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Novelty for Different Prime Partial Bi-Ideals in Non-Commutative Partial Rings and Its Extension

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Abstract: In computer programming languages, partial additive semantics are used. Since partial functions under disjoint-domain sums and functional composition do not constitute a field, linear algebra cannot be applied. A partial ring can be viewed as an algebraic structure that can process natural partial orderings, infinite partial additions, and binary multiplications. In this paper, we introduce the notions of a one-prime partial bi-ideal, a two-prime partial bi-ideal, and a three-prime partial bi-ideal, as well as their extensions to partial rings, in addition to some characteristics of various prime partial bi-ideals. In this paper, we demonstrate that two-prime partial bi-ideal is a generalization of a one-prime partial bi-ideal, and three-prime partial bi-ideal is a generalization of a two-prime partial bi-ideal and a one-prime partial bi-ideal. A discussion of the m_{pb1}, (m_{pb2}, m_{pb3}) systems is presented. In general, the m_{pb2} system is a generalization of the m_{pb1} system, while the m_{pb3} system is a generalization of both m_{pb2} and m_{pb1} systems. If Φ is a prime bi-ideal of $\mho$, then Φ is a one-prime partial bi-ideal (two-prime partial bi-ideal, three-prime partial bi-ideal) if and only if $\mho \setminus \Phi$ is a m_{pb1} system (m_{pb2} system, m_{pb3} system) of $\mho$. If Θ is a prime bi-ideal in the complete partial ring $\mho$ and Δ is an m_{pb3} system of $\mho$ with $\Theta \cap \Delta = \phi$, then there exists a three-prime partial bi-ideal Φ of $\mho$, such that $\Theta \subseteq \Phi$ with $\Phi \cap \Delta = \phi$. These are necessary and sufficient conditions for partial bi-ideal Θ to be a three-prime partial bi-ideal of $\mho$. It is shown that partial bi-ideal Θ is a three-prime partial bi-ideal of $\mho$ if and only if H_Θ is a prime partial ideal of $\mho$. If Θ is a one-prime partial bi-ideal (two-prime partial bi-ideal) in $\mho$, then H_Θ is a prime partial ideal of $\mho$. It is guaranteed that a three-prime partial bi-ideal Φ with a prime bi-ideal Θ does not meet the m_{pb3} system. In order to strengthen our results, examples are provided.

Keywords: partial ring; prime bi-ideal; one-prime partial bi-ideal; two-prime partial bi-ideal; three-prime partial bi-ideal; m_{pb1} system; m_{pb2} system; m_{pb3} system

MSC: 06B10; 20M25; 16Y60

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1. Introduction

Mathematical structures have several applications. It is important to generalize the ideals of algebraic structures and ordered algebraic structures and to make them available for further study and application. Between 1950 and 1980, mathematicians studied bi-ideals, quasi ideals, and interior ideals. During 1950–2019, however, only mathematicians studied their applications. The notions of one-sided ideals of rings and semigroups, as well as the notions of quasi ideals of rings and semigroups, can all be considered generalizations of the notion of ideals of rings and semigroups. Semigroups are generalizations of rings and groups. Semigroup structure can be studied using certain band decompositions in semigroup theory. This research uses bi-ideals of semirings with additively reduced semilattices to open a new area of mathematics. In mathematics, various types of ideals have been discussed in various structures, including semiring [1] and ring [2]. In the theory of algebraic numbers, Dedekind introduced the idea of ideals that included associative rings. In 1952, Good et al. [3] introduced the notion of bi-ideals for semigroups. Furthermore, this is a special case of the (m,n)-ideal discussed by Lajos, i.e., it is a special case of the (m,n)-ideal. Lajos provided both regular and intra-regular semigroups as a result of the bi-ideals [4]. Furthermore, Lajos developed generalized bi-ideals and quasi ideals to analyze regular and intra regular semigroups. As an example, bi-ideals that describe different classes of semigroups [5–7]. Lajos et al. [8] define associative rings in terms of bi-ideals. quasi ideals are generalizations of left ideals and right ideals, and are therefore special cases of bi-ideals. Steinfeld introduced the concept of quasi ideals based on semigroups and rings in 1956. In semirings, prime ideals can be described in a variety of ways [1]. In the theory of commutative rings, the prime ideal has been extensively used. In contrast with commutative rings, its application to non-commutative rings has been less extensive. A few aspects of prime ideals in general rings have been discussed by McCoy [2]. Prime ideals for rings and semirings can be found in [1,9,10]. The concepts of prime bi-ideal and semiprime bi-ideal were introduced by Van der Walt. Specifically, with regard to the subsets X_1 and X_2 of $\mho$ and the product X_1X_2, what we mean is that the subring of $\mho$ is generated by the set of all products x_1x_2, where $x_1 \in X_1, x_2 \in X_2$. In order to define a bi-ideal Θ of $\mho$, we must satisfy the condition $\Theta\mho\Theta \subseteq \Theta$ [8]. If Θ is any bi-ideal of $\mho$, then $\Lambda_\Theta = \{c \in \Theta | \mho c \subseteq \Theta\}$ and $H_\Theta = \{c^{'} \in \Lambda_\Theta | c^{'}\mho \subseteq \Lambda_\Theta\}$ [11]. An ideal Φ of $\mho$ is prime ideal if and only if $\Psi\Theta \subseteq \Phi$, for ideals Ψ and Θ of $\mho$ implies $\Psi \subseteq \Phi$ or $\Theta \subseteq \Phi$ [2]. Palanikumar et al. addressed semigroups, semirings, rings, and ternary semirings in their recent work [12–17]. Recently, Badmaev et al. [18–22] discussed various application for Boolean functions generated by maximal partial ultraclones.

