mathematics

Article

On a Conjecture of Cai–Zhang–Shen for Figurate Primes

Junli Zhang [1] **and Pengcheng Niu** [2,*]

[1] School of Mathematics and Data Science, Shaanxi University of Science and Technology, Xi'an 710021, China
[2] School of Mathematics and Statistics, Northwestern Polytechnical University, Xi'an 710129, China
* Correspondence: pengchengniu@nwpu.edu.cn

Abstract: A conjecture of Cai–Zhang–Shen for figurate primes says that every integer $k > 1$ is the sum of two figurate primes. In this paper, we give an equivalent proposition to the conjecture. By considering extreme value problems with constraints about the conjecture in the cases of odd and even integers and using the method of Lagrange multipliers, the Cardano formula for cubic equations, and the contradiction, we prove the conjecture.

Keywords: figurate prime; Cai–Zhang–Shen conjecture; extreme value problem; constraint

MSC: 11N05

Citation: Zhang, J.; Niu, P. On a Conjecture of Cai–Zhang–Shen for Figurate Primes. *Mathematics* **2023**, *11*, 1532. https://doi.org/10.3390/math11061532

Academic Editor: Patrick Solé

Received: 25 October 2022
Revised: 1 March 2023
Accepted: 1 March 2023
Published: 22 March 2023

1. Introduction

Since 18th century, the so-called Goldbach's binary conjecture has been known, which says that every even number greater than two can be written as the sum of two primes. This problem has received attention from many mathematicians, but, unfortunately, it is still unsolved up to the present day; see Apostol [1], Chen [2], Oliveira e Silva, Herzog and Pardi [3], Pan and Pan [4], and Wang [5].

A binomial coefficient of the form $\binom{p^r}{s}$ is called a figurate prime, where p is a prime, and $r \geq 1$ and $s \geq 0$ are integers. The collection of figurate primes includes one, all primes, and their powers, see [6]. It is well known that numbers of figurate primes and usual primes not larger than x have the same density. In 2015, Cai, Zhang and Shen in [7] proposed a conjecture (we call it Cai–Zhang–Shen conjecture):

$$\text{every integer } k > 1 \text{ is the sum of two figurate primes}$$

and pointed out that the conjecture is true for integers up to 10^7. In this paper, we will discuss the conjecture and confirm that it is true.

Denote the characteristic function of figurate primes i by $\delta(i)$, i.e., $\delta(i) = 1$, when i is a figurate prime; $\delta(i) = 0$, when i is not a figurate prime. We claim that the Cai–Zhang–Shen conjecture for every integer $k \geq 3$ is equivalent to

$$\sum_{i=1}^{k-1} \delta(i)\delta(k-i) > 0, \quad k \geq 3. \tag{1}$$

In fact, if (1) holds, then there exists i such that

$$\delta(i)\delta(k-i) > 0,$$

that is, $\delta(i) = \delta(k-i) = 1$, which implies that i and $k-i$ are figurate primes, and the sum is k. Cai–Zhang–Shen conjecture is true. Conversely, if Cai–Zhang–Shen conjecture is true, that is, every integer k can be expressed as the sum of two figurate primes i and $k-i$, then $\delta(i)\delta(k-i) > 0$ by $\delta(i) = \delta(k-i) = 1$, i.e., (1) is proved.

We can also give the equivalent descriptions for odd and even integers, respectively. Let

$$I_n = \{i|\delta(i) = 1 \text{ for some integer } i = 1, 2, \ldots, n-1\},$$

and by l the number of figurate primes not be greater than $n-1$. We always let

$$l > 10^4.$$

For odd integer $k = 2n-1$, we take $N > 2n-1$ satisfying $\delta(N) = 0$. Then, Cai–Zhang–Shen conjecture is equivalent to

$$\sum_{i \in I_n} \delta(i)\delta(2n-1-i) + \delta(N)^2 > 0, \; n \geq 3. \tag{2}$$

For even integer $k = 2n$, Cai–Zhang–Shen conjecture is equivalent to

$$\sum_{i \in I_n} \delta(i)\delta(2n-i) + \delta(n)^2 > 0, \; n \geq 3. \tag{3}$$

The main result of the paper is

Theorem 1. *Cai–Zhang–Shen conjecture is true.*

We will divide odd integers and even integers to prove Theorem 1. The detailed proof is given only in the case of odd integers, which can be similarly obtained in the case of even integers. Based on the properties satisfied by the characteristic function of the figurate primes, we introduce the objective function $f(x)$ $(x \in \mathbb{R}^{2l+1})$, and two constraints $g(x) = 0$ and $h(x) = 0$. By testing that the set A constructed by constraints is bounded, and the Jacobi determinant of two functions $g(x)$ and $h(x)$ is not 0, and then using the method of Lagrange multipliers, one shows $f(x) > 0$ on the set A. Under the assumption that Cai–Zhang–Shen conjecture is not true, the contradiction is obtained.

We emphasize the difficulties here: one is how to select the applicable objective function and constraints, especially the constraints, and the other is how to prove $f(x) > 0$ on A. Here, the application of the Cardano formula is successful.

