$$

With $z_{r2d}(k)$ being the discrete representation of the second derivative of the road disturbance, obtained by Euler discretization. Therefore, the optimal control action $u(k)$ to be inputted into the system can be obtained through $v(k)$.

It is noticeable that the optimization problem shown in (14) is a Quadratic Programming (QP) problem. Thus, it can be solved in a short time by using any QP-solver. Solving the optimization problem as a function of the auxiliary control variable $v(k)$ instead of the actual control input $u(k)$ allows the differential flatness representation to build a QP problem instead of a nonlinear optimization problem, which will result in longer optimization times and restrict real-time implementation.

Figure 2 presents the block diagram of the proposed differential flatness-based MPC control strategy.

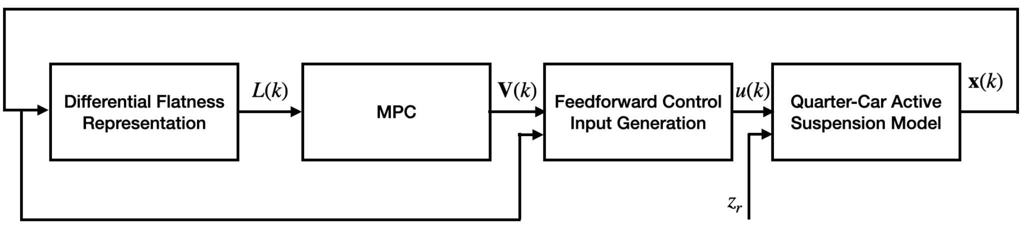

Figure 2. Block diagram of the proposed differential flatness-based Model Predictive Controller.

5. Results and Discussion

In order to observe the advantages and performance of the proposed differential flatness-based Model Predictive Control strategy described in the previous section, the following simulations were performed. The simulations were made using the quarter-car nonlinear active suspension model described in Section 2. Table 1 shows the physical values of the different elements of the suspension model obtained from [45].

Table 1. Constant Values of the Active Suspension system.

Symbol	Value	Units
m_s	216.75	kg
m_u	28.85	kg
k_s	21,700	N/m
c_s	1200	N·s/m
k_{ns}	2170	N/m
c_{ns}	120	N·s/m
k_t	184,000	N/m

The simulations were performed in the Matlab-Simulink® environment. A discretization of the active suspension model was done using a sampling time of $T_s = 5$ ms. A prediction horizon of $N_p = 5$ was defined after several simulations with different prediction horizons were performed. The control objective is to steer the states to the origin while maintaining passenger comfort (chassis acceleration) and keeping safe driving conditions through proper road holding, measured through the suspension deflection $(z_s - z_u)$ while complying with the following constraint:

$$-3000 \leq \mathbf{V}(k) \leq 3000$$

To test the performance of the control strategy, a road profile representing a realistic road disturbance consisting of two bumps of 10 cm in opposite directions. This road profile is defined using the following piece-wise function found in [46] and is shown in Figure 3:

