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Article

New Applications of Faber Polynomial Expansion for Analytical Bi-Close-to-Convex Functions Defined by Using q-Calculus

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Abstract: In this investigation, the q-difference operator and the Sălăgean q-differential operator are utilized to establish novel subclasses of analytical bi-close-to-convex functions. We determine the general Taylor–Maclaurin coefficient of the functions in this class using the Faber polynomial method. We demonstrate the unpredictable behaviour of initial coefficients $|a_2|$, $|a_3|$ and investigate the Fekete–Szegő problem $|a_3 - a_2^2|$ for the subclasses of bi-close-to-convex functions. To highlight the connections between existing knowledge and new research, certain known and unknown corollaries are also highlighted.

Keywords: analytic functions; quantum (or q-) calculus; q-derivative operator; close-to-convex functions; bi-univalent functions; Faber polynomial expansion

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1. Introduction and Definitions

Assume that $\mathfrak{A}$ denotes the set of all analytical functions $f(z)$ in the open unit disk

$$\mathcal{U} = \{z : |z| < 1\},$$

which are normalized by

$$f(0) = 0 \text{ and } f'(0) = 1.$$

Thus, every function $f \in \mathfrak{A}$ can be expressed in the form given in Equation (1):

$$f(z) = z + \sum_{j=2}^{\infty} a_j z^j. \tag{1}$$

An analytical function f is considered univalent if

$$f(z_1) \neq f(z_2) \Rightarrow z_1 \neq z_2,\ \forall\, z_1, z_2 \in \mathcal{U}.$$

We let $\mathcal{S}$ denote the set of all analytical functions in $\mathfrak{A}$ that are univalent in $\mathcal{U}$. For $f_1, f_2 \in \mathfrak{A}$, and f_1 are subordinate to f_2 in $\mathcal{U}$, denoted by

$$f_1(z) \prec f_2(z),\ \ z \in \mathcal{U},$$

if we have a function u, such that

$$u \in \mathcal{B} = \{u : u \in \mathfrak{A},\ |u(z)| < 1 \text{ and } u(0) = 0,\ z \in \mathcal{U}\}$$

and

$$f(z) = y(u(z)),\ z \in \mathcal{U}.$$

For $0 \le \alpha < 1$, let $\mathcal{S}^*(\alpha)$ stand for the class of functions $g \in \mathcal{S}$, which are star-like of the order α in $\mathcal{U}$, such that

$$\mathrm{Re}\left(\frac{zg'(z)}{g(z)}\right) > \alpha,\ \ z \in \mathcal{U}.$$

The family of close-to-convex functions $f \in \mathcal{S}$ of the order α in $\mathcal{U}$ are denoted by $\mathcal{C}(\alpha)$, and defined as (see [1]):

$$\mathrm{Re}\left(\frac{zf'(z)}{g(z)}\right) > \alpha,\ \ z \in \mathcal{U},$$

where

$$g \in \mathcal{S}^*(0) = \mathcal{S}^*.$$

We note that

$$\mathcal{S}^*(\alpha) \subset C(\alpha) \subset \mathcal{S}$$

and

$$\left|a_j\right| < j,\ \ (\text{ for all } f \in \mathcal{S},\ j \in \mathbb{N}\backslash\{1\})$$

by the De Branges Theorem, also known as the Bieberbach Conjecture (see [2,3]). The Koebe one-quarter theorem (see [3]) states that the disk of radius $\frac{1}{4}$ is contained in the image of under every univalent function $f \in \mathcal{S}$. Each $f \in \mathcal{S}$ therefore has an inverse $f^{-1} = F$, defined as:

$$F(f(z)) = z,\ \ z \in \mathcal{U}$$

and

$$f(F(w)) = w,\ |w| < r_0(f),\ r_0(f) \ge \frac{1}{4}.$$

The series of inverse function F is given by

$$F(w) = w - a_2w^2 + (2a_2^2 - a_3)w^3 - \mathcal{Q}(a)w^4 + \ldots, \tag{2}$$

where

$$\mathcal{Q}(a) = (5a_2^3 - 5a_2a_3 + a_4).$$

An analytical function f is called bi-univalent in $\mathcal{U}$ if f and $f^{-1}$are univalent in $\mathcal{U}$. Σ stand for the class of all normalized analytical bi-univalent functions. In 1967, Levin [4] introduced the class of analytical and bi-univalent functions in $\mathcal{U}$ and showed that

$$|a_2| < 1.51.$$

After this, Branan and Clunie [5] enhanced the Levin result of the following form

