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Third Hankel Determinant for a Subfamily of Holomorphic Functions Related with Lemniscate of Bernoulli

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Abstract: The main goal of this investigation is to obtain sharp upper bounds for Fekete-Szegö functional and the third Hankel determinant for a certain subclass $\mathcal{SL}^*(u, v, \alpha)$ of holomorphic functions defined by the Carlson-Shaffer operator in the unit disk. Finally, for some special values of parameters, several corollaries were presented.

Keywords: Hankel determinant; Carlson–Shaffer operator; Lemniscate of Bernoulli; holomorphic function; univalent function; Fekete-Szegö problem; starlike function; Zalcman functional

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1. Introduction and Definitions

Denote by $\mathcal{A}$ the family of holomorphic functions defined in the unit disk $\Omega = \{\varsigma \in \mathbb{C} : |\varsigma| < 1\}$, with expansion

$$l(\varsigma) = \varsigma + \sum_{k=2}^{\infty} m_k \varsigma^k \tag{1}$$

and let $\mathcal{S}$ be the subset of $\mathcal{A}$, consisting of functions which are univalent in Ω.

Let $\mathcal{P}$ be a family of the holomorphic functions t of the form

$$t(\varsigma) = 1 + \sum_{k=1}^{\infty} t_k \varsigma^k, \quad (\varsigma \in \Omega) \tag{2}$$

satisfying $\mathrm{Re}(t(\varsigma)) > 0$ in Ω. The family of starlike functions in Ω are represented by the symbol $\mathcal{S}^*$, which satisfies

$$\frac{\varsigma l'(\varsigma)}{l(\varsigma)} \in \mathcal{P}, \quad (\text{for all } \varsigma \in \Omega).$$

In addition, the symbol $\mathcal{SL}^*$ represents the family of functions that satisfy

$$\left| \left(\frac{\varsigma l'(\varsigma)}{l(\varsigma)} \right)^2 - 1 \right| < 1, \ (\varsigma \in \Omega).$$

As a result, $l \in \mathcal{SL}^*$ can be expressed by

$$\left| w^2 - 1 \right| < 1.$$

if and only if $\frac{\varsigma l'(\varsigma)}{l(\varsigma)}$ is the inside region bounded by the right half of the Bernoulli lemniscate.

This class was introduced by Sokól [1] and Sokól et al. [2]. If there is a Schwarz function w that is holomorphic in Ω, with $w(0) = 0$, $|w(\varsigma)| < 1$, such that $l(\varsigma) = h(w(\varsigma))$, $\varsigma \in \Omega$, then the function l is subordinate to h, denoted by the notation $l \prec h$. If the function h is univalent in Ω, then $l \prec h$ if

$$l(0) = h(0) \quad \text{and} \quad l(\Omega) \subset h(\Omega).$$

A function $l \in \mathcal{A}$ is said to be starlike of order α if and only if

$$\mathrm{Re}\left\{\frac{\varsigma l'(\varsigma)}{l(\varsigma)}\right\} > \alpha, \quad (\varsigma \in \Omega)$$

for some $\alpha (0 \leq \alpha < 1)$. We denote the class of all starlike functions of order α by $\mathcal{S}^*(\alpha)$. We also note that $\mathcal{S}^*(0) = \mathcal{S}^*$ is the well-known class of all normalized starlike functions in Ω.

Now, the function

$$\mathcal{K}_\alpha(\varsigma) = \frac{\varsigma}{(1-\varsigma)^{2(1-\alpha)}} \tag{3}$$

is a well known extremal function for the class $\mathcal{S}^*(\alpha)$, (see [3–5]).

Setting

$$\psi(\alpha, k) = \frac{\Pi_{k=2}^{k}(k - 2\alpha)}{(k-1)!}, \quad (k \geq 2), \tag{4}$$

the function $\mathcal{K}_\alpha$ can be written in the form as follows:

$$\mathcal{K}_\alpha(\varsigma) = \varsigma + \sum_{k=2}^{\infty} \psi(\alpha, k)\varsigma^k. \tag{5}$$

We denote by $\mathcal{F}(\alpha,k,\psi)$ the class of functions $\mathcal{K}_\alpha$. Then, we note that $\psi(\alpha,k)$ is a decreasing function in α and satisfies

$$\lim_{k\to\infty} \psi(\alpha, k) = \begin{cases} \infty & \left(\alpha < \frac{1}{2}\right) \\ 1 & \left(\alpha = \frac{1}{2}\right) \\ 0 & \left(\alpha > \frac{1}{2}\right) \end{cases}.$$

Let $(l * h)(\varsigma)$ be the Hadamard product (or convolution) of two functions l and h, that is, if l given by (1) and h is given by

$$h(\varsigma) = \varsigma + \sum_{k=2}^{\infty} n_k \varsigma^k.$$

Then,

$$(l * h)(\varsigma) = \varsigma + \sum_{k=2}^{\infty} m_k n_k \varsigma^k = (h * l)(\varsigma), \quad (\varsigma \in \Omega). \tag{6}$$

Let $\Theta(u, v, \varsigma)$ be defined by

$$\Theta(u, v, \varsigma) = \varsigma + \sum_{k=2}^{\infty} \frac{(u)_{k-1}}{(v)_{k-1}}\varsigma^k, \ \ (u \in \mathbb{C}, \ v \in \mathbb{C}\backslash\mathbb{Z}_0^-, \ \mathbb{Z}_0^- = \{... - 2, -1, 0\}; \ \varsigma \in \Omega).$$