There are several closely related structures that have been introduced in other contexts which have partially additive semantics, such as those in [23–25]. It is suggested that $\sum$ should be emphasized in computing science, according to the flowchart enclosed with [23]. It is possible to find partially defined infinity operations in many contexts. A wide variety of contexts can be found here, ranging from the semantics of programming languages to the integration theory of systems. Computer scientists try to make programs more understandable by changing the function that was computed without changing their function. A program transformation algebraic theory is clearly required to solve this problem [26]. A positive partial monoid can be explained as follows: If $\sum(c_i : i \in Q)$ is defined and equals 0, then each x_i must be zero. An Abelian monoid satisfies the positivity requirement that $c + c^{'} = 0$ implies $c = 0 = c^{'}$. Due to the partition-association property, a subset of the summable families has finite support and a usual sum. Assume that M is the fixed set. Functions are Q indexed families in M. The notation for this function is $c = (c_i | i \in Q)$. Here, we have used c_i instead of $c(i)$. Instead of making the co-domain explicit, as with the function notation $c : Q \to M$, the family notation suppresses it. As a result, family notation is useful when there is a stable co-domain relationship. In technical terms, "meaning" can be called "semantics". Programming languages use semantics to explain what programs written in a given language mean when they are run. A semantic function is a function whose input is a syntactically correct program. The output is a description of the function calculated by the program based on the input. The partial addition that will be automated for some Q-indexed families in M is expected to include an element of $\sum_i(c_i | i \in Q)$. In the semantic concepts we want to capture, there are no uncountable sums involved, so we can only deal with countable families. Using an axiom, I will demonstrate the ineffectiveness of subdividing a sum based on an axiom.

For example, $c_1 + c_2 + c_3 + c_4 + c_5 + c_6 = c_2 + (c_1 + c_5) + (c_3 + c_4) + c_6$. If $Q = 1, 2, ..., 6$, $Q_a = \{2, 6\}$, $Q_b = \{1, 5\}$, $Q_c = \{3, 4\}$ and $R = \{a, b, c\}$, we may write this result as $\sum(c_i : i \in Q) = \sum(\sum(c_i : i \in Q_j) : j \in R)$. Here, $(Q_j | j \in R)$ is a partition of Q; that is, if $j \neq k$ then $Q_j \cap Q_k = \phi$ and also $Q = \cup(Q_j : j \in R)$. In our definition of $Q_j = \phi$, we would like to emphasize that any number of j (even an infinite number of j) is appropriate if the partition properties are true. Various ideals of partial semirings and gamma partial semirings are discussed by Rao et al. [27–29]. Amarendra Babu et al. [30] discussed the bi-ideals of sum ordered partial semirings. Partial addition and a ternary product-based so-semiring is discussed by Bhagyalakshmi et al. [31]. The theory of partial semirings of continuous valued functions is explained by Shalaginova et al. [32]. Throughout this paper, there are five sections that are organized differently. The following Section 2 contains some basic definitions that need to be briefly explained. The different types of prime partial bi-ideals and their extensions are discussed in Section 3. The partial semiprime bi-ideals are discussed in Section 4. The conclusion is drawn in Section 5. In this study, we aim to achieve the following fundamental goals:

1. A one-prime partial bi-ideal implies a two-prime partial bi-ideal which implies a three-prime partial bi-ideal, but the reverse implication does not hold.
2. The m_{pb1} system implies the m_{pb2} system, which implies m_{pb3} system, and the opposite direction does hold with the Example.
3. A one-partial semiprime bi-ideal implies a two-partial semiprime bi-ideal, which implies a three-partial semiprime bi-ideal, and the reverse implication does not match up.
4. The n_{pb1} system implies the n_{pb2} system, which implies the n_{pb3} system, and the opposite direction is not valid based on the Example.

2. Preliminaries

We discuss some of the basic definitions required for the rest of our study in this section.

Definition 1 ([2])**.** *The algebraic structure $(R, +, \cdot)$ is a ring if*

1. *$(R, +)$ is an abelian group.*
2. *$(R, \cdot)$ is a semigroup.*
3. *(i) $a \cdot (b + c) = a \cdot b + a \cdot c$.*
   (ii) $(a + b) \cdot c = a \cdot c + b \cdot c,\ \forall\ a, b, c \in R$.

Definition 2 ([2])**.** *The algebraic structure $(R, +, \cdot)$ is a ring. Then*

1. *An additive subgroup I of R with the property that $ra \in I (ar \in I)$, for $a \in I,\ r \in R$ is called a left ideal (right ideal) of R.*
2. *If I happens to be both a left ideal and right ideal, then we call I a two-sided ideal of R, or simply an ideal of R.*
3. *The quasi ideal Q of a ring R is a subring Q of R, such that $RQ \cap QR \subseteq Q$.*
4. *The bi-ideal B of a ring R is a subring B of R, satisfying $BRB \subseteq B$.*
5. *An ideal P in a ring R is said to be a prime ideal if and only if it has the following property: if A and B are ideals in R such that $A \cdot B \subseteq P$, then $A \subseteq P$ or $B \subseteq P$.*

Definition 3 ([2])**.** *A set M of elements of a ring R is said to be an m system if $a, b \in M$ there exists $x \in R$, such that $axb \in M$.*

Definition 4. *A partial monoid is a pair $(M, \sum)$, $\sum$ is a partial addition defined on some, but not necessarily all families $(c_i : i \in Q)$ in a non-empty set M subject to the conditions:*
(i) If $(c_i : i \in Q)$ in M and $Q = \{j\}$, then $\sum(c_i : i \in Q)$ is defined and equals c_j (unary sum axiom).
(ii) If $(c_i : i \in Q)$ in M and $(Q_j : j \in R)$ is a partition of Q, then $(c_i : i \in Q)$ is summable if and only if $(c_i : i \in Q_j)$ is summable for all $j \in R$, $(\sum(c_i : i \in Q_j) : j \in R)$ is summable and $\sum(c_i : i \in Q) = \sum(\sum(c_i : i \in Q_j) : j \in R)$ (the partition associativity axiom).