Since Cai–Zhang–Shen conjecture is equivalent to (1), we have from Theorem 1 that

Corollary 1. (1) *holds.*

This paper is organized as follows. The proof of Theorem 1 (odd integers) is given in Section 2. We introduce the objective function $f(x)$ and two constraints $g(x) = 0$ and $h(x) = 0$. Using the method of Lagrange multipliers, one solves the minimum point of $f(x)$ on A and infers $f(x) > 0$ on A. Under the assumption that Cai–Zhang–Shen conjecture is not true, the contradiction is derived. Therefore, Theorem 1 (odd integers) is proved. Two propositions used in Section 2 are proved in Section 3. In Section 4, we prove Theorem 1 (even integers). Since the proof is similar to the previous sections, we only describe the related extreme value problem with constraints and omit the details. Some conclusions are given in Section 5.

At the end of this section, let us state the method of Lagrange multipliers (e.g., refer to [8]) which will be used. For seeking the maximum and minimum values of $f(x)(x \in \mathbb{R}^n)$ with constraints

$$g_i(x) = 0 \; (i = 1, 2, \cdots, m, m < n)$$

(assuming that these extreme values exist and the rank of Jacobian matrix

$$\frac{\partial(g_1, \cdots, g_m)}{\partial(x_1, \cdots, x_n)}$$

of $g_i(x)$ $(i = 1, 2, \cdots, m)$ is m):

(a) Find all $x \in \mathbb{R}^n, \lambda_1, \cdots, \lambda_m \in \mathbb{R}$ such that

$$\frac{\partial f}{\partial x_i} + \lambda_1 \frac{\partial g_1}{\partial x_i} + \cdots + \lambda_m \frac{\partial g_m}{\partial x_i} = 0,\ i = 1, \cdots, n,$$

$$g_i(x) = 0,\ i = 1, 2, \cdots, m,$$

where x is the stationary point and $\lambda_1, \cdots, \lambda_m$ are multipliers;

(b) Evaluate f at all the points x that result from (a). The largest of these values is the maximum value of f, and the smallest is the minimum value of f.

2. Proof of Theorem 1 (Odd Integers)

The following is the Cardano formula for cubic equations:

Lemma 1. *Given the equation*

$$y^3 + 3py + 2q = 0,$$

if $D = p^3 + q^2 > 0$, then there is a real solution

$$y = u_+ + u_-,$$

where

$$u_+ = \left(-q + \sqrt{D}\right)^{1/3},\ u_- = \left(-q - \sqrt{D}\right)^{1/3}.$$

Proof of Theorem 1 (odd integers). Suppose that Cai–Zhang–Shen conjecture for odd integers is not true, namely there exists an odd integer $2n-1$ such that $2n-1$ can not be expressed as the sum of two figurate primes. Denote figurate primes not larger than $n-1$ by $i_1, i_2, \cdots, i_l$ $(i_1 < i_2 < \cdots < i_l)$, and so

$$I_n = \{i_1, i_2, \cdots, i_l\},$$

and $\delta(i_1) = 1,\ \delta(i_2) = 1,\ \cdots\ \delta(i_l) = 1$; let

$$P = (\delta(i_1), \cdots, \delta(i_l), \delta(2n-1-i_l), \cdots, \delta(2n-1-i_1), \delta(N)),$$

i.e., components of P are of

$$\delta(i_1) = \cdots = \delta(i_l) = 1,\ \delta(2n-1-i_1) = \cdots = \delta(2n-1-i_l) = \delta(N) = 0,$$

Clearly, $P \in \mathbb{R}^{2l+1}$.

We introduce a function on $\mathbb{R}^{2l+1}$:

$$f(x) = \sum_{i \in I_n} x_i x_{2n-1-i} + s x_N^2, \tag{4}$$

where

$$s = \frac{3}{8} l^{\frac{1}{3}}.$$

Since P satisfies

$$\sum_{i \in I_n} \left(\delta(i)^2 + \delta(2n-1-i)^2\right) + \varepsilon\delta(N) = l,$$

$$\sum_{i \in I_n} \delta(i)\delta(2n-1-i) + \gamma\delta(N)^3 + \frac{1}{2}\varepsilon\delta(N) = 0,$$

we define two functions on $\mathbb{R}^{2l+1}$:

$$g(x) = \sum_{i \in I_n} \left(x_i^2 + x_{2n-1-i}^2 \right) + \varepsilon x_N - l, \tag{5}$$

$$h(x) = \sum_{i \in I_n} x_i x_{2n-1-i} + \gamma x_N^3 + \frac{1}{2} \varepsilon x_N, \tag{6}$$

where

$$\varepsilon = \frac{3l^{\frac{2}{3}}}{4\sqrt{2}}, \gamma = -\frac{1}{4\sqrt{2}}.$$

Consider the extreme values of $f(x)$ with constraints

$$g(x) = 0 \text{ and } h(x) = 0. \tag{7}$$

Denote

$$A = \left\{ x \in \mathbb{R}^{2l+1} | g(x) = 0, h(x) = 0 \right\}. \tag{8}$$

We describe two propositions whose proofs will be included in Section 3.

