mathematics

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Two New Modified Regularized Methods for Solving the Variational Inclusion and Null Point Problems

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Abstract: In this article, based on the regularization techniques, we construct two new algorithms combining the forward-backward splitting algorithm and the proximal contraction algorithm, respectively. Iterative sequences of the new algorithms can converge strongly to a common solution of the variational inclusion and null point problems in real Hilbert spaces. Multi-inertial extrapolation steps are applied to expedite their convergence rate. We also give some numerical experiments to certify that our algorithms are viable and efficient.

Keywords: variational inclusion; null point; regularized method; multi-step inertial iteration; strong convergence

MSC: 47H04; 47H05; 47H10; 65K10

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Citation: Wang, Y.; Li, M.; Yao, C.; Jiang, B. Two New Modified Regularized Methods for Solving the Variational Inclusion and Null Point Problems. *Mathematics* **2023**, *11*, 1469. https://doi.org/10.3390/math11061469

Academic Editors: Wei-Shih Du, Ravi P. Agarwal and Feng Qi

Received: 8 February 2023
Revised: 10 March 2023
Accepted: 13 March 2023
Published: 17 March 2023

1. Introduction

Let H be a real Hilbert space such that norm is $\|\cdot\|$ and the inner product is $\langle\cdot,\cdot\rangle$, respectively. We recall that the variational inclusion problem (VIP):

$$\text{Find } v^* \in H \text{ such that } 0 \in A(v^*) + B(v^*), \tag{1}$$

where $A : H \to 2^H$ is a set-valued operator and $B : H \to H$ is a single-valued operator. We denote the solution set of (1) by Φ. The variational inclusion problem is a crucial extension of the variational inequality problem. Many nonlinear problems such as problems of saddle point, minimization, and split feasibility can be transformed into variational inclusion problems which can be applied to signal processing, neural networks, medical image reconstruction, machine learning, and data mining, etc., see [1–7].

As we all know, (1) can be converted to the fixed point equation $v^* = J_{\lambda A}(v^* - \lambda B v^*)$ for some $\lambda > 0$, where $J_{\lambda A} = (I + \lambda A)^{-1}$ is the resolvent operator of A. The famous forward–backward splitting method (FBSM) was proposed by Lions and Mercier [8] in 1979:

$$x_{n+1} = J_{\lambda A}(I - \lambda B)x_n,$$

where A and B are maximally monotone and η-inverse strongly monotone, respectively, $\lambda \in (0, 2\eta)$. Note that the Lipschitz continuity of an operator is a weaker property than the inverse strong monotonicity. So the algorithm has a shortcoming: the convergence requires a strong hypothesis. In order to overcome this difficulty, Tseng [9] constructed a modified forward–backward splitting algorithm (TFBSM) in 2000:

$$\begin{cases} y_n = J_{\lambda A}(I - \lambda B)x_n, \\ x_{n+1} = y_n - \lambda(By_n - Bx_n), \end{cases}$$

where B is monotone and Lipschitz continuous.

On the other hand, a famous method for solutions of variational inequalities is the projection and contraction method which was first introduced by He [10] for the variational inequality problem in Euclidean space. Inspired by this, the following proximal contraction method (PCM) was proposed by Zhang and Wang [11] in 2018:

$$\begin{cases} y_n = J_{\lambda_n A}(x_n - \lambda_n B x_n), \\ h_n = x_n - y_n - \lambda_n(Bx_n - By_n), \\ x_{n+1} = x_n - r\beta_n h_n, \end{cases}$$

where $r \in (0,2)$,

$$\beta_n = \begin{cases} 0, & h_n = 0, \\ \frac{\phi(x_n,y_n)}{\|h_n\|^2}, & h_n \neq 0, \end{cases}$$

$\phi(x_n, y_n) = \langle x_n - y_n, h_n \rangle$, and the sequence of variable stepsizes $\{\lambda_n\}$ satisfies some conditions. Notice that both (TFBSM) and (PCM) can only get weak convergent results in real Hilbert spaces. In general, weakly convergent results are obviously less popular than strongly convergent ones. In order to get the strong convergence, Hieu et al. [12] gave an algorithm named the regularization proximal contraction method (RPCM), for solving (1) in 2021:

$$\begin{cases} y_n = J_{\lambda_n A}(x_n - \lambda_n(B + \alpha_n F)x_n), \\ h_n = x_n - y_n - \lambda_n(Bx_n - By_n), \\ x_{n+1} = x_n - r\beta_n h_n, \end{cases}$$

where $r \in (0,2)$, $\phi(x_n, y_n) = \langle w_n - y_n, h_n \rangle$,$\beta_n = \min\left\{\beta, \frac{\phi(x_n,y_n)}{\|h_n\|^2}\right\}$ and $\{\lambda_n\}$ satisfies some appropriate conditions. Before this, some scholars successfully applied this technique to the variational inequality problem. Very recently, Song and Bazighifan [13] introduced an inertial regularized method for solving the variational inequality and null point problem.

In recent years, there has been interest in methods with inertia which are considered effective methods to expedite the convergence. The inertial method is favored by many scholars because of its simple structure and easy operation, which is promoted by many scholars and in-depth research. In 2003, Moudafi and Oliny [14] combined (FBSM) with the inertial method to construct a new algorithm:

$$\begin{cases} y_n = x_n + \vartheta_n(x_n - x_{n-1}), \\ x_{n+1} = J_{\lambda_n A}(y_n - \lambda_n B x_n), \end{cases}$$

where $\{\lambda_n\}$ is a positive real sequence. Furthermore, some scholars have proposed multi-step inertial methods. In 2021, Wang et al. [15] proposed the multi-step inertial hybrid method to solve the problem (1).

Inspired by [12,13,15], we consider the variational inclusion and null point problem:

$$\text{Find } x^{\S} \in \Phi \cap G^{-1}(0) \text{ such that } \langle Fx^{\S}, x - x^{\S} \rangle \geq 0, \quad \forall x \in \Phi \cap G^{-1}(0), \tag{2}$$

where G and F are nonlinear operators. We propose two modified regularized multi-step inertial methods to solve the above problem. These two algorithms are the modified forward-backward splitting algorithm and the proximal contraction algorithm. Using regularization techniques, the new algorithms converge strongly under mild conditions. Some numerical examples are given to show that our algorithms are efficient.

This article is arranged as follows: we introduce some notations, fundamental definitions, and results that are used in later proofs in Section 2. In Section 3, we present the new algorithms and discuss their convergence. In Section 4, we report some numerical experiments to support our theoretical results obtained.

2. Preliminaries

Let H be a real Hilbert space. The weak convergence and strong convergence of sequence $\{x_n\}$ are denoted by $x_n \rightharpoonup x$ and $x_n \to x$, respectively.

Definition 1 ([16]). *The mapping $T : H \to H$ is called*

(i) *monotone, if*
$$\langle Ty - Tx, y - x\rangle \geq 0, \quad \forall x, y \in H;$$

(ii) *γ-strongly monotone ($\gamma > 0$), if*
$$\langle Ty - Tx, y - x\rangle \geq \gamma\|y - x\|^2, \quad \forall x, y \in H;$$

(iii) *δ-inverse strongly monotone ($\delta > 0$), if*
$$\langle Ty - Tx, y - x\rangle \geq \delta\|Ty - Tx\|^2, \quad \forall x, y \in H;$$

(iv) *l-Lipschitz continuous ($l > 0$), if*
$$\|Ty - Tx\| \leq l\|y - x\|, \quad \forall x, y \in H;$$

(v) *firmly nonexpansive, if*
$$\langle Ty - Tx, y - x\rangle \geq \|Ty - Tx\|^2, \quad \forall x, y \in H;$$

(vi) *nonexpansive, if*
$$\|Ty - Tx\| \leq \|y - x\|, \quad \forall x, y \in H.$$

Definition 2 ([16]). *Let $T : H \to 2^H$ be a set-valued mapping. The graph of T is defined by* $\mathrm{Graph}(T) = \{(x, u) : x \in H, u \in Tx\}$. *The mapping T is said to be*

(i) *monotone, if*
$$\langle v - u, y - x\rangle \geq 0, \quad \forall u \in Tx, v \in Ty;$$

(ii) *maximally monotone, if T is monotone on H and for any $(y, v) \in H \times H$,*
$$\langle v - u, y - x\rangle \geq 0, \quad \forall (x, u) \in \mathrm{Graph}(T) \text{ indicates } (y, v) \in \mathrm{Graph}(T).$$

Lemma 1 ([17]). *Let $A : H \to 2^H$ be a maximally monotone operator, and $B : H \to H$ be a monotone Lipschitz continuous operator. Then $A + B$ is maximally monotone.*

Lemma 2 ([18]). *Let $\{t_n\}$ be a of nonnegative real sequence satisfying*
$$t_{n+1} \leq (1 - \beta_n)t_n + \beta_n d_n + \varrho_n, \quad \forall x, y \in H,$$
where $\{\beta_n\}, \{d_n\}$ and $\{\varrho_n\}$ satisfying the conditions:

(i) $\{\beta_n\} \subset (0, 1)$, $\sum_{n=1}^{\infty} \beta_n = \infty$;
(ii) $\limsup_{n\to\infty} d_n \leq 0$;
(iii) $\varrho_n \geq 0$ *with* $\sum_{n=1}^{\infty} \varrho_n < \infty$.
Then $\lim_{n\to\infty} t_n = 0$.

Lemma 3 ([19]). *Let C be a nonempty closed convex subset of H and $T : C \to C$ be a nonexpansive mapping. Then, the mapping $I - T$ is demiclosed at zero, i.e., if $x_n \rightharpoonup x$ and $(I - T)x_n \to 0$, then $x \in Fix(T)$.*

3. Main Results

We mainly introduce our new algorithms and analyze their convergence in this section. Let H be a real Hilbert space. The following assumptions will be needed throughout the paper:

(A1) $A : H \to 2^H$ is maximally monotone.
(A2) $B : H \to H$ is monotone and L-Lipschitz continuous.
(A3) $F : H \to H$ is ξ-strongly monotone and k-Lipschitz continuous.
(A4) $G : H \to H$ is γ-inverse strongly monotone.
(A5) $\Omega := \Phi \cap G^{-1}(0) \neq \emptyset$, where Φ is the solution set of (1).

To solve (2), we construct a auxiliary problem:

$$\text{Find } x \in H, \text{ such that } 0 \in A(x) + B(x) + \alpha^{\omega} G(x) + \alpha F(x), \tag{3}$$

for each $\alpha > 0$ and $0 < \omega < 1$, the solution of the problem (3) denoted by x_α.