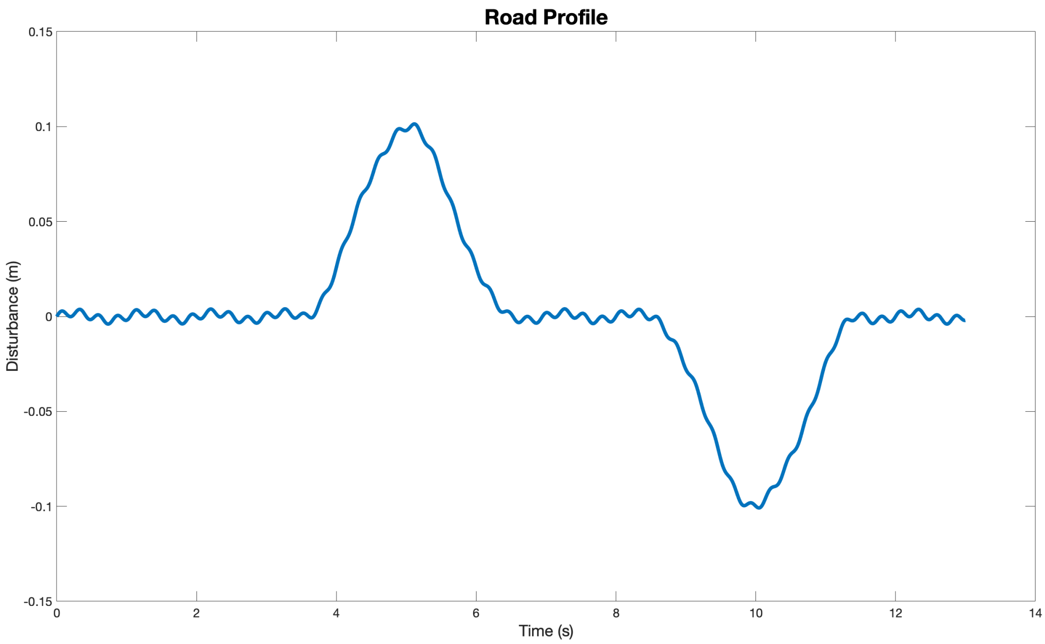

Figure 3. Road Profile.

$$z_r(t) = \begin{cases} f_1(t) + f(t) & t \in [3.5, 5) \\ f_2(t) + f(t) & t \in [5, 6.5) \\ f_3(t) + f(t) & t \in [8.5, 10) \\ f_4(t) + f(t) & t \in [10, 11.5) \\ f(t) & \text{else} \end{cases}$$

with:

$$
\begin{aligned}
f(t) &= 0.002\sin(2\pi t) + 0.002\sin(7.5\pi t) \\
f_1(t) &= -0.0592(t-3.5)^3 + 0.1332(t-3.5)^2 \\
f_2(t) &= 0.0592(t-6.5)^3 + 0.1332(t-6.5)^2 \\
f_3(t) &= 0.0592(t-8.5)^3 - 0.1332(t-8.5)^2 \\
f_4(t) &= -0.0592(t-11.5)^3 - 0.1332(t-11.5)^2
\end{aligned}
$$

The effect of the previous road disturbance on the quarter-car active suspension system is shown in Figures 4–7. Results obtained by a differential flatness-based feedforward (DF-FF) control strategy presented in [34] are also included to compare the performance of the proposed strategy. Figure 4 presents the vertical displacement of the chassis while Figure 5 presents the acceleration of the chassis. Figure 6 shows the suspension deflection $(z_s - z_u)$ while Figure 7 presents the tire deflection $(z_u - z_r)$. Finally, Figure 8 presents the control action $u(k)$ used throughout the simulation.

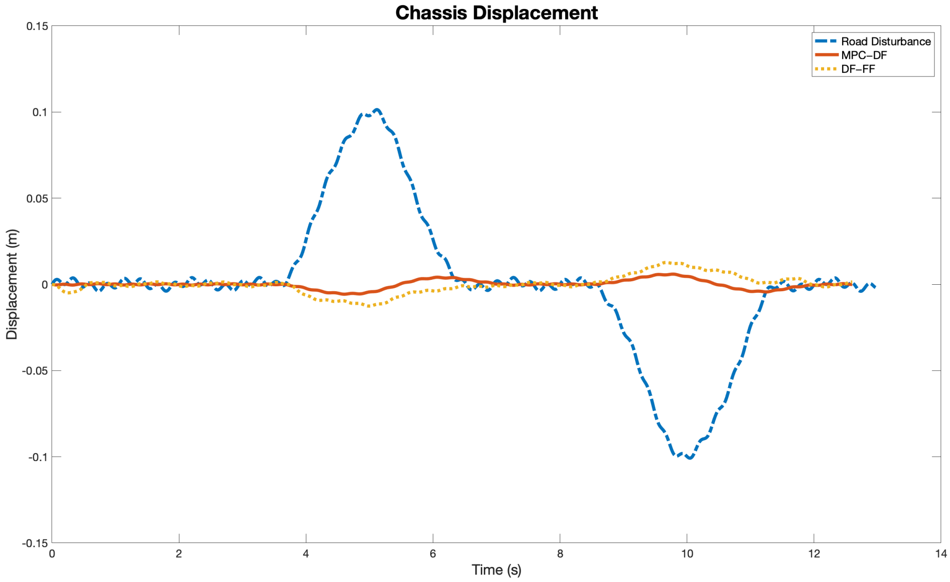

Figure 4. Chassis Displacement (Blue–Road Disturbance, Red–MPC-DF, Yellow–DF-FF).