Proposition 1. *The set A is bounded and closed in $\mathbb{R}^{2l+1}$.*

Proposition 2. *The rank of the Jacobian matrix for functions $g(x)$ and $h(x)$ on A is* 2*.*

Remark 1. *Under the assumption that CZS conjecture is not true, we see that $P \in \mathbb{R}^{2l+1}$ belongs to A because P satisfies* (7)*.*

Remark 2. *By Proposition 2, there are infinite points in A, since there are $2l-1$ independent variables in A.*

Remark 3. *If Cai–Zhang–Shen conjecture is not true, then*

$$\delta(2n-1-i) = \delta(N) = 0 \; (i \in I_n)$$

and

$$f(P) = \sum_{i \in I_n} \delta(i)\delta(2n-1-i) + \delta(N)^2 = 0. \tag{9}$$

We write the Lagrange function

$$Q(x, \lambda, \mu) = f(x) + \lambda g(x) + \mu h(x) \tag{10}$$

and use the method of Lagrange multipliers to find all stationary points of $f(x)$ on A, and then prove

$$f(x) > 0 \text{ at these points,}$$

which show

$$f(x) > 0 \text{ on } A.$$

(1) For $i \in I_n$, we have

$$\left\{ \begin{array}{l} Q_{x_i} = x_{2n-1-i} + 2\lambda x_i + \mu x_{2n-1-i} = 0, \\ Q_{x_{2n-1-i}} = x_i + 2\lambda x_{2n-1-i} + \mu x_i = 0, \end{array} \right. \tag{11}$$

i.e.,

$$\left\{ \begin{array}{l} 2\lambda x_i + (1+\mu) x_{2n-1-i} = 0, \\ (1+\mu) x_i + 2\lambda x_{2n-1-i} = 0. \end{array} \right.$$

The determinant of coefficients is

$$\begin{vmatrix} 2\lambda & 1+\mu \\ 1+\mu & 2\lambda \end{vmatrix} = (2\lambda)^2 - (1+\mu)^2, \tag{12}$$

hence
(a_1) $(2\lambda)^2 - (1+\mu)^2 \neq 0$, $x_i = x_{2n-1-i} = 0$;
for

$$(2\lambda)^2 - (1+\mu)^2 = 0,$$

we have
(a_2) $\lambda \neq 0$, $2\lambda = -(1+\mu)$, $x_i - x_{2n-1-i} = 0$;
(a_3) $\lambda \neq 0$, $2\lambda = 1+\mu$, $x_i + x_{2n-1-i} = 0$;
(a_4) $\lambda = 0$, $\mu = -1$, x_i and x_{2n-1-i} are arbitrary.

(2) For $i = N$, we have $Q_{x_N} = 2sx_N + \varepsilon\lambda + 3\gamma\mu x_N^2 + \frac{1}{2}\varepsilon\mu = 0$, so

$$3\gamma\mu x_N^2 + 2sx_N + \varepsilon\lambda + \frac{1}{2}\varepsilon\mu = 0, \tag{13}$$

and its discriminant is

$$\Delta = (2s)^2 - 12\gamma\mu\left(\varepsilon\lambda + \frac{1}{2}\varepsilon\mu\right); \tag{14}$$

therefore,
(b_1) $\mu = 0$, $2sx_N + \varepsilon\lambda = 0$ and $x_N = \frac{-\varepsilon\lambda}{2s}$;
(b_2) $\mu \neq 0$, $\Delta = 0$, $x_N = -\frac{s}{3\gamma\mu}$;
(b_3) $\mu \neq 0$, $\Delta > 0$, $x_N = \frac{-2s+\sqrt{\Delta}}{6\gamma\mu}$;
(b_4) $\mu \neq 0$, $\Delta < 0$, $x_N = \frac{-2s-\sqrt{\Delta}}{6\gamma\mu}$.

Remark 4. *Note that P is not a stationary point. In fact, components of P do not satisfy (a_1), (a_2), (a_3). If P satisfies (a_4), it knows $\mu = -1$, which contradicts $\mu = 0$ by (b_1); it gives $x_N \neq 0$ by (b_2), which contradicts the component $\delta(N) = 0$ of P; if P satisfies (b_3), then $x_N = \frac{-2s+\sqrt{\Delta}}{6\gamma\mu} = 0$ and $2s = \sqrt{\Delta}$, so $-12\gamma\mu\left(\varepsilon\lambda + \frac{1}{2}\varepsilon\mu\right) = 0$ from (14), but $-12\gamma\mu\left(\varepsilon\lambda + \frac{1}{2}\varepsilon\mu\right) = -6\gamma\varepsilon \neq 0$ by $\lambda = 0$ and $\mu = -1$ in (a_4), a contradiction; if P satisfies (b_4), then $x_N = \frac{-2s-\sqrt{\Delta}}{6\gamma\mu} = 0$ and $2s = -\sqrt{\Delta}$, and $(2s)^2 = \Delta$, it obtains $-12\gamma\mu\left(\varepsilon\lambda + \frac{1}{2}\varepsilon\mu\right) = 0$ by (14), but $-12\gamma\mu\left(\varepsilon\lambda + \frac{1}{2}\varepsilon\mu\right) = -6\gamma\varepsilon \neq 0$ by $\lambda = 0$ and $\mu = -1$ in (a_4), a contradiction. Hence, P does not satisfy (b_1)–(b_4), which shows that P is not a stationary point.*

Let us discuss all combinations of (a_1)–(a_4) and (b_1)–(b_4) and prove $f(x) > 0$ at all stationary points.

Case (a_1), (b_1)**:** Note that $x_i = x_{2n-1-i} = 0$ $(i \in I_n)$ from (a_1). Using

$$0 = g(x) = \varepsilon x_N - l,$$

it solves

$$x_N = \frac{l}{\varepsilon} = \frac{4\sqrt{2}}{3} l^{\frac{1}{3}}. \tag{15}$$

Since

$$0 = h(x) = \gamma x_N^3 + \frac{\varepsilon}{2} x_N = x_N\left(\gamma x_N^2 + \frac{\varepsilon}{2}\right),$$

we have

$$x_N = 0 \ \text{ or } \ x_N^2 = -\frac{\varepsilon}{2\gamma} = \frac{3}{2} l^{\frac{2}{3}}. \tag{16}$$

It is different from x_N in (15), a contradiction.