$$|a_2| \le \sqrt{2}.$$

Furthermore, for the bi-univalent functions, Netanyahu [6] proved that

$$\max|a_2| = \frac{4}{3}.$$

It is indeed essential to mention that the following functions are bi-univalent:

$$f_1(z) = \frac{z}{1-z},\ \ f_2(z) = \log\left(\sqrt{\frac{1+z}{1-z}}\right)$$

and

$$f_3(z) = \log\left(\frac{1}{1-z}\right).$$

The inverse functions that correspond to these:

$$f_1^{-1}(w) = \frac{w}{1+z}, \quad f_2^{-1}(w) = \frac{e^{2w}-1}{e^{2w}+1}$$

and

$$f_3^{-1}(w) = \frac{e^{w}-1}{e^{w}}$$

are also univalent functions. Thus, the functions f_1, f_2 and f_3 are bi-univalent.

The interesting subclass of analytical and bi-univalent functions was proposed and investigated by Branan and Taha [7], who also obtained estimates for the coefficient of the functions in this class. Similarly to the well-known $\mathcal{S}^*(\alpha)$ and $C(\alpha)$ subclasses of star-like and convex functions of the order α $(0 \leq \alpha < 1)$ introduced by Brannan et al. in [8]. The $\mathcal{S}^*_\Sigma(\alpha)$ of bi-star-like functions and the $\mathcal{C}^*_\Sigma(\alpha)$ of bi-convex functions of the order α were in fact introduced by Brannan and Taha [9]. Recently, the exploration of numerous subclasses of the analytical and bi-univalent function class Σ was basically revitalized by the pioneering work of Srivastava et al. [10]. A new subclass of class Σ was created by Xu et al. [11] in 2012, and they investigated coefficient estimates for the functions contained in the new subclass of class Σ. Recently, a number of authors introduced and explored a number of subclasses of class Σ (see for details [12–14]). Only non-sharp estimates of the initial coefficients were examined in these recent works. Two new subclasses of analytical bi-univalent functions are introduced using a Sălăgean–Erdély–Kober operator in [15] and coefficient-related problems are solved regarding this class, including the Fekete–Szegő problem. Three new classes of bi-univalent functions are introduced as generalizations of previously introduced classes and estimates on the coefficients and the Fekete–Szegő problem are obtained in [16]. A new family of holomorphic and bi-univalent functions is introduced using a new operator joining Poisson distribution with a Ruscheweyh derivative operator and upper bounds for the second and third coefficients are discussed in [17]. Other similar very recent studies can be seen in [18–20].

Faber introduced Faber polynomials [21], and first utilized them to establish the general coefficient bounds $|a_j|$ for $j \geq 3$. Gong [22] explained the role of Faber polynomials in the field of mathematics, notably in the context of geometric function theory (GFT). Hamidi and Jahangiri [23,24] discovered some new coefficient bounds for analytical bi-close-to-convex functions by taking the Faber polynomial expansion method into consideration. Additionally, numerous authors [25–30] who implemented the same methodology discovered some interesting and beneficial characteristics for analytical bi-univalent functions. Only a few works have been performed so far by utilizing Faber polynomial expansion methods, and we found very little in the literature that went beyond the bounds of the Maclaurin's series coefficient $|a_j|$ for $j \geq 4$. The general coefficient bounds $|a_j|$ for $j \geq 4$ were recently found by a small number of authors using the Faber polynomial expansion technique (see for detail [29,31–35]).

In the area of GFT, the q-calculus and fractional q-calculus have been widely utilized by scholars who have developed and examined a number of different subclasses of analytical, univalent, and bi-univalent functions. The q-calculus operator was first proposed by Jackson [36,37] in 1909, and the q-difference operator (D_q) was first used by Ismail et al. [38] to establish a class of q-star-like functions in open unit disc $\mathcal{U}$. The most significant applications of q-calculus were essentially given in GFT, and Srivastava was the first to use fundamental (or q-) hypergeometric functions in a book chapter (see for details, [39]). Very recent investigations embedding q-calculus in GFT can be seen in [40–43]. See the following articles for more work on GFT associated with q-calculus operator theory [44–46].

There are numerous disciplines of mathematics and physics where the q-calculus is used, also having many applications in other fields of science such as special polynomials, analytical number theory, quantum group theory, numerical analysis, fractional calculus, and other related theories. Recently, Faber polynomials and special functions have become extremely important in the fields of mathematics, physics, engineering, and other research fields.

We provide some fundamental q-calculus definitions and ideas that will be utilized to establish some new subclasses in this paper.