Definition 5. *Let $\mho$ be the partial semiring if there exists a mapping $\mho \times \mho \to \mho$ which satisfies the following axioms:*
(i) $c_1(c_2c_3) = (c_1c_2)c_3$,
(ii) $(c_i : i \in Q)$ is summable in $\mho$ implies $(c\, c_i : i \in Q)$ is summable in $\mho$ and $c[\sum(c_i : i \in Q)] = \sum(cc_i : i \in Q)$.

(iii) $(c_i : i \in Q)$ *is summable in* $\mho$ *implies* $(c_i\, c : i \in Q)$ *is summable in* $\mho$ *and* $[\sum(c_i : i \in Q]c = \sum(c_i c : i \in Q)$.

Definition 6. *Let* Ψ *be the nonempty subset of* $\mho$. *Then,* Ψ *is said to be a partial left ideal (partial right ideal) of* $\mho$ *if*
(i) $(c_i : i \in Q)$ *is summable in* $\mho$ *and* $c_i \in \Psi,\ \ \forall i \in Q$ *implies* $\sum_i c_i \in \Psi$.
(ii) For all $c \in \mho,\ c' \in \Psi, cc' \in \Psi\ (c'c \in \Psi)$.

Definition 7. *A partial ring is said to be complete if every family within it can be summed up.*

Remark 1. *Let* $\mho$ *be the complete partial ring and* $b \in \mho$. *The right ideal and left ideal of* $\mho$ *are generated by "b"*
(i) $< b >_r = \{c \in \mho | c = nb + \sum_i br_i,\ \ r_i \in \mho, n \in \mathbb{N}\}$.
(ii) $< b >_l = \{c \in \mho | c = nb + \sum_i r_i b,\ \ r_i \in \mho, n \in \mathbb{N}\}$.
(iii) $< b > = \{c \in \mho | c = nb + \sum_i br_i + \sum_j r_j b + \sum_k br_k b,\ \ r_i, r_j, r_k \in \mho, n \in \mathbb{N}\}$.

Definition 8. *Let* $\mho$ *be the partial rings and let* Ψ *and* Θ *be the subsets of* $\mho$. *We define* $\Psi\Theta = \{c \in \mho | c = \sum_i l_i m_i, l_i \in \Psi\ \ and\ \ m_i \in \Theta\}$.

Definition 9 ([11]). *(i) The bi-ideal* Θ *of* $\mho$ *is the prime bi-ideal if* $u_1 \mho u_2 \subseteq \Theta$ *implies* $u_1 \in \Theta$ *or* $u_2 \in \Theta$.
(ii) The bi-ideal Θ *of* $\mho$ *is a semiprime bi-ideal if* $u_1 \mho u_1 \subseteq \Theta$ *implies* $u_1 \in \Theta$.

Theorem 1 ([11]). *(i) The bi-ideal* Θ *of* $\mho$ *is a prime bi-ideal if and only if* $\Psi_1\Psi_2 \subseteq \Theta$, *while* Ψ_1 *as a right ideal of* $\mho$ *and* Ψ_2 *as a left ideal of* $\mho$ *implies* $\Psi_1 \subseteq \Theta$ *or* $\Psi_2 \subseteq \Theta$.
(ii) The bi-ideal Θ *of* $\mho$ *is semiprime bi-ideal if and only if* $\Psi_1^2 \subseteq \Theta$ *(or* $\Psi_2^2 \subseteq \Theta$*) implies* $\Psi_1 \subseteq \Theta$ *(or* $\Psi_2 \subseteq \Theta$*) for any left ideal* Ψ_1 *(or right ideal* Ψ_2*) of* $\mho$.

3. Different Prime Partial Bi-Ideals

In this section, three different prime partial bi-ideals and their corresponding partial m systems were introduced.

Definition 10. *(i) A proper prime bi-ideal* Φ *of* $\mho$ *is called a one-prime partial bi-ideal if* $\Theta_1\Theta_2 \subseteq \Phi$ *implies* $\Theta_1 \subseteq \Phi$ *or* $\Theta_2 \subseteq \Phi$ *for any prime bi-ideals* Θ_1 *and* Θ_2 *of* $\mho$.
(ii) A two-prime partial bi-ideal if $a_1 \mho a_2 \subseteq \Phi$ *implies* $a_1 \in \Phi$ $a_2 \in \Phi$.
(iii) A three-prime partial bi-ideal if $\Upsilon_1\Upsilon_2 \subseteq \Phi$ *implies* $\Upsilon_1 \subseteq \Phi$ *or* $\Upsilon_2 \subseteq \Phi$ *for any prime ideals* Υ_1 *and* Υ_2 *of* $\mho$.