Case (a_1), (b_2)**:** It leads to a contradiction as in Case (a_1), (b_1).

Case $(a_1), (b_3)$**:** It leads to a contradiction as in Case $(a_1), (b_1)$.

Case $(a_1), (b_4)$**:** It leads to a contradiction as in Case $(a_1), (b_1)$.

Case $(a_2), (b_1)$**:** Noting $2\lambda = -(1+\mu)$ and $x_i = x_{2n-1-i}$ by (a_2), and $\mu = 0$ by (b_1), we obtain $\lambda = -\frac{1}{2}$ and also by (b_1),

$$x_N = \frac{-\varepsilon\lambda}{2s} = \frac{\varepsilon}{4s} = \frac{l^{\frac{1}{3}}}{2\sqrt{2}}. \tag{17}$$

Applying $x_i = x_{2n-1-i}$, we see

$$0 = g(x) = 2\sum_{i\in I_n} x_i^2 + \varepsilon x_N - l,$$

$$0 = 2h(x) = 2\sum_{i\in I_n} x_i^2 + 2\gamma x_N^3 + \varepsilon x_N,$$

and so

$$2\gamma x_N^3 + l = 0,$$

then

$$x_N = \left(\frac{-l}{2\gamma}\right)^{\frac{1}{3}} = \sqrt{2}l^{\frac{1}{3}}. \tag{18}$$

It is different from x_N in (17), a contradiction.

Case $(a_2), (b_2)$**:** In virtue of $x_i = x_{2n-1-i}$ by (a_2), similarly to Case $(a_2), (b_1)$, we have

$$x_N = \left(\frac{-l}{2\gamma}\right)^{\frac{1}{3}} = \sqrt{2}l^{\frac{1}{3}}.$$

It follows that

$$f(x) = \sum_{i\in I_n} x_i^2 + s x_N^2 \geq s x_N^2 = \frac{3}{8} l^{\frac{1}{3}} \left(\sqrt{2}l^{\frac{1}{3}}\right)^2 = \frac{3}{4} l > 0.$$

Case $(a_2), (b_3)$**:** We use $x_i = x_{2n-1-i}$ to derive $f(x) > 0$ as in Case $(a_2), (b_2)$.

Case $(a_2), (b_4)$**:** We use $x_i = x_{2n-1-i}$ to derive $f(x) > 0$ as in Case $(a_2), (b_2)$.

Case $(a_3), (b_1)$**:** It gives $2\lambda = 1+\mu$ and $x_i = -x_{2n-1-i}$ by (a_3) and $\mu = 0$ by (b_1); then, $\lambda = \frac{1}{2}$ and by (b_1),

$$x_N = \frac{-\varepsilon\lambda}{2s} = \frac{-\varepsilon}{4s} = -\frac{l^{\frac{1}{3}}}{2\sqrt{2}}. \tag{19}$$

On the other hand, using $x_i = -x_{2n-1-i}$, it yields

$$0 = g(x) = 2\sum_{i\in I_n} x_i^2 + \varepsilon x_N - l,$$

$$0 = 2h(x) = -2\sum_{i\in I_n} x_i^2 + 2\gamma x_N^3 + \varepsilon x_N,$$

so

$$x_N^3 + \frac{\varepsilon}{\gamma} x_N - \frac{l}{2\gamma} = 0. \tag{20}$$

Since

$$3p = \frac{\varepsilon}{\gamma}, \quad 2q = -\frac{l}{2\gamma},$$

and

$$p = \frac{\varepsilon}{3\gamma}, \quad q = \frac{-l}{4\gamma},$$

we have from Lemma 1 and

$$D = p^3 + q^2 = \left(\frac{\varepsilon}{3\gamma}\right)^3 + \left(\frac{-l}{4\gamma}\right)^2 = -l^2 + 2l^2 = l^2,$$

$$u_+ = \left(-q + \sqrt{D}\right)^{\frac{1}{3}} = \left(\frac{l}{4\gamma} + l\right)^{\frac{1}{3}} = \left(-\sqrt{2}l + l\right)^{\frac{1}{3}},$$

$$u_- = \left(-q - \sqrt{D}\right)^{\frac{1}{3}} = \left(\frac{l}{4\gamma} - l\right)^{\frac{1}{3}} = \left(-\sqrt{2}l - l\right)^{\frac{1}{3}}$$

that a real solution to (20) is

$$x_N = u_+ + u_- = \left(-\left(\sqrt{2}-1\right)^{\frac{1}{3}} - \left(\sqrt{2}+1\right)^{\frac{1}{3}}\right) l^{\frac{1}{3}} \approx -2.087 l^{\frac{1}{3}}. \tag{21}$$

It is different from x_N in (19), a contradiction.