Lemma 4. *Under the assumptions (A1)–(A4), for each $\alpha > 0$ and $0 < \omega < 1$, the problem (3) has a unique solution x_α.*

Proof. Since the properties of A, B, G, and F in the hypothesis, we can conclude that $A + B + \alpha^{\omega} G + \alpha F$ is strongly monotone. It is well known that strong monotone operators have unique solutions (see [20]). Therefore, the problem (3) has a unique solution x_α. □

Lemma 5. *The net $\{x_\alpha\}$ is bounded.*

Proof. For each $p \in \Omega$ and $\alpha > 0$, we have $0 \in Ap + Bp$, $Gp = 0$ and $0 \in Ax_\alpha + Bx_\alpha + \alpha^{\omega} Gx_\alpha + \alpha F x_\alpha$. Thus,

$$-\alpha F x_\alpha \in Ax_\alpha + Bx_\alpha + \alpha^{\omega} Gx_\alpha,$$

and

$$0 \in Ap + Bp + \alpha^{\omega} Gp.$$

Using the monotonic property of A, B and G, we derive

$$\langle p - x_\alpha, \alpha F x_\alpha \rangle \geq 0. \tag{4}$$

By (4) and the ξ-strong monotonicity, it follows that

$$\begin{aligned} \langle p - x_\alpha, Fp \rangle &= \langle p - x_\alpha, F x_\alpha \rangle + \langle p - x_\alpha, Fp - F x_\alpha \rangle \\ &\geq \xi \|p - x_\alpha\|^2. \end{aligned} \tag{5}$$

Consequently (5) and the Cauchy-Schwarz inequality, we find $\|Fp\| \|p - x_\alpha\| \geq \xi \|p - x_\alpha\|^2$, then $\|p - x_\alpha\| \leq \|Fp\| / \xi$, we get

$$\begin{aligned} \|x_\alpha\| &\leq \|p\| + \|p - x_\alpha\| \\ &\leq \|p\| + \frac{\|Fp\|}{\xi}. \end{aligned}$$

So the net $\{x_\alpha\}$ is bounded. □

Lemma 6. *For all $\alpha_1, \alpha_2 \in (0, 1)$, there exists $M > 0$ such that,*

$$\|x_{\alpha_1} - x_{\alpha_2}\| \leq \frac{|\alpha_2 - \alpha_1|}{\alpha_1 \alpha_2} M.$$

Proof. According to the assumption, $x_{\alpha_1}, x_{\alpha_2}$ are solutions of the problem (3), let us suppose that $0 < \alpha_2 < \alpha_1 < 1$. Then,

$$0 \in Ax_{\alpha_1} + Bx_{\alpha_1} + \alpha_1^{\omega} Gx_{\alpha_1} + \alpha_1 Fx_{\alpha_1}$$

and

$$0 \in Ax_{\alpha_2} + Bx_{\alpha_2} + \alpha_2^{\omega} Gx_{\alpha_2} + \alpha_2 Fx_{\alpha_2},$$

which implies

$$-\alpha_1^{\omega} Gx_{\alpha_1} - \alpha_1 Fx_{\alpha_1} \in (A+B)x_{\alpha_1}$$

and

$$-\alpha_2^{\omega} Gx_{\alpha_2} - \alpha_2 Fx_{\alpha_2} \in (A+B)x_{\alpha_2}.$$

By Lemma 1, we know that

$$\langle x_{\alpha_1} - x_{\alpha_2}, -\alpha_1 Fx_{\alpha_1} - \alpha_1^{\omega} Gx_{\alpha_1} + \alpha_2 Fx_{\alpha_2} + \alpha_2^{\omega} Gx_{\alpha_2} \rangle \geq 0,$$

or, equivalently,

$$\begin{aligned} &\langle x_{\alpha_1} - x_{\alpha_2}, (\alpha_2 - \alpha_1) Fx_{\alpha_2} \rangle + \langle x_{\alpha_1} - x_{\alpha_2}, \alpha_1 (Fx_{\alpha_2} - Fx_{\alpha_1}) \rangle \\ &+ \langle x_{\alpha_1} - x_{\alpha_2}, (\alpha_2^{\omega} - \alpha_1^{\omega}) Gx_{\alpha_2} \rangle + \langle x_{\alpha_1} - x_{\alpha_2}, \alpha_1^{\omega} (Gx_{\alpha_2} - Gx_{\alpha_1}) \rangle \geq 0. \end{aligned}$$

The properties of G and F and the Cauchy-Schwarz inequality imply that

$$\begin{aligned} \alpha_1 \xi \|x_{\alpha_1} - x_{\alpha_2}\|^2 &\leq (\alpha_2^{\omega} - \alpha_1^{\omega}) \langle x_{\alpha_1} - x_{\alpha_2}, Gx_{\alpha_2} \rangle + (\alpha_2 - \alpha_1) \langle x_{\alpha_1} - x_{\alpha_2}, Fx_{\alpha_2} \rangle \\ &\leq |\alpha_2^{\omega} - \alpha_1^{\omega}| \|x_{\alpha_1} - x_{\alpha_2}\| \|Gx_{\alpha_2}\| + |\alpha_2 - \alpha_1| \|x_{\alpha_1} - x_{\alpha_2}\| \|Fx_{\alpha_2}\| \end{aligned}$$

which equal to

$$\|x_{\alpha_1} - x_{\alpha_2}\| \leq \frac{|\alpha_2^{\omega} - \alpha_1^{\omega}| \|Gx_{\alpha_2}\| + |\alpha_2 - \alpha_1| \|Fx_{\alpha_2}\|}{\alpha_1 \xi}. \tag{6}$$

The Lipschitz continuity of the mapping F and G imply they are bounded. Combining the Lagrange's mean-value theorem, we deduce that

$$|\alpha_2^{\omega} - \alpha_1^{\omega}| = \alpha_1^{\omega} - \alpha_2^{\omega} \leq \omega \alpha_2^{\omega - 1} (\alpha_1 - \alpha_2) \leq \omega \alpha_2^{-1} (\alpha_1 - \alpha_2) \leq \alpha_2^{-1} (\alpha_1 - \alpha_2),$$

this together with (6), implies that

$$\|x_{\alpha_1} - x_{\alpha_2}\| \leq \frac{|\alpha_2 - \alpha_1|}{\alpha_1 \alpha_2} \frac{\|Gx_{\alpha_2}\|}{\xi} + \frac{|\alpha_2 - \alpha_1|}{\alpha_1 \alpha_2} \frac{\|Fx_{\alpha_2}\|}{\xi} \leq \frac{|\alpha_2 - \alpha_1|}{\alpha_1 \alpha_2} M, \tag{7}$$

where $M = \frac{1}{\xi} \sup_{\alpha \in (0,1)} \{\|Gx_\alpha\| + \|Fx_\alpha\|\}$. Indeed, since F and G are Lipschitz continuous, the net $\{\|Gx_\alpha\|\}$ and $\{\|Fx_\alpha\|\}$ is bounded. If $0 < \alpha_1 \leq \alpha_2 < 1$, we can also get the same results. □

Lemma 7. $\lim_{\alpha \to 0^+} x_\alpha = x^{\S}$.

Proof. According to the conclusion of Lemma 5, there exists a subsequence $\{x_{\alpha_m}\}$ of the net $\{x_\alpha\}$ such that $x_{\alpha_m} \rightharpoonup \overline{x}$ and $\alpha_m \to 0^+$ as $m \to \infty$. From RVI, we have that $-Bx_\alpha - \alpha^{\omega} Gx_\alpha - \alpha Fx_\alpha \in Ax_\alpha$. Let us take a point (u, v) in Graph$(A+B)$, that is, $v \in Au + Bu$. Thus, we derive by the assumption (A1),

$$\langle u - x_\alpha, v - Bu + Bx_\alpha + \alpha^{\omega} Gx_\alpha + \alpha Fx_\alpha \rangle \geq 0.$$

Replace α with α_m, we deduce from the monotonicity of B that

$$
\begin{aligned}
0 &\leq \langle u - x_{\alpha_m}, v - Bu + Bx_{\alpha_m} + \alpha_m^{\omega} G x_{\alpha_m} + \alpha_m F x_{\alpha_m} \rangle \\
&= \langle u - x_{\alpha_m}, \alpha_m^{\omega} G x_{\alpha_m} + \alpha_m F x_{\alpha_m} \rangle + \langle u - x_{\alpha_m}, v \rangle - \langle x_{\alpha_m} - u, Bx_{\alpha_m} - Bu \rangle \\
&\leq \langle u - x_{\alpha_m}, \alpha_m^{\omega} G x_{\alpha_m} + \alpha_m F x_{\alpha_m} \rangle + \langle u - x_{\alpha_m}, v \rangle.
\end{aligned} \tag{8}
$$

It obtains that the sequence $\{Fx_{\alpha_m}\}$ is bounded by the boundedness of the sequence $\{x_{\alpha_m}\}$ and the Lipschitz continuity of F. Letting $m \to \infty$ in relation (8) and we infer that

$$\langle u - \overline{x}, v \rangle \geq 0, \quad \forall (u, v) \in \mathrm{Graph}(A + B),$$

$$\overline{x} \in (A + B)^{-1}(0). \tag{9}$$

For every $q \in \Omega$, $0 \in Aq + Bq$ and $Gq = 0$. By (3), we obtain

$$-\alpha_m^{\omega} G x_{\alpha_m} - \alpha_m F x_{\alpha_m} \in Ax_{\alpha_m} + Bx_{\alpha_m},$$

due to the definition of $A + B$, we know that

$$\langle x_{\alpha_m} - q, -\alpha_m^{\omega} G x_{\alpha_m} - \alpha_m F x_{\alpha_m} \rangle \geq 0,$$

by the monotonicity of F,

$$
\begin{aligned}
\alpha_m^{\omega} \langle G x_{\alpha_m}, x_{\alpha_m} - q \rangle &\leq \alpha_m \langle F x_{\alpha_m}, q - x_{\alpha_m} \rangle \\
&\leq \alpha_m \langle Fq, q - x_{\alpha_m} \rangle,
\end{aligned}
$$

which leads to

$$\langle G x_{\alpha_m}, x_{\alpha_m} - q \rangle \leq \alpha_m^{1-\omega} \langle Fq, q - x_{\alpha_m} \rangle \to 0. \tag{10}$$