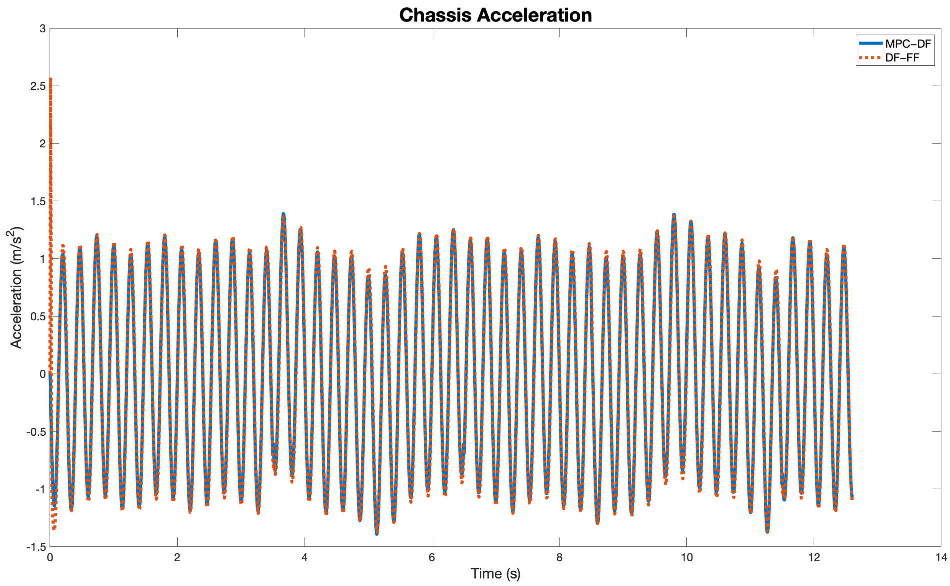

Figure 5. Chassis Acceleration (Blue–MPC-DF, Red–DF-FF).

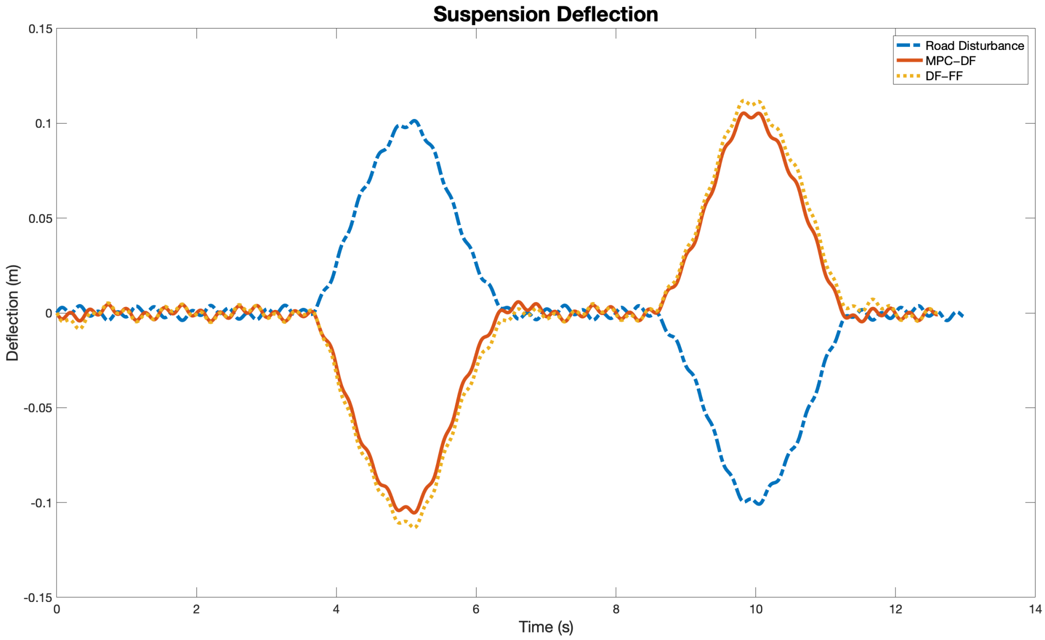

Figure 6. Suspension Deflection (Blue–Road Disturbance, Red–MPC-DF, Yellow–DF-FF).