Case $(a_3), (b_2)$**:** Noting $x_i + x_{2n-1-i} = 0$ by (a_3), it follows as in Case $(a_3), (b_1)$ that

$$x_N = \left(-\left(\sqrt{2}-1\right)^{\frac{1}{3}} - \left(\sqrt{2}+1\right)^{\frac{1}{3}}\right) l^{\frac{1}{3}} \approx -2.087 l^{\frac{1}{3}}.$$

Using

$$0 = h(x) = \sum_{i \in I_n} x_i x_{2n-1-i} + \gamma x_N^3 + \frac{1}{2}\varepsilon x_N,$$

it implies

$$\begin{aligned}
f(x) &= -\gamma x_N^3 - \frac{1}{2}\varepsilon x_N + s x_N^2 \\
&= -\left(\frac{-1}{4\sqrt{2}}\right)(-2.087)^3 l - \frac{1}{2}\frac{3l^{\frac{2}{3}}}{4\sqrt{2}}(-2.087) l^{\frac{1}{3}} + \frac{3}{8} l^{\frac{1}{3}} (-2.087)^2 l^{\frac{2}{3}} \\
&= -\frac{(2.087)^3}{4\sqrt{2}} l + \frac{3\cdot(2.087)}{8\sqrt{2}} l + \frac{3\cdot(2.087)^2}{8} l = 2.087 l \left(-\frac{(2.087)^2}{4\sqrt{2}} + \frac{3}{8\sqrt{2}} + \frac{3\cdot(2.087)}{8}\right) \\
&= 2.087 l(-0.769 + 0.265 + 0.75) = 2.087 l(-0.769 + 1.015) > 0.
\end{aligned}$$

Case $(a_3), (b_3)$**:** It follows $f(x) > 0$ as in Case $(a_3), (b_2)$.
Case $(a_3), (b_4)$**:** It follows also $f(x) > 0$ as in Case $(a_3), (b_2)$.
Case $(a_4), (b_1)$**:** It knows $\mu = -1$ by (a_4), which contradicts $\mu = 0$ by (b_1).
Case $(a_4), (b_2)$**:** Note by (a_4) and (b_2), we have $\mu = -1$ and

$$x_N = -\frac{s}{3\gamma\mu} = \frac{s}{3\gamma} = -\frac{1}{\sqrt{2}} l^{\frac{1}{3}}.$$

Using

$$h(x) = 0,$$

it derives

$$\begin{aligned}
f(x) = -\gamma x_N^3 - \frac{1}{2}\varepsilon x_N + s x_N^2 &= -\frac{-1}{4\sqrt{2}}\left(\frac{-l^{\frac{1}{3}}}{\sqrt{2}}\right)^3 - \frac{1}{2}\frac{3l^{\frac{2}{3}}}{4\sqrt{2}}\left(\frac{-l^{\frac{1}{3}}}{\sqrt{2}}\right) + \frac{3}{8} l^{\frac{1}{3}}\left(\frac{-l^{\frac{1}{3}}}{\sqrt{2}}\right)^2 \\
&= -\frac{1}{4\sqrt{2}} \cdot \frac{l}{2\sqrt{2}} + \frac{3l}{16} + \frac{3}{8} \cdot \frac{l}{2} = -\frac{l}{16} + \frac{3l}{16} + \frac{3l}{16} > 0.
\end{aligned}$$

Case $(a_4), (b_3)$**:** Notes $\lambda = 0$ and $\mu = -1$ by (a_4) and so $\Delta = 0$, which contradicts $\Delta > 0$ by (b_3).

Case $(a_4), (b_4)$**:** As in Case $(a_4), (b_3)$, a contradiction also follows.

Noting that A is a bounded closed set in $\mathbb{R}^{2l+1}$ and $f(x)$ is continuous in $\mathbb{R}^{2l+1}$, we know that $f(x)$ achieves the minimum value on A. Summing up the above discussions, we indeed prove that the minimum of $f(x)$ on A is positive, and so

$$f(x) > 0, \ x \in A.$$

Since one supposes that Cai–Zhang–Shen conjecture is not true, it follows that $f(x) > 0$ $(x \in A)$ from the above analysis and so

$$f(P) > 0$$

because of $P \in A$. However, it contradicts (9). Theorem 1 (odd integers) is proved. □

3. Proofs of Propositions 1 and 2

Proof of Proposition 1. The closeness of A in (8) is evident. We divide two steps to prove that A is bounded, i.e., first prove that, when the set $\{x_N\}$ constructed by components x_N of $x \in A$ is bounded, it concludes that A is bounded; next, prove that the set $\{x_N\}$ must be bounded by the contradiction.

Step 1. Suppose that the set $\{x_N\}$ is bounded; then, there exists a constant $C > 0$, such that $|x_N| \leq C$. It uses $g(x) = 0$ to show

$$\sum_{j=1}^{l}\left(x_{i_j}^2 + x_{2n-1-i_j}^2\right) + x_N^2 = x_N^2 - \varepsilon x_N + l \leq C^2 + \varepsilon C + l.$$

Hence, A is bounded.

Step 2. Let us prove the boundedness of $\{x_N\}$ by the contradiction. Assume that $\{x_N\}$ is unbounded; then, for any positive integer α, there exists $x_{N,\alpha}$ in $\{x_N\}$, such that $|x_{N,\alpha}| \geq \alpha$. Thus, $x_{N,\alpha} \to \infty$ as $\alpha \to \infty$. For convenience, we simply denote $x_N \to \infty$. It follows from $g(x) = 0$ that

$$-\varepsilon x_N + l = \sum_{j=1}^{l}\left(x_{i_j}^2 + x_{2n-1-i_j}^2\right) \tag{22}$$

and $x_N \to \infty$ should be

$$x_N \to -\infty,$$

so there exists one or several components in $x_{i_j}, x_{2n-1-i_j} (j = 1, 2, \cdots, l)$ tending to ∞. We consider the following subcases.