By the property of G, noting (10) and $Gq = 0$, we obtain

$$
\begin{aligned}
\gamma \| G x_{\alpha_m} \|^2 &= \gamma \| G x_{\alpha_m} - Gq \|^2 \\
&= \langle G x_{\alpha_m} - Gq, x_{\alpha_m} - q \rangle \\
&\leq \langle G x_{\alpha_m}, x_{\alpha_m} - q \rangle \to 0,
\end{aligned}
$$

which yields that

$$\lim_{m \to \infty} G x_{\alpha_m} = 0.$$

For any $\iota \in (0, 2\gamma]$, $G_\iota = I - \iota G$ is nonexpansive obviously holds. Owing to Lemma 3, we obtain that $\overline{x} \in \mathrm{Fix}(G_\iota)$,

$$\overline{x} \in G^{-1}(0),$$

together with (9), implies

$$\overline{x} \in \Omega.$$

Noting (5), we obtain $\langle Fp, p - x_\alpha \rangle \geq 0$ for all $p \in \Omega$. Letting $\alpha = \alpha_m \to 0^+$, we have

$$\langle Fp, p - \overline{x} \rangle \geq 0, \quad \forall p \in \Omega.$$

By Minty lemma [21], we get

$$\langle F\overline{x}, p - \overline{x} \rangle \geq 0, \quad \forall p \in \Omega.$$

Due to uniqueness of the solution $x^{\S}$ to the problem (2), we have $\overline{x} = x^{\S}$. Since $\overline{x}$ is any point in $\omega_w(x_\alpha)$, $\omega_w(x_\alpha) = \{x^{\S}\}$, that is, the net $\{x_\alpha\}$ converges weakly to $x^{\S}$. After that, applying (5) for $p = x^{\S}$, we get

$$\xi\|x^{\S} - x_\alpha\|^2 \leq \langle Fx^{\S}, x^{\S} - x_\alpha\rangle. \tag{11}$$

Taking limit in (11) as $\alpha \to 0^+$, we obtain

$$\lim_{\alpha\to 0^+} \xi\|x^{\S} - x_\alpha\|^2 \leq \lim_{\alpha\to 0^+} \langle Fx^{\S}, x^{\S} - x_\alpha\rangle = 0.$$

Thus, $\lim_{\alpha\to 0^+} \|x^{\S} - x_\alpha\| = 0$. $\square$

Remark 1. *α_n can be chosen as $\alpha_n = \frac{1}{n^p}$, where $0 < p < \frac{1}{2}$.*

Lemma 8. *Under the condition (A2), the sequence $\{\lambda_n\}$ generated by Algorithm 1 or Algorithm 2 is convergent and*

$$\lim_{n\to\infty} \lambda_n = \lambda > 0.$$

To be more precise, we have $\lambda \geq \min\{\lambda_1, \frac{\mu}{L}\} > 0$.

Algorithm 1 Modified multi-steps inertial forward-backward splitting method with regularization

Initialization: Let $x_0, x_1 \in H$ be arbitrary, $\mu \in (0,1)$, $\lambda_1 \in (0,(1-\mu)\gamma)$ and set $n := 1$. Choose a sequence $\{\tau_n\} \subset [0,+\infty)$ such that $\sum_{n=1}^{\infty} \tau_n = \tau < \infty$ and $0 < \mu + \frac{\lambda_1+\tau}{\gamma} < 1$. Choose a sequence$\{\alpha_n\} \subset [0,+\infty)$ satisfying:

$$\sum_{n=1}^{\infty} \alpha_n = \infty, \quad \lim_{n\to\infty} \alpha_n = 0, \quad \lim_{n\to\infty} \frac{\alpha_{n+1} - \alpha_n}{\alpha_{n+1}\alpha_n^2} = 0.$$

For a given positive integer N, choose a sequence $\{\epsilon_{i,n}\} \subset [0,+\infty)$ $(i = 1,2,\ldots,N)$ satisfying

$$\lim_{n\to+\infty} \frac{\epsilon_{i,n}}{\alpha_n} = 0.$$

Iterative steps: Calculate x_{n+1} as follows:
Step 1. Compute

$$w_n = x_n + \sum_{i=1}^{\min\{n,N\}} \theta_{i,n}(x_{n-i+1} - x_{n-i}),$$

where $0 \leq \theta_{i,n} \leq \theta_i$ for some $\theta_i \in \mathbb{R}$ with

$$\theta_{i,n} = \begin{cases} \min\Big\{\theta_i, \frac{\epsilon_{i,n}}{\|x_{n-i+1}-x_{n-i}\|}\Big\}, & \text{if } x_{n-i+1} \neq x_{n-i}, \\ \theta_i, & \text{otherwise.} \end{cases}$$

Step 2. Compute

$$y_n = J_{\lambda_n A}\Big(w_n - \lambda_n(B + \alpha_n^{\omega} G + \alpha_n F)w_n\Big).$$

Step 3. Compute

$$x_{n+1} = y_n - \lambda_n(By_n - Bw_n + \alpha_n^{\omega} Gy_n - \alpha_n^{\omega} Gw_n),$$

and

$$\lambda_{n+1} = \begin{cases} \min\Big\{\lambda_n + \tau_n, \frac{\mu\|w_n - y_n\|}{\|Bw_n - By_n\|}\Big\}, & \text{if } Bw_n \neq By_n, \\ \lambda_n + \tau_n, & \text{otherwise.} \end{cases}$$

Set $n = n + 1$ and go to **Step 1**.

Algorithm 2 Modified multi-steps inertial proximal contraction method with regularization

Initialization: Let $x_0, x_1 \in H$ be arbitrary, $r \in (0,2)$, $\beta > 0$, $\mu \in (0,1)$, and $\lambda_1 \in (0,(1-\mu)\gamma)$ and set $n := 1$. Choose a sequence $\{\tau_n\} \subset [0,+\infty)$ such that $\sum_{n=1}^{\infty} \tau_n = \tau < \infty$ and $0 < \mu + \frac{\lambda_1+\tau}{\gamma} < 1$. Choose a sequence $\{\alpha_n\} \subset [0,+\infty)$ satisfying:

$$\sum_{n=1}^{\infty} \alpha_n = \infty, \quad \lim_{n\to\infty} \alpha_n = 0, \quad \lim_{n\to\infty} \frac{\alpha_{n+1}-\alpha_n}{\alpha_{n+1}\alpha_n^2} = 0.$$

For a given positive integer N, choose a sequence $\{\epsilon_{i,n}\} \subset [0,+\infty)$ $(i = 1,2,\ldots,N)$ satisfying

$$\lim_{n\to\infty} \frac{\epsilon_{i,n}}{\alpha_n} = 0.$$

Iterative steps: Calculate x_{n+1} as follows:
Step 1. Compute

$$w_n = x_n + \sum_{i=1}^{\min\{N,n\}} \theta_{i,n}(x_{n-i+1} - x_{n-i}),$$

where $0 \le \theta_{i,n} \le \theta_i$ for some $\theta_i \in \mathbb{R}$ with

$$\theta_{i,n} = \begin{cases} \min\Big\{\theta_i, \frac{\epsilon_{i,n}}{\|x_{n-i+1}-x_{n-i}\|}\Big\}, & \text{if } x_{n-i+1} \neq x_{n-i}, \\ \theta_i, & \text{otherwise.} \end{cases}$$

Step 2. Compute

$$y_n = J_{\lambda_n A}\Big(w_n - \lambda_n(B + \alpha_n^{\omega} G + \alpha_n F)w_n\Big).$$

and

$$\lambda_{n+1} = \begin{cases} \min\Big\{\lambda_n + \tau_n, \frac{\mu\|w_n-y_n\|}{\|Bw_n-By_n\|}\Big\}, & \text{if } Bw_n \neq By_n, \\ \lambda_n + \tau_n, & \text{otherwise.} \end{cases}$$

step 3. Compute

$$\begin{cases} h_n = w_n - y_n - \lambda_n\Big((Bw_n - By_n) + \alpha_n^{\omega}(Gw_n - Gy_n)\Big), \\ \phi(w_n, y_n) = \langle w_n - y_n, h_n\rangle. \end{cases}$$

Step 4. Compute

$$x_{n+1} = w_n - r\beta_n h_n,$$

where

$$\beta_n = \begin{cases} \frac{\phi(w_n,y_n)}{\|h_n\|^2}, & \text{if } \|h_n\| \neq 0, \\ \beta, & \text{otherwise.} \end{cases}$$

Set $n = n+1$ and go to **Step 1**.

Proof. Since

$$\|Bw_n - By_n\| \le L\|w_n - y_n\|,$$

in the case of $Bw_n \neq By_n$,

$$\frac{\mu\|w_n - y_n\|}{\|Bw_n - By_n\|} \ge \frac{\mu\|w_n - y_n\|}{L\|w_n - y_n\|} = \frac{\mu}{L}.$$

By induction, can draw the sequence $\{\lambda_n\}$ has the lower bound $\min\{\lambda_1, \frac{\mu}{L}\}$. Since the computation of λ_{n+1}, we can get

$$\lambda_{n+1} \le \lambda_n + \tau_n,$$

that is

$$\lambda_{n+1} - \lambda_n \leq \tau_n.$$

Let $[a]_+$ represent $\max\{a, 0\}$ for all $a \in \mathbb{R}$. And we know $\tau_n \geq 0$, then

$$[\lambda_{n+1} - \lambda_n]_+ \leq \tau_n.$$

Because $\sum_{n=1}^{\infty} \tau_n < \infty$, obviously

$$\sum_{n=1}^{\infty} [\lambda_{n+1} - \lambda_n]_+ < \infty.$$

Besides $[a]_+ = \frac{1}{2}a + \frac{1}{2}|a|$, we infer

$$|\lambda_{n+1} - \lambda_n| = 2[\lambda_{n+1} - \lambda_n]_+ - \lambda_{n+1} + \lambda_n,$$

then,

$$\sum_{n=1}^{k} |\lambda_{n+1} - \lambda_n| = 2\sum_{n=1}^{k} [\lambda_{n+1} - \lambda_n]_+ - \lambda_{k+1} + \lambda_1.$$

Since $\{\lambda_n\}$ has the lower bound $\min\{\lambda_1, \frac{\mu}{L}\}$, we know $\lambda_{k+1} > 0$. So we have

$$\sum_{n=1}^{k} |\lambda_{n+1} - \lambda_n| < 2\sum_{n=1}^{k} [\lambda_{n+1} - \lambda_n]_+ + \lambda_1,$$

furthermore,

$$\sum_{n=1}^{\infty} |\lambda_{n+1} - \lambda_n| < \infty.$$

Therefore, $\{\lambda_n\}$ is convergent. □

Theorem 1. *If the conditions (A1)–(A5) hold, $x^{\S}$ is the unique solution of problem (2) and the sequence $\{x_n\}$ is generated by Algorithm 1, then x_n converges strongly to $x^{\S}$.*

Proof. Setting $s_n = By_n - Bw_n + \alpha_n^{\omega} Gy_n - \alpha_n^{\omega} Gw_n$,

$$\begin{aligned} \|x_{n+1} - x_{\alpha_n}\|^2 &= \|y_n - \lambda_n s_n - x_{\alpha_n}\|^2 \\ &= \|y_n - x_{\alpha_n}\|^2 + \lambda_n^2 \|s_n\|^2 - 2\lambda_n \langle y_n - x_{\alpha_n}, s_n \rangle. \end{aligned} \tag{12}$$

