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Performance of a New Sixth-Order Class of Iterative Schemes for Solving Non-Linear Systems of Equations

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Abstract: This manuscript is focused on a new parametric class of multi-step iterative procedures to find the solutions of systems of nonlinear equations. Starting from Ostrowski's scheme, the class is constructed by adding a Newton step with a Jacobian matrix taken from the previous step and employing a divided difference operator, resulting in a triparametric scheme with a convergence order of four. The convergence order of the family can be accelerated to six by setting two parameters, resulting in a uniparametric family. We performed dynamic and numerical development to analyze the stability of the sixth-order family. Previous studies for scalar functions allow us to isolate those elements of the family with stable performance for solving practical problems. In this regard, we present dynamical planes showing the complexity of the family. In addition, the numerical properties of the class are analyzed with several test problems.

Keywords: nonlinear systems of equations; multipoint iterative methods; analysis of convergence; real discrete dynamics; chaos and stability

MSC: 65H10; 37C20

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1. Introduction

A large number of problems in Computer Science and related disciplines are mathematically characterized by a nonlinear equation or a nonlinear system of equations $F(x) = 0$, where $F : D \subset \mathbb{R}^n \to \mathbb{R}^n$ is a sufficiently Frechet-differentiable function over an open convex set D. Finding the value of a solution ξ is a problem that has been tackled via multiple strategies in fields such as Numerical Analysis, Applied Mathematics, and Engineering.

Newton's scheme is the best known scheme for finding the root $\xi \in D$ of F:

$$x^{(k+1)} = x^{(k)} - [F'(x^{(k)})]^{-1}F(x^{(k)}),$$

where $k \geq 0$ and the Jacobian matrix of F at $x^{(k)}$ is denoted by $F'(x^{(k)})$.

In recent years, this problem has attracted the attention of many scientists, highlighting the following techniques. The extension of scalar to vector iterative methods [1–4] is a common practice, provided the extension is feasible, that affords solutions to n-dimensional problems. To improve the convergence order without compromising the computational cost, new steps are included with only one new evaluation of F, keeping F' frozen [5–8].

We propose in this manuscript a new parametric class of multi-step iterative procedure (1) for solving systems of nonlinear equations. This family is a multidimensional extension of the set of methods defined in [9] for nonlinear equations. The starting point of this family is Ostrowski's scheme, appending a Newton-type step with a "frozen" Jacobian matrix. Thus, it has an iterative expression with three arbitrary parameters and three steps:

$$
\begin{cases}
y^{(k)} = x^{(k)} - [F'(x^{(k)})]^{-1}F(x^{(k)}), \\
z^{(k)} = y^{(k)} - [2[x^{(k)}, y^{(k)}; F] - F'(x^{(k)})]^{-1}F(y^{(k)}), \\
x^{(k+1)} = z^{(k)} - (\alpha I + \beta u^{(k)} + \gamma v^{(k)})[F'(x^{(k)})]^{-1}F(z^{(k)}),
\end{cases} \tag{1}
$$

where α, β, and γ are arbitrary parameters, $v^{(k)} = [x^{(k)}, y^{(k)}; F]^{-1}F'(x^{(k)})$, $u^{(k)} = I - [F'(x^{(k)})]^{-1}[x^{(k)}, y^{(k)}; F]$, and $k = 0, 1, 2, \ldots$. The definition of the divided difference operator can be found in [10], where it is defined as the map $[\cdot, \cdot; F] : D \times D \subset \mathbb{R}^n \times \mathbb{R}^n \to \mathcal{L}(\mathbb{R}^n)$ that satisfies

$$
[x, y; F](x - y) = F(x) - F(y), \quad \forall x, y \in D. \tag{2}
$$

Starting from (1), a uniparametric family is constructed that reaches a convergence order of six, which is corroborated by a supporting convergence analysis. The objective of the new family is to increase the convergence order without significantly increasing the computational cost.

The dynamic behavior of the rational operator obtained from iterative schemes applied to low-degree nonlinear polynomial systems is an effective tool for analyzing the stability and reliability of these numerical methods [6,11]. The stability of the family is analyzed using a real multidimensional discrete dynamical system. Here, we construct dynamical planes that show the complexity of this class. It should be noted that we extend the complex analysis presented in [9] for scalar functions to vector functions in order to choose stable members from the parameter spaces. Several numerical tests are performed to illustrate the efficiency and stability of the iterative schemes.

The outline of this manuscript is as follows: we introduce the proposed class of iterative procedures in Section 1; its convergence is analyzed in Section 2, finding that with appropriate selection of the parameters it is possible to find a single-parametric family for the sixth order of convergence. Section 3 is devoted to the dynamical analysis of this family in order to find those members with best and worst performance in terms of stability. The numerical performance is checked in Section 4, and our conclusions are stated in Section 5.

2. Convergence Analysis of the Family

In this section, we analyze the convergence properties of the new triparametric iterative family. Although the order of the triparametric family is four, in the proof we use higher-order Taylor expansions, as they are useful for proving the order of the uniparametric family.

Theorem 1 (Tri-parametric class)**.** *Consider a sufficiently differentiable function $F : D \subseteq \mathbb{R}^n \to \mathbb{R}^n$ in an convex open set D. Let $\xi \in D$ be a solution of the nonlinear system $F(x) = 0$. Assume that $F'(x)$ is continuous and nonsingular at ξ and that $x^{(0)}$ is a seed that is sufficiently close to ξ. Then, sequence $\{x^{(k)}\}_{k \geq 0}$ can be obtained using expression (1), converges to solution ξ, and has a convergence order of four. Under this hypothesis, its error equation is*

$$
e^{(k+1)} = (1 - \alpha - \gamma)\left(C_2^3 - C_3 C_2\right)e^{(k)^4} + \mathcal{O}(e^{(k)^5}),
$$

where α, β, and γ are arbitrary parameters, $C_q = \frac{1}{q!}[F'(\xi)]^{-1}F^{(q)}(\xi)$, $q = 2, 3, \ldots$, and $e^{(k)} = x^{(k)} - \xi$.

Proof. Next, let us consider ξ such that $F(\xi) = 0$ and $F'(\xi) \neq 0$); here, we let be $x^{(k)} = \xi + e^{(k)}$. Using Taylor expansion series of $F(x^{(k)})$ and $F'(x^{(k)})$ around ξ, we have

$$
F(x^{(k)}) = F'(\xi)\left[e^{(k)} + C_2 e^{(k)^2} + C_3 e^{(k)^3} + C_4 e^{(k)^4}\right] + \mathcal{O}(e^{(k)^5}) \tag{3}
$$

and

$$
F'(x^{(k)}) = F'(\xi)\left[I + 2C_2 e^{(k)} + 3C_3 e^{(k)^2} + 4C_4 e^{(k)^3}\right] + \mathcal{O}(e^{(k)^4}), \tag{4}
$$

where the coefficients C_q are defined as $C_q = \dfrac{1}{q!}[F'(\xi)]^{-1}F^{(q)}(\xi)$, $q = 2, 3, \ldots$

Now, the Taylor expansion of the inverse $[F'(x^{(k)})]^{-1}$ can be stated as follows:

$$^{-1} = \left[I + X_2 e^{(k)} + X_3 e^{(k)^2} + X_4 e^{(k)^3} + X_5 e^{(k)^4} + X_6 e^{(k)^5}\right]\left[F'(\xi)\right]^{-1} + \mathcal{O}(e^{(k)^7}), \tag{5}$$

where $X_2, X_3, \ldots, X_6$ are unknowns such that

$$^{-1}F'(x^{(k)}) = I. \tag{6}$$

Then, we have

$$\begin{aligned}
X_2 &= -2C_2,\\
X_3 &= 4C_2^2 - 3C_3,\\
X_4 &= -8C_2^3 + 6C_2C_3 + 6C_3C_2 - 4C_4,\\
X_5 &= 16C_2^4 - 12C_2^2C_3 - 12C_2C_3C_2 + 8C_2C_4 + 9C_3^2 - 12C_3C_2^2 + 8C_4C_2 - 5C_5,\\
X_6 &= -32C_2^5 + 24C_2^3C_3 + 24C_2^2C_3C_2 - 16C_2^2C_4 + 24C_2C_3C_2^2 - 16C_2C_4C_2 - 18C_2C_3^2 + 10C_2C_5\\
&\quad - 18C_3^2C_2 + 24C_3C_2^3 - 18C_3C_2C_3 + 12C_3C_4 - 16C_4C_2^2 + 12C_4C_3 + 10C_5C_2 - 6C_6.
\end{aligned} \tag{7}$$