(1) If $x_{i_1} \to \infty$ and $x_{2n-1-i_1}, x_{i_j}, x_{2n-1-i_j} (j = 2, \cdots, l)$ are bounded, then we have $x_{i_1}^2 \to +\infty$ and from (22) that

$$x_{i_1}^2 = -x_{2n-1-i_1}^2 - \sum_{j=2}^{l}\left(x_{i_j}^2 + x_{2n-1-i_j}^2\right) - \varepsilon x_N + l := -\varepsilon x_N + C_1, \tag{23}$$

where C_1 is finite, so

$$\frac{x_{i_1}^2}{-x_N} \to \varepsilon, \quad \frac{|x_{i_1}|}{(-x_N)^{\frac{1}{2}}} \to \sqrt{\varepsilon}.$$

It yields from $h(x) = 0$ that

$$x_{i_1}x_{2n-1-i_1} = -\sum_{j=2}^{l} x_{i_j}x_{2n-1-i_j} - \gamma x_N^3 - \frac{1}{2}\varepsilon x_N := -\gamma x_N^3 - \frac{1}{2}\varepsilon x_N + C_2, \tag{24}$$

where C_2 is finite.

When $x_{2n-1-i_1} = 0$, we have by (24) that

$$0 = -\gamma x_N^3 - \frac{1}{2}\varepsilon x_N + C_2,$$

and the right-hand side tends to $-\infty$ (noting $\gamma < 0$), a contradiction.
When $x_{2n-1-i_1} \neq 0$, it follows from (24) to see

$$\begin{aligned} \frac{x_{i_1} x_{2n-1-i_1}}{(-x_N)^{\frac{1}{2}}} &= -\gamma \frac{x_N^3}{(-x_N)^{\frac{1}{2}}} - \frac{1}{2}\varepsilon \frac{x_N}{(-x_N)^{\frac{1}{2}}} + \frac{C_2}{(-x_N)^{\frac{1}{2}}} \\ &= \gamma \frac{(-x_N)^3}{(-x_N)^{\frac{1}{2}}} + \frac{1}{2}\varepsilon \frac{-x_N}{(-x_N)^{\frac{1}{2}}} + \frac{C_2}{(-x_N)^{\frac{1}{2}}} \to -\infty, \quad (\gamma < 0) \end{aligned}$$

but the left-hand side tends to $\pm\sqrt{\varepsilon} x_{2n-1-i_1}$, a contradiction.

(2) If $x_{i_1} \to \infty$ and $x_{2n-1-i_1} \to \infty$ and $x_{i_j}, x_{2n-1-i_j} (j = 2, \cdots, l)$ are bounded, then

$$x_{i_1}^2 + x_{2n-1-i_1}^2 \to +\infty,$$

It shows by (22) that

$$x_{i_1}^2 + x_{2n-1-i_1}^2 = -\sum_{j=2}^{l} \left(x_{i_j}^2 + x_{2n-1-i_j}^2 \right) - \varepsilon x_N + l := -\varepsilon x_N + C_3, \tag{25}$$

where C_3 is finite, so

$$\frac{x_{i_1}^2 + x_{2n-1-i_1}^2}{-x_N} \to \varepsilon > 0.$$

It gives from $h(x) = 0$ that

$$x_{i_1} x_{2n-1-i_1} = -\sum_{j=2}^{l} x_{i_j} x_{2n-1-i_j} - \gamma x_N^3 - \frac{1}{2}\varepsilon x_N := -\gamma x_N^3 - \frac{1}{2}\varepsilon x_N + C_4, \tag{26}$$

where C_4 is finite. We have by (26) that

$$\frac{x_{i_1} x_{2n-1-i_1}}{-x_N} = -\gamma \frac{x_N^3}{-x_N} - \frac{1}{2}\varepsilon \frac{x_N}{-x_N} + \frac{C_4}{-x_N} = \gamma \frac{-x_N^3}{-x_N} + \frac{1}{2}\varepsilon \frac{-x_N}{-x_N} + \frac{C_4}{-x_N} \to -\infty;$$

then,

$$+\infty \leftarrow \frac{2|x_{i_1} x_{2n-1-i_1}|}{-x_N} \leq \frac{x_{i_1}^2 + x_{2n-1-i_1}^2}{-x_N} \to \varepsilon,$$

a contradiction.

(3) If $x_{i_1} \to \infty$ and $x_{i_2} \to \infty$ and $x_{2n-1-i_1}, x_{2n-1-i_2}, x_{i_j}, x_{2n-1-i_j} (j = 3, \cdots, l)$ are bounded, then

$$x_{i_1}^2 + x_{i_2}^2 \to \infty$$

and, from (22),

$$x_{i_1}^2 + x_{i_2}^2 = -\varepsilon x_N - x_{2n-1-i_1}^2 - x_{2n-1-i_2}^2 - \sum_{j=3}^{l} \left(x_{i_j}^2 + x_{2n-1-i_j}^2 \right) + l := -\varepsilon x_N + C_5, \tag{27}$$

where C_5 is finite. Hence,

$$\frac{x_{i_1}^2 + x_{i_2}^2}{-x_N} \to \varepsilon,$$

and

$$\frac{|x_{i_1}|}{-x_N} = \frac{x_{i_1}^2}{-x_N}\frac{1}{|x_{i_1}|} \leq \frac{x_{i_1}^2 + x_{i_2}^2}{-x_N}\frac{1}{|x_{i_1}|} \to 0,$$

$$\frac{|x_{i_2}|}{-x_N} = \frac{x_{i_2}^2}{-x_N}\frac{1}{|x_{i_2}|} \leq \frac{x_{i_1}^2 + x_{i_2}^2}{-x_N}\frac{1}{|x_{i_2}|} \to 0.$$