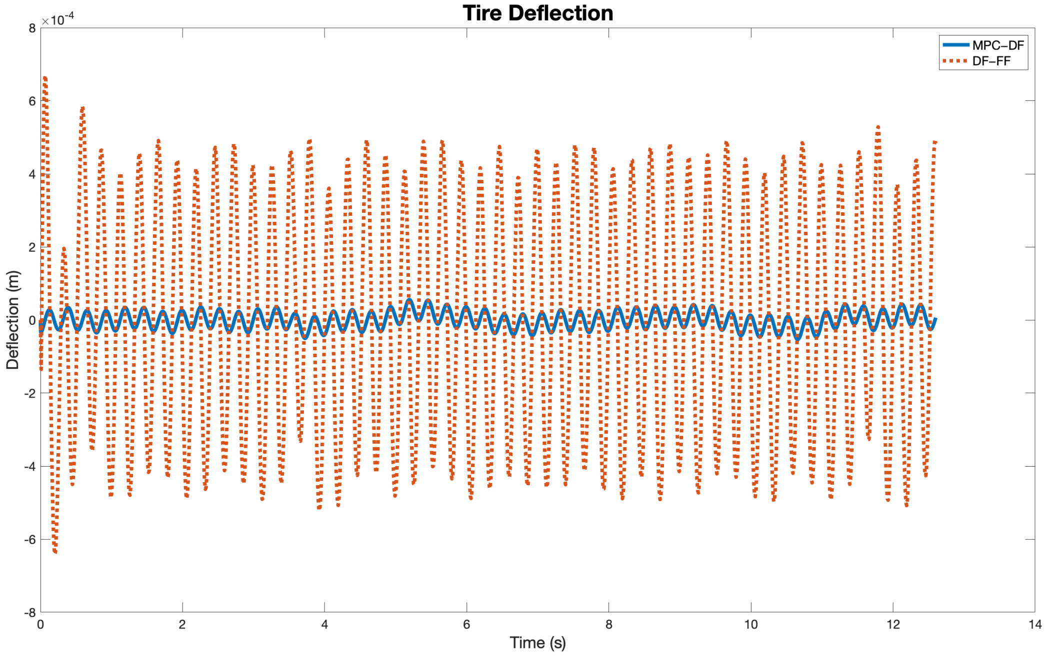

Figure 7. Tire Deflection (Blue–MPC-DF, Red–DF-FF).

The performance of the proposed flatness-based MPC is better in terms of displacement of the chassis while maintaining similar values on the acceleration when compared to the flatness-based feedforward linearization control strategy, thus resulting in better comfort for the passengers of the vehicle. Suspension deflection also shows similar performance, while the MPC-DF significantly outperforms on tire deflection, which will result in the maintainence of proper road-holding conditions, ensuring safety for drivers in case of sudden changes in steering direction and also reducing tire damage, which enhances its lifespan.

The performance of both control strategies is shown in Table 2 in terms of the peak values of the different state values, while Table 3 presents its RMS values. It is noticeable that the proposed MPC strategy outperforms the flatness-based feedforward linearization control strategy, in terms of both the peak values and the RMS values it exhibits an improvement of around 60% in the chassis displacement, 7% in suspension deflection, 90% in tire deflection with a detriment in suspension acceleration of less than 1%.