Thus, by multiplying (5) by (3) and replacing them in the first step of (1), we have

$$y^{(k)} = \xi - \left[-C_2 e^{(k)^2} + (-2C_3 + 2C_2^2)e^{(k)^3} + A_4 e^{(k)^4} + A_5 e^{(k)^5} + A_6 e^{(k)^6}\right] + \mathcal{O}(e^{(k)^7}), \tag{8}$$

where

$$\begin{aligned}
A_4 &= -3C_4 + 4C_2C_3 - 4C_2^3 + 3C_3C_2,\\
A_5 &= -4C_5 + 6C_2C_4 - 8C_2^2C_3 + 6C_3^2 + 8C_2^4 - 6C_2C_3C_2 - 6C_3C_2^2 + 4C_4C_2,\\
A_6 &= -5C_6 + 8C_2C_5 - 12C_2^2C_4 + 9C_3C_4 + 16C_2^3C_3 - 12C_2C_3^2 - 12C_3C_2C_3 + 8C_4C_3 - 16C_2^5\\
&\quad + 12C_2^2C_3C_2 + 12C_2C_3C_2^2 - 8C_2C_4C_2 - 9C_3^2C_2 + 12C_3C_2^3 - 8C_4C_2^2 + 5C_5C_2.
\end{aligned} \tag{9}$$

Again, by means of the Taylor series, we develop $F(y^{(k)})$ around ξ, with $e_y^{(k)} = y^{(k)} - \xi$, meaning that we have

$$F(y^{(k)}) = F'(\xi)\left[C_2 e^{(k)^2} + (2C_3 - 2C_2^2)e^{(k)^3} + B_4 e^{(k)^4} + B_5 e^{(k)^5} + B_6 e^{(k)^6}\right] + \mathcal{O}(e^{(k)^7}), \tag{10}$$

where

$$\begin{aligned}
B_4 &= -A_4 + C_2A_2^2,\\
B_5 &= -A_5 + C_2A_2A_3 + C_2A_3A_2,\\
B_6 &= -A_6 + C_2A_2A_4 + C_2A_3^2 + C_2A_4A_2 - C_3A_2^3.
\end{aligned} \tag{11}$$

In order to prove the order of convergence of the second step of (1), we can use the Genocchi–Hermite formula (see [12]):

$$= \int_0^1 F'(x + th)\,dt \tag{12}$$

Expanding $F'(x + th)$ in the Taylor series around x, we have

$$\int_0^1 F'(x + th)\,dt = F'(x) + \frac{1}{2!}F''(x)h + \frac{1}{3!}F'''(x)h^2 + \frac{1}{4!}F^{(iv)}(x)h^3 + \frac{1}{5!}F^{(v)}(x)h^4 + \mathcal{O}(h^5). \tag{13}$$

Denoting $e = x - \xi$ and taking into account that $F'(\xi)$ is nonsingular, we have

$$= F'(\xi)\left[I + P_1 e^{(k)} + P_2 e^{(k)^2} + P_3 e^{(k)^3} + P_4 e^{(k)^4}\right] + \mathcal{O}(e^{(k)^5}), \tag{14}$$

where the error at the first step is denoted by $e_y^{(k)} = y^{(k)} - \xi$. In this expression,

$$\begin{aligned} P_1 &= C_2, \\ P_2 &= C_2^2 + C_3, \\ P_3 &= C_4 + 2C_2C_3 + C_3C_2 - 2C_2^3, \\ P_4 &= C_5 + 3C_2C_4 - 4C_2^2C_3 + 4C_2^4 - 3C_2C_3C_2 + 2C_3^2 - C_3C_2^2 + C_4C_2. \end{aligned} \tag{15}$$

Now, denoting $M = 2[x^{(k)}, y^{(k)}; F] - F'(x^{(k)})$, we have

$$M = F'(\xi)\left[I + M_2 e^{(k)^2} + M_3 e^{(k)^3} + M_4 e^{(k)^4}\right] + \mathcal{O}(e^{(k)^5}), \tag{16}$$

where

$$\begin{aligned} M_2 &= 2C_2^2 - C_3, \\ M_3 &= 2\left(-C_4 + 2C_2C_3 + C_3C_2 - 2C_2^3\right), \\ M_4 &= -3C_5 + 6C_2C_4 - 8C_2^2C_3 + 8C_2^4 - 6C_2C_3C_2 + 4C_3^2 - 2C_3C_2^2 + 2C_4C_2. \end{aligned} \tag{17}$$

The inverse of M must satisfy

$$M^{-1}M = I, \tag{18}$$

where

$$M^{-1} = \left[I + Y_1 e^{(k)} + Y_2 e^{(k)^2} + Y_3 e^{(k)^3} + Y_4 e^{(k)^4}\right]\left[F'(\xi)\right]^{-1} + \mathcal{O}(e^{(k)^5}), \tag{19}$$

where $Y_1, \ldots, Y_4$ are unknowns. Then, replacing M^{-1} and M in (18), we have

$$\begin{aligned} Y_1 &= 0, \\ Y_2 &= -2C_2^2 + C_3, \\ Y_3 &= 2C_4 - 4C_2C_3 - 2C_3C_2 + 4C_2^3, \\ Y_4 &= 3C_5 - 6C_2C_4 + 6C_2^2C_3 - 4C_2^4 + 6C_2C_3C_2 - 3C_3^2 - 2C_4C_2. \end{aligned} \tag{20}$$

Next, we denote $L = M^{-1}F(y^{(k)})$ and obtain

$$L = C_2 e^{(k)^2} + 2\left(C_3 - C_2^2\right)e^{(k)^3} + L_4 e^{(k)^4} + L_5 e^{(k)^5} + L_6 e^{(k)^6} + \mathcal{O}(e^{(k)^7}), \tag{21}$$

where

$$\begin{aligned} L_4 &= 3C_4 - 4C_2C_3 - 2C_3C_2 + 3C_2^3, \\ L_5 &= 4C_5 - 6C_2C_4 + 6C_2^2C_3 - 4C_3^2 - 4C_2^4 + 4C_2C_3C_2 + 2C_3C_2^2 - 2C_4C_2, \\ L_6 &= 5C_6 - 8C_2C_5 + 9C_2^2C_4 - 6C_3C_4 - 8C_2^3C_3 + 8C_2C_3^2 + 4C_3C_2C_3 - 4C_4C_3 \\ &\quad + 6C_2^5 - 7C_2^2C_3C_2 - 5C_2C_3C_2^2 + 5C_2C_4C_2 + 3C_3^2C_2 - 2C_3C_2^3 + 2C_4C_2^2 - 2C_5C_2. \end{aligned} \tag{22}$$

Therefore,

$$z^{(k)} = y^{(k)} - L = \xi - \left[K_4 e^{(k)^4} + K_5 e^{(k)^5} + K_6 e^{(k)^6}\right] + \mathcal{O}(e^{(k)^7}), \tag{23}$$

where

$$
\begin{aligned}
K_4 &= -C_2^3 + C_3C_2, \\
K_5 &= -2C_2^2C_3 + 2C_3^2 + 4C_2^4 - 2C_2C_3C_2 - 4C_3C_2^2 + 2C_4C_2, \\
K_6 &= -3C_2^2C_4 + 3C_3C_4 + 8C_2^3C_3 - 4C_2C_3^2 - 8C_3C_2C_3 + 4C_4C_3 - 10C_2^5 \\
&\quad + 5C_2^2C_3C_2 + 7C_2C_3C_2^2 - 3C_2C_4C_2 - 6C_3^2C_2 + 10C_3C_2^3 - 6C_4C_2^2 + 3C_5C_2.
\end{aligned} \tag{24}
$$