Definition 1. *The definition of the q-number $[t]_q$ for $q \in (0,1)$ is:*

$$[t]_q = \frac{1-q^t}{1-q}, \qquad (t \in \mathbb{C}).$$

More specifically, $t = j \in \mathbb{N}$,

$$\begin{aligned} [j]_q &= \frac{1-q^j}{1-q}, \\ &= \sum_{k=0}^{j-1} q^k, \end{aligned} \tag{3}$$

and the q-factorial $[j]_q!$ can be defined as:

$$[j]_q! = \prod_{k=1}^{j} [k]_q, \quad (j \in \mathbb{N}).$$

In particular, $[0]_q! = 1$.

Definition 2 ([36]). *For $f \in \mathfrak{A}$, the q-difference operator, usually referred to as the q-derivative operator, is defined by*

$$\begin{aligned} D_q f(z) &= \frac{f(z) - f(qz)}{(1-q)z}, \quad z \neq 0, q \neq 1, \\ &= 1 + \sum_{j=2}^{\infty} [j]_q a_j z^{j-1}. \end{aligned} \tag{4}$$

Definition 3 ([47]). *For $m \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$. The definition of a Sălăgean q-differential operator for $f \in \mathfrak{A}$ is given as:*

$$\begin{aligned} \mathcal{S}_q^0 f(z) &= f(z),\ \mathcal{S}_q^1 f(z) = z D_q f(z) = \frac{f(qz) - f(z)}{(q-1)}, \cdots, \\ \mathcal{S}_q^m f(z) &= z D_q \left(\mathcal{S}_q^{m-1} f(z) \right) = f(z) * \left(z + \sum_{j=2}^{\infty} \left([j]_q \right)^m z^j \right), \\ &= z + \sum_{j=2}^{\infty} [j]_q^m a_j z^j. \end{aligned}$$

Very recent results were published involving a Sălăgean q-differential operator as it can be seen in [48–50].

Recent papers [23,24,29,30] encouraged us to use this method to build new subclasses of bi-close-to-convex functions of class Σ associated with q-calculus operator theory.

Definition 4. *The function f of the Equation (1) is known as a close-to-convex function class $C_\Sigma(m, \alpha, q)$ if there is a $g \in \mathcal{S}^*$ satisfying*

$$\operatorname{Re}\left(\frac{\mathcal{S}_q^m f(z)}{g(z)}\right) > \alpha$$

and

$$\operatorname{Re}\left(\frac{\mathcal{S}_q^m F(w)}{G(w)}\right) > \alpha,$$

where, $m \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}, 0 \leq \alpha < 1, z, w \in \mathcal{U}$ *and* $F = f^{-1}$ *given by Equation* (2).

Remark 1. *For* $m = 1$ *and* $q \to 1-$ *in Definition* (4)*, we have a known class* $C_\Sigma(\alpha)$ *proven by Hamidi and Jahangiri in* [23].

Definition 5. *The function* f *of Equation* (1) *is known as a close-to-convex function class* $C_\Sigma(\alpha, q)$ *in* $\mathcal{U}$ *if* $g \in \mathcal{S}^*$ *satisfies*

$$\operatorname{Re}\left(\frac{z\mathcal{D}_q f(z)}{g(z)}\right) > \alpha$$

and

$$\operatorname{Re}\left(\frac{z\mathcal{D}_q F(w)}{G(w)}\right) > \alpha,$$

where, $0 \leq \alpha < 1, z, w \in \mathcal{U}$ *and* $F = f^{-1}$ *given by Equation* (2).

The quantum (q) operator theory is associated with a wide range of problems in important areas of mathematical physics and engineering and is used in the solutions of heat transfer and other problems with cylindrical and spherical coordinates. Several new subclasses of convex and star-like functions have been defined, and many of their interesting properties have been obtained, using the q-analogous of difference and the Sălăgean operator. Studying certain subclasses of star-like functions and their generalizations is one of the classical area of the field of GFT. In this paper, we try to investigate new geometric properties of close-to-convex functions by using Faber polynomial methods and the well-known q-operator.

2. The Faber Polynomial Expansion Method and Its Applications

Using the Faber polynomial method, the coefficients of the inverse map $f^{-1} = F$ of an analytical function f can be expressed as follows (see [27,51]):

$$F(w) = f^{-1}(w) = w + \sum_{j=2}^{\infty} \frac{1}{j} \mathfrak{R}_{j-1}^{j}(a_2, a_3, ..., a_j) w^j,$$