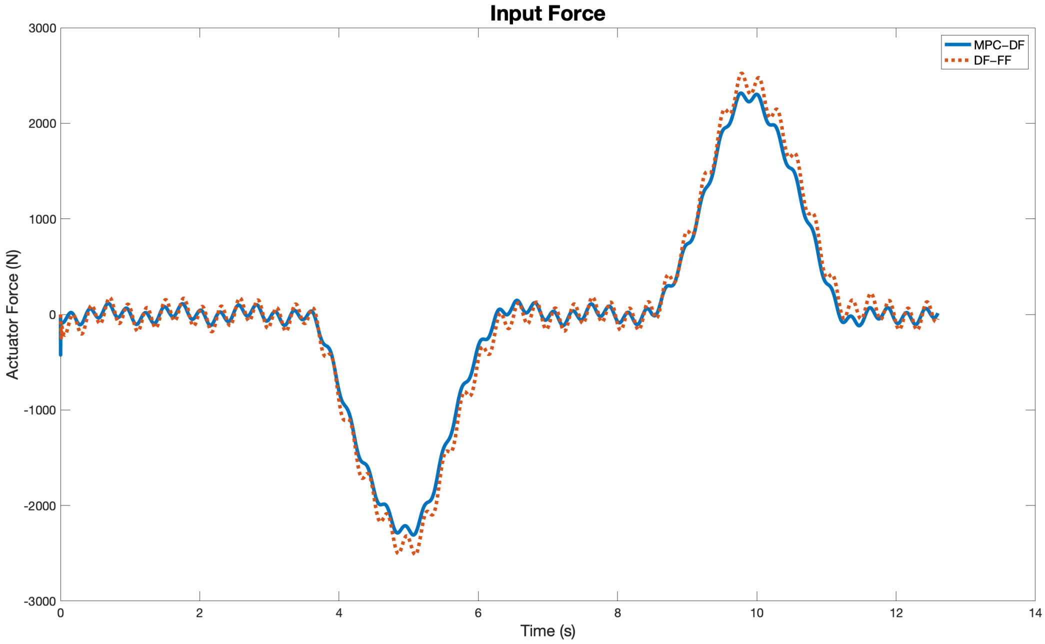

Figure 8. Input Force (Blue–MPC-DF, Red–DF-FF).

Table 2. Peak Value Performances.

Controller	Chassis Displacement (m)	Suspension Deflection (m)	Chassis Acceleration (m/s^2)	Tire Deflection (cm)
MPC-DF	0.0050	0.1054	1.3896	0.0056
DF-FF	0.0126	0.1129	1.3872	0.0549

Table 3. RMS Value Performances.

Controller	Chassis Displacement (m)	Suspension Deflection (m)	Chassis Acceleration (m/s^2)	Tire Deflection (cm)
MPC-DF	0.0021	0.0436	0.7935	0.0024
DF-FF	0.0053	0.0471	0.7903	0.0269

The proposed controller also exhibits fast optimization times, allowing real-time execution. The average optimization time is 1.98 ms on a Matlab-Simulink® R2019-b running on a Macbook Air with a dual-core Intel® Core i5 1.8 GHz processor and RAM of 8 GB 1600 MHz DDR3. These execution times are suitable for the proposed sampling time of 5 ms.

6. Conclusions

In this research work, a novel differential flatness-based model predictive control strategy for a nonlinear quarter-car active suspension system was presented. This controller is based on a flatness representation of the nonlinear active suspension model in order to provide fast optimization times to allow real-time implementations. The nonlinearities of the active suspension system were embedded in a linear representation of the flat output of the system and its derivatives to build a cost function that can be solved by QP. The results proved that the proposed controller is better in terms of road holding, reducing suspension deflection by 7% and tire deflection by 90% when compared to a flatness-based feedforward controller. Passenger comfort is also improved by a reduction in the chassis displacement by 60% while maintaining similar values for chassis acceleration when compared to the flatness-based feedforward controller. The execution times averaged 1.98 ms per iteration, which allows real-time implementation while sampling every 5 ms. Future research on stability and robustness conditions for the proposed algorithm is considered as well as a generalization of the strategy for every nonlinear system with flatness properties. Experimental tests on real active suspension test benches are also considered as future work.