Similarly, denoting $e_z^{(k)} = z^{(k)} - \xi$,

$$F(z^{(k)}) = F'(\xi)\left[-K_4e^{(k)^4} - K_5e^{(k)^5} - K_6e^{(k)^6}\right] + \mathcal{O}(e^{(k)^7}). \tag{25}$$

Using (5) and (25) and denoting $N = [F'(x^{(k)})]^{-1}F(z^{(k)})$, we have

$$N = (C_2^3 - C_3C_2)e^{(k)^4} + N_5e^{(k)^5} + N_6e^{(k)^6} + \mathcal{O}(e^{(k)^7}), \tag{26}$$

$$
\begin{aligned}
N_5 &= 2C_2^2C_3 - 2C_3^2 - 6C_2^4 + 4C_2C_3C_2 + 4C_3C_2^2 - 2C_4C_2, \\
N_6 &= 3C_2^2C_4 - 3C_3C_4 - 12C_2^3C_3 + 8C_2C_3^2 + 8C_3C_2C_3 - 4C_4C_3 + 22C_2^5 \\
&\quad - 13C_2^2C_3C_2 - 15C_2C_3C_2^2 + 7C_2C_4C_2 + 9C_3^2C_2 - 13C_3C_2^3 + 6C_4C_2^2 - 3C_5C_2.
\end{aligned} \tag{27}
$$

Then, replacing (5) and (14) in $u^{(k)}$,

$$u^{(k)} = C_2e^{(k)} + (-3C_2^2 + 2C_3)e^{(k)^2} + \mathcal{O}(e^{(k)^3}). \tag{28}$$

Now, we can find the Taylor series expansion of $[x^{(k)}, y^{(k)}; F]^{-1}$ as follows:

$${}^{-1} = \left[I + R_1e^{(k)} + R_2e^{(k)^2}\right][F'(\xi)]^{-1} + \mathcal{O}(e^{(k)^3}), \tag{29}$$

where R_1 and R_2 are unknowns such that

$${}^{-1}[x^{(k)}, y^{(k)}; F] = I. \tag{30}$$

Thus, we have

$$
\begin{aligned}
R_1 &= -C_2, \\
R_2 &= -C_3.
\end{aligned} \tag{31}
$$

By substituting (29) and (4) in $v^{(k)}$,

$$v^{(k)} = I + v_1e^{(k)} + v_2e^{(k)^2} + \mathcal{O}(e^{(k)^3}), \tag{32}$$

where

$$
\begin{aligned}
v_1 &= C_2, \\
v_2 &= 2C_3 - 2C_2^2.
\end{aligned} \tag{33}
$$

Denoting $T = (\alpha I + \beta u^{(k)} + \gamma v^{(k)})N$ and using (28) and (32), we have

$$T = (\alpha + \gamma)(C_2^3 - C_3C_2)e^{(k)^4} + T_5e^{(k)^5} + T_6e^{(k)^6} + \mathcal{O}(e^{(k)^7}), \tag{34}$$

$$
\begin{aligned}
T_5 &= (\alpha + \gamma)N_5 + (\beta + \gamma)u_1N_4, \\
T_6 &= (\alpha + \gamma)N_6 + (\beta + \gamma)u_1N_5 + (\beta u_2 + \gamma v_2)N_4.
\end{aligned} \tag{35}
$$

Finally, using (23) and (34),

$$x^{(k+1)} = \xi - \left[W_4e^{(k)^4} + W_5e^{(k)^5} + W_6e^{(k)^6}\right] + \mathcal{O}(e^{(k)^7}), \tag{36}$$

where

$$
\begin{aligned}
W_4 &= (\alpha+\gamma-1)\left(C_2^3 - C_3C_2\right),\\
W_5 &= 2(\alpha+\gamma-1)\left(C_2^2C_3 - C_3^2 + 2C_3C_2^2 - C_4C_2\right) - (6\alpha-\beta+5\gamma-4)C_2^4\\
&\quad + (4\alpha-\beta+3\gamma-2)C_2C_3C_2,\\
W_6 &= -3C_2^2C_4 + 3C_3C_4 + 8C_2^3C_3 - 4C_2C_3^2 - 8C_3C_2C_3 + 4C_4C_3 - 10C_2^5 + 5C_2^2C_3C_2\\
&\quad + 7C_2C_3C_2^2 - 3C_2C_4C_2 - 6C_3^2C_2 + 10C_3C_2^3 - 6C_4C_2^2 + 3C_5C_2\\
&\quad + (\alpha+\gamma)\Big(3C_2^2C_4 - 3C_3C_4 - 12C_2^3C_3 + 8C_2C_3^2 + 8C_3C_2C_3 - 4C_4C_3 + 22C_2^5\\
&\quad -13C_2^2C_3C_2 - 15C_2C_3C_2^2 + 7C_2C_4C_2 + 9C_3^2C_2 - 13C_3C_2^3 + 6C_4C_2^2 - 3C_5C_2\Big)\\
&\quad + (\beta+\gamma)C_2\left(2C_2^2C_3 - 2C_3^2 - 6C_2^4 + 4C_2C_3C_2 + 4C_3C_2^2 - 2C_4C_2\right)\\
&\quad + \left(\beta\left(-3C_2^2 + 2C_3\right) + \gamma\left(2C_3 - 2C_2^2\right)\right)\left(C_2^3 - C_3C_2\right),
\end{aligned}
\tag{37}
$$

and the error equation is

$$
\begin{aligned}
e^{(k+1)} &= -W_4 e^{(k)^4} - W_5 e^{(k)^5} - W_6 e^{(k)^6} + \mathcal{O}(e^{(k)^7})\\
&= (1-\alpha-\gamma)\left(C_2^3 - C_3C_2\right)e^{(k)^4} + \mathcal{O}(e^{(k)^5}).
\end{aligned}
\tag{38}
$$

This finishes the proof. □

From Theorem 1, the triparametric family is fourth-order convergent for any α, β, and γ. Nevertheless, the order of convergence can be accelerated by reducing the number of parameters, resulting in a uniparametric family.

Theorem 2 (Uni-parametric family)**.** *Consider a sufficiently differentiable function $F : D \subseteq \mathbb{R}^n \to \mathbb{R}^n$ defined in a convex open set D, and let $\xi \in D$ be a solution of the nonlinear system $F(x) = 0$. Assuming that $F'(x)$ is nonsingular and continuous at ξ and $x^{(0)}$ is a seed close enough to ξ, sequence $\{x^{(k)}\}_{k\geq 0}$ obtained using (1) converges to ξ with the sixth order of convergence only if $\gamma = 1-\alpha$ and $\beta = 1+\alpha$. Therefore, its error equation is*

$$
e^{(k+1)} = \left(C_3^2C_2 - C_3C_2^3 + 6C_2^5 - 6C_2^2C_3C_2\right)e^{(k)^6} + \mathcal{O}(e^{(k)^7}),
$$

where $C_q = \frac{1}{q!}[F'(\xi)]^{-1}F^{(q)}(\xi)$, $q = 2, 3, \ldots$, and $e^{(k)} = x^{(k)} - \xi$.