Since x_{α_n} is the solution of (3), we get

$$x_{\alpha_n} = J_{\lambda_n A}\Big(x_{\alpha_n} - \lambda_n (Bx_{\alpha_n} + \alpha_n^{\omega} Gx_{\alpha_n} + \alpha_n Fx_{\alpha_n})\Big),$$

and $J_{\lambda_n A}$ is firmly nonexpansive,

$$\begin{aligned} &\langle y_n - x_{\alpha_n}, w_n - x_{\alpha_n} - \lambda_n (Bw_n + \alpha_n^{\omega} Gw_n + \alpha_n Fw_n \\ &-Bx_{\alpha_n} - \alpha_n^{\omega} Gx_{\alpha_n} - \alpha_n Fx_{\alpha_n})\rangle \geq \|y_n - x_{\alpha_n}\|^2, \end{aligned}$$

which implies

$$\begin{aligned} &\langle y_n - x_{\alpha_n}, w_n - x_{\alpha_n}\rangle - \lambda_n \langle y_n - x_{\alpha_n}, Bw_n + \alpha_n^{\omega} Gw_n - Bx_{\alpha_n} - \alpha_n^{\omega} Gx_{\alpha_n}\rangle \\ &-\alpha_n \lambda_n \langle y_n - x_{\alpha_n}, Fw_n - Fx_{\alpha_n}\rangle \geq \|y_n - x_{\alpha_n}\|^2. \end{aligned} \tag{13}$$

Since the monotony of B and G, we find

$$\lambda_n \langle y_n - x_{\alpha_n}, By_n - Bx_{\alpha_n} + \alpha_n^{\omega} Gy_n - \alpha_n^{\omega} Gx_{\alpha_n}\rangle \geq 0, \tag{14}$$

combining (13) and (14), we derive

$$\begin{aligned} & \lambda_n \langle y_n - x_{\alpha_n}, By_n - Bw_n + \alpha_n^\omega G y_n - \alpha_n^\omega G w_n \rangle \\ \geq \quad & \langle y_n - x_{\alpha_n}, x_{\alpha_n} - w_n \rangle + \alpha_n \lambda_n \langle y_n - x_{\alpha_n}, F w_n - F x_{\alpha_n} \rangle + \|y_n - x_{\alpha_n}\|^2, \end{aligned}$$

or, equivalently,

$$\begin{aligned} & \langle y_n - x_{\alpha_n}, s_n \rangle \\ \geq \quad & \frac{1}{\lambda_n} \langle y_n - x_{\alpha_n}, x_{\alpha_n} - w_n \rangle + \alpha_n \langle y_n - x_{\alpha_n}, F w_n - F x_{\alpha_n} \rangle \\ & + \frac{1}{\lambda_n} \|y_n - x_{\alpha_n}\|^2. \end{aligned} \tag{15}$$

Combining (12) and (15), we get that

$$\begin{aligned} \|x_{n+1} - x_{\alpha_n}\|^2 \leq \lambda_n^2 \|s_n\|^2 - 2\langle y_n - x_{\alpha_n}, x_{\alpha_n} - w_n \rangle \\ - 2\alpha_n \lambda_n \langle y_n - x_{\alpha_n}, F w_n - F x_{\alpha_n} \rangle - \|y_n - x_{\alpha_n}\|^2, \end{aligned} \tag{16}$$

and the fact of

$$2\langle y_n - x_{\alpha_n}, x_{\alpha_n} - w_n \rangle = \|y_n - w_n\|^2 - \|w_n - x_{\alpha_n}\|^2 - \|x_{\alpha_n} - y_n\|^2, \tag{17}$$

imply that

$$\begin{aligned} \|x_{n+1} - x_{\alpha_n}\|^2 \quad \leq \quad & \|x_{\alpha_n} - w_n\|^2 - \|w_n - y_n\|^2 + \lambda_n^2 \|s_n\|^2 \\ & -2\alpha_n \lambda_n \langle y_n - x_{\alpha_n}, F w_n - F x_{\alpha_n} \rangle. \end{aligned} \tag{18}$$

Since

$$\begin{aligned} \lambda_n^2 \|s_n\|^2 \quad = \quad & \lambda_n^2 \|By_n - Bw_n\|^2 + \lambda_n^2 \alpha_n^{2\omega} \|G y_n - G w_n\|^2 \\ & +2\lambda_n^2 \alpha_n^\omega \langle By_n - Bw_n, G y_n - G w_n \rangle \\ \leq \quad & \frac{\lambda_n^2 \mu^2}{\lambda_{n+1}^2} \|w_n - y_n\|^2 + \frac{\lambda_n^2 \alpha_n^{2\omega}}{\gamma^2} \|w_n - y_n\|^2 + \frac{2\lambda_n^2 \mu \alpha_n^\omega}{\gamma \lambda_{n+1}} \|w_n - y_n\|^2 \\ \leq \quad & \left(\frac{\mu \lambda_n}{\lambda_{n+1}} + \frac{\lambda_1 + \tau}{\gamma} \right)^2 \|w_n - y_n\|^2. \end{aligned} \tag{19}$$

Let t_1, t_2 and t_3 be three positive numbers such that

$$2\xi - k t_1 - t_2 - t_3 > 0.$$

By virtue of Lemma 8, $\alpha_n \to 0$ and $\frac{\epsilon_{i,n}}{\alpha_n} \to 0$, there exists $n_0 \geq 1$, $\forall n \geq n_0$ such that

$$\begin{aligned} & 1 - \frac{\alpha_n \lambda_n k}{t_1} - \left(\frac{\mu \lambda_n}{\lambda_{n+1}} + \frac{\lambda_1 + \tau}{\gamma} \right)^2 > 0, \\ & 1 - t_3 \alpha_n \lambda_n > 0, \\ & \sum_{i=1}^{N} \epsilon_{i,n} \leq t_3 \lambda_n \alpha_n. \end{aligned}$$

Because F is strongly monotone,

$$\begin{aligned}
& \langle y_n - x_{\alpha_n}, Fw_n - Fx_{\alpha_n} \rangle \\
= \; & \langle w_n - x_{\alpha_n}, Fw_n - Fx_{\alpha_n} \rangle + \langle y_n - w_n, Fw_n - Fx_{\alpha_n} \rangle \\
\geq \; & \xi \| w_n - x_{\alpha_n} \|^2 - k \| w_n - x_{\alpha_n} \| \| y_n - w_n \| \\
\geq \; & \xi \| w_n - x_{\alpha_n} \|^2 - \frac{kt_1}{2} \| w_n - x_{\alpha_n} \|^2 - \frac{k}{2t_1} \| y_n - w_n \|^2 \\
= \; & \left(\xi - \frac{kt_1}{2} \right) \| w_n - x_{\alpha_n} \|^2 - \frac{k}{2t_1} \| y_n - w_n \|^2.
\end{aligned} \tag{20}$$

In views of (18)–(20), we get

$$\begin{aligned}
& \| x_{n+1} - x_{\alpha_n} \|^2 \\
\leq \; & (1 - \alpha_n \lambda_n (2\xi - kt_1)) \| w_n - x_{\alpha_n} \|^2 \\
& - \left(1 - \frac{\alpha_n \lambda_n k}{t_1} - \left(\frac{\lambda_1 + \tau}{\gamma} + \frac{\mu \lambda_n}{\lambda_{n+1}} \right)^2 \right) \| w_n - y_n \|^2.
\end{aligned} \tag{21}$$

which implies,

$$\| x_{n+1} - x_{\alpha_n} \|^2 \leq (1 - \alpha_n \lambda_n (2\xi - kt_1)) \| w_n - x_{\alpha_n} \|^2, \ \forall n \geq n_0. \tag{22}$$

By Lemma 6, for all $n \geq n_0$, we have

$$\begin{aligned}
& \| x_{n+1} - x_{\alpha_{n+1}} \|^2 \\
= \; & 2 \langle x_{\alpha_n} - x_{\alpha_{n+1}}, x_{n+1} - x_{\alpha_n} \rangle + \| x_{\alpha_n} - x_{\alpha_{n+1}} \|^2 + \| x_{n+1} - x_{\alpha_n} \|^2 \\
\leq \; & 2 \| x_{\alpha_n} - x_{\alpha_{n+1}} \| \| x_{n+1} - x_{\alpha_n} \| + \| x_{\alpha_{n+1}} - x_{\alpha_n} \|^2 + \| x_{n+1} - x_{\alpha_n} \|^2 \\
\leq \; & \frac{1}{t_2 \alpha_n \lambda_n} \| x_{\alpha_n} - x_{\alpha_{n+1}} \|^2 + t_2 \alpha_n \lambda_n \| x_{n+1} - x_{\alpha_n} \|^2 + \| x_{n+1} - x_{\alpha_{n+1}} \|^2 \\
& + \| x_{\alpha_n} - x_{\alpha_{n+1}} \|^2 \\
= \; & \left(1 + \frac{1}{t_2 \alpha_n \lambda_n} \right) \| x_{\alpha_n} - x_{\alpha_{n+1}} \|^2 + (1 + t_2 \alpha_n \lambda_n) \| x_{n+1} - x_{\alpha_n} \|^2 \\
\leq \; & \left(1 + \frac{1}{t_2 \alpha_n \lambda_n} \right) \left(\frac{\alpha_{n+1} - \alpha_n}{\alpha_n \alpha_{n+1}} \right)^2 M^2 + (1 + t_2 \alpha_n \lambda_n) \| x_{n+1} - x_{\alpha_n} \|^2,
\end{aligned} \tag{23}$$

where M appears in Lemma 6. Substituting (23) into (22), for all $n \geq n_0$, we deduce

$$\begin{aligned}
& \| x_{n+1} - x_{\alpha_{n+1}} \|^2 \\
\leq \; & (1 + t_2 \alpha_n \lambda_n)(1 - \alpha_n \lambda_n (2\xi - kt_1)) \| w_n - x_{\alpha_n} \|^2 \\
& + \left(1 + \frac{1}{t_2 \alpha_n \lambda_n} \right) \left(\frac{\alpha_{n+1} - \alpha_n}{\alpha_n \alpha_{n+1}} \right)^2 M^2 \\
= \; & (1 - (2\xi - kt_1 - t_2) \alpha_n \lambda_n - (2\xi - kt_1) t_2 \alpha_n^2 \lambda_n^2) \| w_n - x_{\alpha_n} \|^2 \\
& + \left(1 + \frac{1}{t_2 \alpha_n \lambda_n} \right) \left(\frac{\alpha_{n+1} - \alpha_n}{\alpha_n \alpha_{n+1}} \right)^2 M^2 \\
\leq \; & (1 - (2\xi - kt_1 - t_2) \alpha_n \lambda_n) \| w_n - x_{\alpha_n} \|^2 \\
& + \frac{(1 + t_2 \alpha_n \lambda_n)(\alpha_{n+1} - \alpha_n)^2}{t_2 \lambda_n \alpha_n^3 \alpha_{n+1}^2} M^2.
\end{aligned} \tag{24}$$