The function, $\Theta(u, v, \varsigma)$ is known as the incomplete beta function. The term $(\varkappa)_k$ is the Pochhammer symbol that can be expanded in Gamma functions as

$$(\varkappa)_k = \frac{\Gamma(\varkappa + k)}{\Gamma(\varkappa)} = \begin{cases} 1, & k = 0 \\ \varkappa(\varkappa+1)(\varkappa+2)...(\varkappa+k-1), & k \in \mathbb{N} = \{1, 2, 3, \cdots\}. \end{cases}$$

Corresponding to the $\Theta(u,v,\varsigma)$ Carlson-Shaffer function [6], an operator $\mathcal{L}(u,v)$ is introduced for $l \in \mathcal{A}$ using the Hadamard product as follows:

$$\mathcal{L}(u,v)l(\varsigma) = \Theta(u,v,\varsigma) * l(\varsigma) = \varsigma + \sum_{k=2}^{\infty} \frac{(u)_{k-1}}{(v)_{k-1}} m_k \varsigma^k \quad (\varsigma \in \Omega).$$

Further, for the function $\mathcal{L}(u,v)l(\varsigma)$

$$\tau(\varsigma) = \mathcal{L}(u,v)l(\varsigma) * \mathcal{K}_\alpha(\varsigma) = \varsigma + \sum_{k=2}^{\infty} \frac{(u)_{k-1}}{(v)_{k-1}} \psi(\alpha,k) m_k \varsigma^k \tag{7}$$

where $\mathcal{L}(u,v)$ is called the Carlson-Shaffer operator [6], and the operator $*$ stands for the Hadamard product (or convolution product) of two power series as given by (6). We will show by $\widetilde{\mathcal{F}}(\alpha,k,\psi)$ the family of functions $\tau(\varsigma)$.

Definition 1. *We consider that $\mathcal{SL}^*(u,v,\alpha)$ is the family of holomorphic functions given by*

$$\mathcal{SL}^*(u,v,\alpha) = \left\{ \tau(\varsigma) \in \widetilde{\mathcal{F}}(\alpha,k,\psi) : \left| \left(\frac{\varsigma\tau'(\varsigma)}{\tau(\varsigma)} \right)^2 - 1 \right| < 1 \right\}, \tag{8}$$

$$\frac{\varsigma\tau'(\varsigma)}{\tau(\varsigma)} \prec \sqrt{1+\varsigma}, \quad (\varsigma \in \Omega), \tag{9}$$

where

$$\tau(\varsigma) = \varsigma + \sum_{k=2}^{\infty} \frac{(u)_{k-1}}{(v)_{k-1}} \psi(\alpha,k) m_k \varsigma^k. \tag{10}$$

Hankel matrices arise naturally in a wide range of applications in science, engineering, and other related areas, such as signal processing and control theory. For a survey of Hankel matrices and polynomials, the reader is referred to [7,8] and the references therein.

The Hankel determinant $\mathcal{H}_{q,k}(l)$ $(q,k \in \mathbb{N})$ for a function $l \in \mathcal{S}$ of the form (1) was defined by Pommerenke (see [9,10]) as

$$\mathcal{H}_{q,k}(l) = \begin{vmatrix} m_k & m_{k+1} & \cdots & m_{k+q-1} \\ m_{k+1} & m_{k+2} & \cdots & m_{k+q} \\ \vdots & \vdots & & \vdots \\ m_{k+q-1} & m_{k+q} & \cdots & m_{k+2q-2} \end{vmatrix} \quad (m_1 = 1).$$

For fixed integer q and k, the growth of $\mathcal{H}_{q,k}(l)$ has been studied for different subfamilies of univalent functions. These studies focus on the main subclasses of certain holomorphic functions. In fact, the majority of papers discuss the determinants $\mathcal{H}_{2,2}(l)$ and $\mathcal{H}_{3,1}(l)$. Case $\mathcal{H}_{2,1}(l) = m_3 - m_2^2$ is also very well known. In the year 1933, Fekete and Szegö (see [11]) obtained a sharp bound of the function $m_3 - \mu m_2^2$ with real $\mu \in \mathbb{R}$ for a univalent function l. For $\mu \in \mathbb{C}$ this functional was generalized as $\left|m_3 - \mu m_2^2\right|$. Estimating for the upper bound of $\left|m_3 - \mu m_2^2\right|$ is known as the Fekete-Szegö problem, (see [12–14]). The second Hankel determinant $\mathcal{H}_{2,2}(l)$ is given by $\mathcal{H}_{2,2}(l) = m_2 m_4 - m_3^2$. In recent years, the research on Hankel determinants has focused on the estimation of $|\mathcal{H}_{2,2}(l)|$. Several authors obtained results for different classes of univalent functions. For example, the sharp bounds for the second Hankel determinant $\mathcal{H}_{2,2}(l)$ were obtained for the classes of starlike and convex functions in [15–18]. Lee et al. [19] established the sharp bound for $|\mathcal{H}_{2,2}(l)|$ by generalizing their classes by means of the principle of subordination between holomorphic functions. Our main focus in this investigation is for the class $\mathcal{SL}^*(u,v,\alpha)$ on the Hankel determinant $\mathcal{H}_{3,1}(l)$. The calculation of $|\mathcal{H}_{3,1}(l)|$ is far more challenging compared to finding the bound of $|\mathcal{H}_{2,2}(l)|$. Further, in this work, we find the sharp bounds for $|\mathcal{H}_{2,2}(l)|$, when $l \in \mathcal{SL}^*(u,v,\alpha)$, $\alpha \in [0,1)$, together with the sharp bound of the functional

$$\mathcal{Z} = |m_2 m_3 - m_4|,$$

when $l \in \mathcal{SL}^*(u,v,\alpha)$ and $\alpha \in [0,1)$.