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References

1. Phu, D.X.; An, J.-H.; Choi, S.-B. A Novel Adaptive PID Controller with Application to Vibration Control of a Semi-Active Vehicle Seat Suspension. *Appl. Sci.* **2017**, *7*, 1055. [CrossRef]
2. Mohamed, A.H.; Abidou, D.; Maged, S.A. LQR and PID Controllers Performance on a Half Car Active Suspension System. In Proceedings of the 2021 International Mobile, Intelligent, and Ubiquitous Computing Conference (MIUCC), El Cairo, Egypt, 26 May 2021; pp. 48–53.
3. Zhao, L.; Zeng, Z.; Wang, Z.; Ji, C. PID control of vehicle active suspension based on particle Swarm optimization. *J. Phys. Conf. Ser.* **2020**, *1748*, 032028. [CrossRef]
4. Shafiei, B. A Review on PID Control System Simulation of the Active Suspension System of a Quarter Car Model While Hitting Road Bumps. *J. Inst. Eng. India Ser. C* **2022**, *103*, 1001–1011. [CrossRef]
5. Luo, J.; Pang, H.; Wang, M.; Yao, R. PI Observer-Based Fault-Tolerant Tracking Controller for Automobile Active Suspensions. *IEEE Access* **2022**, *10*, 47203–47218. [CrossRef]
6. Afshar, K.K.; Javadi, A.; Jahed-Motlagh, M.R. Robust H∞ control of an active suspension system with actuator time delay by predictor feedback. *IET Control. Theory Appl.* **2018**, *12*, 1012–1023. [CrossRef]
7. Kaleemullah, M.; Faris, W.; Ghazaly, N.M. Analysis of active suspension control policies for vehicle using robust controllers. *Int. J. Adv. Sci. Technol.* **2019**, *28*, 836–855.
8. Pham, T.P.; Tran, G.Q.B.; Sename, O.; Van Phan, T.T.; Hoang, D.; Nguyen, Q.D. Real-time Damper Force Estimation for Automotive Suspension: A Generalized H2/LPV Approach. *Period. Polytech. Transp. Eng.* **2022**, *50*, 309–317. [CrossRef]
9. Ma, X.; Wong, P.K.; Li, W.; Zhao, J.; Ghadikolaei, M.A.; Xie, Z. Multi-objective H 2/H∞ control of uncertain active suspension systems with interval time-varying delay. *Proc. Inst. Mech. Eng. Part I J. Syst. Control. Eng.* **2022**. [CrossRef]
10. Alfadhli, A.; Darling, J.; Hillis, A.J. The Control of an Active Seat Suspension Using an Optimised Fuzzy Logic Controller, Based on Preview Information from a Full Vehicle Model. *Vibration* **2018**, *1*, 20–40. [CrossRef]
11. Avesh, M.; Srivastava, R.; Sharma, R. C.; Sharma, N. Effective design of active suspension system using fuzzy logic control approach. *Int. J. Veh. Struct. Syst.* **2019**, *11*, 536–539. [CrossRef]
12. Yatak, M.Ö.; Şahin, F. Ride comfort-road holding trade-off improvement of full vehicle active suspension system by interval type-2 fuzzy control. *Eng. Sci. Technol. Int. J.* **2021**, *24*, 259–270. [CrossRef]
13. Anh, N.T. Control an active suspension system by using PID and LQR controller. *Int. J. Mech. Prod. Eng. Res. Dev.* **2021**, *10*, 7003–7012.