In the view of, for all $n \geq n_0$,

$$
\begin{aligned}
& \|w_n - x_{\alpha_n}\|^2 \\
= \;& \|x_n + \sum_{i=1}^{N} \theta_{i,n}(x_{n-i+1} - x_{n-i}) - x_{\alpha_n}\|^2 \\
\leq \;& \left(\|x_n - x_{\alpha_n}\| + \sum_{i=1}^{N} \theta_{i,n}\|x_{n-i+1} - x_{n-i}\|\right)^2 \\
= \;& \|x_n - x_{\alpha_n}\|^2 + \sum_{i=1}^{N} \theta_{i,n}^2\|x_{n-i+1} - x_{n-i}\|^2 \\
& +2\sum_{1\leq i<j\leq N} \theta_{i,n}\theta_{j,n}\|x_{n-i+1} - x_{n-i}\|\|x_{n-j+1} - x_{n-j}\| \\
& +2\sum_{i=1}^{N} \theta_{i,n}\|x_n - x_{\alpha_n}\|\|x_{n-i+1} - x_{n-i}\| \\
\leq \;& \left(1 + \sum_{i=1}^{N} \epsilon_{i,n}\right)\|x_n - x_{\alpha_n}\|^2 + \sum_{i=1}^{N} \epsilon_{i,n}^2 + \sum_{i=1}^{N} \epsilon_{i,n} + 2\sum_{1\leq i<j\leq N} \epsilon_{i,n}\epsilon_{j,n} \\
= \;& \left(1 + \sum_{i=1}^{N} \epsilon_{i,n}\right)\|x_n - x_{\alpha_n}\|^2 + \bar{\epsilon}_n \\
\leq \;& (1 + t_3\alpha_n\lambda_n)\|x_n - x_{\alpha_n}\|^2 + \bar{\epsilon}_n,
\end{aligned}
\tag{25}
$$

where $\bar{\epsilon}_n = \sum_{i=1}^{N} \epsilon_{i,n}^2 + \sum_{i=1}^{N} \epsilon_{i,n} + 2\sum_{1\leq i<j\leq N} \epsilon_{i,n}\epsilon_{j,n}$. The condition of $\{\epsilon_{i,n}\}$ implies that $\lim_{n\to\infty} \frac{\bar{\epsilon}_n}{\alpha_n} = 0$. Substituting (25) into (24), for all $n \geq n_0$,

$$
\begin{aligned}
& \|x_{n+1} - x_{\alpha_{n+1}}\|^2 \\
\leq \;& (1 - (2\xi - kt_1 - t_2)\alpha_n\lambda_n)(1 + t_3\alpha_n\lambda_n)\|x_n - x_{\alpha_n}\|^2 \\
& + \frac{(1 + t_2\alpha_n\lambda_n)(\alpha_{n+1} - \alpha_n)^2}{t_2\lambda_n\alpha_n^3\alpha_{n+1}^2}M^2 + \bar{\epsilon}_n \\
\leq \;& (1 - (2\xi - kt_1 - t_2 - t_3)\alpha_n\lambda_n)\|x_n - x_{\alpha_n}\|^2 \\
& + \frac{(1 + t_2\alpha_n\lambda_n)(\alpha_{n+1} - \alpha_n)^2}{t_2\lambda_n\alpha_n^3\alpha_{n+1}^2}M^2 + \bar{\epsilon}_n \\
= \;& (1 - (2\xi - kt_1 - t_2 - t_3)\alpha_n\lambda_n)\|x_n - x_{\alpha_n}\|^2 \\
& +(2\xi - kt_1 - t_2 - t_3)\alpha_n\lambda_n\frac{(1 + t_2\alpha_n\lambda_n)(\alpha_{n+1} - \alpha_n)^2}{(2\xi - kt_1 - t_2 - t_3)t_2\lambda_n^2\alpha_n^4\alpha_{n+1}^2}M^2 + \bar{\epsilon}_n \\
\leq \;& (1 - \varphi_n)\|x_n - x_{\alpha_n}\|^2 + \varphi_n M'\left(\frac{\alpha_{n+1} - \alpha_n}{\alpha_{n+1}\alpha_n^2}\right)^2 + \bar{\epsilon}_n \\
= \;& (1 - \varphi_n)\|x_n - x_{\alpha_n}\|^2 + \varphi_n\zeta_n,
\end{aligned}
$$

where $\varphi_n = (2\xi - kt_1 - t_2 - t_3)\alpha_n\lambda_n$, $M' = \sup_{n\in N}\left\{\frac{(1+t_2\alpha_n\lambda_n)M^2}{(2\xi-kt_1-t_2-t_3)t_2\lambda_n^2}\right\}$ is positive and $\zeta_n = M'\left(\frac{\alpha_{n+1}-\alpha_n}{\alpha_{n+1}\alpha_n^2}\right)^2 + \frac{\bar{\epsilon}_n}{\varphi_n}$. Because the constraints of $\{\lambda_n\}$ and $\{\alpha_n\}$, we know that $\varphi_n \to 0$, $\sum_{n=1}^{\infty} \varphi_n = \infty$, and $\zeta_n \to 0$. We deduce from Lemma 2 that $\|x_n - x_{\alpha_n}\|^2 \to 0$ as $n \to \infty$. □

Theorem 2. *If the conditions (A1)–(A5) hold, $x^{\S}$ is the unique solution of problem (2) and the sequence $\{x_n\}$ is generated by Algorithm 2, then x_n converges strongly to $x^{\S}$.*

Proof. We have $\lim_{n\to\infty} 1 - \frac{\mu\lambda_n}{\lambda_{n+1}} - \frac{\lambda_1+\tau}{\gamma} = 1 - \mu - \frac{\lambda_1+\tau}{\gamma} > 0$ by Lemma 8, so for all $n \geq n_0$, there exists $\delta > 0$ and $n_0 \geq 1$ such that $1 - \frac{\mu\lambda_n}{\lambda_{n+1}} - \frac{\lambda_1+\tau}{\gamma} > \delta > 0$. We can also obtain $\lim_{n\to\infty} 1 + \frac{\mu\lambda_n}{\lambda_{n+1}} + \frac{\lambda_1+\tau}{\gamma} = 1 + \mu + \frac{\lambda_1+\tau}{\gamma} > 0$, then $\left\{1 + \frac{\mu\lambda_n}{\lambda_{n+1}} + \frac{\lambda_1+\tau}{\gamma}\right\}$ is bounded. We will use the letter V to denote $\sup_{n\in\mathbb{N}}\left\{1 + \frac{\mu\lambda_n}{\lambda_{n+1}} + \frac{\lambda_1+\tau}{\gamma}\right\}$, obviously $V > 0$.

In the remainder proof, we assume that $n \geq n_0$. Setting $s_n = By_n - Bw_n + \alpha_n^\omega Gy_n - \alpha_n^\omega Gw_n$, then

$$\begin{aligned} & \|h_n\| \\ \geq\ & \|w_n - y_n\| - \lambda_n\|Bw_n - By_n\| - \lambda_n\alpha_n^\omega\|Gw_n - Gy_n\| \\ \geq\ & \|w_n - y_n\| - \frac{\mu\lambda_n}{\lambda_{n+1}}\|w_n - y_n\| - \frac{\lambda_1+\tau}{\gamma}\|w_n - y_n\| \\ =\ & \left(1 - \frac{\mu\lambda_n}{\lambda_{n+1}} - \frac{\lambda_1+\tau}{\gamma}\right)\|w_n - y_n\| \\ \geq\ & \delta\|w_n - y_n\|. \end{aligned}$$

In the meantime,

$$\begin{aligned} & \|h_n\| \\ \leq\ & \|w_n - y_n\| + \lambda_n\|s_n\| \\ \leq\ & \|w_n - y_n\| + \lambda_n(\|Bw_n - By_n\| + \alpha_n^\omega\|Gw_n - Gy_n\|) \\ \leq\ & \left(1 + \frac{\mu\lambda_n}{\lambda_{n+1}} + \frac{\lambda_1+\tau}{\gamma}\right)\|w_n - y_n\| \\ \leq\ & V\|w_n - y_n\|. \end{aligned} \tag{26}$$

For any $n \geq n_0$, $w_n = y_n$ is equivalent to $h_n = 0$. Since

$$\begin{aligned} & \phi(w_n, y_n) \\ =\ & \langle w_n - y_n, w_n - y_n + \lambda_n s_n\rangle \\ =\ & \|w_n - y_n\|^2 - \langle w_n - y_n, \lambda_n(Bw_n - By_n) + \alpha_n^\omega\lambda_n(Gw_n - Gy_n)\rangle \\ \geq\ & \|w_n - y_n\|^2 - \frac{\mu\lambda_n}{\lambda_{n+1}}\|w_n - y_n\|^2 - \frac{\lambda_1+\tau}{\gamma}\|w_n - y_n\|^2 \\ =\ & \left(1 - \frac{\mu\lambda_n}{\lambda_{n+1}} - \frac{\lambda_1+\tau}{\gamma}\right)\|w_n - y_n\|^2 \\ \geq\ & \delta\|w_n - y_n\|^2, \end{aligned} \tag{27}$$

combining (26) and (27), if $h_n \neq 0$, then

$$\beta_n = \frac{\phi(w_n, y_n)}{\|h_n\|^2} \geq \frac{\delta}{V^2} > 0,$$

hence $\beta_n \geq \min\left\{\beta, \frac{\delta}{V^2}\right\} > 0$. Then observe that

$$\begin{aligned} & \|x_{n+1} - x_{\alpha_n}\|^2 \\ =\ & \|w_n - x_{\alpha_n}\|^2 - r\beta_n h_n \\ =\ & \|w_n - x_{\alpha_n}\|^2 + r^2\beta_n^2\|h_n\|^2 - 2r\beta_n\langle w_n - x_{\alpha_n}, h_n\rangle. \end{aligned} \tag{28}$$

By the definition of β_n,

$$\phi(w_n, y_n) = \beta_n\|h_n\|^2,$$

which and (28) imply

$$\|x_{n+1} - x_{\alpha_n}\|^2 = \|w_n - x_{\alpha_n}\|^2 + r^2\beta_n\phi(w_n, y_n) - 2r\beta_n\langle w_n - x_{\alpha_n}, h_n\rangle. \tag{29}$$