2. Preliminary Lemmas

Some preliminary results required in the following section are now listed.

Lemma 1 ([20]). *Suppose that $\mathcal{P}$ denotes the family of holomorphic functions t normalized by*

$$t(\varsigma) = 1 + t_1\varsigma + t_2\varsigma^2 + \dots \tag{11}$$

and satisfying the condition $Re(t(\varsigma)) > 0$, $\varsigma \in \Omega$. Then, for any $\eta \in \mathbb{R}$,

$$\left|t_2 - \eta t_1^2\right| \leq \begin{cases} -4\eta + 2, & \eta < 0 \\ 2, & 0 \leq \eta \leq 1 \\ 4\eta - 2, & \eta \geq 1 \end{cases} \tag{12}$$

The equality holds true in (12) if and only if

$$t(\varsigma) = \frac{1+\varsigma}{1-\varsigma}$$

or one of its rotations, when $\eta < 0$ or $\eta > 1$. If $0 < \eta < 1$, then the equality holds true in (12) if and only if

$$t(\varsigma) = \frac{1+\varsigma^2}{1-\varsigma^2}$$

or one of its rotations. If $\eta = 0$, the equality holds true in (12) if and only if

$$t(\varsigma) = \left(\frac{1+\delta}{2}\right)\frac{1+\varsigma}{1-\varsigma} + \left(\frac{1-\delta}{2}\right)\frac{1-\varsigma}{1+\varsigma}, \quad 0 \leq \delta \leq 1$$

or one of its rotations. If $\eta = 1$, then the equality in (12) holds true if $t(\varsigma)$ is a reciprocal of one of the functions, such that the equality holds true in the case when $\eta = 0$.

Lemma 2 ([21]). *Assume that $t \in \mathcal{P}$ is the form Equation (2), and $\eta \in \mathbb{C}$, we have*

$$\left|t_2 - \eta t_1^2\right| \leq 2\max\{1, |1-2\eta|\}.$$

Lemma 3 ([22,23]). *If $t \in \mathcal{P}$ and has the form (11) then*

$$2t_2 = t_1^2 + x(4 - t_1^2)$$

for some x, $|x| \leq 1$ and

$$4t_3 = t_1^3 + 2(4-t_1^2)t_1x - (4-t_1^2)t_1x^2 + 2(4-t_1^2)(1-|x|^2)\varsigma$$

for some ς, $|\varsigma| \leq 1$.

Lemma 4 ([24]). *If $t \in \mathcal{P}$ and has the form (11), then*

$$|t_k| \leq 2 \quad (k \in \mathbb{N})$$

and the inequality is sharp.

3. Main Results

In the remainder of this work, we will assume that $u \geq v > 0$ until explicitly stated otherwise.

We now prove our first result asserted by Theorem 1 below.

Theorem 1. *If the function l, given by (1) belongs to the class $\mathcal{S}^*(u,v,\alpha)$, then $\mu \in \mathbb{R}$, we have*

$$\left|m_3 - \mu m_2^2\right| \leq \begin{cases} \frac{1}{16}\left(\frac{v(v+1)}{u(u+1)(1-\alpha)(3-2\alpha)} - \mu \frac{v^2}{u^2(1-\alpha)^2}\right) & \mu \leq -\frac{3u(v+1)(1-\alpha)}{v(u+1)(3-2\alpha)} \\ \frac{1}{4}\frac{v(v+1)}{u(u+1)(1-\alpha)(3-2\alpha)}, & -\frac{3u(v+1)(1-\alpha)}{v(u+1)(3-2\alpha)} \leq \mu \leq \frac{5u(v+1)(1-\alpha)}{v(u+1)(3-2\alpha)} \\ \frac{1}{16}\left(-\frac{v(v+1)}{u(u+1)(1-\alpha)(3-2\alpha)} + \mu \frac{v^2}{u^2(1-\alpha)^2}\right), & \mu \geq \frac{5u(v+1)(1-\alpha)}{v(u+1)(3-2\alpha)} \end{cases}.$$

Proof. From Equation (9), it follows that

$$\frac{\varsigma\tau'(\varsigma)}{\tau(\varsigma)} \prec \Phi(\varsigma).$$

Define the function first,

$$t(\varsigma) = 1 + \sum_{k=1}^{\infty} t_k\varsigma^k = \frac{1+w(\varsigma)}{1-w(\varsigma)}.$$