Proof. Using the results of Theorem 1, in order to cancel W_4 and W_5 the coefficients of $e^{(k)^4}$ and $e^{(k)^5}$ in (38), which are respectively $\alpha+\gamma = 1$, $6\alpha-\beta+5\gamma = 4$ and $4\alpha-\beta+3\gamma = 2$, must be satisfied. This system has infinite solutions for

$$
\beta = 1+\alpha \quad \text{and} \quad \gamma = 1-\alpha, \tag{39}
$$

with α being a disposable parameter. Then, replacing (39) in (37), we have the following:

$$
W_4 = 0, \quad W_5 = 0, \quad \text{and} \quad W_6 = -C_3^2C_2 + C_3C_2^3 - 6C_2^5 + 6C_2^2C_3C_2, \tag{40}
$$

for which the error equation is

$$
\begin{aligned}
e^{(k+1)} &= -W_6 e^{(k)^6} + \mathcal{O}(e^{(k)^7})\\
&= \left(C_3^2C_2 - C_3C_2^3 + 6C_2^5 - 6C_2^2C_3C_2\right)e^{(k)^6} + \mathcal{O}(e^{(k)^7}).
\end{aligned}
\tag{41}
$$

This finishes the proof. □

As follows from Theorem 2, by replacing $\beta = 1+\alpha$ and $\gamma = 1-\alpha$ in (1) the tri-parametric family becomes a uniparametric family with sixth-order convergence. Thus, the iterative expression of the new three-step family dependent on α, denoted henceforth as MCCT(α), is

$$\begin{cases} y^{(k)} = x^{(k)} - [F'(x^{(k)})]^{-1}F(x^{(k)}), \\ z^{(k)} = y^{(k)} - [2[x^{(k)}, y^{(k)}; F] - F'(x^{(k)})]^{-1}F(y^{(k)}), \\ x^{(k+1)} = z^{(k)} - (\alpha I + (1+\alpha)u^{(k)} + (1-\alpha)v^{(k)})[F'(x^{(k)})]^{-1}F(z^{(k)}), \end{cases} \tag{42}$$

where α is an arbitrary parameter $v^{(k)} = [x^{(k)}, y^{(k)}; F]^{-1}F'(x^{(k)})$, $u^{(k)} = I - [F'(x^{(k)})]^{-1}$ $[x^{(k)}, y^{(k)}F]$, and $k = 0, 1, 2, \ldots$.

Next, we analyze the stability of the MCCT(α) family in order to select its best members, for which we use the real dynamical tools presented in Section 3.

3. Real Dynamics for Stability

This section refers to our analysis of the dynamical behavior of the rational operator related with iterative schemes of the MCCT(α) family. It provides significative information about the reliability and stability of the class. We construct rational operators and their dynamical planes in order to identify the performance of particular schemes from the different basins of attraction.

3.1. Rational Operator

Rational operators are built on low-degree nonlinear polynomial systems, as the criterion of stability of a method applied to these systems can be generalized to other multidimensional cases. Thus, we propose two nonlinear systems, one with separated variables $F(x_1, x_2)$ and another with non-separated variables $G(x_1, x_2)$, as follows:

$$F(x_1, x_2) = (x_1^2 - 1, x_2^2 - 1) = (0, 0), \tag{43}$$

$$G(x_1, x_2) = \left(x_1^2 + x_2^2 - 1, x_1^2 - x_2^2 - \frac{1}{2}\right) = (0, 0). \tag{44}$$

Proposition 1 (Rational operator R_F)**.** *Consider the polynomial system $F(x_1, x_2)$ provided in (43) with roots $(-1,-1)$, $(-1,1)$, $(1,-1)$, $(1,1) \in \mathbb{R}^2$. The rational operator associated with the MCCT(α) family and applied on $F(x_1, x_2)$ (with $\alpha \in \mathbb{R}$ being an arbitrary parameter) is*

$$R_F(x_1, x_2, \alpha) = (R_{F_{11}}, R_{F_{12}}), \tag{45}$$

where

$$R_{F_{11}} = \frac{1}{32}\left(\frac{(x_1^2-1)^4(\alpha + (\alpha-19)x_1^4 - 2(\alpha-1)x_1^2 + 1)}{4x_1^5(x_1^2+1)^2(3x_1^2+1)} + \frac{8(x_1^4 + 6x_1^2 + 1)}{x_1^3 + x_1} - \frac{\alpha(x_2^2-1)^4}{x_2^3(x_2^2+1)^2}\right),$$
$$R_{F_{12}} = \frac{1}{32}\left(\frac{(x_2^2-1)^4(\alpha + (\alpha-19)x_2^4 - 2(\alpha-1)x_2^2 + 1)}{4x_2^5(x_2^2+1)^2(3x_2^2+1)} + \frac{8(x_2^4 + 6x_2^2 + 1)}{x_2^3 + x_2} - \frac{\alpha(x_1^2-1)^4}{x_1^3(x_1^2+1)^2}\right).$$

In Proposition 1, note that the rational operator $R_F(x_1, x_2, \alpha)$ is obtained by substituting the nonlinear system $F(x_1, x_2)$ into the iterative scheme of the MCCT(α) family. To simplify R_F, we can select a value of α that cancels terms in the expression in order to reduce it. It is easy to show that the rational operator is simpler for $\alpha = 0$ and that there are fewer fixed and critical points, which can improve the performance of the associated method. In addition, the components of $R_F(x_1, x_2, 0)$ are of separate variables, as shown by

$$R_F(x_1,x_2,0) = \left(\frac{77x_1^{12} + 782x_1^{10} + 775x_1^8 + 404x_1^6 + 11x_1^4 - 2x_1^2 + 1}{128x_1^5(x_1^2+1)^2(3x_1^2+1)}, \frac{77x_2^{12} + 782x_2^{10} + 775x_2^8 + 404x_2^6 + 11x_2^4 - 2x_2^2 + 1}{128x_2^5(x_2^2+1)^2(3x_2^2+1)} \right). \tag{46}$$

Proposition 2 (Rational operator R_G)**.** *Consider the polynomial system $G(x_1, x_2)$ provided in (44) with roots $\left(-\frac{\sqrt{3}}{2}, -\frac{1}{2}\right)$, $\left(-\frac{\sqrt{3}}{2}, \frac{1}{2}\right)$, $\left(\frac{\sqrt{3}}{2}, -\frac{1}{2}\right)$, $\left(\frac{\sqrt{3}}{2}, \frac{1}{2}\right) \in \mathbb{R}^2$. The rational operator associated with the MCCT(α) family and applied on $G(x_1, x_2)$ (with $\alpha \in \mathbb{R}$ being an arbitrary parameter) is*

$$R_G(x_1,x_2,\alpha) = (R_{G_{11}}, R_{G_{12}}), \tag{47}$$

where

$$R_{G_{11}} = \frac{(3-4x_1^2)^4(9(\alpha+1)+16(\alpha-19)x_1^4 - 24(\alpha-1)x_1^2)}{24576x_1^5(4x_1^2+1)(4x_1^2+3)^2} + \frac{16x_1^4+72x_1^2+9}{64x_1^3+48x_1} - \frac{\alpha(1-4x_2^2)^4}{512x_2^3(4x_2^2+1)^2},$$

$$R_{G_{12}} = \frac{(1-4x_2^2)^4(\alpha+16(\alpha-19)x_2^4 - 8(\alpha-1)x_2^2+1)}{8192x_2^5(4x_2^2+1)^2(12x_2^2+1)} + \frac{16x_2^4+24x_2^2+1}{64x_2^3+16x_2} - \frac{\alpha(3-4x_1^2)^4}{512x_1^3(4x_1^2+3)^2}.$$

From Proposition 2, note that the rational operator $R_G(x_1, x_2, \alpha)$ is obtained by substituting the nonlinear system $G(x_1, x_2)$ into the iterative scheme of the MCCT(α) family. In the same way for R_F, it is easy to prove that the rational operator R_G is simpler for $\alpha = 0$. Moreover, the components of $R_G(x_1, x_2, 0)$ are of separate variables, as shown by

$$R_G(x_1,x_2,0) = \left(\frac{(-304x_1^4+24x_1^2+9)(3-4x_1^2)^4}{24576x_1^5(4x_1^2+1)(4x_1^2+3)^2} + \frac{16x_1^4+72x_1^2+9}{64x_1^3+48x_1}, \frac{(-304x_2^4+8x_2^2+1)(1-4x_2^2)^4}{8192x_2^5(4x_2^2+1)^2(12x_2^2+1)} + \frac{16x_2^4+24x_2^2+1}{64x_2^3+16x_2} \right). \tag{48}$$

With these two rational operators $R_F(x_1, x_2, \alpha)$ and $R_G(x_1, x_2, \alpha)$, we can study the stability of the MCCT(α) family by means of dynamical planes built for different values of α. These planes show the complexity of the iterative class.