where

$$\begin{aligned}
\mathfrak{R}_{j-1}^{-j} &= \frac{(-j)!}{(-2j+1)!(j-1)!} a_2^{j-1} + \frac{(-j)!}{[2(-j+1)]!(j-3)!} a_2^{j-3} a_3 \\
&+ \frac{(-j)!}{(-2j+3)!(j-4)!} a_2^{j-4} a_4 \\
&+ \frac{(-j)!}{[2(-j+2)]!(j-5)!} a_2^{j-5} \left[a_5 + (-j+2) a_3^2\right] \\
&+ \frac{(-j)!}{(-2j+5)!(j-6)!} a_2^{j-6} [a_6 + (-2j+5) a_3 a_4] \\
&+ \sum_{i \geq 7} a_2^{j-i} \mathcal{Q}_i,
\end{aligned}$$

and the variables $a_2, a_3, ...a_j$ make up the homogeneous polynomial $\mathcal{Q}_i$, for $7 \leq i \leq j$. In particular, the initial three terms of $\mathfrak{N}_{j-1}^{-j}$ are

$$\begin{aligned} \frac{1}{2}\mathfrak{N}_1^{-2} &= -a_2,\ \frac{1}{3}\mathfrak{N}_2^{-3} = 2a_2^2 - a_3, \\ \frac{1}{4}\mathfrak{N}_3^{-4} &= -(5a_2^3 - 5a_2a_3 + a_4). \end{aligned}$$

In general, for $v \in \mathbb{Z}$ ($\mathbb{Z} := 0, \pm 1, \pm 2, ...$ and $j \geq 2$, an expansion of $\mathfrak{N}_{j-1}^v$ of the form:

$$\mathfrak{N}_{j-1}^v = va_j + \frac{v(v-1)}{2}H_{j-1}^2 + \frac{v!}{(v-3)!3!}H_{j-2}^3 + ... + \frac{v!}{(v-j+1)!(j-1)!}H_{j-1}^{j-1},$$

where,

$$H_{j-1}^v = H_{j-1}^v(a_2, a_3....a_j)$$

and by [51], we have

$$H_{j-1}^y(a_2, ..., a_j) = \sum_{j=2}^{\infty} \frac{y!(a_2)^{\mu_1}...(a_j)^{\mu_j - 1}}{\mu_1!, ..., \mu_{j-1}!}, \text{ for } a_1 = 1 \text{ and } y \leq j.$$

The sum is taken over all non-negative integers $\mu_1, ..., \mu_{j-1}$ which satisfy

$$\mu_1 + \mu_2 + ... + \mu_{j-1} = y$$

and

$$\mu_1 + 2\mu_2 + ... + (j-1)\mu_j = j - 1.$$

Clearly,

$$H_{j-1}^{j-1}(a_1, ..., a_j) = a_2^{j-1}$$

and

$$H_j^j = a_1^j, \text{ and } H_j^1 = a_j$$

are first and last polynomials.

We shall demonstrate our findings using the subsequent lemma.

Lemma 1. *The Caratheodory Lemma (see [3]). If $p \in \mathcal{P}$ and*

$$p(z) = 1 + \sum_{j=1}^{\infty} c_j z^j,$$

then

$$|c_j| \leq 2,$$

where $\mathcal{P}$ is the family of all analytical functions that have a positive real part and $p(0) = 1$.

Motivated by the recent studies involving Faber polynomial expansion [52–55], the general coefficient $|a_j|$ of bi-close-to-convex functions of class $C_\Sigma(m, \alpha, q)$ are determined using the Faber polynomial expansions in this study under suitable gap series conditions. After this, we demonstrate the unpredictable behaviour of initial coefficients $|a_2|$, $|a_3|$ and investigate the Fekete–Szegő problem $|a_3 - a_2^2|$. We also provide an example of the bi-close-to-convex function in the class $C_\Sigma(m, \alpha, q)$.

3. Main Results

The general Taylor–Maclaurin coefficients of functions in $C_\Sigma(m, \alpha, q)$ are determined in this section using the Faber polynomial expansion method.

Theorem 1. *If $f \in C_\Sigma(m,\alpha,q)$ be given by Equation (1). If $a_k = 0$, and $2 \le k \le j-1$, then*

$$|a_j| \le \frac{2(1-\alpha)+j}{[j]_q^m}, \quad for \quad j \ge 3.$$

Proof. If $f \in C_\Sigma(m,\alpha,q)$, then there exists a function $g(z)$, so that we have the Faber polynomial expansion

$$\begin{aligned}&\frac{\mathcal{S}_q^m f(z)}{g(z)}\\ =\ & 1+\sum_{j=2}^{\infty}\left[\left([j]_q^m a_j - b_j\right)\sum_{l=1}^{j-2}\mathfrak{R}_l^{-1}(b_2,b_3,...b_{l+1})\left(\left([j]_q^m - l\right)a_{j-l} - b_{j-l}\right)\right]z^{j-1}. \end{aligned} \quad (5)$$