Since $t \in \mathcal{P}$,

$$w(\varsigma) = \frac{t(\varsigma)-1}{t(\varsigma)+1}.$$

Using Equation (9), we have

$$\frac{\varsigma\tau'(\varsigma)}{\tau(\varsigma)} = \Phi(w(\varsigma)).$$

Now as

$$\left[\frac{2t(\varsigma)}{1+t(\varsigma)}\right]^{\frac{1}{2}} = \left[2 - \frac{2}{1+t(\varsigma)}\right]^{\frac{1}{2}},$$

so, we have

$$\begin{aligned} \left[\frac{2t(\varsigma)}{1+t(\varsigma)}\right]^{\frac{1}{2}} &= 1 + \frac{1}{4}t_1\varsigma + \left(\frac{1}{4}t_2 - \frac{5}{32}t_1^2\right)\varsigma^2 + \left(\frac{1}{4}t_3 - \frac{5}{16}t_1t_2 + \frac{13}{128}t_1^3\right)\varsigma^3 \\ &\quad + \left(\frac{19}{8}t_1t_3 - \frac{3}{2}t_4 + \frac{361}{512}t_1^4 + \frac{9}{8}t_2^2 - \frac{34}{16}t_1^2t_2\right)\varsigma^4 + \ldots \end{aligned}$$

Similarly,

$$\begin{aligned} \frac{\varsigma\tau'(\varsigma)}{\tau(\varsigma)} &= 1 + Q_2m_2\varsigma + \left(2Q_3m_3 - Q_2^2m_2^2\right)\varsigma^2 + \left(3Q_4m_4 + Q_2^3m_2^3 - 3Q_2Q_3m_2m_3\right)\varsigma^3 \\ &\quad + \left(4Q_5m_5 - 4Q_2Q_4m_2m_4 - 2Q_3^2m_3^2 - Q_2^4m_2^4 + 4Q_2^2Q_3m_2^2m_3\right)\varsigma^4 + \ldots \end{aligned}$$

where

$$\begin{aligned}
Q_2 &= \frac{u}{v}(2-2\alpha), \\
Q_3 &= \frac{u(u+1)}{2v(v+1)}(2-2\alpha)(3-2\alpha), \\
Q_4 &= \frac{u(u+1)(u+2)}{6v(v+1)(v+2)}(2-2\alpha)(3-2\alpha)(4-2\alpha), \\
Q_5 &= \frac{u(u+1)(u+2)(u+2)}{24v(v+1)(v+2)(v+2)}(2-2\alpha)(3-2\alpha)(4-2\alpha)(5-2\alpha).
\end{aligned}$$

Thus,

$$m_2 = \frac{v}{4u(2-2\alpha)}t_1, \tag{13}$$

$$m_3 = \frac{v(v+1)}{u(u+1)(2-2\alpha)(3-2\alpha)}\left[\frac{1}{4}t_2 - \frac{3}{32}t_1^2\right], \tag{14}$$

$$m_4 = \frac{v(v+1)(v+2)}{u(u+1)(u+2)(2-2\alpha)(3-2\alpha)(4-2\alpha)}\left[\frac{1}{2}t_3 - \frac{7}{16}t_1t_2 + \frac{13}{128}t_1^3\right] \tag{15}$$

and

$$m_5 = \frac{v(v+1)(v+2)(v+3)}{u(u+1)(u+2)(u+3)(2-2\alpha)(3-2\alpha)(4-2\alpha)(5-2\alpha)}\left[\frac{19}{8}t_1t_3 - \frac{3}{2}t_4 + \frac{361}{512}t_1^4 + \frac{9}{8}t_2^2 - \frac{34}{16}t_1^2t_2\right]. \tag{16}$$

We now have the following using the Equations (13) and (14):

$$m_3 - \mu m_2^2 = \frac{v(v+1)}{u(u+1)(2-2\alpha)(3-2\alpha)}\left[\frac{1}{4}t_2 - \frac{3}{32}t_1^2\right] - \mu\frac{v^2}{16u^2(2-2\alpha)^2}t_1^2,$$

$$\left|m_3 - \mu m_2^2\right| \leq \frac{v(v+1)}{8u(u+1)(1-\alpha)(3-2\alpha)}\left|t_2 - \frac{1}{8}\left(\mu\frac{v(u+1)(3-2\alpha)}{u(1-\alpha)(v+1)} + 3\right)t_1^2\right|. \tag{17}$$

We obtained the required result by applying Lemma 1 to Equation (17). This completes the proof of Theorem 1. □

Theorem 2. *If the function l, given by (1), belongs to the class $\mathcal{SL}^*(u,v,\alpha)$, then $\mu \in \mathbb{C}$, we have*

$$\left|m_3 - \mu m_2^2\right| \leq \frac{v(v+1)}{4u(u+1)(1-\alpha)(3-2\alpha)}\max\left\{1, \left|\frac{1}{4}\mu\frac{v(u+1)(3-2\alpha)}{u(v+1)(1-\alpha)} - \frac{1}{4}\right|\right\}.$$

Proof. By making use of Equations (13) and (14), we have

$$m_3 - \mu m_2^2 = \frac{v(v+1)}{u(u+1)(2-2\alpha)(3-2\alpha)}\left[\frac{1}{4}t_2 - \frac{3}{32}t_1^2\right] - \mu\frac{v^2}{16u^2(2-2\alpha)^2}t_1^2,$$

$$\left|m_3 - \mu m_2^2\right| \leq \frac{v(v+1)}{8u(u+1)(1-\alpha)(3-2\alpha)}\left|t_2 - \frac{1}{8}\left(\mu\frac{v(u+1)(3-2\alpha)}{u(v+1)(1-\alpha)} + 3\right)t_1^2\right|$$

therefore, using Lemma 2, we obtain the result,

$$\left|m_3 - \mu m_2^2\right| \leq \frac{v(v+1)}{4u(u+1)(2-2\alpha)(3-2\alpha)}\max\left\{1, \left|\frac{1}{4}\mu\frac{v(u+1)(3-2\alpha)}{u(v+1)(1-\alpha)} - \frac{1}{4}\right|\right\}.$$

Thus, the proof of Theorem 2 is completed. □

For the case $\mu \in \mathbb{C}$ and $u = v$ in Theorem 2, this reduces to the following result.