Example 1. *Consider* $\mho = \left\{ \begin{pmatrix} a_1 & a_2 \\ 0 & a_3 \end{pmatrix} \middle| a_1, a_2, a_3 \in \mathbb{Z}_2 \right\}$ *with "*$\sum$*" is defined on* $\mho$ *by*

$$\sum_i (c_i : i \in Q) = \begin{cases} \sum_i c_i & if\ \ c_i \in Q\ isfinite \\ \\ undefined & otherwise \end{cases}$$

and " $\cdot$ *" is defined by the usual multiplication.*

Example 2. *Consider* $\mho = \left\{ \begin{pmatrix} a_1 & a_2 \\ a_3 & a_4 \end{pmatrix} \middle| a_1, a_2, a_3, a_4 \in \mathbb{Z}_2 \right\}$ *with "*$\sum$*" is defined on* $\mho$ *by*

$$\sum_i (c_i : i \in Q) = \begin{cases} \sum_i c_i & if\ \ c_i \in Q\ isfinite \\ \\ undefined & otherwise \end{cases}$$

and "$\cdot$" is defined by the usual multiplication.

Remark 2. *Every left ideal, right ideal and bi-ideal are a partial left ideal, partial right ideal and partial bi-ideal of $\mho$.*

Proof. Straightforward. □

The converse of the Remark 2 cannot be proved by the following example.

Example 3. *In Example 1, (i)* $\Psi_1 = \left\{ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right\}$ *is a partial right ideal of $\mho$.*
Since $\Psi_1 + \Psi_1 = \left\{ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix} \right\} \nsubseteq \Psi_1$ *implies Ψ_1 is not a right ideal of $\mho$.*
(ii) $\Lambda_1 = \left\{ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \right\}$ *is a partial left ideal of $\mho$.*
Since $\Lambda_1 + \Lambda_1 = \left\{ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} \right\} \nsubseteq \Lambda_1$ *implies Λ_1 is not a left ideal of $\mho$.*

Example 4. *In Example 2, (iii)* $\Upsilon = \left\{ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \right\}$ *is a partial ideal of $\mho$. Since $\Upsilon + \Upsilon = \mho \nsubseteq \Upsilon$ implies Υ is not an ideal of $\mho$.*
(iv) $\Theta = \left\{ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} \right\}$ *is a partial bi-ideal of $\mho$.*
Since $\Theta + \Theta = \left\{ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} \right\} \nsubseteq \Theta$ *implies Θ is not a bi-ideal of $\mho$.*

Lemma 1. *If Φ is a one-prime partial bi-ideal of $\mho$, then Φ is a two-prime partial bi-ideal of $\mho$.*

Proof. Straightforward. □

In the example below, we can see that the converse of the Lemma 1 is not true.

Example 5. *In Example 2,*
$\Phi = \left\{ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \right\}$, $\Theta_1 = \left\{ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix} \right\}$ *and*
$\Theta_2 = \left\{ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix} \right\}$. *Here, Φ is a two-prime partial bi-ideal, but not a one-prime partial bi-ideal. Now,* $\Theta_1 \cdot \Theta_2 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \subseteq \Phi$, *but* $\Theta_1 \nsubseteq \Phi$ *and* $\Theta_2 \nsubseteq \Phi$.

Lemma 2. *If Φ is a two-prime partial bi-ideal of $\mho$, then Φ is a three-prime partial bi-ideal of $\mho$.*

Proof. Straightforward. □

We cannot prove the converse of Lemma 2 based on the following example.

Example 6. *Based on the Example 2,* $\Phi = \left\{ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix} \right\}$. *Here,* Φ *is a three-prime partial bi-ideal, but not a two-prime partial bi-ideal. Now,* $\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \mho \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = \left\{ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix} \right\} \subseteq \Phi$, *but* $\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \nsubseteq \Phi$ *and* $\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \nsubseteq \Phi$.

Definition 11. *(i) A subset* Δ *of* $\mho$ *is called a* m_{pb1} *system if, for any* $a_1, b_1 \in \Delta$, *there exists* $a_1^1 \in < a_1 >_b$ *and* $b_1^1 < b_1 >_b$, *such that* $a_1^1 b_1^1 \in \Delta$.
(ii) A subset Δ *of* $\mho$ *is called a* m_{pb2} *system if, for any* $a_1, b_1 \in \Delta$, *there exists* $a_1^1 \in < a_1 >_r$ *and* $b_1^1 \in < b_1 >_l$, *such that* $a_1^1 b_1^1 \in \Delta$.
(iii) A subset Δ *of* $\mho$ *is called a* m_{pb3} *system if, for any* $a_1, b_1 \in \Delta$, *there exists* $a_1^1 \in < a_1 >$ *and* $b_1^1 \in < b_1 >$ *such that* $a_1^1 b_1^1 \in \Delta$.

Lemma 3. *If* Φ *is a partial bi-ideal of* $\mho$, *then* Φ *is a one-prime partial bi-ideal (two-prime partial bi-ideal, three-prime partial bi-ideal) if and only if* $\mho \setminus \Phi$ *is a* m_{pb1} *system (*m_{pb2} *system,* m_{pb3} *system) of* $\mho$.