By the definition of h_n,

$$\phi(w_n, y_n) = \|w_n - y_n\|^2 + \lambda_n \langle w_n - y_n, s_n\rangle, \tag{30}$$

and

$$\begin{aligned} & \langle w_n - x_{\alpha_n}, h_n\rangle \\ = \;& \langle w_n - x_{\alpha_n}, w_n - y_n\rangle + \lambda_n \langle w_n - y_n, s_n\rangle + \lambda_n \langle y_n - x_{\alpha_n}, s_n\rangle. \end{aligned} \tag{31}$$

Substituting (30) and (31) into (29), we infer

$$\begin{aligned} \|x_{n+1} - x_{\alpha_n}\|^2 = \|w_n - x_{\alpha_n}\|^2 + r^2\beta_n \|w_n - y_n\|^2 - (2-r) r\beta_n \lambda_n \langle w_n - y_n, s_n\rangle \\ - 2r\beta_n \lambda_n \langle y_n - x_{\alpha_n}, s_n\rangle - 2r\beta_n \langle w_n - x_{\alpha_n}, w_n - y_n\rangle. \end{aligned} \tag{32}$$

And then, by the properties of B and G, we infer that

$$\begin{aligned} \langle w_n - y_n, s_n\rangle &= -\langle w_n - y_n, Bw_n - By_n\rangle - \alpha_n^{\omega} \langle w_n - y_n, Gw_n - Gy_n\rangle \\ &\geq -\left(\frac{\mu}{\lambda_{n+1}} + \frac{1}{\gamma}\right) \|w_n - y_n\|^2. \end{aligned} \tag{33}$$

Using the same method in the Theorem 1, we get

$$\begin{aligned} & \|x_{n+1} - x_{\alpha_n}\|^2 \\ \leq \;& \Big(1 - r\beta_n \alpha_n \lambda_n (2\xi - kt_1)\Big) \|w_n - x_{\alpha_n}\|^2 \\ & - r\beta_n \left(2 - r - (2-r)\lambda_n \left(\frac{\mu}{\lambda_{n+1}} + \frac{1}{\gamma}\right) - \frac{\alpha_n k \lambda_n}{t_1}\right) \|w_n - y_n\|^2 \\ \leq \;& \Big(1 - r\beta_n \alpha_n \lambda_n (2\xi - kt_1)\Big) \|w_n - x_{\alpha_n}\|^2 \\ & - r\beta_n \left((2-r)\left(1 - \frac{\mu}{\lambda_{n+1}} - \frac{\lambda_1 + \tau}{\gamma}\right) - \frac{\alpha_n k \lambda_n}{t_1}\right) \|w_n - x_{\alpha_n}\|^2 \\ \leq \;& \Big(1 - r\beta_n \alpha_n \lambda_n (2\xi - kt_1)\Big) \|w_n - x_{\alpha_n}\|^2 \\ & - r\beta_n \left((2-r)\eta - \frac{\alpha_n k \lambda_n}{t_1}\right) \|w_n - x_{\alpha_n}\|^2. \end{aligned}$$

where $t_1 \in (0, \frac{2\xi}{k})$. Cause $\alpha_n \to 0$, we assume $(2-r)\eta - \frac{\alpha_n k \lambda_n}{t_1} > 0$. Hence

$$\begin{aligned} & \|x_{n+1} - x_{\alpha_n}\|^2 \\ \leq \;& \Big(1 - r\beta_n \alpha_n \lambda_n (2\xi - kt_1)\Big) \|w_n - x_{\alpha_n}\|^2. \end{aligned}$$

The remaining proofs are the same as Theorem 1. □

4. Numerical Experiments

Three examples are given to show the performances of our algorithms. When the coefficients of inertia are equal to zero, let us use MFBMR and MPCMR for Algorithms 1 and 2, respectively. We denote Algorithm 1 for $N = 1, 2, 3$ by MIFBMR, 2-MMIFBMR and 3-MMIFBMR, respectively. Similarly denote Algorithm 2 for $N = 1, 2, 3$ by MIPCMR, 2-MMIPCMR and 3-MMIPCMR, respectively. All the programmes are written in Matlab 9.0 and performed on PC Desktop Intel(R) Core(TM) i5-1035G1 CPU @ 1.00 GHz 1.19 GHz, RAM 16.0 GB.

Example 1. *Suppose $H = \mathbb{R}$. Let $A : \mathbb{R} \to 2^{\mathbb{R}}$ be a mapping defined as*

$$Ax := \left\{\frac{1}{4}x\right\}, \quad \forall x \in \mathbb{R},$$

and $B : \mathbb{R} \to \mathbb{R}$ as

$$Bx := x \arctan x - \frac{1}{2}\ln(1+x^2) + \frac{\pi}{2}x, \quad \forall x \in \mathbb{R}.$$

Set mapping $G : \mathbb{R} \to \mathbb{R}$ as

$$Gx := x - \sin x, \quad \forall x \in \mathbb{R}.$$

It is obvious that A is maximally monotone. We can prove that B is monotone and Lipschitz continuous. We know G is $\frac{1}{2}$-inverse strongly monotone by calculation. Let $F = 0.4I$.

Choose $\theta_i = 0.1$,$x_0 = 1$ and $\epsilon_{i,n} = n^{-2}$ for MIFBMR, 2-MMIFBMR, 3-MMIFBMR, MIPCMR, 2-MMIPCMR and 3-MMIPCMR. Choose $x_1 = 1$, $\omega = 0.6$, $\lambda_1 = 0.08$, $\mu = 0.6$, $\tau_n = 0.1(n+1)^{-4}$ and $\alpha_n = n^{-1/3}$ for each algorithm. Choose $r = 1$, $\beta = 2$ for MPCMR, MIPCMR, 2-MMIPCMR, and 3-MMIPCMR. It is obvious that $\Omega = \{0\}$ and $x^{\S} = 0$ is the only one solution of problem (2). The numerical results of this example are represented in Figures 1 and 2.

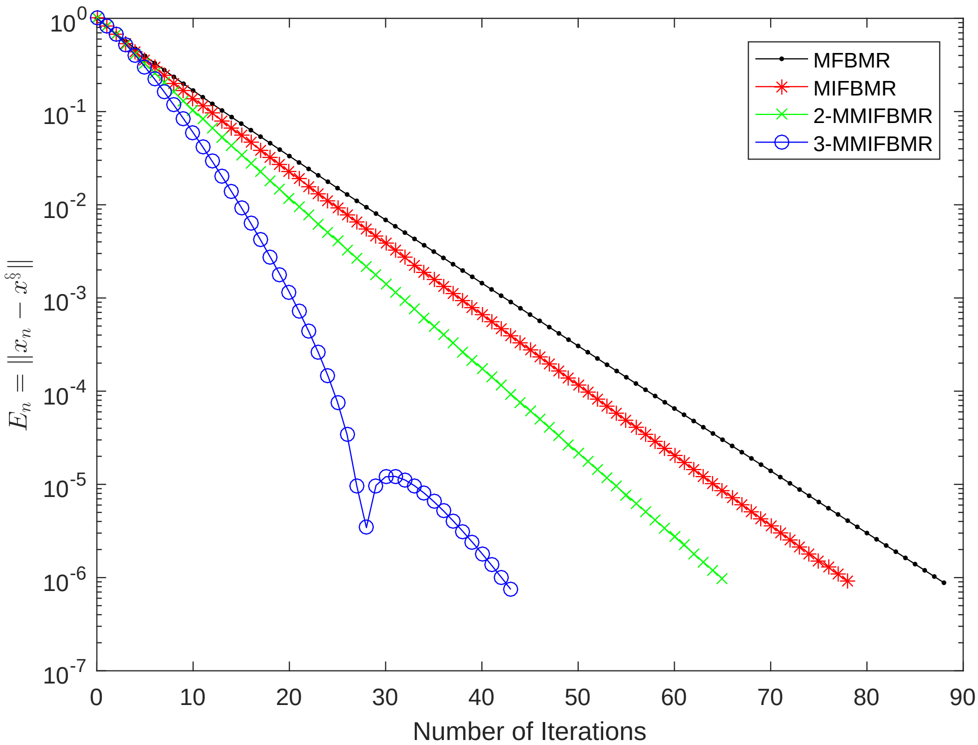

Figure 1. Comparison of MFBMR, MIFBMR, 2-MMIFBMR and 3-MMIFBMR in Example 1.

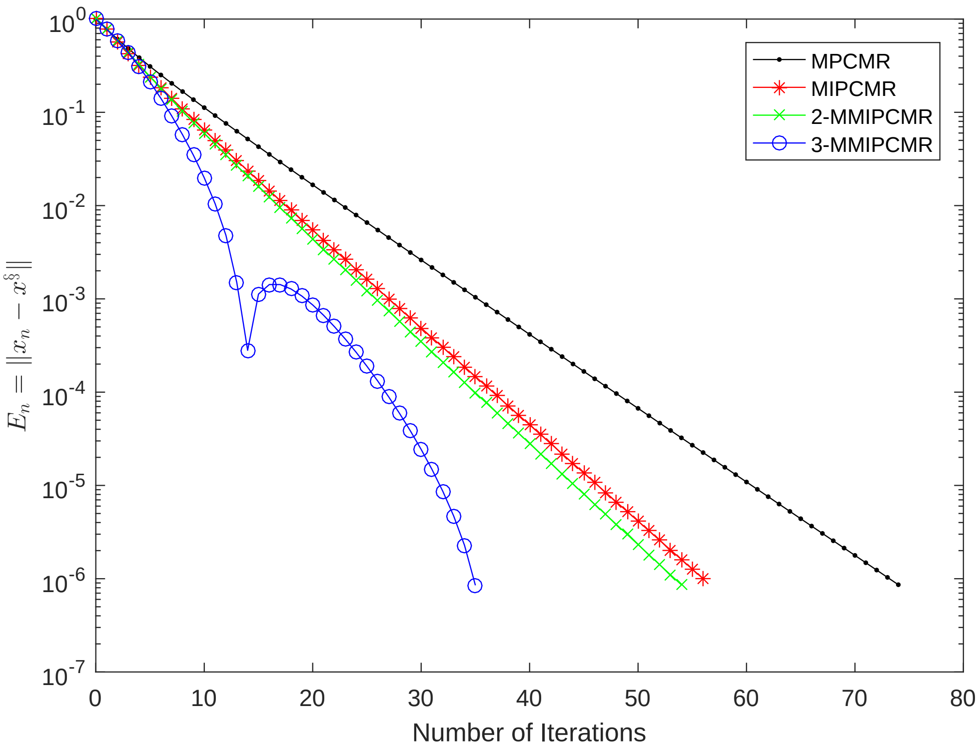

Figure 2. Comparison of MPCMR, MIPCMR, 2-MMIPCMR and 3-MMIPCMR in Example 1.