Corollary 1. *Let $\alpha \in [0,1)$ and $\mu \in \mathbb{C}$. If the function l, given by (1), belongs to the class $\mathcal{SL}^*(u,u,\alpha) = \mathcal{SL}^*(u,\alpha)$, then*

$$\left|m_3 - \mu m_2^2\right| \leq \frac{1}{4(1-\alpha)(3-2\alpha)} \max\left\{1, \left|\frac{1}{4}\mu\frac{(3-2\alpha)}{(1-\alpha)} - \frac{1}{4}\right|\right\}$$

and the inequality is sharp.

4. The Hankel Determinant $\mathcal{H}_{2,2}(l)$

In this section, we find the sharp bound for the modulus of the second Hankel determinant $\mathcal{H}_{2,2}(l) = m_2m_4 - m_3^2$, when $l \in \mathcal{SL}^*(u,v,\alpha)$.

Theorem 3. *If the function l, given by (1), belongs to the class $\mathcal{SL}^*(u,v,\alpha)$, then*

$$\left|m_2m_4 - m_3^2\right| \leq \frac{v^2(v+1)^2}{16u^2(u+1)^2(1-\alpha)^2(3-2\alpha)^2}.$$

Proof. Using the Equations (13)–(15), we obtain the following

$$\begin{aligned} m_2m_4 - m_3^2 \quad &= \quad \left(\frac{v}{4u(2-2\alpha)}t_1\right)\left\{\frac{v(v+1)(v+2)}{u(u+1)(u+2)(2-2\alpha)(3-2\alpha)(4-2\alpha)}\left[\frac{1}{2}t_3 - \frac{7}{16}t_1t_2 + \frac{13}{128}t_1^3\right]\right\} \\ &\quad -\left[\frac{v(v+1)}{u(u+1)(2-2\alpha)(3-2\alpha)}\left(\frac{1}{4}t_2 - \frac{3}{32}t_1^2\right)\right]^2. \end{aligned}$$

After simplification, we have

$$\begin{aligned} m_2m_4 - m_3^2 \quad &= \quad \frac{v^2(v+1)}{12{,}288u^2(u+1)(2-2\alpha)^2(3-2\alpha)}\left\{\frac{1536(v+2)}{(u+2)(4-2\alpha)}t_1t_3 - \frac{768(v+1)}{(u+1)(3-2\alpha)}t_2^2\right. \\ &\quad \left.+\left(\frac{576(v+1)}{(u+1)(3-2\alpha)} - \frac{1344(v+2)}{(u+2)(4-2\alpha)}\right)t_1^2t_2 + \left(\frac{312(v+2)}{(u+2)(4-2\alpha)} - \frac{108(v+1)}{(u+1)(3-2\alpha)}\right)t_1^4\right\}. \end{aligned}$$

By substituting values of t_2 and t_3 from Lemma 3, after some simplification, we arrive at

$$\begin{aligned} m_2m_4 - m_3^2 \quad &= \quad \frac{v^2(v+1)}{12{,}288u^2(u+1)(2-2\alpha)^2(3-2\alpha)}\left\{\left(\frac{312(v+2)}{(u+2)(4-2\alpha)} - \frac{108(v+1)}{(u+1)(3-2\alpha)}\right)t_1^4\right. \\ &\quad +\frac{384(v+2)}{(u+2)(4-2\alpha)}t_1\left[t_1^3 + 2t_1\left(4-t_1^2\right)x - t_1\left(4-t_1^2\right)x^2 + 2\left(4-t_1^2\right)\left(1-|x|^2\varsigma\right)\right] \\ &\quad +\left(\frac{288(v+1)}{(u+1)(3-2\alpha)} - \frac{672(v+2)}{(u+2)(4-2\alpha)}\right)t_1^2\left[t_1^2 + \left(4-t_1^2\right)x\right] \\ &\quad \left.-\frac{192(v+1)}{(u+1)(3-2\alpha)}\left[t_1^2 + \left(4-t_1^2\right)x\right]^2\right\}. \end{aligned}$$

Now, taking the module and replacing $|x|$ by ρ and t_1 by t, we have

$$\begin{aligned} \left|m_2m_4 - m_3^2\right| \quad &\leq \quad \frac{v^2(v+1)}{12{,}288u^2(u+1)(2-2\alpha)^2(3-2\alpha)}\left\{\left(\frac{12(v+2)}{(u+2)(2-\alpha)} - \frac{12(v+1)}{(u+1)(3-2\alpha)}\right)t^4\right. \\ &\quad +\left(\frac{96(v+1)}{(u+1)(3-2\alpha)} - \frac{48(v+2)}{(u+2)(2-\alpha)}\right)t^2\left(4-t^2\right)\rho + \frac{384(v+2)}{(u+2)(2-\alpha)}t\left(4-t^2\right) \\ &\quad \left.+\left(\frac{192(v+2)}{(u+2)(2-\alpha)}t^2 + \frac{384(v+2)}{(u+2)(2-\alpha)}t + \frac{192(v+1)}{(u+1)(3-2\alpha)}\left(4-t^2\right)\right)\rho^2\left(4-t^2\right)\right\} \\ &= \quad F(t,\rho). \end{aligned} \tag{18}$$