3.2. Fixed Points and Their Stability

The fixed points are calculated from the rational operators $R_F(x_1, x_2, \alpha)$ and $R_G(x_1, x_2, \alpha)$ provided in (45) and (47), respectively. Using these points, we can perform a stability analysis.

Proposition 3 (R_F fixed points)**.** *The real fixed points of $R_F(x_1, x_2, \alpha)$ are the roots of the equation $R_F(x_1, x_2, \alpha) = (x_1, x_2)$, that is,*

$$fp_1 = (-1,-1), fp_2 = (-1,1), fp_3 = (1,-1), fp_4 = (1,1),$$

corresponding to the roots of the polynomial system $F(x_1, x_2)$ provided in (43); moreover, they are superattracting. While other strange fixed points may appear, their components are roots of polynomials of degree 120.

Proposition 4 (R_G fixed points)**.** *The real fixed points of $R_G(x_1, x_2, \alpha)$ are the roots of the equation $R_G(x_1, x_2, \alpha) = (x_1, x_2)$, that is,*

$$fp_1 = \left(-\frac{\sqrt{3}}{2}, -\frac{1}{2}\right), fp_2 = \left(-\frac{\sqrt{3}}{2}, \frac{1}{2}\right), fp_3 = \left(\frac{\sqrt{3}}{2}, -\frac{1}{2}\right), fp_4 = \left(\frac{\sqrt{3}}{2}, \frac{1}{2}\right),$$

corresponding to the roots of the polynomial system $G(x_1, x_2)$ provided in (44)*; these are superattracting as well. Again, for other strange fixed points that may appear, their components are roots of polynomials of degree 120.*

Propositions 3 and 4 establish a minimum of four fixed points for polynomial systems $F(x_1, x_2)$ and $G(x_1, x_2)$. Of these, fp_1 to fp_4 correspond to the roots of the original systems, and are attractive and critical points.

3.3. Dynamical Planes

We can perform a stability analysis of the MCCT(α) family by representing dynamical planes of the rational operators $R_F(x_1, x_2, \alpha)$ and $R_G(x_1, x_2, \alpha)$. Two values of α with different behavior in the parameter space of Figure 1 have been chosen; value $\alpha = 0$ is in the red zone, which implies convergence, while value $\alpha = 200$ is in the black zone, which does not guarantee convergence. These parameter spaces were obtained from the MCCT(α) family for scalar cases [9], with their results then extrapolated for vector cases.

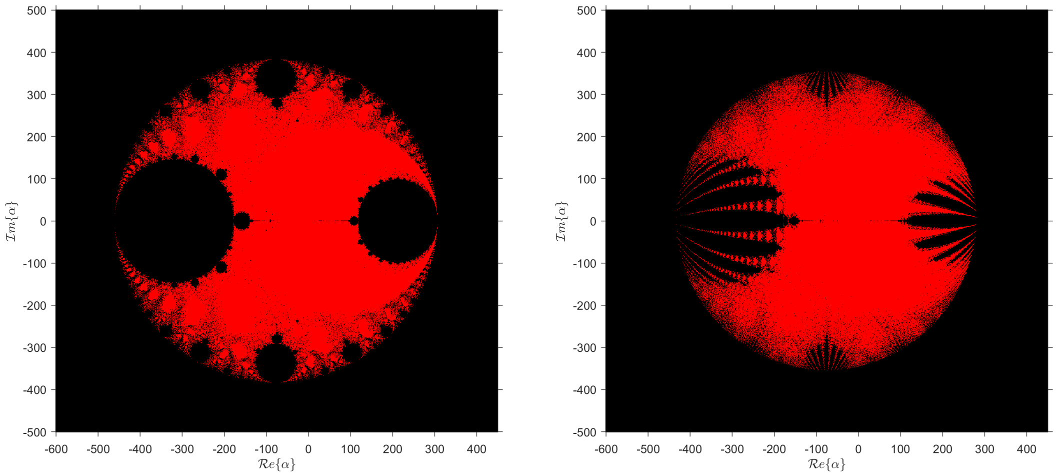
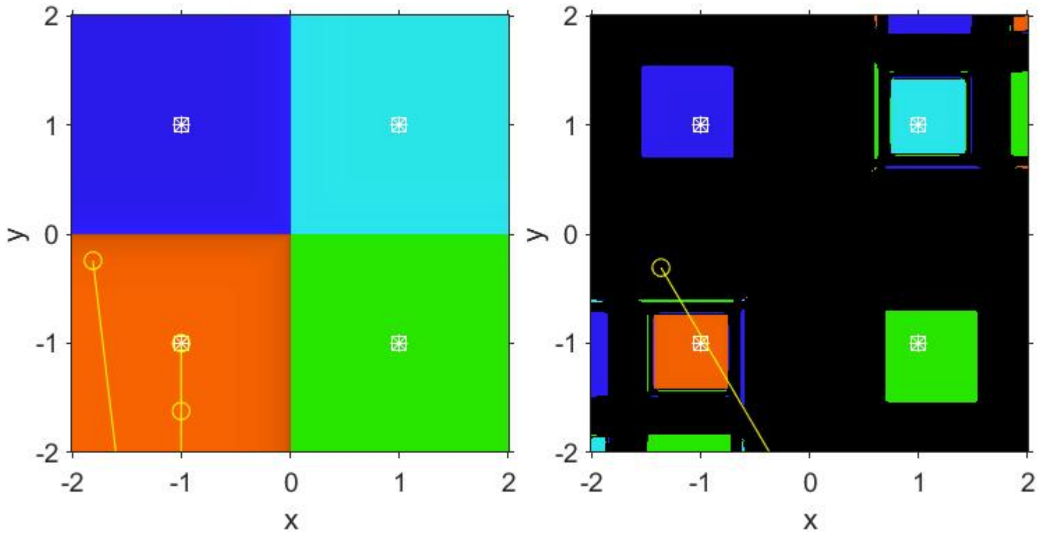


Figure 1. Parameter spaces of free critical points of the MCCT(α) family applied to a nonlinear polynomial equation $(x - a)(x - b) = 0$, where $a, b \in \mathbb{C}$.

A dynamical plane is represented by a mesh of 400×400 points in $\mathbb{R}^2$. Each point of the mesh is a seed of the iterative process. The convergence of the scheme is shown with a maximum of 50 iterations and a stopping criterion of $||x^{(k+1)} - x^{(k)}|| < 10^{-3}$. Each root is color assigned. The color of the mesh points indicates which root it converges to, with black being the points at which the maximum number of iterations is reached and brighter colors indicating a lower number of iterations. Fixed points are represented in white by a circle ('$\bigcirc$'), critical points by a square ('$\square$'), and attractors by an asterisk ('$*$'). The resulting plane was generated using Matlab R2020b.

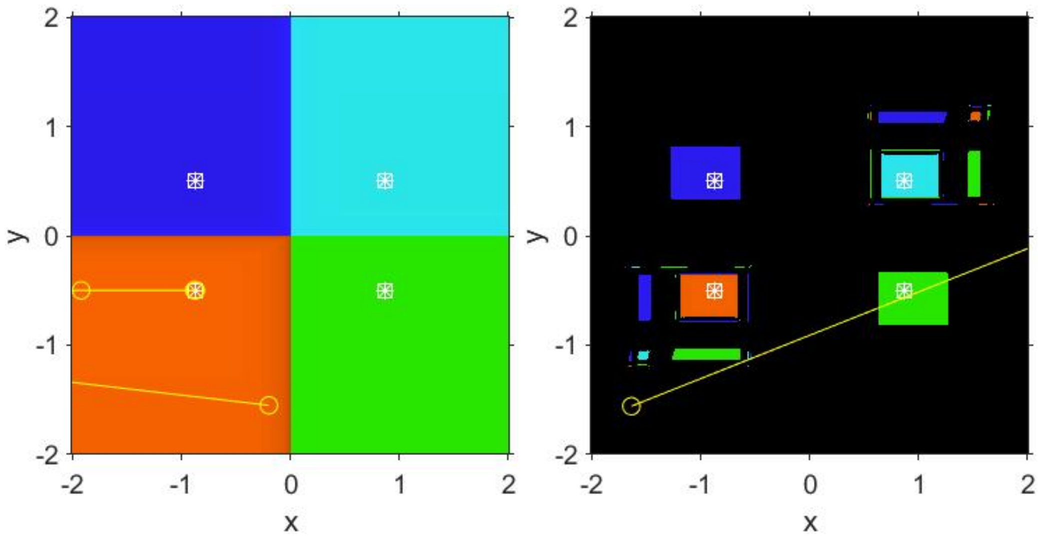
The dynamical planes corresponding to $R_F(x_1, x_2, 0)$ and $R_F(x_1, x_2, 200)$ on the one hand and $R_G(x_1, x_2, 0)$ and $R_G(x_1, x_2, 200)$ on the other are shown in Figures 2 and 3, respectively. In both cases, yellow convergence orbits can be observed.