Example 2. *Let $H = \mathbb{R}^s$. Let $F = I$. Let $A : \mathbb{R}^s \to 2^{\mathbb{R}^s}$ be defined by*

$$Ax := \{Jx\}, \quad \forall x \in \mathbb{R}^s,$$

where J is an upper triangular matrix whose nonzero elements are all 1 in $\mathbb{R}^{s\times s}$. Let $B : \mathbb{R}^s \to \mathbb{R}^s$ be a mapping defined as

$$Bx := Ex, \forall x \in \mathbb{R}^s,$$

where

$$E = CC^{\mathrm{T}} + S + D,$$

here C is a matrix, S is a skew-symmetric matrix and D is a diagonal matrix whose diagonal entries are positive. They all in $\mathbb{R}^{s\times s}$. Therefore E is positive definite. Obviously, B is monotone and Lipschitz continuous. Define $G : \mathbb{R}^s \to \mathbb{R}^s$ as

$$Gx := x - \frac{1}{\|Q\|}Qx, \quad \forall x \in \mathbb{R}^s,$$

where Q is a nonzero matrix in $\mathbb{R}^{s\times s}$. We know G is $\frac{1}{2}$-inverse strongly monotone by calculation.

Choose $x_0 = (1, 1, \cdots, 1)^{\mathrm{T}}$, $\epsilon_{i,n} = n^{-2}$ and $\theta_i = 0.1$ for MIFBMR, 2-MMIFBMR, 3-MMIFBMR, MIPCMR, 2-MMIPCMR and 3-MMIPCMR. Choose $x_1 = (1, 1, \cdots, 1)^{\mathrm{T}}$, $\omega = 0.5$, $\mu = 0.5$, $\lambda_1 = 0.2$, $\tau_n = 0.1(n+1)^{-4}$ and $\alpha_n = n^{-1/4}$ for each algorithm. Choose $r = 1$, $\beta = 2$ for MPCMR, MIPCMR, 2-MMIPCMR and 3-MMIPCMR. All the diagonal elements of D are arbitrary in $(0, 2)$, the elements of C, S and Q are generated randomly in $(-2, 2)$, $(-2, 2)$ and $(0, 1)$, respectively. It is obvious that $\Omega = \{(0, 0, \cdots, 0)^{\mathrm{T}}\}$ and hence the solution of (2) $x^{\S} = (0, 0, \cdots, 0)^{\mathrm{T}}$ is unique. The numerical results are represented in Figures 3 and 4.

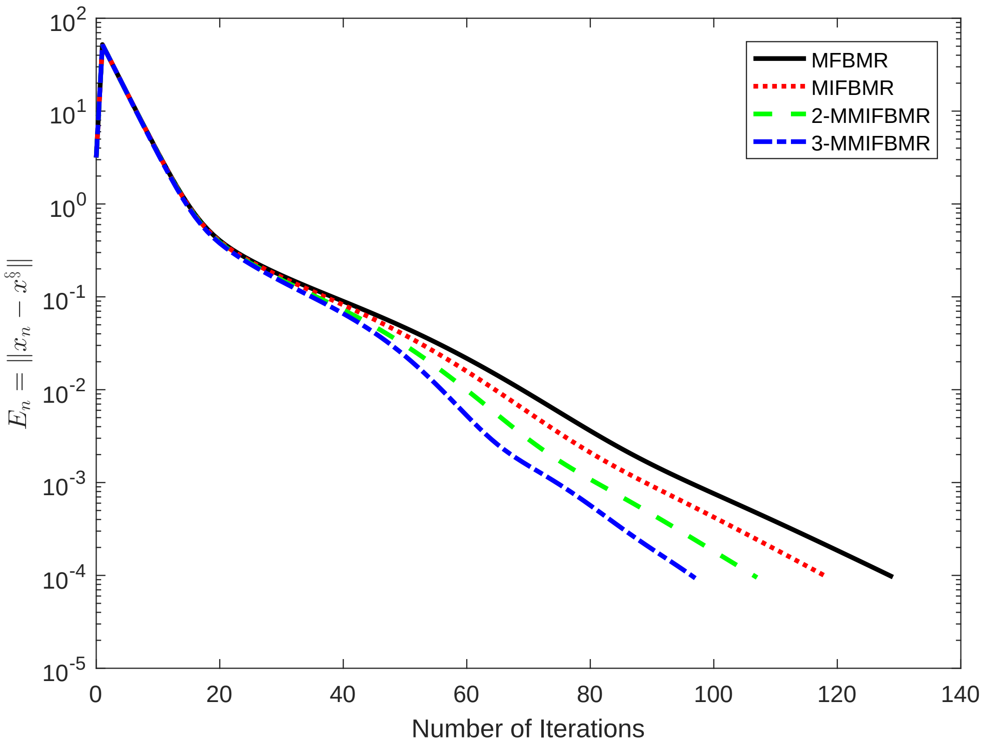

Figure 3. Comparison of MFBMR, MIFBMR, 2-MMIFBMR and 3-MMIFBMR in Example 2 with $s = 10$.

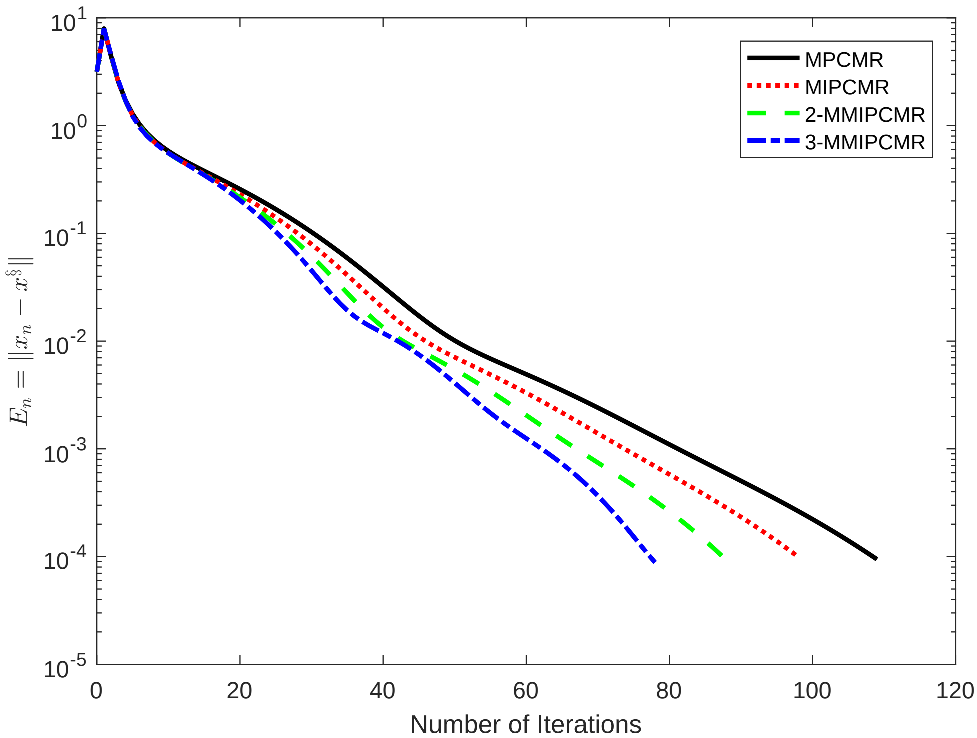

Figure 4. Comparison of MPCMR, MIPCMR, 2-MMIPCMR and 3-MMIPCMR in Example 2 with $s = 10$.

Example 3. *Let $H = \mathbb{R}^2$. Let $A : \mathbb{R}^2 \to 2^{\mathbb{R}^2}$ be a mapping defined as*

$$A(u,v)^{\mathrm{T}} := \left\{ \begin{pmatrix} 2 & -5 \\ -5 & 13 \end{pmatrix} (u,v)^{\mathrm{T}} \right\}, \quad \forall (u,v)^{\mathrm{T}} \in \mathbb{R}^2,$$

$B : \mathbb{R}^2 \to \mathbb{R}^2$ be a mapping defined as

$$B(u,v)^{\mathrm{T}} := (u + v + \sin u, -u + v + \sin v)^{\mathrm{T}}, \quad \forall (u,v)^{\mathrm{T}} \in \mathbb{R}^2,$$

and $F : \mathbb{R}^2 \to \mathbb{R}^2$ be a mapping defined as

$$F(u,v)^{\mathrm{T}} := (2u + 2v + \sin u, -2u + 2v + \sin v)^{\mathrm{T}}, \quad \forall (u,v)^{\mathrm{T}} \in \mathbb{R}^2.$$

Define $G : \mathbb{R}^2 \to \mathbb{R}^2$ as

$$G(u,v)^{\mathrm{T}} := \frac{3}{28} \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} (u,v)^{\mathrm{T}}, \quad \forall (u,v)^{\mathrm{T}} \in \mathbb{R}^2.$$

We can claim that B is monotone and $\sqrt{10}$-Lipschitz continuous, F is 1-strongly monotone and $\sqrt{26}$-Lipschitz continuous. We know G is 2-inverse strongly monotone by calculation. Choose $\theta_i = 0.1$,$x_0 = (1,1)^{\mathrm{T}}$ and $\epsilon_{i,n} = n^{-2}$ for MIFBMR, 2-MMIFBMR, 3-MMIFBMR, MIPCMR, 2-MMIPCMR and 3-MMIPCMR. Choose $x_1 = (1,1)^{\mathrm{T}}$, $\omega = 0.8$, $\lambda_1 = 0.05$, $\mu = 0.2$, $\tau_n = 0.1(n+1)^{-6}$ and $\alpha_n = n^{-2/5}$ for each algorithm. Choose $r = 1$, $\beta = 2$ for MPCMR, MIPCMR, 2-MMIPCMR and 3-MMIPCMR. It is obvious that $\Omega = \{(0,0)^{\mathrm{T}}\}$ and $x^{\S} = (0,0)^{\mathrm{T}}$ is the only solution of problem (2). The numerical results are represented in Figures 5–8.