Upon differentiating both sides (18) with respect to ρ, we obtain

$$\begin{aligned}\frac{\partial F(t,\rho)}{\partial \rho} &= \frac{v^2(v+1)}{12{,}288u^2(u+1)(2-2\alpha)^2(3-2\alpha)}\left\{\left(\frac{96(v+1)}{(u+1)(3-2\alpha)}-\frac{48(v+2)}{(u+2)(2-\alpha)}\right)t^2\left(4-t^2\right)\right.\\ &\left.+\left(\frac{192(v+2)}{(u+2)(2-\alpha)}t^2+\frac{384(v+2)}{(u+2)(2-\alpha)}t+\frac{192(v+1)}{(u+1)(3-2\alpha)}\left(4-t^2\right)\right)2\rho\left(4-t^2\right)\right\}.\end{aligned}$$

It is clear that

$$\frac{\partial F(t,\rho)}{\partial \rho} > 0,$$

which show that $F(t,\rho)$ is an increasing function of ρ on the closed interval $[0,1]$. This implies that the maximum value occurs at $\rho = 1$. This implies that

$$\max\{F(t,\rho)\} = F(t,1) = G(t).$$

We now observe that

$$\begin{aligned}G(t) &= \frac{v^2(v+1)}{12{,}288u^2(u+1)(2-2\alpha)^2(3-2\alpha)}\left\{\left(\frac{12(v+2)}{(u+2)(2-\alpha)}-\frac{12(v+1)}{(u+1)(3-2\alpha)}\right)t^4\right.\\ &+\left(\frac{96(v+1)}{(u+1)(3-2\alpha)}-\frac{48(v+2)}{(u+2)(2-\alpha)}\right)t^2\left(4-t^2\right)+\frac{384(v+2)}{(u+2)(2-\alpha)}t\left(4-t^2\right)\\ &\left.+\left(\frac{192(v+2)}{(u+2)(2-\alpha)}t^2+\frac{384(v+2)}{(u+2)(2-\alpha)}t+\frac{192(v+1)}{(u+1)(3-2\alpha)}\left(4-t^2\right)\right)\left(4-t^2\right)\right\}.\end{aligned} \tag{19}$$

Differentiating (19) with respect to t, we obtain

$$\begin{aligned}G'(t) &= \frac{v^2(v+1)}{12{,}288u^2(u+1)(2-2\alpha)^2(3-2\alpha)}\left\{4\left(\frac{84(v+1)}{(u+1)(3-2\alpha)}-\frac{132(v+2)}{(u+2)(2-\alpha)}\right)t^3\right.\\ &\left.-\frac{96(v+2)}{(u+2)(2-\alpha)}t^2+16\left(\frac{72(v+2)}{(u+2)(2-\alpha)}-\frac{144(v+1)}{(u+1)(3-2\alpha)}\right)t+\frac{3072(v+2)}{(u+2)(2-\alpha)}\right\}.\end{aligned}$$

Differentiating again above equation with respect to t, we have

$$\begin{aligned}G''(t) &= \frac{v^2(v+1)}{12{,}288u^2(u+1)(2-2\alpha)^2(3-2\alpha)}\left\{12\left(\frac{84(v+1)}{(u+1)(3-2\alpha)}-\frac{132(v+2)}{(u+2)(2-\alpha)}\right)t^2\right.\\ &\left.-\frac{192(v+2)}{(u+2)(2-\alpha)}t+16\left(\frac{72(v+2)}{(u+2)(2-\alpha)}-\frac{144(v+1)}{(u+1)(3-2\alpha)}\right)\right\}.\end{aligned}$$

For $t = 0$, $(t \in [0,2])$ shows that the maximum value of $G(t)$ occurs at $t = 0$. Hence, we obtain,

$$\left|m_2m_4 - m_3^2\right| \leq \frac{v^2(v+1)^2}{16u^2(u+1)^2(1-\alpha)^2(3-2\alpha)^2}.$$

Thus, the proof of Theorem 3 is completed. □

Upon setting $u = v$ in Theorem 3, we are led to the following results, respectively:

Corollary 2. *Let $\alpha \in [0,1)$. If the function l, given by (1), belongs to the class $\mathcal{SL}^*(u,u,\alpha) = \mathcal{SL}^*(u,\alpha)$, then*

$$\left|m_2m_4 - m_3^2\right| \leq \frac{1}{16(1-\alpha)^2(3-2\alpha)^2}$$

and the inequality is sharp.

If we choose $\alpha = 0$ in Corollary 2, we obtain the following corollary.

Corollary 3. *Let $\alpha \in [0,1)$. If the function l, given by (1), belongs to the class $\mathcal{SL}^*(u,u,0) = \mathcal{SL}^*(u)$, then*

$$\left|m_2m_4 - m_3^2\right| \leq \frac{1}{144}$$

and the inequality is sharp.

5. The Zalcman Functional

In this section, we prove the following theorem on the upper bound estimate of the Zalcman functional $|m_2m_3 - m_4|$, noting that a non-sharp inequality was found in [25–29].