For the inverse map $F = f^{-1} \in C_\Sigma(m,\alpha,q)$, there exists a function

$$G(w) = w + \sum_{j=2}^{\infty} B_j w^j \in \mathcal{S}^*(0),$$

so that

$$\mathrm{Re}\left(\frac{\mathcal{S}_q^m F(w)}{G(w)}\right) > \alpha$$

in $\mathcal{U}$. Presumed from Equation (5), the Faber polynomial expansion of $F = f^{-1}$ is

$$F(w) = w + \sum_{j=2}^{\infty} A_j w^j.$$

Thus, the Faber polynomial expansion of $\frac{\mathcal{S}_q^m F(w)}{G(w)}$ is given by

$$\begin{aligned}&\frac{\mathcal{S}_q^m F(w)}{G(w)}\\ =\ & 1+\sum_{j=2}^{\infty}\left[\left([j]_q^m A_j - B_j\right)\sum_{l=1}^{j-2}\mathfrak{R}_l^{-1}(B_2,B_3,...B_{l+1})\left(\left([j]_q^m - l\right)A_{j-l} - B_{j-l}\right)\right]w^{j-1}. \end{aligned} \quad (6)$$

Since $f \in C_\Sigma(m,\alpha,q)$ is in $\mathcal{U}$ and there exists a positive real part function

$$p(z) = 1 + \sum_{j=1}^{\infty} c_j z^j$$

so that

$$\begin{aligned}\frac{\mathcal{S}_q^m f(z)}{g(z)} &= 1+(1-\alpha)p(z)\\ &= 1+(1-\alpha)\sum_{j=1}^{\infty} c_j z^j. \end{aligned} \quad (7)$$

Similarly, for $F \in C_\Sigma(m,\alpha,q)$ in $\mathcal{U}$ there exists a positive real part function

$$q(w) = 1 + \sum_{j=1}^{\infty} d_j w^j$$

so that

$$\begin{aligned}\frac{\mathcal{S}_q^m F(z)}{G(z)} &= 1+(1-\alpha)q(w) \\ &= 1+(1-\alpha)\sum_{j=1}^{\infty} d_j w^j. \end{aligned} \tag{8}$$

Evaluating the coefficients of the Equations (5) and (7), for any $j \geq 2$, yields

$$[j]_q^m a_j - b_j + \sum_{l=1}^{j-2} \mathfrak{R}_l^{-1}(b_2, b_3, \ldots b_{l+1})\left(\left([j]_q^m - l\right)a_{j-l} - b_{j-l}\right) = (1-\alpha)c_{j-1}. \tag{9}$$

Evaluating the coefficients of the Equations (6) and (8), for any $j \geq 2$, yields

$$[j]_q^m A_j - B_j + \sum_{l=1}^{j-2} \mathfrak{R}_l^{-1}(B_2, B_3, \ldots B_{l+1})\left(\left([j]_q^m - l\right)A_{j-l} - B_{j-l}\right) = (1-\alpha)d_{j-1}. \tag{10}$$

For special cases where $j = 2$, from (9) and (10), we obtain

$$\begin{aligned}[2]_q^m a_2 - b_2 &= (1-\alpha)c_1, \\ [2]_q^m A_2 - B_2 &= (1-\alpha)d_1. \end{aligned}$$

Solving for a_2 and taking the absolute values and using the Lemma 1, we can obtain

$$|a_2| \leq \frac{2}{[2]_q^m}(2-\alpha).$$

But under the assumption, $2 \leq i \leq j-1$, and $a_i = 0$, respectively, we yield:

$$A_j = -a_j$$

and

$$[j]_q^m a_j - b_j = (1-\alpha)c_{j-1}, \tag{11}$$

$$[j]_q^m a_j - B_j = (1-\alpha)d_{j-1}. \tag{12}$$

By solving Equations (11) and (12) for a_j and determining the absolute values, and by the Carathéodory Lemma 1, we obtain

$$|a_j| \leq \frac{2(1-\alpha)+j}{[j]_q^m},$$

upon noticing that

$$|b_j| \leq j \text{ and } |B_j| \leq j.$$

This completes Theorem 1. □

Theorem 2. *If $f \in C_\Sigma(\alpha, q)$ is given by Equation (1), and if $a_i = 0$, and $2 \leq i \leq j-1$, then*

$$|a_j| \leq \frac{2(1-\alpha)+j}{[j]_q} \quad \textit{for} \quad j \geq 3.$$

For $m = 1$, and $q \to 1-$, we obtain a known corollary in Theorem 1 that was proven in [23].