It follows by $h(x) = 0$ that

$$x_{i_1}x_{2n-1-i_1} + x_{i_2}x_{2n-1-i_2} = -\sum_{j=3}^{l} x_{i_j}x_{2n-1-i_j} - \gamma x_N^3 - \frac{1}{2}\varepsilon x_N := -\gamma x_N^3 - \frac{1}{2}\varepsilon x_N + C_6, \tag{28}$$

where C_6 is finite, so

$$\frac{x_{i_1}x_{2n-1-i_1}}{-x_N} + \frac{x_{i_2}x_{2n-1-i_2}}{-x_N} = -\gamma\frac{x_N^3}{-x_N} - \frac{1}{2}\varepsilon\frac{x_N}{-x_N} + \frac{C_6}{-x_N} = \gamma\frac{-x_N^3}{-x_N} + \frac{1}{2}\varepsilon + \frac{C_6}{-x_N}.$$

The left-hand side tends to 0, and the right-hand side tends to $-\infty$, a contradiction. The remaining cases can be treated similarly. Then, $\{x_N\}$ must be bounded.

Proposition 1 is proved. □

Remark 5. *(a)* *In the proof of Proposition 1, if $x_{i_1} \to \infty$ in (1) is changed to that one of $x_{i_2}, \cdots, x_{i_l}, x_{2n-1-i_1}, x_{2n-1-i_2}, \cdots, x_{2n-1-i_l}$ tends to ∞; then, one can solve as in (1).*

(b) *As a generalized case of (2) in the proof of Proposition 1, if components $x_i, x_{2n-1-i} (i \in I_n)$ tend to ∞, then*

$$\sum_{i \in I_n}\left(x_i^2 + x_{2n-1-i}^2\right) \to \infty.$$

It follows by $g(x) = 0$ that

$$\sum_{i \in I_n}\left(x_i^2 + x_{2n-1-i}^2\right) = -\varepsilon x_N - l,$$

so

$$\frac{\sum_{i \in I_n}\left(x_i^2 + x_{2n-1-i}^2\right)}{-x_N} \to \varepsilon > 0.$$

We have from $h(x) = 0$ that

$$\sum_{i \in I_n} x_i x_{2n-1-i} = -\gamma x_N^3 - \frac{1}{2}\varepsilon x_N \to -\infty,$$

hence

$$\frac{\sum_{i \in I_n} x_i x_{2n-1-i}}{-x_N} = \frac{-\gamma x_N^3}{-x_N} - \frac{1}{2}\varepsilon\frac{x_N}{-x_N} \to -\infty,$$

and by the Cauchy inequality,

$$+\infty \leftarrow \frac{\left|\sum_{i \in I_n} x_i x_{2n-1-i}\right|}{-x_N} \leq \frac{\left(\sum_{i \in I_n} x_i^2\right)^{\frac{1}{2}}\left(\sum_{i \in I_n} x_{2n-1-i}^2\right)^{\frac{1}{2}}}{-x_N} \leq \frac{\left(\sum_{i \in I_n}\left(x_i^2 + x_{2n-1-i}^2\right)\right)^{\frac{1}{2}}\left(\sum_{i \in I_n}\left(x_i^2 + x_{2n-1-i}^2\right)\right)^{\frac{1}{2}}}{-x_N}$$

$$= \frac{\sum_{i \in I}\left(x_i^2 + x_{2n-1-i}^2\right)}{-x_N} \to \varepsilon,$$

a contradiction.

(c) *To the generalized case of (3) in the proof of Proposition 1, if $x_i (i \in I_n)$ tends to ∞ and $x_{2n-1-i} (i \in I_n)$ are bounded, then*

$$\sum_{i\in I_n} x_i^2 \to \infty \text{ and } \sum_{i\in I_n} x_{2n-1-i}^2 \text{ is bounded.}$$

It uses $g(x) = 0$ to have

$$\sum_{i\in I_n} x_i^2 = -\sum_{i\in I_n} x_{2n-1-i}^2 - \varepsilon x_N + l,$$

and

$$\frac{\sum_{i\in I_n} x_i^2}{-x_N} \to \varepsilon > 0.$$

It follows from $h(x) = 0$ that

$$\sum_{i\in I_n} x_i x_{2n-1-i} = -\gamma x_N^3 - \frac{1}{2}\varepsilon x_N \to -\infty,$$

and

$$\frac{\sum_{i\in I_n} x_i x_{2n-1-i}}{(-x_N)^{\frac{1}{2}}} = \frac{-\gamma x_N^3}{(-x_N)^{\frac{1}{2}}} - \frac{1}{2}\varepsilon \frac{x_N}{(-x_N)^{\frac{1}{2}}} \to -\infty;$$

then, by the Cauchy inequality,

$$+\infty \leftarrow \frac{\left|\sum_{i\in I_n} x_i x_{2n-1-i}\right|}{(-x_N)^{\frac{1}{2}}} \leq \frac{\left(\sum_{i\in I_n} x_i^2\right)^{\frac{1}{2}} \left(\sum_{i\in I_n} x_{2n-1-i}^2\right)^{\frac{1}{2}}}{(-x_N)^{\frac{1}{2}}} \to \sqrt{\varepsilon}\left(\sum_{i\in I_n} x_{2n-1-i}^2\right)^{\frac{1}{2}},$$

a contradiction.