Theorem 4. *If the function l, given by (1), belongs to the class $\mathcal{SL}^*(u,v,\alpha)$, then*

$$|m_2m_3 - m_4| \leq \frac{v(v+1)(v+2)}{4u(u+1)(u+2)(1-\alpha)(2-\alpha)(3-2\alpha)}.$$

Proof. Using the values given in (13)–(15) we have

$$\begin{aligned} m_2m_3 - m_4 &= \left(\frac{v}{4u(2-2\alpha)}t_1\right)\left[\frac{v(v+1)}{u(u+1)(2-2\alpha)(3-2\alpha)}\left(\frac{1}{4}t_2 - \frac{3}{32}t_1^2\right)\right] \\ &\quad -\left\{\frac{v(v+1)(v+2)}{u(u+1)(u+2)(2-2\alpha)(3-2\alpha)(4-2\alpha)}\left[\frac{1}{2}t_3 - \frac{7}{16}t_1t_2 + \frac{13}{128}t_1^3\right]\right\}. \end{aligned}$$

By substituting values of t_2 and t_3 from Lemma 3, after some simplification, we have

$$\begin{aligned} m_2m_3 - m_4 &= \frac{v(v+1)}{u(u+1)(2-2\alpha)(3-2\alpha)}\left\{\frac{v}{4u(2-2\alpha)}\left(\frac{1}{4}t_1t_2 - \frac{3}{32}t_1^3\right)\right. \\ &\quad \left. -\frac{(v+2)}{(u+2)(4-2\alpha)}\left[\frac{1}{2}t_3 - \frac{7}{16}t_1t_2 + \frac{13}{128}t_1^3\right]\right\} \\ &= \frac{v(v+1)}{u(u+1)(2-2\alpha)(3-2\alpha)}\left\{\frac{1}{32}\frac{v}{u(2-2\alpha)}t_1\left[t_1^2 + \left(4-t_1^2\right)x\right] - \frac{3v}{128u(2-2\alpha)}t_1^3\right. \\ &\quad -\frac{1}{8}\frac{(v+2)}{(u+2)(4-2\alpha)}\left[t_1^3 + 2t_1\left(4-t_1^2\right)x - t_1\left(4-t_1^2\right)x^2 + 2\left(4-t_1^2\right)\left(1-|x|^2\varsigma\right)\right] \\ &\quad \left. +\frac{7(v+2)}{32(u+2)(4-2\alpha)}t_1\left[t_1^2 + \left(4-t_1^2\right)x\right] - \frac{13(v+2)}{128(u+2)(4-2\alpha)}t_1^3\right\}. \end{aligned}$$

Using Lemma 3, and since $t_1 \leq 2$ by Lemma 4, let $t_1 = t$ and assume, without restriction, that $t \in [0,2]$. By using the triangle inequality with $\rho = |x|$, we arrive at

$$\begin{aligned} |m_2m_3 - m_4| &\leq \frac{v(v+1)}{768u(u+1)(2-2\alpha)(3-2\alpha)}\left\{\left(\frac{3v}{u(1-\alpha)} - \frac{3v}{(u+2)(2-\alpha)}\right)t^3\right. \\ &\quad +\left(\frac{12v}{u(1-\alpha)} - \frac{12(v+2)}{(u+2)(2-\alpha)}\right)t\left(4-t^2\right)\rho \\ &\quad \left. +\left(\frac{96(v+2)}{(u+2)(2-\alpha)} + \frac{96(v+2)}{(u+2)(2-\alpha)}\rho^2 + \frac{48(v+2)}{(u+2)(2-\alpha)}t\rho^2\right)\left(4-t^2\right)\right\} \\ &= F_1(t,\rho). \end{aligned}$$

Differentiating $F_1(t,\rho)$ with respect to ρ, we have

$$\begin{aligned} F_1'(\rho) &= \frac{v(v+1)}{768u(u+1)(2-2\alpha)(3-2\alpha)}\left\{\left(\frac{12v}{u(1-\alpha)} - \frac{12(v+2)}{(u+2)(2-\alpha)}\right)t\left(4-t^2\right)\right. \\ &\quad \left. +\frac{192(v+2)}{(u+2)(2-\alpha)}\rho\left(4-t^2\right) + \frac{96(v+2)}{(u+2)(2-\alpha)}t\rho\left(4-t^2\right)\right\} \\ &> 0. \end{aligned}$$

This implies that $F_1(t,\rho)$ is an increasing function of ρ on the closed interval $[0,1]$. Hence, $F_1(\rho) \leq F_1(0)$ for all $\rho \in [0,1]$, that is

$$\begin{aligned} F_1(\rho) &= \frac{v(v+1)}{768u(u+1)(2-2\alpha)(3-2\alpha)} \left\{ \left(\frac{3v}{u(1-\alpha)} - \frac{3v}{(u+2)(2-\alpha)} \right) t^3 \right. \\ &\quad \left. + \frac{96(v+2)}{(u+2)(2-\alpha)} \left(4-t^2\right) \right\} \\ &= G_1(t). \end{aligned}$$

Differentiating $G_1(t)$ with respect to t, we have

$$\begin{aligned} G_1'(t) &= \frac{v(v+1)}{768u(u+1)(2-2\alpha)(3-2\alpha)} \left\{ \left(\frac{9v}{u(1-\alpha)} - \frac{9v}{(u+2)(2-\alpha)} \right) t^2 \right. \\ &\quad \left. - \frac{192(v+2)}{(u+2)(2-\alpha)} t \right\}. \end{aligned}$$