Corollary 1 ([23])**.** *If $f \in C_\Sigma(\alpha)$ is given by Equation (1), and if $a_i = 0$, and $2 \le i \le j-1$, then*

$$\left|a_j\right| \le 1 + \frac{2(1-\alpha)}{j}, \quad \text{for } j \ge 3.$$

The following theorem for initial coefficients $|a_2|$ and $|a_3|$, as well as $\left|a_3 - a_2^2\right|$ for $C_\Sigma(m, \alpha, q)$ is obtained as a particular case to Theorem 1.

Theorem 3. *If $f \in C_\Sigma(m, \alpha, q)$ is given by Equation (1), then*

$$|a_2| \le \begin{cases} \sqrt{\frac{2(1-\alpha)}{[3]_q^m - [2]_q^m}} & \text{if } 0 \le \alpha < 1 - \frac{\left([2]_q^m - 1\right)^2}{2\left([3]_q^m - [2]_q^m\right)}, \\ \frac{2(1-\alpha)}{[2]_q^m - 1} & \text{if } 1 - \frac{\left([2]_q^m - 1\right)^2}{2\left([3]_q^m - [2]_q^m\right)} \le \alpha < 1, \end{cases}$$

$$|a_3| \le \begin{cases} \frac{2(1-\alpha)\left(2[3]_q^m - [2]_q^m\right)}{\left(2[3]_q^m - [2]_q^m - 2\right)\left([3]_q^m - 1\right)} & \text{if } 0 \le \alpha < 1 - \frac{\left([2]_q^m - 1\right)^2}{2\left([3]_q^m - [2]_q^m\right)}, \\ \frac{2(1-\alpha)}{[3]_q^m - 1}(3 - 2\alpha) & \text{if } 1 - \frac{\left([2]_q^m - 1\right)^2}{2\left([3]_q^m - [2]_q^m\right)} \le \alpha < 1, \end{cases}$$

$$\left|a_3 - a_2^2\right| \le \frac{2(1-\alpha)}{[3]_q^m - 1}.$$

Proof. In the proof of Theorem 1, considering $g(z) = f(z)$, then we obtain $a_j = -b_j$. For $j = 2$ in Equations (9) and (10) we thus yield

$$a_2 = \frac{(1-\alpha)c_1}{[2]_q^m - 1}$$

and

$$-a_2 = \frac{(1-\alpha)d_1}{[2]_q^m - 1}.$$

Taking the absolute values, we obtain

$$|a_2| \le \frac{2(1-\alpha)}{[2]_q^m - 1}.$$

For $j = 3$, in Equations (9) and (10), we have

$$\left([3]_q^m - 1\right)a_3 - \left([2]_q^m - 1\right)a_2^2 = (1-\alpha)c_2 \tag{13}$$

and

$$\left(2a_2^2 - a_3\right)\left([3]_q^m - 1\right) - \left([2]_q^m - 1\right)a_2^2 = (1-\alpha)d_2. \tag{14}$$

By adding Equations (13) and (14) and find $|a_2|$, we arrive at

$$\left|a_2^2\right| = \frac{(1-\alpha)|d_2 + c_2|}{2\left([3]_q^m - [2]_q^m\right)}.$$

By applying the Carathéodory's Lemma 1, we have

$$|a_2| \le \sqrt{\frac{2(1-\alpha)}{[3]_q^m - [2]_q^m}}.$$

As a result, we obtain the estimate

$$\sqrt{\frac{2(1-\alpha)}{[3]_q^m-[2]_q^m}}<\frac{2(1-\alpha)}{[3]_q^m-[2]_q^m}.$$

Reducing Equation (14) to its simplest form, we obtain

$$\left(2[3]_q^m-[2]_q^m-1\right)a_2^2-\left([3]_q^m-1\right)a_3=(1-\alpha)d_2. \tag{15}$$

Multiplying Equation (13) by

$$\frac{2[3]_q^m-[2]_q^m-1}{[2]_q^m-1}$$

and adding it to Equation (15), we obtain

$$|a_3|=\frac{(1-\alpha)}{\left(2[3]_q^m-[2]_q^m-2\right)\left([3]_q^m-1\right)}\left|\left(2[3]_q^m-[2]_q^m-1\right)c_2+d_2\right|.$$

Applying the Carathéodory's Lemma 1, we obtain

$$|a_3|\leq\frac{2(1-\alpha)\left(2[3]_q^m-[2]_q^m\right)}{\left(2[3]_q^m-[2]_q^m-2\right)\left([3]_q^m-1\right)}.$$