Proof. Let Φ be the one-prime partial bi-ideal of $\mho$ and let $a_1, b_1 \in \mho \setminus \Phi$. Hence, $< a_1 >_b < b_1 >_b \nsubseteq \Phi$. Then there exists $a_1^{'} \in < a_1 >_b$ and $b_1^{'} \in < b_1 >_b$, such that $a_1^{'} \cdot b_1^{'} = \{n_1 a_1 + n_2 a_1^2 + l_1\} \cdot \{n_1^{'} b_1 + n_2^{'} b_1^2 + l_2\}$, where $l_1 = \sum_i a_1 r_i a_1$ and $l_2 = \sum_i b_1 r_i^{'} b_1$ for $n_1, n_2, n_1^{'}, n_2^{'} \in \mathbb{N}$ and $r_i, r_i^{'} \in \mho$. Since $l_1 l_2 \in < a_1 >_b < b_1 >_b \nsubseteq \Phi$. Thus, $a_1 b_1 \notin \Phi$. Hence, $\mho \setminus \Phi$ is a m_{pb1} system.

Conversely, let Θ_1 and Θ_2 be the partial bi-ideals of $\mho$, such that $\Theta_1 \Theta_2 \subseteq \Phi$. Suppose that $\Theta_1 \nsubseteq \Phi$ and $\Theta_2 \nsubseteq \Phi$. Then, there exists $b_1 \in \Theta_1$ and $b_2 \in \Theta_2$, such that $b_1 \notin \Phi$ and $b_2 \notin \Phi$. Let $b_1, b_2 \in \mho \setminus \Phi$. Since $\mho \setminus \Phi$ is an m_{pb1} system, then there exists $b_1^{'} \in < b_1 >_b$ and $b_2^{'} \in < b_2 >_b$ such that $b_1^{'} b_2^{'} \in \mho \setminus \Phi$. However, $b_1^{'} b_2^{'} \in < b_1 >_b \cdot < b_2 >_b \subseteq \Theta_1 \Theta_2 \subseteq \Phi$, which is a contradiction. Thus, $\Theta_1 \subseteq \Phi$ or $\Theta_2 \subseteq \Phi$. Hence, Φ is a one-prime partial bi-ideal of $\mho$. □

Remark 3. *Every* m_{pb1} *system is a* m_{pb2} *system.*

As illustrated by the following example, the converse may not be true.

Example 7. *In Example 2, Clearly* $\Delta = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \right\}$ *is an* m_{pb2} *system, but not a* m_{pb1} *system.*
Put $z_1 = \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}, z_2 = \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix}$. *Now,* $z_1^{'} \in \langle z_1 \rangle_b$ *and* $z_2^{'} \in \langle z_2 \rangle_b$ *such that* $z_1^{'} \cdot z_2^{'} \notin \Delta$.

Remark 4. *Every* m_{pb2} *system is a* m_{pb3} *system.*

In the following example, however, the converse may not be true.

Example 8. *In Example 2, clearly,* $\Delta = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \right\}$ *is an* m_{pb3} *system, but not a* m_{pb2} *system.*

Let $z_1 = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, z_2 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$. *Now,* $z_1' \in \langle z_1 \rangle_r$ *and* $z_2' \in \langle z_2 \rangle_l$, *such that* $z_1' \cdot z_2' = \left\{ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix} \right\} \notin \Delta$.

Remark 5. *For any bi-ideal* Θ *of* $\mho$, $\Lambda_\Theta = \{c \in \Theta \mid \mho c \subseteq \Theta\}$ *and* H_Θ *is defined as* $H_\Theta = \{c' \in \Lambda_\Theta \mid c'\mho \subseteq \Lambda_\Theta\}$.

Remark 6. *For any bi-ideal* Θ *of* $\mho$, $\Psi_\Theta = \{c \in \Theta \mid c\mho \subseteq \Theta\}$ *and* H_Θ *is defined as* $H_\Theta = \{c' \in \Psi_\Theta \mid \mho c' \subseteq \Psi_\Theta\}$.

Lemma 4. *Let* Θ *be the partial bi-ideal of* $\mho$. *Then,* Λ_Θ *is a partial left ideal of* $\mho$, *such that* $\Lambda_\Theta \subseteq \Theta$.

Proof. Let $c_i \in \Lambda_\Theta$. Then, $c_i \in \Theta$ and $\mho c_i \subseteq \Theta$, $\forall i$. Since Θ is a partial bi-ideal of $\mho$, then $\sum_i c_i \in \Theta$ and $(c_1 \cdot c_2 \cdot ... \cdot c_n) \in \Theta$. Now, $\mho(\sum_i c_i) \subseteq \Theta$. Thus, $\sum_i c_i \in \Lambda_\Theta$. Now, $\mho(c_1 \cdot c_2 \cdot ... \cdot c_n) = (\mho c_1) \cdot (c_2 \cdot ... \cdot c_n) \subseteq (\mho c_1) \cdot (\mho c_2) \cdot (c_3 ... \cdot c_n) \subseteq (\mho c_1) \cdot (\mho c_2) \cdot ... \cdot (\mho c_n) \subseteq \Theta$. Thus, $(c_1 \cdot c_2 \cdot ... \cdot c_n) \in \Lambda_\Theta$. Let $c \in \Lambda_\Theta$ and $r \in \mho$. Since $rc \in \mho c \subseteq \Theta$, we have $rc \in \Theta$ and $\mho rc \subseteq \mho\mho c \subseteq \mho c \subseteq \Theta$. Thus, $rc \in \Lambda_\Theta$. Hence Λ_Θ is a partial left ideal of $\mho$ and $\Lambda_\Theta \subseteq \Theta$. □

Lemma 5. *Let* Θ *be the partial bi-ideal of* $\mho$. *Then* H_Θ *is a partial subring of* $\mho$.