In both cases, the method for $\alpha = 0$ presents four basins of attraction associated with the roots. No black areas are observed. Consequently, this method shows good dynamic behavior. In contrast, in R_F and R_G the method for $\alpha = 200$ presents the same four basins of attraction associated with the roots, except now with reduced size, which minimizes the possibility of convergence to the solution. Black areas of slow convergence are observed. In consequence, this method shows poor dynamic performance.



(**a**) (**b**)

Figure 2. Dynamical planes for $R_F(x_1, x_2, \alpha)$: (**a**) convergence to $pf_1 = (-1, -1)$ for $\alpha = 0$ and an initial estimation $x^{(0)}$ close to the roots; (**b**) no convergence to any pf for $\alpha = 200$ and an initial estimation $x^{(0)}$ close to the roots.



(**a**) (**b**)

Figure 3. Dynamical planes for $R_G(x_1, x_2, \alpha)$: (**a**) convergence to $pf_1 \approx (-0.87, -0.5)$ for $\alpha = 0$ and an initial estimation $x^{(0)}$ close to the roots; (**b**) no convergence to any pf for $\alpha = 200$ and an initial estimation $x^{(0)}$ close to the roots.

From Figures 2 and 3, it is apparent that the basins of attraction have similar behavior for the rational operators R_F and R_G with $\alpha = 0$. However, for $\alpha = 200$, these basins are reduced and the associated iterative methods do not easily converge to the solution.

If we consider nonlinear systems that involve logarithmic, trigonometric, and exponential as well as polynomial functions, the behavior of the representative members of the MCCT(α) family for $\alpha = 0$ and $\alpha = 200$ is similar to what has already been studied. For example, when analyzing the systems shown in Table 1, we can observe in their dynamical planes (see Figures 4–6) that the regions of the basins of attraction for $\alpha = 0$ are much larger than for $\alpha = 200$, increasing the chances of converging to the solution for the first case.

In addition, more regions of slow convergence or non-convergence are observed for the MCCT(200) iterative method as compared to the MCCT(0) method.

Table 1. Tested nonlinear systems for dynamical analysis.

Non-Linear System	Some Roots
$M(x_1, x_2) = (e^{x_1}e^{x_2} + x_1\cos(x_2), x_1 + x_2 - 1) = (0, 0)$	$\xi \approx (-6.4165, 7.4165; -4.3816, 5.3816; 3.4706, -2.4706; 5.1572, -4.1572; 9.1554, -8.1554)^T$
$N(x_1, x_2) = \left(\ln(x_1^2) - 2\ln(\cos(x_2)), x_1\tan(x_2)\right) = (0, 0)$	$\xi = (-1, 0; 1, 0)^T$
$O(x_1, x_2) = (x_1 + e^{x_2} - \cos(x_2) + 0.5, 3x_1 - x_2 - \sin(x_2)) = (0, 0)$	$\xi \approx (-0.2535, -0.3851; -0.9389, -1.8576; -1.0935, -4.0974)^T$

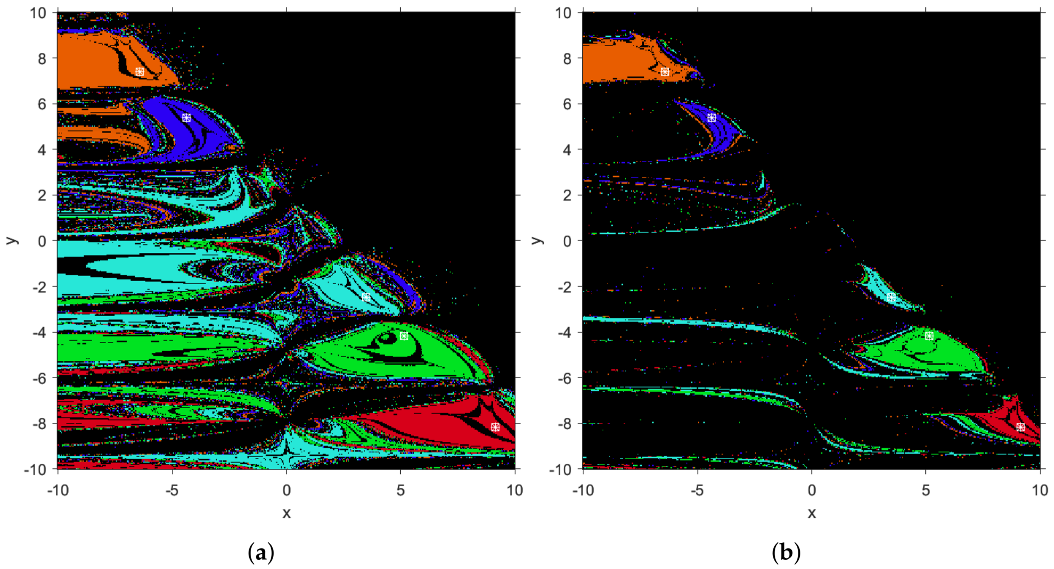


Figure 4. Dynamical planes for system $M(x_1, x_2)$: (**a**) considering the MCCT(0) method and (**b**) considering the MCCT(200) method.

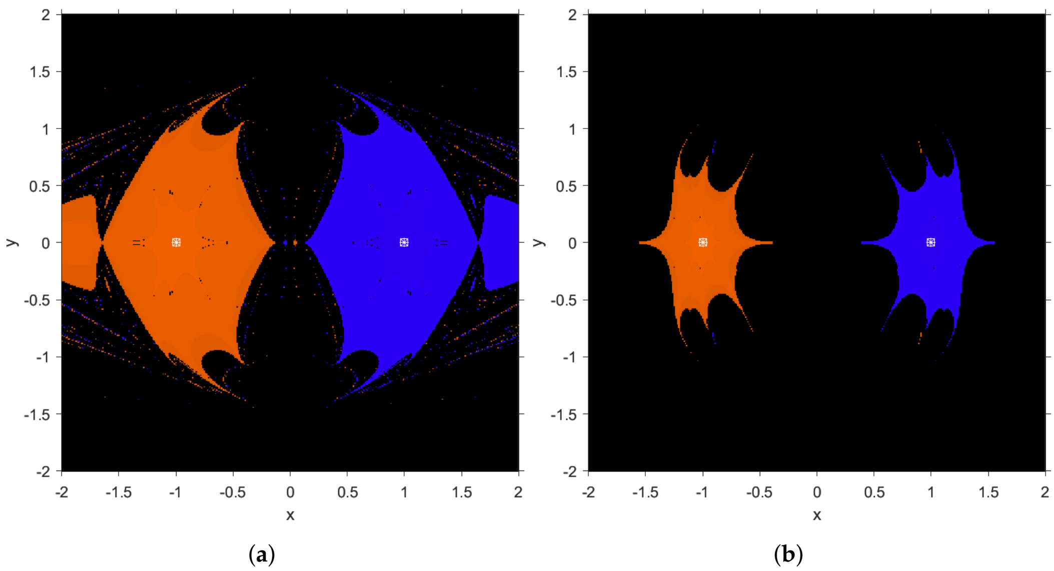


Figure 5. Dynamical planes for system $N(x_1, x_2)$: (**a**) considering the MCCT(0) method and (**b**) considering the MCCT(200) method.