Substituting $a_2=\frac{c_1(1-\alpha)}{[2]_q^m-1}$ into Equation (13), we have

$$a_3=\frac{(1-\alpha)}{[3]_q^m-1}\left\{c_2+(1-\alpha)c_1^2\right\}.$$

Taking the modulus and applying the Carathéodory's Lemma 1

$$|a_3|\leq\frac{2(1-\alpha)}{[3]_q^m-1}(3-2\alpha).$$

Lastly, subtracting Equations (13) from (14), we have

$$\left|a_3-a_2^2\right|\leq\frac{2(1-\alpha)}{[3]_q^m-1}.$$

□

Theorem 4. *If $f\in C_\Sigma(\alpha,q)$ is given by Equation (1). Then,*

$$|a_2|\leq\begin{cases}\sqrt{\frac{2(1-\alpha)}{q^2}} & \text{if } 0\leq\alpha<\frac{1}{2},\\ \frac{2(1-\alpha)}{q} & \text{if } \frac{1}{2}\leq\alpha<1,\end{cases}$$

$$|a_3|\leq\begin{cases}\frac{2(1-\alpha)\left(1+q+2q^2\right)}{q(1+q)^2(2q-1)} & \text{if } 0\leq\alpha<\frac{1}{2},\\ \frac{2(1-\alpha)}{q(1+q)}(3-2\alpha) & \text{if } \frac{1}{2}\leq\alpha<1,\end{cases}$$

$$\left|a_3-a_2^2\right|\leq\frac{2(1-\alpha)}{q(1+q)}.$$

In Theorem 3, we obtain the known corollary proven in [23] for $m=1$ and $q\to1-$.

Corollary 2 ([23])**.** *Let $f \in C_\Sigma(\alpha)$ be given by Equation(1). Then,*

$$|a_2| \leq \begin{cases} \sqrt{2(1-\alpha)} & \text{if } 0 \leq \alpha < \frac{1}{2}, \\ 2(1-\alpha) & \text{if } \frac{1}{2} \leq \alpha < 1, \end{cases}$$
$$|a_3| \leq \begin{cases} 2(1-\alpha) & \text{if } 0 \leq \alpha < \frac{1}{2}, \\ (1-\alpha)(3-2\alpha) & \text{if } \frac{1}{2} \leq \alpha < 1. \end{cases}$$

Example 1. *For $j \geq 3$, we will demonstrate that $f(z) = z + \frac{1-\alpha}{j-1} z^j$ is bi-close-to-convex of the order α, $0 \leq \alpha < 1$ in $\mathcal{U}$. For the function $g(z) = z - \frac{1-\alpha}{j-\alpha} z^j$ star-like in $\mathcal{U}$; we have*

$$\begin{aligned} \frac{\mathcal{S}_q^m f(z)}{g(z)} &= \frac{1 + \frac{[j]_q^m (1-\alpha)}{j-\alpha} z^{j-1}}{1 - \frac{(1-\alpha)}{j-\alpha} z^{j-1}} \\ &= 1 + \sum_{k=1}^{\infty} \left(\frac{(1-\alpha)^k}{(j-\alpha)^k} + \frac{[j]_q^m (1-\alpha)^k}{(j-1)(j-\alpha)^{k-1}} \right) z^{(j-1)k}. \end{aligned}$$

Therefore,

$$\begin{aligned} \frac{\frac{\mathcal{S}_q^m f(z)}{g(z)} - \alpha}{1-\alpha} &= 1 + \sum_{k=1}^{\infty} \left(\frac{[j]_q^m j + \left(j - [j]_q^m \alpha \right) - 1}{(j-1)(j-\alpha)} \right) \frac{(1-\alpha)^k}{(j-\alpha)^{k-1}} z^{(j-1)k} \\ &= \frac{\mathfrak{X}_0}{2} + \sum_{k=1}^{\infty} \mathfrak{X}_k z^{(j-1)k}. \end{aligned}$$

Since $\lim_{k \to \infty} \mathfrak{X}_k = 0$*, we note that $\mathfrak{X}_k$ is a convex null sequence and*

$$\mathfrak{X}_0 - \mathfrak{X}_1 \geq \mathfrak{X}_1 - \mathfrak{X}_2 \geq \ldots \geq \mathfrak{X}_k - \mathfrak{X}_{k-1} \geq \ldots \geq 0.$$

Therefore,

$$\mathrm{Re}\left(\frac{\mathcal{S}_q^m f(z)}{g(z)} - \alpha \right) > 0 \text{ in } \mathcal{U}.$$