Proof. Let $c_i \in H_\Theta$. Then, $c_i \in \Lambda_\Theta$ and $c_i\mho \subseteq \Lambda_\Theta$, $\forall i$. Since $c_i \in \Lambda_\Theta$, $c_i \in \Theta$ and $\mho c_i \subseteq \Theta$, $\forall i$. Since $c_i \in \Theta$ and Θ is a partial subring of $\mho$, we have $\sum_i c_i \in \Theta$ and $(c_1 \cdot c_2 \cdot \ldots \cdot c_n) \in \Theta$. Now, $\mho(\sum_i c_i) \subseteq \Theta$ implies $\sum_i c_i \in \Lambda_\Theta$. Now, $(\sum_i c_i)\mho \subseteq \Lambda_\Theta$ implies $\sum_i c_i \in H_\Theta$. Now, $\mho(c_1 \cdot c_2 \cdot \ldots \cdot c_n) = (\mho c_1) \cdot (c_2 \cdot \ldots \cdot c_n) \subseteq (\mho c_1) \cdot (\mho c_2) \cdot (c_3 \ldots c_n) \subseteq (\mho c_1) \cdot (\mho c_2) \cdot \ldots \cdot (\mho c_n) \subseteq \Theta$ implies $(c_1 \cdot c_2 \cdot \ldots \cdot c_n) \in \Lambda_\Theta$ and $(c_1 \cdot c_2 \cdot \ldots \cdot c_n)\mho = (c_1 \cdot c_2 \cdot \ldots \cdot c_{n-1}) \cdot (c_n\mho) \subseteq (c_1\mho) \cdot (c_2\mho) \cdot \ldots \cdot (c_n\mho) \subseteq \Lambda_\Theta$. Thus, $(c_1 \cdot c_2 \cdot \ldots \cdot c_n) \in H_\Theta$. Hence, H_Θ is a partial subring of $\mho$. □

Lemma 6. *Let* Θ *be the partial left ideal of* $\mho$. *Then,* $\Lambda_\Theta = \Theta$.

Proof. Clearly, $\Lambda_\Theta \subseteq \Theta$. Let $c \in \Theta$, since Θ is a partial left ideal of $\mho$. We have $\mho c \subseteq \Theta$ implies $c \in \Lambda_\Theta$. Thus, $\Theta \subseteq \Lambda_\Theta$. Hence, $\Lambda_\Theta = \Theta$. □

Theorem 2. *If* Θ *is any partial bi-ideal of a complete partial ring* $\mho$, *then* H_Θ *is the unique largest two sided partial ideal of* $\mho$ *contained in* Θ.

Proof. Let Θ be the any partial bi-ideal of $\mho$. First, we prove that H_Θ is the two sided partial ideal of $\mho$. Since $\Lambda_\Theta \subseteq \Theta$ and $H_\Theta \subseteq \Lambda_\Theta$, $H_\Theta \subseteq \Lambda_\Theta \subseteq \Theta$. Let $c_i \in H_\Theta, \forall i \in \Upsilon$ and $c' \in \mho$. Then, $c_i \in H_\Theta \subseteq \Theta \implies c_i \in \Theta$. Since $c_i \in \Lambda_\Theta$, we have $\mho c_i \subseteq \Theta$ and $c_i\mho \subseteq \Lambda_\Theta$, $\forall i \in \Upsilon$. Since Θ is a partial bi-ideal of $\mho$, then $\sum_i c_i \in \Theta$. Since $c_i \in \Lambda_\Theta$, $\sum_i c_i \in \Lambda_\Theta$, $\mho(\sum_i c_i) \subseteq \Lambda_\Theta \subseteq \Theta$ and $(\sum_i c_i)\mho \subseteq \Lambda_\Theta$. Hence, $\sum_i c_i \in H_\Theta$. Since $c \in \Lambda_\Theta$, then $c'c \in \mho c \subseteq \Theta$ and $\mho c'c \subseteq \mho\mho c \subseteq \mho c \subseteq \Theta \implies c'c \in \Lambda_\Theta$. Moreover, $cc' \in c\mho \subseteq \Lambda_\Theta$. Therefore, $cc' \in \Lambda_\Theta$ and $c'c \in \Lambda_\Theta$. To prove that $cc' \in H_\Theta$ and $c'c \in H_\Theta$. Now, $cc'\mho \subseteq c\mho\mho \subseteq c\mho \subseteq \Lambda_\Theta \implies cc' \in H_\Theta$. Moreover, $c'c\mho \subseteq \mho c\mho \subseteq \mho\Lambda_\Theta \subseteq \Lambda_\Theta \implies c'c \in H_\Theta$, since Λ_Θ is a partial left ideal of $\mho$. Hence, H_Θ is a two-sided partial ideal of $\mho$. To prove H_Θ is the largest two-sided partial ideal of $\mho$l let Υ be any partial ideal of $\mho$ and $\Upsilon \subseteq \Theta$. Let $i \in \Upsilon$. Then, $i \in \Theta$ and $\mho i \subseteq \Upsilon \subseteq \Theta$. Hence, $\mho i \subseteq \Theta \implies i \in \Lambda_\Theta$. Hence, $\Upsilon \subseteq \Lambda_\Theta$. Next, $i \in \Lambda_\Theta$ and $i\mho \subseteq \Upsilon \subseteq \Lambda_\Theta \implies i \in H_\Theta$. Hence, $\Upsilon \subseteq H_\Theta$. □

Theorem 3. *Let* Θ *be any partial bi-ideal of a complete partial ring* $\mho$. *If* Θ *is a one-prime partial bi-ideal (two-prime partial bi-ideal) of* $\mho$, *then* H_Θ*is a prime partial ideal of* $\mho$.