Proof of Proposition 2. Let us apply the contradiction. Assume that the rank of the Jacobian matrix for $g(x)$ and $h(x)$ is smaller than 2, then there exists $\kappa \neq 0$, such that

$$\nabla g(x) = \kappa \nabla h(x). \tag{29}$$

For $i \in I_n$, it has $g_{x_i} = 2x_i$, $g_{x_{2n-1-i}} = 2x_{2n-1-i}$, $h_{x_i} = x_{2n-1-i}$, $h_{x_{2n-1-i}} = x_i$, and by (29) that

$$\begin{cases} 2x_i = \kappa x_{2n-1-i}, \\ 2x_{2n-1-i} = \kappa x_i, \end{cases} \tag{30}$$

i.e., $\left(1 - \left(\frac{\kappa}{2}\right)^2\right) x_i = 0$; then,

$$x_i = 0 \text{ or } \kappa = 2, -2.$$

For $i = N$, it follows $g_{x_N} = \varepsilon$, $h_{x_N} = 3\gamma x_N^2 + \frac{1}{2}\varepsilon$, and by (29) that

$$3\kappa\gamma x_N^2 = -\frac{1}{2}\kappa\varepsilon + \varepsilon. \tag{31}$$

We can show that all cases above yield contradictions. Actually, when $x_i = 0$, we have $x_{2n-1-i} = 0$ from $2x_{2n-1-i} = \kappa x_i$ in (30), and so

$$0 = g(x) = \varepsilon x_N - l,$$

$$0 = h(x) = \gamma x_N^3 + \frac{1}{2}\varepsilon x_N.$$

It yields a contradiction as in Case (a_1), (b_1).

When $\kappa = 2$, we have from (30) and (31), respectively, that $x_i = x_{2n-1-i}$ and $x_N^2 = \frac{-\frac{1}{2}\kappa\varepsilon+\varepsilon}{3\kappa\gamma} = \frac{-\varepsilon+\varepsilon}{6\gamma} = 0$, i.e., $x_N = 0$; then,

$$0 = h(x) = \sum_{i \in I_n} x_i^2,$$

and

$$x_i = x_{2n-1-i} = 0,$$

so

$$0 = g(x) = -l < 0,$$

a contradiction.

When $\kappa = -2$, it yields from (30) and (31), respectively, that $x_i = -x_{2n-1-i}$ and

$$x_N^2 = \frac{\varepsilon + \varepsilon}{-6\gamma} = -\frac{\varepsilon}{3\gamma} = l^{\frac{2}{3}};$$

then,

$$x_N = \pm l^{\frac{1}{3}}. \tag{32}$$

Using

$$0 = g(x) = 2\sum_{i \in I_n} x_i^2 + \varepsilon x_N - l,$$

$$0 = 2h(x) = -2\sum_{i \in I_n} x_i^2 + 2\gamma x_N^3 + \varepsilon x_N,$$

we have

$$2\gamma x_N^3 + 2\varepsilon x_N - l = 0,$$

and obtain as in Case $(a_3), (b_1)$ that

$$x_N = \left(-\left(\sqrt{2}-1\right)^{\frac{1}{3}} - \left(\sqrt{2}+1\right)^{\frac{1}{3}}\right) l^{\frac{1}{3}}.$$

It is different from x_N in (32), a contradiction.

Proposition 2 is proved. □

4. Proof of Theorem 1 (Even Integers)

For the even integers, supposing that the Cai–Zhang–Shen conjecture is not true, then there exists an even integer $2n$ such that $2n$ can not be expressed as the sum of two figurate primes. Let us take, respectively,

$$P = (\delta(i_1), \cdots, \delta(i_l), \delta(n), \delta(2n - i_l), \cdots, \delta(2n - i_1)),$$

$$f(x) = \sum_{i \in I_n} x_i x_{2n-i} + s x_n^2,$$

$$g(x) = \sum_{i \in I_n} \left(x_i^2 + x_{2n-i}^2\right) + \varepsilon x_n - l,$$

$$h(x) = \sum_{i \in I_n} x_i x_{2n-i} + \gamma x_n^3 + \frac{1}{2}\varepsilon x_n.$$

Similarly to the proof for odd integers in Section 2, we also reach a contradiction.

5. Conclusions

In previous sections, we prove Cai–Zhang–Shen conjecture for figurate primes. The way of proving this really provides a new approach to confirm Goldbach's binary conjecture. It is worth trying, and we will further consider the well-known and difficult conjecture.

Author Contributions: Writing—original draft, P.N. and J.Z.; Writing—review & editing, P.N. All authors have read and agreed to the published version of the manuscript.

Funding: This research received no external funding.

Data Availability Statement: No data was used.

Acknowledgments: We are especially indebted to the anonymous referees for their careful reading and many useful suggestions.

Conflicts of Interest: The authors declare that there is no conflict of interest regarding the publication of this paper.

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