For $F = f^{-1}$, we have

$$F(w) = w - \frac{1-\alpha}{j-1} w^j$$

and let

$$G(z) = z + \frac{1-\alpha}{j-\alpha} w^j$$

which is star-like in $\mathcal{U}$. As a result, we have

$$\frac{\frac{\mathcal{S}_q^m F(z)}{G(z)} - \alpha}{1-\alpha} = \frac{2}{2} + \sum_{k=1}^{\infty} \left((-1)^k \frac{[j]_q^m j + \left(j - [j]_q^m \alpha \right) - 1}{(j-1)(j-\alpha)} \right) \frac{(1-\alpha)^k}{(j-\alpha)^{k-1}} w^{(j-1)k}.$$

Obviously,

$$\mathrm{Re}\left(\frac{\mathcal{S}_q^m F(z)}{G(z)} - \alpha \right) > 0 \text{ in } \mathcal{U}.$$

Since its coefficients are dominated by the convex null sequence $\mathfrak{X}_k$.

Example 2. *For $j \geq 3$, we will demonstrate that $f(z) = z + \frac{1-\alpha}{j-1} z^j$ is bi-close-to-convex of the order α, $0 \leq \alpha < 1$ in $\mathcal{U}$. For the function $g(z) = z - \frac{1-\alpha}{j-\alpha} z^j$ star-like in $\mathcal{U}$; we have*

$$\begin{aligned}
\frac{zD_q f(z)}{g(z)} &= \frac{1 + \frac{[j]_q(1-\alpha)}{j-\alpha} z^{j-1}}{1 - \frac{(1-\alpha)}{j-\alpha} z^{j-1}} \\
&= 1 + \sum_{k=1}^{\infty} \left(\frac{(1-\alpha)^k}{(j-\alpha)^k} + \frac{[j]_q (1-\alpha)^k}{(j-1)(j-\alpha)^{k-1}} \right) z^{(j-1)k}.
\end{aligned}$$

Therefore,

$$\begin{aligned}
\frac{\frac{zD_q f(z)}{g(z)} - \alpha}{1-\alpha} &= 1 + \sum_{k=1}^{\infty} \left(\frac{[j]_q j + (j - [j]_q \alpha) - 1}{(j-1)(j-\alpha)} \right) \frac{(1-\alpha)^k}{(j-\alpha)^{k-1}} z^{(j-1)k} \\
&= \frac{w_0}{2} + \sum_{k=1}^{\infty} w_k z^{(j-1)k}.
\end{aligned}$$

Since $\lim_{k\to\infty} w_k = 0$, we note that w_k is a convex null sequence and

$$w_0 - w_1 \geq w_1 - w_2 \geq \ldots \geq w_k - w_{k-1} \geq \ldots \geq 0.$$

Therefore,

$$\mathrm{Re}\left(\frac{zD_q f(z)}{g(z)} - \alpha \right) > 0 \text{ in } \mathcal{U}.$$

For $F = f^{-1}$, we have

$$F(w) = w - \frac{1-\alpha}{j-1} w^j$$

and let

$$G(z) = z + \frac{1-\alpha}{j-\alpha} w^j$$

which is star-like in $\mathcal{U}$. As a result, we have

$$\frac{\frac{zD_q f(z)}{g(z)} - \alpha}{1-\alpha} = \frac{2}{2} + \sum_{k=1}^{\infty} \left((-1)^k \frac{[j]_q j + (j - [j]_q \alpha) - 1}{(j-1)(j-\alpha)} \right) \frac{(1-\alpha)^k}{(j-\alpha)^{k-1}} w^{(j-1)k}.$$

Obviously,

$$\mathrm{Re}\left(\frac{zD_q F(z)}{G(z)} - \alpha \right) > 0 \text{ in } \mathcal{U}.$$

Since its coefficients are dominated by the convex null sequence $\mathfrak{X}_k$.

4. Conclusions

In this paper, we used the q-difference operator and the Sălăgean q-differential operator to systematically define subclasses of analytical bi-close-to-convex functions, which were prompted mainly by recent research in GFT. We determined the general Taylor–Maclaurin coefficient of the functions of newly defined classes by using the Faber polynomial method and investigated the unpredictable behaviour of initial coefficients $|a_2|$, $|a_3|$ and Fekete–Szegő problem $\left|a_3 - a_2^2\right|$ of subclasses of bi-close-to-convex functions. In addition, the paper showed how the findings are improved and generalized many interesting corollaries by appropriate specialization of the parameters m and q factors, including some recently published results.

Making use of the definition of q-difference and Sălăgean q-differential operators could inspire researchers to create new, different subclasses of bi-close-to-convex functions.

A number of new subclasses of bi-close-to-convex functions can be defined by using the symmetrical q-calculus operator theory, and the unpredictable behaviour of coefficient bounds can be discussed by using the Faber polynomial expansion technique.

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