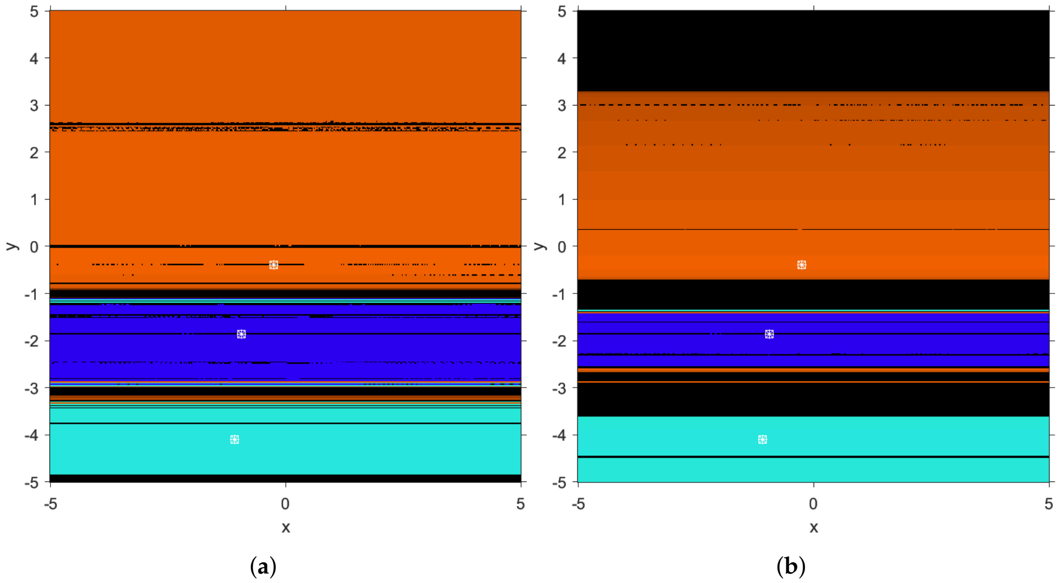


Figure 6. Dynamical planes for system $O(x_1, x_2)$: (**a**) considering the MCCT(0) method and (**b**) considering the MCCT(200) method.

4. Numerical Results

Several numerical tests were carried out to check the performance of MCCT(α) family, with the aim of verifying our theoretical results for convergence and stability. We employed two members of the class used above as representatives, namely, MCCT(0) and MCCT(200). These methods were applied to the same two-by-two non-linear test systems seen above and to new three-by-three and four-by-four systems. Along with the corresponding roots, they are summarized in Table 2.

Table 2. Test nonlinear systems and their roots.

Non-Linear Test System	Roots
$F(x_1, x_2) = \left(x_1^2 - 1, x_2^2 - 1\right) = (0, 0)$	$\xi = (1, 1)^T$
$G(x_1, x_2) = \left(x_1^2 + x_2^2 - 1, x_1^2 - x_2^2 - \frac{1}{2}\right) = (0, 0)$	$\xi = \left(\frac{\sqrt{3}}{2}, \frac{1}{2}\right)^T$
$M(x_1, x_2) = (e^{x_1} e^{x_2} + x_1 \cos(x_2), x_1 + x_2 - 1) = (0, 0)$	$\xi \approx (3.4706, -2.4706)^T$
$N(x_1, x_2) = \left(\ln(x_1^2) - 2\ln(\cos(x_2)), x_1 \tan(x_2)\right) = (0, 0)$	$\xi = (1, 0)^T$
$O(x_1, x_2) = (x_1 + e^{x_2} - \cos(x_2) + 0.5, 3x_1 - x_2 - \sin(x_2)) = (0, 0)$	$\xi \approx (-0.2535, -0.3851)^T$
$P(x_1, x_2, x_3) = \left(\cos(x_2) - \sin(x_1), x_3^{x_1} - \frac{1}{x_2}, e^{x_1} - x_3^2\right) = (0, 0)$	$\xi \approx (0.9096, 0.6612, 1.5758)^T$
$Q(x_1, x_2, x_3, x_4) = (x_2x_3 + x_4(x_2 + x_3), x_1x_3 + x_4(x_1 + x_3),$ $x_1x_2 + x_4(x_1 + x_2), x_1x_2 + x_1x_3 + x_2x_3 - 1) = (0, 0)$	$\xi \approx (0.5774, 0.5774,$ $0.5774, -0.2887)^T$

A comparison of MCCT(0) was conducted against three methods from the literature: Newton's [10], Ostrowski's [13], and HMT's methods [14]. Table 3 collects the numerical results, using initial guesses for $x^{(0)}$ close to ξ solutions.

The computations were performed in Matlab R2020b using variable precision arithmetic, with a mantissa of 200 digits. For each scheme, the amount of iterations (iter) needed to converge to the solution was analyzed in such a way that the stopping criteria $|||x^{(k+1)} - x^{(k)}|| < 10^{-100}$ or $|||F(x^{(k+1)})|| < 10^{-100}$ were satisfied.

The approximate computational order of convergence (ACOC) [15] was obtained. The ACOC column is 'nc' if the number of iterations reaches 50 or '-' if the ACOC does not stabilize.

Table 3 indicates that MCCT(0) converges to ξ in fewer iterations than the other methods in five of the seven nonlinear systems. The theoretical order of convergence is achieved by ACOC as well, being close to six. This method was analyzed for seeds both near the solution and far from it, i.e., for $x^{(0)} \approx 3\xi$ and $x^{(0)} > 10\xi$, respectively. The obtained results are collected in Tables 4 and 5.

The results in Tables 4 and 5 show that MCCT(0) converges to the solution in six of the seven nonlinear test systems, regardless of the initial estimates used. The ACOC does not stabilize its value in several cases; however, when it does its value approaches six.

Our analysis of the MCCT(200) method is shown below. The numerical results for $x^{(0)} \approx \xi$) and $x^{(0)} \approx 3\xi$) are presented in Tables 6 and 7.

MCCT(200) presents convergence problems for $x^{(0)} \approx 3\xi$, as it does not converge to the solution in three of the seven cases, establishing a dependence on the initial estimate and the nonlinear test system used. In addition, the number of iterations is increased with respect to the MCCT(0) method for the same conditions in those systems in which the solution is reached.

Consequently, we conclude that the method for $\alpha = 0$ is robust, and is able to converge to the solution in few iterations and for any seed and system used. Nevertheless, the method for $\alpha = 200$ is unstable, as it does not tend to the solution according to the seed and the system used. It can be observed that both methods converge to the solution with order six. Therefore, the theoretical results of the dynamical behavior and convergence analysis of the MCCT(α) family can be considered verified.

Table 3. Numerical results of MCCT(0) and known schemes on test problems for $x^{(0)} \approx \xi$.