14. Yu, W.; Li, J.; Yuan, J.; Ji, X. LQR controller design of active suspension based on genetic algorithm. In Proceedings of the 2021 IEEE 5th Information Technology, Networking, Electronic and Automation Control Conference (ITNEC), Xi'an, China, 15 October 2021; Volume 5, pp. 1056–1060.
15. Thi Thu Huong, T.; Tuan Anh, N.; Thang Binh, H.; Duc Ngoc, N.; Ngoc Duyen, D. Optimizing the Parameter of the LQR Controller for Active Suspension System. In Proceedings of the The AUN/SEED-Net Joint Regional Conference in Transportation, Energy, and Mechanical Manufacturing Engineering, RCTEMME 2021, Singapore, 1 June 2022.
16. Taghavifar, H.; Mardani, A.; Hu, C.; Qin, Y. Adaptive robust nonlinear active suspension control using an observer-based modified sliding mode interval type-2 fuzzy neural network. *IEEE Trans. Intell. Veh.* **2019**, *5*, 53–62. [CrossRef]
17. Aljarbouh, A.; Fayaz, M. Hybrid Modelling and Sliding Mode Control of Semi-Active Suspension Systems for Both Ride Comfort and Road-Holding. *Symmetry* **2020**, *12*, 1286. [CrossRef]
18. Aljarbouh, A.; Fayaz, M.; Qureshi, M.S.; Boujoudar, Y. Hybrid sliding mode control of full-car semi-active suspension systems. *Symmetry* **2021**, *13*, 2442. [CrossRef]
19. Theunissen, J.; Sorniotti, A.; Gruber, P.; Fallah, S.; Ricco, M.; Kvasnica, M.; Dhaens, M. Regionless explicit model predictive control of active suspension systems with preview. *IEEE Trans. Ind. Electron.* **2019**, *67*, 4877–4888. [CrossRef]
20. Narayan, J.; Gorji, S.A.; Ektesabi, M.M. Power reduction for an active suspension system in a quarter car model using MPC. In Proceedings of the IEEE International Conference on Energy Internet (ICEI), Sydney, Australia, 24–28 August 2020; pp. 140–146.
21. Piñón, A.; Favela-Contreras, A.; Félix-Herrán, L.C.; Beltran-Carbajal, F.; Lozoya, C. An ARX Model-Based Predictive Control of a Semi-Active Vehicle Suspension to Improve Passenger Comfort and Road-Holding. *Actuators* **2021**, *10*, 47. [CrossRef]
22. Yao, J.; Wang, M.; Li, Z.; Jia, Y. Research on model predictive control for automobile active tilt based on active suspension. *Energies* **2021**, *14*, 671. [CrossRef]
23. Enders, E.; Burkhard, G.; Munzinger, N. Analysis of the Influence of Suspension Actuator Limitations on Ride Comfort in Passenger Cars Using Model Predictive Control. *Actuators* **2020**, *9*, 77. [CrossRef]
24. Göhrle, C.; Schindler, A.; Wagner, A.; Sawodny, O. Model predictive control of semi-active and active suspension systems with available road preview. In Proceedings of the 2013 European Control Conference (ECC), Zurich, Switzerland, 17–19 July 2013; pp. 1499–1504.
25. Rathai, K.M.M.; Alamir, M.; Sename, O.; Tang, R. A parameterized NMPC scheme for embedded control of semi-active suspension system. *IFAC-PapersOnLine* **2018**, *51*, 301–306. [CrossRef]
26. Dalboni, M.; Tavernini, D.; Montanaro, U.; Soldati, A.; Concari, C.; Dhaens, M.; Sorniotti, A. Nonlinear model predictive control for integrated energy-efficient torque-vectoring and anti-roll moment distribution. *IEEE/ASME Trans. Mechatronics* **2021**, *26*, 1212–1224. [CrossRef]
27. Pedro, J.O.; Nhlapo, S.M.; Mpanza, L.J. Model predictive control of half-car active suspension systems using particle swarm optimisation. *IFAC-PapersOnLine* **2020**, *53*, 14438–14443. [CrossRef]
28. Wang, D.; Zhao, D.; Gong, M.; Yang, B. Research on robust model predictive control for electro-hydraulic servo active suspension systems. *IEEE Access* **2017**, 6, 3231–3240. [CrossRef]
29. Rodriguez-Guevara, D.; Favela-Contreras, A.; Beltran-Carbajal, F.; Sotelo, D.; Sotelo, C. Active Suspension Control Using an MPC-LQR-LPV Controller with Attraction Sets and Quadratic Stability Conditions. *Mathematics* **2021**, *9*, 2533. [CrossRef]
30. Rodriguez-Guevara, D.; Favela-Contreras, A.; Beltran-Carbajal, F.; Sotelo, C.; Sotelo, D. An MPC-LQR-LPV Controller with Quadratic Stability Conditions for a Nonlinear Half-Car Active Suspension System with Electro-Hydraulic Actuators. *Machines* **2022**, *10*, 137. [CrossRef]
31. Morato, M.M.; Normey-Rico, J.E.; Sename, O. Novel qLPV MPC design with least-squares scheduling prediction. *IFAC-PapersOnLine* **2019**, *52*, 158–163. [CrossRef]
32. Morato, M.M.; Normey-Rico, J.E.; Sename, O. Short-sighted robust lpv model predictive control: Application to semi-active suspension systems. In Proceedings of the 2021 European Control Conference (ECC), Delft, The Netherlands, 29 June 2021; pp. 1525–1530.
33. Fliess, M.; Lévine, J.; Martin, P.; Rouchon, P. Flatness and defect of non-linear systems: Introductory theory and examples. *Int. J. Control* **1995**, *61*, 1327–1361. [CrossRef]
34. Beltran-Carbajal, F.; Chavez-Conde, E.; Favela-Contreras, A.; Chavez-Bracamontes, R. Active nonlinear vehicle suspension control based on real-time estimation of perturbation signals. In Proceedings of the 2011 IEEE International Conference on Industrial Technology, Auburn, AL, USA, 14 March 2011; pp. 437–442.
35. Diwakar, A.D.; Manivannan, P.V. Differential flatness based LQR control of a magnetorheological damper in a quarter-car semi-active suspension system. *Int. J. Mech. Eng. Robot Res.* **2020**, *9*. [CrossRef]
36. Darus, R.B. *Modeling and Control of Active Suspension for a Full Car Model*; Faculty of Electrical Engineering, University of Technology Malaysia: Skudai, Malaysia, 2008.
37. Chávez Conde, E.; Beltrán Carbaja, F.; Valderrábano González, A.; Favela Contreras, A. Active vibration control of vehicle suspension systems using sliding modes, differential flatness and generalized proportional-integral control. *Rev. Fac. Ing. Univ. Antioq.* **2011**, *61*, 104–113.
38. Velagić, J.; Hrusto, A. Design of coupled disturbance observers for nonlinear half-car active suspension system. In Proceedings of the 2017 IEEE International Conference on Advanced Intelligent Mechatronics (AIM), Munich, Germany, 3 July 2017; pp. 547–552.