Again, differentiating the above equation with respect to t, we have

$$\begin{aligned} G_1''(t) &= \frac{v(v+1)}{768u(u+1)(2-2\alpha)(3-2\alpha)} \left\{ \left(\frac{18v}{u(1-\alpha)} - \frac{18v}{(u+2)(2-\alpha)} \right) t \right. \\ &\quad \left. - \frac{192(v+2)}{(u+2)(2-\alpha)} \right\} \\ &< 0. \end{aligned}$$

Since $t \in [0,2]$, by the assumption, it follows that $G_1(t)$ attains maximum at $t = 0$, which corresponds to $\rho = 0$, and it is the desired upper bound. Hence, we obtain

$$|m_2 m_3 - m_4| \leq \frac{v(v+1)(v+2)}{4u(u+1)(u+2)(1-\alpha)(2-\alpha)(3-2\alpha)}.$$

The proof of Theorem 4 is thus completed. □

If we put $u = v$ in Theorem 4, we have the following results, respectively:

Corollary 4. *Let $\alpha \in [0,1)$. If the function l, given by (1), belongs to the class $\mathcal{SL}^*(u,u,\alpha) = \mathcal{SL}^*(u,\alpha)$, then*

$$|m_2 m_3 - m_4| \leq \frac{1}{4(1-\alpha)(2-\alpha)(3-2\alpha)}$$

and the inequality is sharp.

If we choose $\alpha = 0$ in Corollary 4, we arrive at the following result.

Corollary 5. *Let $\alpha \in [0,1)$. If the function l, given by (1), belongs to the class $\mathcal{SL}^*(u,u,0) = \mathcal{SL}^*(u)$, then*

$$|m_2 m_3 - m_4| \leq \frac{1}{24}$$

and the inequality is sharp.

Theorem 5. *If the function l, given by (1), belongs to the class $\mathcal{SL}^*(u,v,\alpha)$, then*

$$\begin{aligned} |H_3(1)| &\leq \frac{v^2(v+1)^2}{512u^2(u+1)^2(1-\alpha)^2(3-2\alpha)^2} \\ &\quad \times \left\{ \frac{2v(v+1)}{u(u+1)(1-\alpha)(3-2\alpha)} + \frac{50(v+2)^2}{(u+2)^2(2-\alpha)^2} + \frac{169(v+2)(v+3)}{(u+2)(u+3)(2-\alpha)(5-2\alpha)} \right\}. \end{aligned}$$

Proof. Since

$$|H_3(1)| \leq |m_3|\left|m_2m_4 - m_3^2\right| + |m_4||m_2m_3 - m_4| + |m_5|\left|m_3 - m_2^2\right|$$

using the fact that $m_1 = 1$, with Theorems 1, 3 and 4 and Lemma 4, we have the required result

$$\begin{aligned}
|H_3(1)| &\leq |m_3|\left|m_2m_4 - m_3^2\right| + |m_4||m_2m_3 - m_4| + |m_5|\left|m_1m_3 - m_2^2\right| \\
&= \frac{v(v+1)}{16u(u+1)(1-\alpha)(3-2\alpha)}\left[\frac{1}{16}\left(\frac{v(v+1)}{u(u+1)(1-\alpha)(3-2\alpha)}\right)^2\right] \\
&\quad + \frac{25v(v+1)(v+2)}{64u(u+1)(u+2)(1-\alpha)(3-2\alpha)(2-\alpha)}\left[\frac{1}{4}\left(\frac{v(v+1)(v+2)}{u(u+1)(u+2)(1-\alpha)(2-\alpha)(3-2\alpha)}\right)\right] \\
&\quad + \frac{169v(v+1)(v+2)(v+3)}{128u(u+1)(u+2)(u+3)(1-\alpha)(3-2\alpha)(2-\alpha)(5-2\alpha)}\left[\frac{1}{4}\frac{v(v+1)}{u(u+1)(1-\alpha)(3-2\alpha)}\right] \\
&= \frac{v^2(v+1)^2}{512u^2(u+1)^2(1-\alpha)^2(3-2\alpha)^2} \\
&\quad \times \left\{\frac{2v(v+1)}{u(u+1)(1-\alpha)(3-2\alpha)} + \frac{50(v+2)^2}{(u+2)^2(2-\alpha)^2} + \frac{169(v+2)(v+3)}{(u+2)(u+3)(2-\alpha)(5-2\alpha)}\right\}.
\end{aligned}$$

The proof of Theorem 5 is thus completed. □

If we set $u = v$ in Theorem 5, we establish the below inequality.

Corollary 6. *Let $\alpha \in [0,1)$. If the function l, given by (1), belongs to the class $\mathcal{SL}^*(u,u,\alpha) = \mathcal{SL}^*(u,\alpha)$, then*

$$|H_3(1)| \leq \frac{1}{512(1-\alpha)^2(3-2\alpha)^2}\left\{\frac{2}{(1-\alpha)(3-2\alpha)} + \frac{50}{(2-\alpha)^2} + \frac{169}{(2-\alpha)(5-2\alpha)}\right\}$$

and the inequality is sharp.

6. Conclusions

In the present investigation, we have estimated smaller upper bounds and more accurate estimations for the functionals $\left|m_3 - \mu m_2^2\right|$ and $\left|m_2m_4 - m_3^2\right|$ for the class $\mathcal{SL}^*(u,v,\alpha)$ of holomorphic functions associated with the Carlson-Shaffer operator in the unit disk.

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