System	Method	$\|\|x^{(k+1)} - x^{(k)}\|\|$	$\|\|F(x^{(k+1)})\|\|$	Iter	ACOC
$F(x_1, x_2)$	MCCT(0)	4.1590×10^{-41}	1.0578×10^{-162}	3	6.0326
$x^{(0)} = (0.90, 0.90)^T$	Newton	4.0862×10^{-82}	1.1806×10^{-163}	7	2.0000
	Ostrowski	2.3572×10^{-61}	6.5488×10^{-183}	4	-
	HMT	2.1362×10^{-52}	5.5061×10^{-208}	3	-
$G(x_1, x_2)$	MCCT(0)	1.6140×10^{-29}	3.8389×10^{-115}	3	5.9785
$x^{(0)} = (0.80, 0.40)^T$	Newton	8.4816×10^{-62}	1.0174×10^{-122}	7	2.0000
	Ostrowski	3.1433×10^{-46}	8.7844×10^{-137}	4	-
	HMT	2.5671×10^{-36}	1.9467×10^{-208}	3	-
$M(x_1, x_2)$	MCCT(0)	1.1444×10^{-49}	1.3224×10^{-131}	3	5.5845
$x^{(0)} = (3.40, -2.40)^T$	Newton	2.4421×10^{-57}	2.1989×10^{-114}	6	2.0000
	Ostrowski	3.9750×10^{-66}	3.0486×10^{-148}	4	-
	HMT	8.1589×10^{-54}	7.7869×10^{-208}	3	5.9851
$N(x_1, x_2)$	MCCT(0)	1.0160×10^{-76}	5.8602×10^{-308}	4	-
$x^{(0)} = (0.90, 0.10)^T$	Newton	1.6691×10^{-73}	1.5673×10^{-146}	7	2.0000
	Ostrowski	6.5957×10^{-87}	1.4347×10^{-259}	5	-
	HMT	3.0359×10^{-41}	3.8934×10^{-208}	3	6.1133
$O(x_1, x_2)$	MCCT(0)	1.2709×10^{-37}	6.3818×10^{-107}	3	5.9417
$x^{(0)} = (-0.20, -0.30)^T$	Newton	2.3676×10^{-73}	3.4799×10^{-146}	7	2.0000
	Ostrowski	7.1310×10^{-54}	3.2193×10^{-123}	4	-
	HMT	2.3769×10^{-43}	4.0133×10^{-208}	3	5.9289
$P(x_1, x_2, x_3)$	MCCT(0)	1.3178×10^{-66}	1.2841×10^{-162}	4	-
$x^{(0)} = (0.80, 0.60, 1.50)^T$	Newton	1.4817×10^{-63}	2.0520×10^{-126}	7	1.9802
	Ostrowski	2.3811×10^{-82}	2.8239×10^{-179}	5	-
	HMT	6.1154×10^{-24}	5.9378×10^{-139}	3	6.2016
$Q(x_1, x_2, x_3, x_4)$	MCCT(0)	2.4839×10^{-22}	2.0342×10^{-128}	3	5.6492
$x^{(0)} = (0.50, 0.50, 0.50, -0.20)^T$	Newton	3.2002×10^{-72}	7.8134×10^{-145}	7	2.0156
	Ostrowski	4.1778×10^{-49}	3.0779×10^{-157}	4	4.0962
	HMT	2.0321×10^{-44}	1.6859×10^{-208}	3	-

Table 4. Numerical performance of MCCT(0) on test problems for $x^{(0)} \approx 3\xi$.

System	$x^{(0)}$	$\|\|x^{(k+1)} - x^{(k)}\|\|$	$\|\|F(x^{(k+1)})\|\|$	Iter	ACOC
$F(x_1, x_2)$	$(3.00, 3.00)^T$	2.9511×10^{-49}	2.6815×10^{-195}	4	5.8233
$G(x_1, x_2)$	$(2.60, 1.50)^T$	1.0829×10^{-49}	6.7038×10^{-196}	4	5.8689
$M(x_1, x_2)$	$(10.41, -7.41)^T$	2.3053×10^{-41}	1.0439×10^{-113}	4	5.9611
$N(x_1, x_2)$	$(3.00, 0.00)^T$	nc	nc	nc	nc
$O(x_1, x_2)$	$(-0.76, -1.16)^T$	7.0387×10^{-71}	7.9136×10^{-174}	5	-
$P(x_1, x_2, x_3)$	$(2.73, 1.98, 4.73)^T$	6.8830×10^{-58}	4.1779×10^{-146}	5	-
$Q(x_1, x_2, x_3, x_4)$	$(1.73, 1.73, 1.73, -0.87)^T$	1.2880×10^{-33}	8.1032×10^{-180}	4	-

Table 5. Numerical performance of MCCT(0) on test problems for $x^{(0)} > 10\xi$.

System	$x^{(0)}$	$\|\|x^{(k+1)} - x^{(k)}\|\|$	$\|\|F(x^{(k+1)})\|\|$	Iter	ACOC
$F(x_1, x_2)$	$(11.00, 11.00)^T$	3.4914×10^{-55}	0	5	-
$G(x_1, x_2)$	$(9.53, 5.50)^T$	1.2350×10^{-55}	0	5	-
$M(x_1, x_2)$	$(38.18, -27.18)^T$	4.9654×10^{-57}	3.4199×10^{-145}	5	5.4814
$N(x_1, x_2)$	$(11.00, 0.00)^T$	nc	nc	nc	nc
$O(x_1, x_2)$	$(-2.79, -4.24)^T$	3.7780×10^{-39}	2.3868×10^{-110}	3	-
$P(x_1, x_2, x_3)$	$(10.01, 7.27, 17.33)^T$	1.6228×10^{-61}	2.8246×10^{-153}	14	-
$Q(x_1, x_2, x_3, x_4)$	$(6.35, 6.35, 6.35, -3.18)^T$	1.0412×10^{-45}	1.9467×10^{-208}	5	-

Table 6. Numerical performance of MCCT(200) on test problems for $x^{(0)} \approx \xi$.

System	$x^{(0)}$	$\|\|x^{(k+1)} - x^{(k)}\|\|$	$\|\|F(x^{(k+1)})\|\|$	Iter	ACOC
$F(x_1, x_2)$	$(0.90, 0.90)^T$	1.9038×10^{-29}	4.6447×10^{-116}	3	6.0626
$G(x_1, x_2)$	$(0.80, 0.40)^T$	5.1091×10^{-67}	1.9467×10^{-208}	4	-
$M(x_1, x_2)$	$(3.40, -2.40)^T$	1.6761×10^{-43}	2.8365×10^{-119}	3	5.9400
$N(x_1, x_2)$	$(0.90, 0.10)^T$	2.0202×10^{-48}	7.7869×10^{-208}	4	-
$O(x_1, x_2)$	$(-0.20, -0.30)^T$	3.8365×10^{-85}	6.7625×10^{-202}	4	-
$P(x_1, x_2, x_3)$	$(0.80, 0.60, 1.50)^T$	3.2604×10^{-41}	5.4472×10^{-112}	4	-
$Q(x_1, x_2, x_3, x_4)$	$(0.50, 0.50, 0.50, -0.20)^T$	8.7884×10^{-87}	1.9467×10^{-208}	4	5.6358

Table 7. Numerical performance of MCCT(200) on test problems for $x^{(0)} \approx 3\xi$.

System	$x^{(0)}$	$\|\|x^{(k+1)} - x^{(k)}\|\|$	$\|\|F(x^{(k+1)})\|\|$	Iter	ACOC
$F(x_1, x_2)$	$(3.00, 3.00)^T$	9.6219×10^{-49}	3.0304×10^{-193}	5	5.7669
$G(x_1, x_2)$	$(2.60, 1.50)^T$	3.4103×10^{-49}	7.5761×10^{-194}	5	5.8239
$M(x_1, x_2)$	$(10.41, -7.41)^T$	5.0005×10^{-75}	1.0922×10^{-182}	9	-
$N(x_1, x_2)$	$(3.00, 0.00)^T$	nc	nc	nc	nc
$O(x_1, x_2)$	$(-0.76, -1.16)^T$	nc	nc	nc	nc
$P(x_1, x_2, x_3)$	$(2.73, 1.98, 4.73)^T$	nc	nc	nc	nc
$Q(x_1, x_2, x_3, x_4)$	$(1.73, 1.73, 1.73, -0.87)^T$	5.9657×10^{-32}	4.5520×10^{-179}	5	5.8489

5. Conclusions

In conclusion, the designed class MCCT(α) for solving systems of nonlinear equations proves to be a highly efficient class with a convergence order of six.

We analyzed the convergence of the class of iterative schemes, assessed its stability using a real multidimensional discrete dynamical system, and verified its throughput performance numerically using several test problems.

The stable members of the MCCT(α) family exhibited outstanding numerical performance. The method for $\alpha = 0$ proved to be robust (stable) according to the real dynamics analysis performed. The method for $\alpha = 200$ was shown to be unstable and chaotic, and it may not converge to the searched solution. The theoretical order of convergence was verified by ACOC, and is close to six. Finally, numerical experiments were conducted to confirm the theoretical results.

Future lines of research consist of introducing a new step with similar characteristics to increase the order of convergence without considerable penalty to its computational cost, then analyzing its effect on the stability of the resulting family of methods.

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