39. Mauledoux, M.M.; Caldas, O.I.; Avilés, O.F.; Mejía-Ruda, E.; Jiménez, S. A hybrid differential flatness and sliding modes controller for dynamical structural testing on lower limp prostheses. *Appl. Mech. Mater.* **2015**, *713*, 777–780. [CrossRef]
40. Conde, E.C.; Carbajal, F.B.; González, A.V.; Bracamontes, R.C. Robust control of active vehicle suspension systems using sliding modes and differential flatness with MATLAB. In *MATLAB for Engineers-Applications in Control*; Electrical Engineering, IT and Robotics; IntechOpen: Rijeka, Croatia, 2011.
41. Lin, B.; Su, X.; Li, X. Fuzzy sliding mode control for active suspension system with proportional differential sliding mode observer. *Asian J. Control* **2019**, *21*, 264–276. [CrossRef]
42. Xia, Y.; Lin, M.; Zhang, J.; Fu, M.; Li, C.; Li, S.; Yang, Y. Trajectory planning and tracking for four-wheel steering vehicle based on differential flatness and active disturbance rejection controller. *Int. J. Adapt. Control Signal Process.* **2021**, *35*, 2214–2244. [CrossRef]
43. Beltran-Carbajal, F.; Valderrabano-Gonzalez, A.; Favela-Contreras, A. R.; Rosas-Caro, J. C. Active disturbance rejection control of a magnetic suspension system. *Asian J. Control* **2015**, *17*, 842–854. [CrossRef]
44. Rigatos; Gerasimos G. *Nonlinear Control and Filtering Using Differential Flatness Approaches: Applications to Electromechanical Systems*; Springer: Berlin/Heidelberg, Germany, 2015; Volume 25.
45. Tahboub, K.A. Active nonlinear vehicle-suspension variable-gain control. In Proceedings of the 2005 IEEE International Symposium on, Mediterranean Conference on Control and Automation Intelligent Control, Limassol, Cyprus, 27 June 2005; pp. 569–574.
46. Chen, P.C.; Huang, A.C. Adaptive sliding control of non-autonomous active suspension systems with time-varying loadings. *J. Sound Vib.* **2005**, *282*, 1119–1135. [CrossRef]