Next, we provide an example showing that the converse of Theorem 3 does not hold.

Example 9. *In Example 2,* $H_{\Theta_1} = H_{\Theta_2} = \left\{ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \right\}$ *is a prime partial ideal.*

Let $\Theta_1 = \left\{ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix} \right\}$ *and* $\Theta_2 = \left\{ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \right\}$, Θ_1 *is not a one-prime partial bi-ideal and* Θ_2 *is not a two-prime partial bi-ideal of* $\mho$*. Since*
$\left\{ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \right\} \cdot \left\{ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} \right\} = \left\{ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \right\} \subseteq \Theta_1$ *and*
$\left\{ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} \right\} \mho \left\{ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix} \right\} = \left\{ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \right\} \subseteq \Theta_2.$

Theorem 4. *Let* Θ *be any partial bi-ideal of a complete partial ring* $\mho$*.* Θ *is a three-prime partial bi-ideal of* $\mho$ *if and only if* H_Θ *is a prime partial ideal of* $\mho$*.*

Proof. Let Θ be a three-prime partial bi-ideal of $\mho$. Let C and $C^{'}$ be partial ideals of $\mho$, such that $CC^{'} \subseteq H_\Theta$. Since $H_\Theta \subseteq \Theta$, $CC^{'} \subseteq \Theta$. Since Θ is an three-prime partial bi-ideal of $\mho$, $C \subseteq \Theta$ or $C^{'} \subseteq \Theta$. By Theorem 2, H_Θ is the unique largest partial ideal of $\mho$, such that $H_\Theta \subseteq \Theta$. Thus, $C \subseteq H_\Theta$ or $C^{'} \subseteq H_\Theta$. Hence, H_Θ is a prime partial ideal of $\mho$.

Conversely, let H_Θ be the prime partial ideal of $\mho$. Suppose that $\Upsilon_1\Upsilon_2 \subseteq \Theta$, for any partial ideals Υ_1 and Υ_2 of $\mho$, since $H_\Theta \subseteq \Theta$ and Υ_1, Υ_2 are partial ideals of $\mho$. Hence, $\Upsilon_1\Upsilon_2 \subseteq H_\Theta$, since H_Θ is a prime partial ideal of $\mho$. Hence $\Upsilon_1 \subseteq H_\Theta$ or $\Upsilon_2 \subseteq H_\Theta$, since $H_\Theta \subseteq \Theta$. Thus, $\Upsilon_1 \subseteq \Theta$ or $\Upsilon_2 \subseteq \Theta$. Hence, Θ is a three-prime partial bi-ideal of $\mho$. □

Theorem 5. *Let* Δ *be the* m_{pb3} *system and* Θ *be the prime bi-ideal of a complete partial ring* $\mho$ *with* $\Theta \cap \Delta = \phi$*. Then, there exists a three-prime partial bi-ideal* Φ *of* $\mho$ *containing* Θ *with* $\Phi \cap \Delta = \phi$*.*

Proof. Let $\mathbb{X} = \{R | R$ be a partial bi-ideal with $\Theta \subseteq R$ and $R \cap \Delta = \phi\}$. Clearly $\mathbb{X} \neq \{\phi\}$ and is an ideal of $\mho$. According to Zorn's lemma, $\mathbb{X}$ contains a maximal element Φ with X and $\Phi \cap \Delta = \{\phi\}$. Let us show that Φ is a three-prime partial bi-ideal. Using Theorem 4, we show that H_Φ is a prime partial ideal in $\mho$. Since $H_\Phi \subseteq \Phi$ and $\Phi \cap \Delta = \{\phi\}$, this implies that $H_\Phi \cap \Delta = \{\phi\}$.

Case-(i) Suppose that H_Φ is a maximal prime ideal, such that $H_\Phi \cap \Delta = \{\phi\}$. Suppose $< a >< b > \subseteq H_\Phi$. This implies that $< a > \subseteq H_\Phi$ or $< b > \subseteq H_\Phi$. Suppose that $< a > \nsubseteq H_\Phi$ and $< b > \nsubseteq H_\Phi$. Let us show that $< a >< b > \nsubseteq H_\Phi$. Since $< a > \nsubseteq H_\Phi$, hence $c \in < a >$ but $c \notin H_\Phi$ and $< b > \nsubseteq H_\Phi$, hence $c^{'} \in < b >$ but $c^{'} \notin H_\Phi$. Then, $< c > \subseteq < a >$ and $< c^{'} > \subseteq < b >$. Based on the maximal property of Φ, $(H_\Phi + < c >) \cap \Delta \neq \phi$ and $(H_\Phi + < c^{'} >) \cap \Delta \neq \phi$. Since Δ is an m_{pb3} system for $z_1, z_2 \in \Delta$, then there exist $z_1 \in (H_\Phi + < c >) \cap \Delta$ and $z_2 \in (H_\Phi + < c^{'} >) \cap \Delta$, such that $z_1^{'} z_2^{'} \in < z_1 >< z_2 >$. If $z_1 \in (H_\Phi + < c >)$, then $z_1 = l^{'} + l_1$ for some $l^{'} \in H_\Phi$ and $l_1 \in < c >$. If $z_2 \in (H_\Phi + < c^{'} >)$, then $z_2 = l^{''} + l_2$ for some $l^{''} \in H_\Phi$ and $l_2 \in < c^{'} >$. Now, $z_1{'}z_2{'} \in (l^{'} + l_1)(l^{''} + l_2) = l^{'}l^{''} + l^{'}l_2 + l_1 l^{''} + l_1 l_2 \in H_\Phi + < a >< b >$. If $< a >< b > \subseteq H_\Phi$, then $z_1^{'} z_2^{'} \in H_\Phi$. So $H_\Phi \cap \Delta \neq \phi$, which is a contradiction. Hence, $< a >< b > \nsubseteq H_\Phi$. Hence, H_Φ is a prime partial ideal of $\mho$.