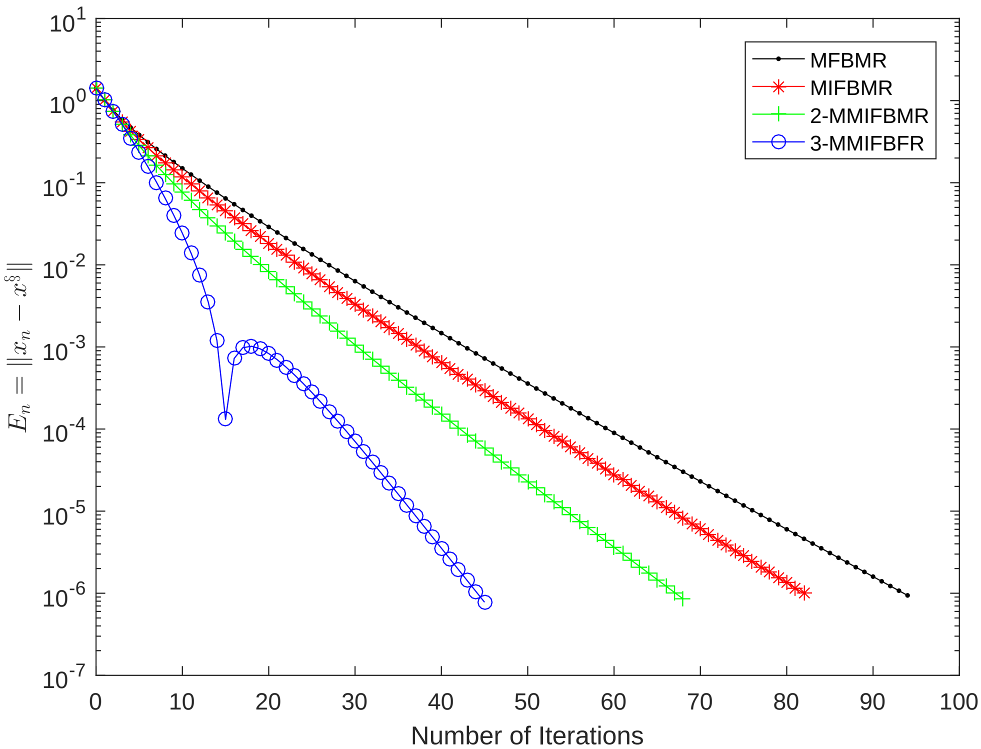

Figure 5. Comparison of MFBMR, MIFBMR, 2-MMIFBMR and 3-MMIFBMR in Example 3.

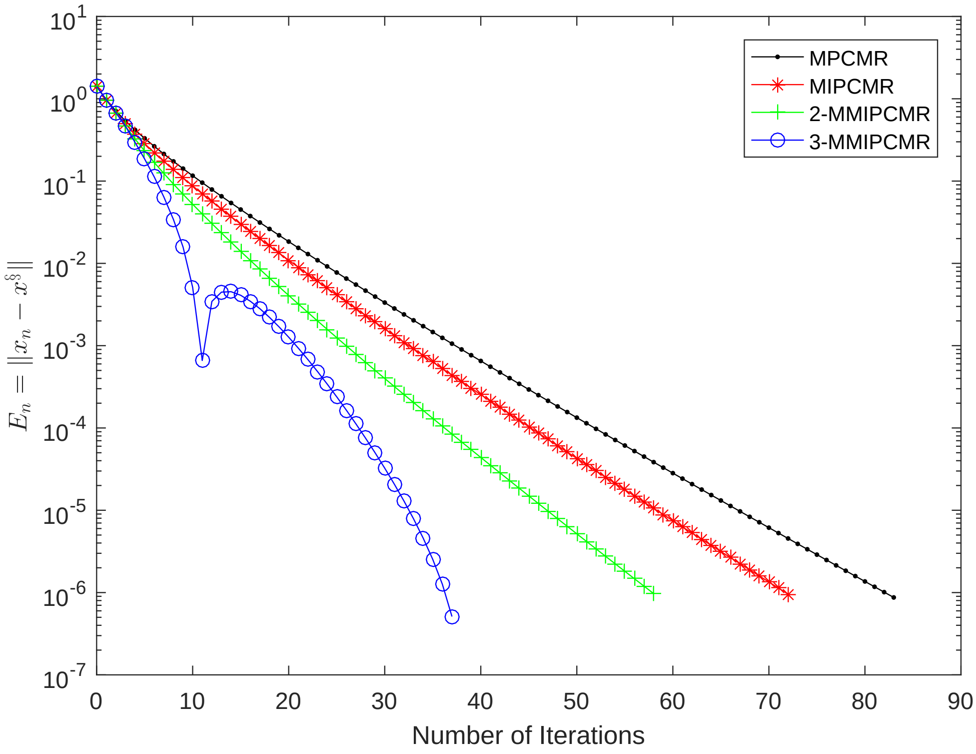

Figure 6. Comparison of MPCMR, MIPCMR, 2-MMIFBMR and 3-MMIPCMR in Example 3.

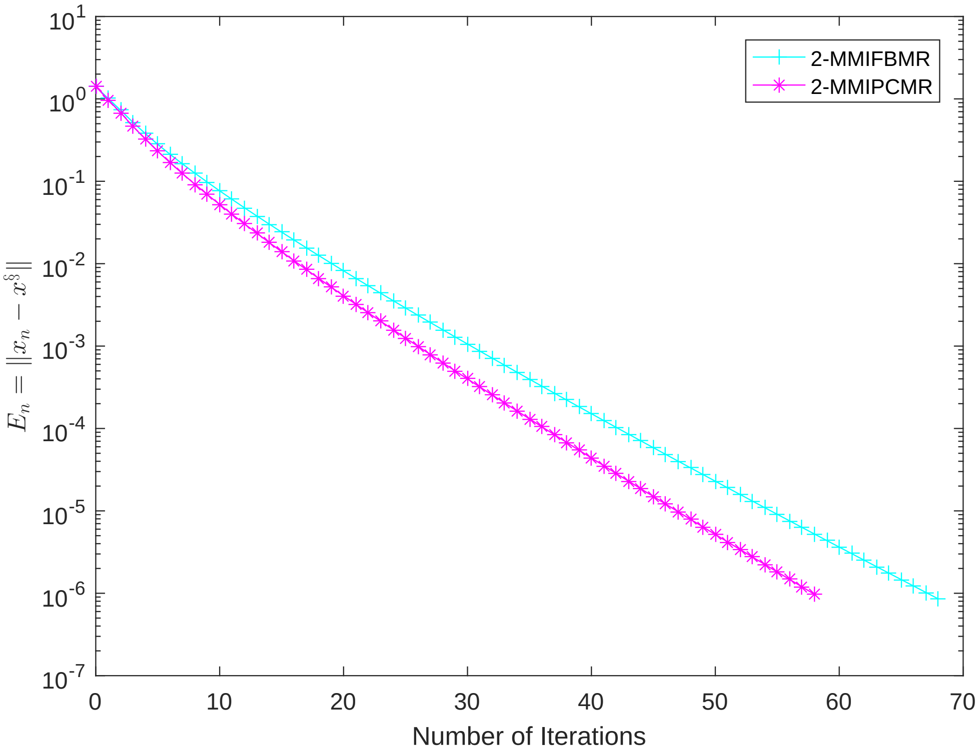

Figure 7. Comparison of 2-MMIFBMR and 2-MMIPCMR in Example 3.

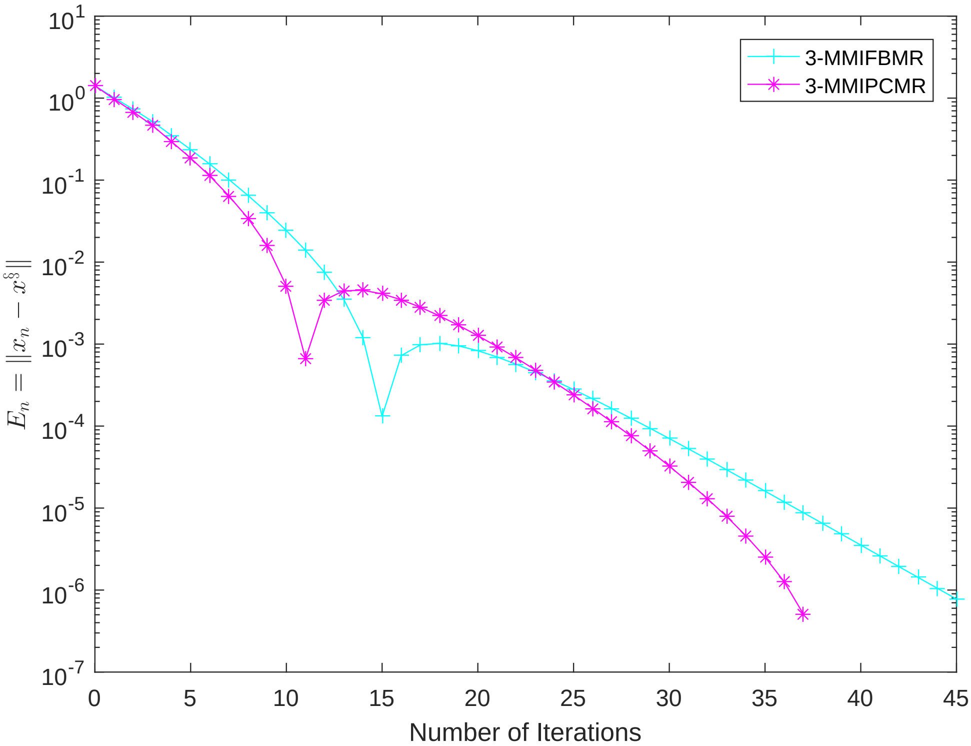

Figure 8. Comparison of 3-MMIFBMR and 3-MMIPCMR in Example 3.

Remark 2. *In Algorithms 1 and 2, the values of L, k and ξ are not necessary to be known.*

5. Conclusions

We have introduce two improved regularized algorithms with multi-step inertia to solve the variational inclusion and null point problem in Hilbert spaces. Then we can get strong convergence without using the inverse strongly monotone assumption. Another advantage of our algorithms is that the stepsizes do not need to use the Lipschitz constant of the operator. In addition, the values of k, L, and ξ are not needed in the calculation process, and the choice of α_n seems harsh but is actually available, such as $\alpha_n = n^{-p}, 0 < p < 1/2$. Finally, the feasibility and effectiveness of our algorithms can be seen in the figures of the numerical experiments. After this, a question is how to get strong convergence under weaker conditions. We will discuss and study this issue in the future.

Author Contributions: Conceptualization, M.L.; Methodology, Y.W. and B.J.; Validation, Y.W., M.L. and C.Y.; Formal analysis, Y.W.; Investigation, M.L. and C.Y.; Resources, C.Y. and B.J.; Data curation, C.Y. and B.J.; Writing—original draft, M.L.; Writing—review & editing, B.J.; Project administration, Y.W., M.L. and C.Y.; Funding acquisition, Y.W. All authors have read and agreed to the published version of the manuscript.

Funding: This work was supported by the National Natural Science Foundation of China (Grant no. 11671365).

Data Availability Statement: Not applicable.

Conflicts of Interest: The authors declare that they have no competing interest.

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