Case-(ii) If H_Φ is not a maximal prime ideal, then there exists a prime partial ideal Υ, such that $\Upsilon \subseteq \Phi \cap \Delta = \phi$, to apply case-(i), we get the proof. □

4. Different Partial Semiprime Bi-Ideals

Throughout this section, we will introduce three different partial semiprime bi-ideals and their corresponding partial n systems.

Definition 12. *(i) A proper prime bi-ideal Φ of $\mho$ is called a one-partial semiprime bi-ideal if $\Theta^2 \subseteq \Phi$ implies $\Theta \subseteq \Phi$, for any prime bi-ideal Θ of $\mho$.*
(ii) It is called a two-partial semiprime bi-ideal if $a_1 \mho a_1 \subseteq \Phi$ implies $a_1 \in \Phi$.
(iii) It is called a three-partial semiprime bi-ideal if $\Upsilon^2 \subseteq \Phi$ implies $\Upsilon \subseteq \Phi$, for any ideal Υ of $\mho$.

Lemma 7. *If Φ is a one-partial semiprime bi-ideal of $\mho$, then Φ is a two-partial semiprime bi-ideal of $\mho$.*

The following example suggests that Lemma 7 could not have a converse.

Example 10. *In Example 2, $\Theta = \left\{ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \right\}$ and $\Phi = \left\{ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right\}$. Here, Φ is a two-partial semiprime bi-ideal, but not a one-partial semiprime bi-ideal by $\Theta \cdot \Theta \subseteq \Phi$, but $\Theta \nsubseteq \Phi$.*

Lemma 8. *If Φ is a two-partial semiprime bi-ideal of $\mho$, then Φ is a three-partial semiprime bi-ideal of $\mho$.*

In the following example, the converse of Lemma 8 may not be true.

Example 11. *In Example 2, $\Phi = \left\{ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \right\}$ is a three-partial semiprime bi-ideal, but not a two-prime partial bi-ideal. Since $\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \mho \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} = \left\{ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix} \right\} \subseteq \Phi$, but $\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \nsubseteq \Phi$ and $\begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} \nsubseteq \Phi$.*

Definition 13. *(i) A subset N of $\mho$ is called a n_{pb1} system if, for any $a_1 \in N$, there exist $a_1^{'}, a_1^{''} \in < a_1 >_b$ such that $a_1^{'} a_1^{''} \in N$.*
(ii) A subset N of $\mho$ is called a n_{pb2} system if for any $a_1 \in N$, there exist $a_1^{'}, a_1^{''} \in < a_1 >_r$ ($a_1^{'}, a_1^{''} \in < a_1 >_l$) such that $a_1^{'} a_1^{''} \in N$.
(iii) A subset N of $\mho$ is called a n_{pb3} system if for any $a_1 \in N$, there exist $a_1^{'}, a_1^{''} \in < a_1 >$ such that $a_1^{'} a_1^{''} \in N$.

Lemma 9. *If Φ is a partial bi-ideal of $\mho$, then Φ is a one-partial semiprime bi-ideal, two-partial semiprime bi-ideal and three-partial semiprime bi-ideal if and only if $\mho \setminus \Phi$ is an n_{pb1} system (n_{pb2} system, n_{pb3} system).*

Remark 7. *Every n_{pb1} system is a n_{pb2} system.*

From the following example, it can be seen that the converse may not be true.

Example 12. *In Example 2, Clearly, $N = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \right\}$ is a n_{pb2} system, but not a n_{pb1} system.*

Theorem 6. *Let Θ be any partial bi-ideal of $\mho$. If Θ is a one-partial semiprime bi-ideal (two-partial semiprime bi-ideal) of $\mho$, then H_Θ is a partial semiprime ideal of $\mho$.*

In the following example, the converse of Theorem 6 might not be true.

Example 13. *In Example 2,* $H_\Theta = \left\{ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \right\}$ *is a partial semiprime ideal but not a one-partial semiprime bi-ideal. For the prime bi-ideal* $\Theta = \left\{ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \right\}$*. Since* $\Theta \cdot \Theta \subseteq H_\Theta$*.*

Theorem 7. *Let* Θ *be any partial bi-ideal of* $\mho$*. If* Θ *is a three-partial semiprime bi-ideal of* $\mho$ *if and only if* H_Θ *is a partial semiprime ideal of* $\mho$*.*

5. Conclusions

In this paper, we introduce several prime partial bi-ideals of partial rings and identify prime partial bi-ideals and partial semiprime bi-ideals. We also introduced and characterized three partial *m* systems. Every one-prime partial bi-ideal is a two-prime partial bi-ideal and every two-prime partial bi-ideal is a three-prime partial bi-ideal. As far as examples are concerned, the reverse is not true. A further development of semirings, ternary semirings and hyper semirings based on quasi ideals, tri-ideals and bi-quasi ideals will be the focus of future research.

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